

Treatment of a system with explicitly broken gauge symmetries

F. A. Chishtie^{1,*}, T. Hanif², and D. G. C. McKeon^{3,4}

¹*Department of Space Science, Institute of Space Technology, Islamabad 44000, Pakistan*

²*Department of Theoretical Physics, University of Dhaka, Dhaka-1000, Bangladesh*

³*Department of Applied Mathematics, The University of Western Ontario, London, ON N6A 5B7, Canada*

⁴*Department of Mathematics and Computer Science, Algoma University, Sault Ste. Marie, ON P6A 2G4, Canada*

*E-mail: farrukh.chishtie@gmail.com; farrukh.chishtie@ist.edu.pk

Received November 6, 2013; Revised January 3, 2014; Accepted January 4, 2014; Published February 21, 2014

.....
A system in which the free part of the action possesses a gauge symmetry that is not respected by the interacting part presents problems when quantized. We illustrate how the Dirac constraint formalism can be used to address this difficulty by considering an antisymmetric tensor field interacting with a spinor field.
.....

Subject Index A13, B00, B05, B06, B30

1. Introduction

The Fadeev–Popov (FP) approach to the quantization of the massless Yang–Mills (YM) gauge field [1–4] is quite useful. It provides a way of eliminating non-physical degrees of freedom that are merely gauge artifacts while allowing for the introduction of a variety of gauge choices, both covariant and non-covariant. (In fact, it is possible to extend the FP procedure to accommodate more than one gauge-fixing condition [5,6].)

A practical problem overcome by the FP technique is the difficulty in obtaining the free-field propagator for a gauge field. Naively, the propagator for a massless vector gauge field V_μ involves inverting the operator $(\partial^2 g_{\mu\nu} - \partial_\mu \partial_\nu)$, but this is impossible because the gauge invariance $V_\mu \rightarrow V_\mu + \partial_\mu \theta$ means that this operator has a vanishing eigenvalue. In the FP approach, such bilinears are supplemented by a gauge-breaking term such as $-\frac{1}{2}(\partial \cdot V)^2$, making it possible to obtain the propagator. Of course, if the classical action also has a bilinear term that explicitly breaks gauge invariance (such as $\frac{1}{2}m^2 V_\mu V^\mu$) this problem does not arise.

However, it is possible to have a gauge invariance present in the bilinear part of the Lagrangian that is broken explicitly by the interaction. (For example, the interaction $-\lambda(V_\mu V^\mu)^2$ could occur in addition to the Maxwell action for V_μ .) In this case, the FP procedure is not directly applicable, and yet the free field propagator cannot be obtained from the bilinear part of the action as by itself it possesses a gauge invariance.

In order to address this problem, it is necessary to keep in mind that the FP procedure is equivalent to the path integral (PI) as derived from canonical quantization for YM gauge theories [7,8] but that is not always the case, as has been illustrated in Refs. [9–11]. A system that involves first- and/or second-class constraints (as introduced by Dirac [12,13]) and thereby possesses a gauge invariance

can only be quantized using the PI, if the measure of the PI is modified by the appropriate functional determinants and delta functions [7,14]. Only with these functional determinants can the PI be related to what is obtained from canonical quantization. These modifications are equivalent to having the FP measure for the PI for YM theory, but this need not always be the case.

Recalling this, we examine the problem of quantizing an antisymmetric tensor field $\phi_{\mu\nu}$ interacting with a spinor ψ through a magnetic moment interaction. We consider both the massless, gauge invariant, free field action for $\phi_{\mu\nu}$, and also supplement it with a scalar and/or pseudoscalar mass term. If these mass terms vanish, we encounter the problem mentioned above of defining the free propagator when the interaction is not gauge invariant. It is shown that this model has constraints that modify the measure of the PI so that the functional integral is well defined and there is a free field propagator for $\phi_{\mu\nu}$. (We are not considering this model to have direct physical relevance, but rather as a way of illustrating how the Dirac constraint formalism can be used to overcome a field theory problem that cannot be handled using the Fadeev–Popov procedure.) In the next section, we consider the Dirac constraint structure [12,13] of a model in which $\phi_{\mu\nu}$ interacts with ψ and has a mass m^2 and a pseudoscalar mass μ^2 in the limits of vanishing coupling and vanishing m^2 and/or μ^2 . In each of these limits the constraint structure has peculiar features, though it is only in the case $m^2 = \mu^2 = 0$, $g \neq 0$ that we illustrate the situation in which the bilinear part of the action possesses a gauge invariance that is not present in the full action.

An unresolved problem remains, however; it is not clear if the resulting PI is covariant as manifest covariance has been lost. The difficulty originally plagued both quantum electrodynamics [15–18] and YM theory [19,20] but in these theories the FP approach made it possible to retain manifest covariance. In the case of the model being examined here, it is not clear how non-trivial functional determinants arising from second-class constraints can be converted into a form that is manifestly covariant.

We use the notation outlined in the appendix.

2. A spinor–tensor model

The action

$$\mathcal{L}_\phi = \frac{1}{12} (\partial_\mu \phi_{\nu\lambda} + \partial_\nu \phi_{\lambda\mu} + \partial_\lambda \phi_{\mu\nu})^2 \equiv G_{\lambda\mu\nu}^2 \quad (1)$$

for the field $\phi_{\mu\nu} = -\phi_{\nu\mu}$ possesses the gauge invariance

$$\delta\phi_{\mu\nu} = \partial_\mu \theta_\nu - \partial_\nu \theta_\mu. \quad (2)$$

Consequently, if we write

$$\mathcal{L}_\phi = \frac{1}{2} \phi_{\alpha\beta} \left(-\frac{1}{2} \partial^2 I^{\alpha\beta,\gamma\delta} + Q^{\alpha\beta,\gamma\delta} \right) \phi_{\gamma\delta}, \quad (3)$$

where

$$I^{\alpha\beta,\gamma\delta} = \frac{1}{2} (g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma}) \quad (4a)$$

$$Q^{\alpha\beta,\gamma\delta} = \frac{1}{4} (\partial^{\alpha\gamma} g^{\beta\delta} - \partial^{\beta\gamma} g^{\alpha\delta} + \partial^{\beta\delta} g^{\alpha\gamma} - \partial^{\alpha\delta} g^{\beta\gamma}), \quad (4b)$$

we find that

$$M_0^{\alpha\beta,\gamma\delta} = -\frac{1}{2} \partial^2 I^{\alpha\beta,\gamma\delta} + Q^{\alpha\beta,\gamma\delta}; \quad (5)$$

$M_0^{\alpha\beta,\gamma\delta}\partial_\gamma = 0$ and thus $M_0^{\alpha\beta,\gamma\delta}$ has no inverse. We can supplement M_0 with

$$M_{\mu^2}^{\alpha\beta,\gamma\delta} = -\frac{\mu^2}{4}\epsilon^{\alpha\beta\gamma\delta} \quad (6a)$$

and/or

$$M_{m^2}^{\alpha\beta,\gamma\delta} = -\frac{m^2}{2}I^{\alpha\beta,\gamma\delta} \quad (6b)$$

and it is obvious that since neither of these are invariant under that transformation of Eq. (2), one can now find a free propagator for $\phi_{\mu\nu}$. For example, if $\mu^2 = 0$, $m^2 \neq 0$ then the propagator can be obtained from

$$\left[-\frac{1}{2}(\partial^2 + m^2)I^{\alpha\beta,\gamma\delta} + Q^{\alpha\beta,\gamma\delta} \right]^{-1} = -\frac{2}{\partial^2 + m^2} \left[I^{\alpha\beta,\gamma\delta} + \frac{2}{m^2}Q^{\alpha\beta,\gamma\delta} \right], \quad (7)$$

which is well defined.

However, if we simply take $\mathcal{L}_\phi$ and couple $\phi_{\mu\nu}$ to a spinor ψ so that $\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_\psi$ where

$$\mathcal{L}_\psi = \bar{\psi} \left(i\gamma \cdot \partial + g\sigma^{\mu\nu}\gamma^5\phi_{\mu\nu} \right) \psi, \quad (8)$$

then the free Lagrangian for $\phi_{\mu\nu}$ is gauge invariant while the interaction with ψ is not and the problem outlined in the preceding section occurs.

We now recall that if one employs canonical quantization for a system with first-class constraints φ_i , second-class constraints θ_i , and gauge conditions γ_i , then the transition amplitude is given by the PI:

$$\langle \text{out} | \text{in} \rangle = \int dq_i dp_i M \exp i \int_{-\infty}^{\infty} dt (\dot{q}_i p_i - H(q_i, p_i)), \quad (9)$$

where H is the canonical Hamiltonian, $q_i(t \rightarrow \pm\infty) = (q_{\text{out}}, q_{\text{in}})$, and M is the contribution to the functional measure that is a consequence of constraints being present [7,14]:

$$M = \delta(\phi_i)\delta(\theta_i)\delta(\gamma_i)\det\{\phi_i, \gamma_j\}\det^{1/2}\{\theta_i, \theta_j\}, \quad (10)$$

with $\{, \}$ denoting the Poisson bracket (PB).

For YM theory, there is a single gauge invariance and it has been shown [7,8] that for this case the measure of Eq. (9) is the same as the FP measure. However, in other cases (such as the non-Abelian extension of $\mathcal{L}_\phi$ [9], the first-order Einstein–Hilbert action in $d \geq 3$ dimensions [10,11], and supergravity in $2+1$ dimensions [21]) this equivalence does not hold.

We are thus motivated to study the constraint structure of $\mathcal{L}_\phi + \mathcal{L}_\psi$ possibly supplemented by

$$\mathcal{L}_{\mu^2} = -\frac{\mu^2}{8}\epsilon^{\mu\nu\lambda\sigma}\phi_{\mu\nu}\phi_{\lambda\sigma} \quad (11)$$

and/or

$$\mathcal{L}_{m^2} = -\frac{m^2}{4}\phi^{\mu\nu}\phi_{\mu\nu} \quad (12)$$

in order to see how a suitable transition amplitude can be defined by using the PI of Eqs. (9) and (10). Some interesting features of the Dirac constraint formalism become apparent if μ^2 and m^2 are non-zero.

We begin by defining

$$A_i = \phi_{0i}, \quad B_i = \frac{1}{2}\epsilon_{ijk}\phi_{jk} \quad (12a,b)$$

so that

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}\dot{B}_i\dot{B}_i - \epsilon_{ijk}A_i\partial_j\dot{B}_k + \frac{1}{2}A_i\left(\partial_i\partial_j - \partial^2\delta_{ij}\right)A_j - \frac{1}{2}(B_{i,i})^2 \\ & - \mu^2A_iB_i + \frac{m^2}{2}(A_i^2 - B_i^2) + i\psi^\dagger(\dot{\psi} + \alpha^i\psi_{,i}) + g\psi^\dagger(S^iA_i + i\gamma^iB_i)\psi. \end{aligned} \quad (13)$$

From Eq. (13) it is apparent that the canonical momentum associated with the fields A_i , B_i , ψ , and $\psi^\dagger$ are respectively

$$\pi_i^A = 0 \quad (14a)$$

$$\pi_i^B = \dot{B}_i - \epsilon_{ijk}\partial_jA_k \quad (14b)$$

$$\pi^\dagger = -i\psi^\dagger \quad (14c)$$

$$\pi = 0; \quad (14d)$$

Eqs. (14 a,c,d) are primary constraints. Since by Eq. (A6a)

$$\left\{\pi^\dagger + i\psi^\dagger, \pi\right\} = -i \quad (15)$$

we see that there are two primary second-class constraints,

$$\chi_1 = \pi^\dagger + i\psi^\dagger \quad (16a)$$

$$\chi_2 = \pi. \quad (16b)$$

The canonical Hamiltonian is now given by

$$\begin{aligned} \mathcal{H} = & \frac{1}{2}\pi_i^B\pi_i^B + \epsilon_{ijk}\pi_i^B\partial_jA_k + \frac{1}{2}(B_{i,i})^2 + \mu^2A_iB_i - \frac{m^2}{2}(A_i^2 - B_i^2) \\ & - i\psi^\dagger\alpha^i\psi_{,i} - g\psi^\dagger(S^iA_i + i\gamma^iB_i)\psi. \end{aligned} \quad (17)$$

In order to eliminate the two second-class constraints of Eq. (16) we define the Dirac bracket (DB):

$$\{X, Y\}^* = \{X, Y\} - i[\{X, \chi_1\}\{\chi_2, Y\} + \{X, \chi_2\}\{\chi_1, Y\}] \quad (18)$$

if X and Y are fermionic, so that

$$\left\{\psi, \psi^\dagger\right\}^* = -i. \quad (19)$$

The primary constraint of Eq. (14a) now leads to the secondary constraints

$$\Lambda_i = \epsilon_{ijk}\partial_j\pi_k^B + \mu^2B_i - m^2A_i - g\psi^\dagger S^i\psi. \quad (20)$$

Now we first consider the limit $\mu^2 = m^2 = g = 0$. In this case there are three secondary constraints,

$$\lambda_i = \epsilon_{ijk}\partial_j\pi_k^B, \quad (21)$$

but only two of them are independent as $\partial_i\lambda_i = 0$. It is easily shown that there are no tertiary (third-generation) constraints and that the constraints of Eqs. (14) and (21) are all first class. With these five first-class constraints and their five associated gauge conditions, there are ten constraints on the 12 variables in phase space ($\phi_{\mu\nu}$ and the associated momenta); we are left with $12 - 10 = 2$ physical

degrees of freedom in phase space. These correspond to having a scalar and its conjugate momentum. The gauge generator of Henneaux, Teitelboim, and Zanelli [22] is of the form

$$G = v^i \pi_i^A + \mu^i \lambda_i \quad (22)$$

and the equation

$$\dot{v}^i \pi_i^A + \dot{\mu}^i \lambda_i + \left\{ G, \int dx \left(\mathcal{H}_c + U_i \pi_i^A \right) \right\} - \delta U^i \pi_i^A = 0 \quad (23)$$

results in

$$v^i = \dot{\mu}^i. \quad (24)$$

We then find that the gauge generator G will generate the transformation of Eq. (2) with $\theta_i = \mu_i$ and $\theta_0 = 0$.

To see how $\mathcal{L}_\phi$ in Eq. (1) is, by itself, equivalent to a free massless scalar, consider

$$\mathcal{L}_{\nu\phi} = \frac{1}{8} V_\mu V^\mu + \frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} (\partial_\mu V_\nu - \partial_\nu V_\mu) \phi_{\lambda\sigma}. \quad (25)$$

For some scalar ρ , the equation of motion for V_μ , when substituted into Eq. (25), leads to Eq. (1). The equation of motion for $\phi_{\mu\nu}$ leads to $V_\mu = 2\partial_\mu \rho$, which, upon eliminating V_μ in Eq. (25), results in $\mathcal{L}_{\nu\phi} = \frac{1}{2} (\partial_\mu \rho)^2$, which is the action for a massless scalar.

If we next take $\mu^2 = g = 0$ in Eq. (20), we have the secondary constraint

$$\Lambda_i^{(m^2)} = \lambda_i - m^2 A_i, \quad (26)$$

then Eqs. (14) and (26) define a set of six second-class constraints as

$$\left\{ \pi_i^A, \Lambda_j^{(m^2)} \right\} = m^2 \delta_{ij}. \quad (27)$$

There are now six second-class constraints on $\phi_{\mu\nu}$ and its conjugate momenta, leaving $12 - 6 = 6$ physical degrees of freedom in phase space. Thus the presence of a scalar mass term increases the number of degrees of freedom in the system.

Now considering the limit $m^2 = g = 0$ in Eq. (20), we then have the secondary constraint [9]

$$\Lambda_i^{(\mu^2)} = \lambda_i + \mu^2 B_i. \quad (28)$$

As $\left\{ \Lambda_i^{(\mu^2)}, \Lambda_j^{(\mu^2)} \right\} = \left\{ \pi_i^A, \Lambda_j^{(\mu^2)} \right\} = 0$, it is necessary to check if there are any tertiary constraints; one easily finds that there is now the tertiary constraint

$$T_i = \mu^2 \pi_i^B. \quad (29)$$

We see that $\Lambda_i^{(\mu^2)}$ and T_i are second class as

$$\left\{ \Lambda_i^{(\mu^2)}, T_j \right\} = \mu^4 \delta_{ij}. \quad (30)$$

With six second-class constraints, plus the three first-class constraints π_i^A and the associated gauge conditions, there are $6 + 3 + 3 = 12$ constraints in phase space on $\phi_{\mu\nu}$ and its canonical momenta.

This leaves no net degrees of freedom for the field $\phi_{\mu\nu}$, which is consistent with the results of Refs. [23,24]. It is peculiar that adding a pseudoscalar mass term reduces the number of degrees of freedom; this is unlike having a scalar mass $m^2 \neq 0$, or the addition of a Proca mass to the vector gauge field V_μ , in which case V_μ acquires a longitudinal polarization.

Finally, if we take $\mu^2 = 0$ and $m^2 = 0$ in Eq. (20), then we have the constraint

$$L_i = -g\psi^\dagger S^i \psi. \quad (31)$$

From Eq. (19) it follows that

$$\begin{aligned} \{L_i, L_j\}^* &= g^2 \left[\psi^\dagger S^i \left\{ \psi, \psi^\dagger \right\}^* S^j \psi + (S^i \psi)^T \left\{ \psi^\dagger, \psi \right\}^* (\psi^\dagger S^j)^T \right] \\ &= 2g^2 \epsilon^{ijk} \left(\psi^\dagger \Sigma^k \psi \right). \end{aligned} \quad (32)$$

However, this does not mean that all of the constraints are second class as the number of second-class bosonic constraints must be even. If we decompose L_i into longitudinal and transverse parts,

$$L_i^L = \frac{\partial_i \partial_j}{\partial^2} L_j \quad (33a)$$

$$L_i^T = L_i - L_i^L, \quad (33b)$$

then we see that L_i^L is a pair of second-class constraints while

$$\{L_i^L, L_j^T\}^* = 0. \quad (34)$$

We now find that

$$\begin{aligned} &\left\{ L_i^L, \int dy \mathcal{H} \right\}^* \\ &= g \frac{\partial_i \partial_j}{\partial^2} \left[-\psi^\dagger \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \psi_{,j} - 2g\epsilon_{jkl} A_k \psi^\dagger \begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_l \end{pmatrix} \psi + 2gB_j \psi^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi \right] \\ &\equiv K_i^L. \end{aligned} \quad (35)$$

We see that despite Eq. (34) L_i^L is not first class; together L_i^L and K_i^L constitute a pair of second-class constraints as $\{L_i^L, K_j^L\}^* \neq 0$.

We thus see that there are 28 degrees of freedom in phase space ($\phi_{\mu\nu}$, ψ , ψ^* , and their conjugate momenta), eight primary second-class constraints (χ_1 and χ_2), three secondary second-class constraints (L_i), a tertiary second-class constraint (K_i^L), and three primary first-class constraints (π_i^A), which are accompanied by three gauge conditions (γ_i). In total, there are $8 + 3 + 1 + 3 + 3 = 18$ constraints on the 28 degrees of freedom in phase space. The $28 - 18 = 10$ physical degrees of freedom in phase space are the two polarizations of the spinor and also of its antiparticle plus a degree of freedom associated with the tensor; these are all accompanied by a conjugate momentum. A suitable gauge choice associated with the first-class constraint $\pi_i^A = 0$ is

$$A_i = 0. \quad (36)$$

This results in the Lagrangian of Eq. (13) having bilinear terms:

$$\mathcal{L}_\phi^{(2)} = \frac{1}{2} \dot{B}_i \dot{B}_i - \frac{1}{2} B_{i,i} B_{j,j} \quad (37)$$

so that one can find a propagator for the field B_i :

$$(\delta_{ij} \partial_t^2 - \partial_i \partial_j)^{-1} = \frac{1}{\partial_t^2} \left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2 - \partial_t^2} \right). \quad (38)$$

The contribution of the measure M of Eq. (9) coming from the second-class constraints (namely $\det^{1/2} \{\theta_i, \theta_j\}$) is a complicated non-covariant expression, as can be seen from Eqs. (32) and (35).

It is interesting to consider the consequences of the equations of motion when $m^2 = 0$ but with both μ^2 and g being non-zero. If

$$\mathcal{L} = \frac{1}{12} G_{\mu\nu\lambda}^2 - \frac{\mu^2}{8} \epsilon^{\mu\nu\lambda\sigma} \phi_{\mu\nu} \phi_{\lambda\sigma} + \bar{\psi} \left(i \gamma \cdot \partial + g \sigma^{\mu\nu} \gamma^5 \phi_{\mu\nu} \right) \psi \quad (39)$$

then the equation of motion for $\phi_{\mu\nu}$ that follows from Eq. (39) is (with $J^{\mu\nu} = \bar{\psi} \sigma^{\mu\nu} \gamma^5 \psi$)

$$-\frac{1}{2} \partial_\mu G^{\mu\nu\lambda} - \frac{\mu^2}{4} \epsilon^{\alpha\beta\nu\lambda} \phi_{\alpha\beta} + g J^{\nu\lambda} = 0. \quad (40)$$

Upon operating on Eq. (40) with ∂_ν , we obtain

$$\frac{\mu^2}{12} \epsilon^{\lambda\alpha\beta\gamma} G_{\alpha\beta\gamma} + g \partial_\nu J^{\nu\lambda} = 0, \quad (41)$$

which in turn implies that

$$G_{\alpha\beta\gamma} = -\frac{2g}{\mu^2} \epsilon_{\lambda\alpha\beta\gamma} \partial_\nu J^{\nu\lambda}. \quad (42)$$

If $g = 0$, then by Eqs. (40,41) $\phi_{\mu\nu} = 0$; for $g \neq 0$ these equations imply that

$$\phi_{\mu\nu} = -\frac{2g}{\mu^2} \left[\frac{1}{\mu^2} \left(\partial_{\mu\rho}^2 J_\nu^\rho - \partial_{\nu\rho}^2 J_\mu^\rho \right) - \epsilon_{\mu\nu\lambda\sigma} J^{\lambda\sigma} \right], \quad (43)$$

showing that if $m^2 = 0$, $\mu^2 \neq 0$ then the tensor field is fixed by the spinor field.

3. Discussion

An unresolved problem in quantum field theory is that of quantizing a model in which the bilinear part of the action possesses a gauge symmetry that is not present in the interaction. In this paper, we have illustrated how to address this difficulty by considering a model in which a spinor couples to an antisymmetric tensor field. Aspects of this model were considered by Deser and Witten [23] as well as in Ref. [24]. Explicit calculations using conventional quantization were shown to lead to problems in Refs. [9,27–29]. We have shown that a PI can be well defined in this model provided full use is made of the Dirac constraints occurring in this system. However, as the second-class constraints occurring are non-trivial, the PI is no longer manifestly covariant. It is not readily apparent how covariance could be present after this model is quantized. We are currently addressing this problem.

Acknowledgements

Roger Macleod had some insightful comments. T.H. would like to thank Dr Arshad Momen for helpful discussions.

Appendix

We use the Dirac matrices γ^μ where

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad (A1)$$

where σ^i is a Pauli spin matrix. These satisfy the condition

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad (\eta^{\mu\nu} = \text{diag}(+ - - -)). \quad (A2)$$

Furthermore, we employ the matrices

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \sigma^{\mu\nu} = -\frac{1}{4} [\gamma^\mu, \gamma^\nu] = \frac{i}{2} \epsilon^{\mu\nu\lambda\sigma} \sigma_{\lambda\sigma} \gamma^5. \quad (A3)$$

It also is convenient to employ

$$S^i = \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \quad \Sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \quad \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}. \quad (\text{A4})$$

We employ the left derivative for Grassmann variables θ_i :

$$\frac{d}{d\theta_i} (\theta_j \theta_k) = \delta_{ij} \theta_k - \delta_{ik} \theta_j. \quad (\text{A5})$$

Our convention for the Poisson brackets is

$$\{F_1, F_2\} = (F_{1,q} F_{2,p} + F_{2,q} F_{1,p}) - (F_{1,\psi} F_{2,\pi} + F_{2,\psi} F_{1,\pi}) \quad (\text{A6a})$$

$$\{B_1, B_2\} = (B_{1,q} B_{2,p} - B_{2,q} B_{1,p}) + (B_{1,\psi} B_{2,\pi} - B_{2,\psi} B_{1,\pi}) \quad (\text{A6b})$$

$$\{B, F\} = -\{F, B\} = (B_{,q} F_{,p} - F_{,q} B_{,p}) + (F_{,\psi} B_{,\pi} + B_{,\psi} F_{,\pi}) \quad (\text{A6c})$$

where F_i (B_i) are Grassmann odd (even) functions and we have the canonical variables (q_i, p_i) and (ψ_i, π_i) , which are bosonic and fermionic respectively.

If $L = L(q_i, \dot{q}_i, \psi_i, \dot{\psi}_i)$ then

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \pi_i = \frac{\partial L}{\partial \dot{\psi}_i} \quad (\text{A7})$$

and

$$H(q_i, p_i, \psi_i, \pi_i) = \dot{q}_i p_i + \dot{\psi}_i \pi_i - L. \quad (\text{A8})$$

Funding

Open Access funding: SCOAP³.

References

- [1] L. D. Fadeev and V. N. Popov, Phys. Lett. B **25**, 29 (1967).
- [2] B. S. DeWitt, Phys. Rev. **162**, 1195 (1967).
- [3] S. Mandelstam, Phys. Rev. **175**, 1580 (1968).
- [4] R. P. Feynman, Acta. Phys. Pol. **24**, 687 (1963).
- [5] F. T. Brandt, J. Frenkel, and D. G. C. McKeon, Phys. Rev. D **76**, 105029 (2007).
- [6] F. T. Brandt and D. G. C. McKeon, Phys. Rev. D **79**, 087702 (2009).
- [7] L. Fadeev, Theor. Math. Phys. **1**, 3 (1969).
- [8] L. D. Fadeev and A. A. Slavnov, *Gauge Fields* (Benjamin Cummings, Reading, MA, 1980).
- [9] F. Chishtie and D. G. C. McKeon, Int. J. Mod. Phys. A **27**, 1250077 (2012).
- [10] D. G. C. McKeon, Int. J. Mod. Phys. A **25**, 3453 (2010).
- [11] F. Chishtie and D. G. C. McKeon, Classical Quantum Gravity **29**, 235016 (2012).
- [12] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Dover, Mineola, NY, 2001).
- [13] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, NJ, 1992).
- [14] P. Senjanovic, Ann. Phys. **100**, 227 (1976).
- [15] E. C. G. Stueckelberg, Ann. Phys. (Berlin) **21**, 367 (1934).
- [16] R. P. Feynman, Phys. Rev. **76**, 769 (1949).
- [17] J. Schwinger, Phys. Rev. **73**, 416 (1948).
- [18] S. Tomonaga, Prog. Theor. Phys. **1**, 1 (1946).
- [19] I. Khriplovich, Sov. J. Nucl. Phys. **10**, 235 (1970).
- [20] R. N. Mohapatra, Phys. Rev. D **4**, 1007 (1971).
- [21] D. G. C. McKeon, [arXiv:1203.6046](https://arxiv.org/abs/1203.6046) [hep-th].
- [22] M. Henneaux, C. Teitelboim, and J. Zanelli, Nucl. Phys. B **332**, 169 (1990).
- [23] S. Deser and E. Witten, Nucl. Phys. B **178**, 491 (1981).

- [24] S. Kuzmin and D. G. C. McKeon, Phys. Lett. B **596**, 301 (2009).
- [25] D. G. C. McKeon, [arXiv:1209.4908](#) [hep-th].
- [26] D. G. C. McKeon, Can. J. Phys. **56**, 1195 (1978).
- [27] F. Chishtie, M. Gagne-Portelance, T. Hanif, S. Homayouni, and D. G. C. McKeon, Phys. Lett. B **632**, 445 (2006).
- [28] A. Buchel, F. Chishtie, M. Gagne-Portelance, S. Homayouni, and D. G. C. McKeon, Int. J. Mod. Phys. A **23**, 529 (2008).
- [29] A. Buchel, F. A. Chishtie, M. T. Hanif, S. Homayouni, J. Jia, and D. G. C. McKeon, Int. J. Mod. Phys. A **25**, 163 (2010).