

Duality in $\mathcal{N} = 4$ Liouville theory and moonshine phenomena

Tohru Eguchi^{1,*} and Yuji Sugawara^{2,*}

¹*Department of Physics and Research Center for Mathematical Physics, Rikkyo University, Tokyo 171-8501, Japan*

²*Department of Physical Sciences, College of Science and Engineering, Ritsumeikan University, Shiga 525-8577, Japan*

*E-mail: tohru.eguchi@gmail.com, ysugawa@se.ritsumeai.ac.jp

Received March 24, 2016; Accepted April 29, 2016; Published June 30, 2016

.....
We consider the $\mathcal{N} = 4$ Liouville theory by varying the linear dilaton coupling constant Q . It is known that, at two different values of coupling constant $Q = \sqrt{\frac{2}{N}}, -(N-1)\sqrt{\frac{2}{N}}$, the system exhibits two different small $\mathcal{N} = 4$ superconformal symmetries with central charge $c = 6$ and $c = 6(N-1)$, respectively. In the context of string theory these two theories are considered to describe the Coulomb and Higgs branches of the theory and are expected to be dual to each other. We study the Mathieu and umbral moonshine phenomena in these two theories and discuss their dual descriptions. We mainly consider the case of A_N -type modular invariants.
.....

Subject Index A13, B10, B24, B29

1. Introduction

In this paper we study the $\mathcal{N} = 4$ supersymmetric Liouville theory in order to have a deeper understanding of the Mathieu and umbral moonshine phenomena discovered recently [1–3]. It is known that, when one perturbs the large $\mathcal{N} = 4$ theory by varying the strength of the linear dilaton term Q , there are two special values of Q where the theory possesses small $\mathcal{N} = 4$ supersymmetry with the central charges $c = 6$ and $c = 6(N-1)$. These two theories have different $SU(2)_R$ symmetries, i.e., different components of $SO(4) = SU(2) \times SU(2)$ of large $\mathcal{N} = 4$ Liouville theory. In the context of string theory, these describe the Higgs and Coulomb branches of the theory and are expected to have descriptions dual to each other.

While the values of the central charges ($c = 6, 6(N-1)$) appear to differ greatly at large N , it is known that the value of the effective central charge does not vary much in Liouville theory as dilaton coupling is varied. Thus we have dual pairs of $\mathcal{N} = 4$ conformal field theories (CFTs) (cases 1 and 2) with comparable degrees of freedom. We introduce various assumptions on the elliptic genera of the case 1,2 theories and derive algebraic identities and inequalities. We can determine the shadows and elliptic genera of the case 1,2 theories that exhibit nice dual descriptions. We also discuss the relation of Ref. [4] to case 1 theories.

We work mainly with A -type modular invariants in this paper; there are some subtleties in the case of D, E -type modular invariants of umbral moonshine, which will be discussed in a future publication.

2. Preliminaries: Free-field realizations of $\mathcal{N} = 4$ superconformal systems

In this preliminary section we summarize the basic properties of the large/small $\mathcal{N} = 4$ superconformal algebras and their free-field realizations.

2.1. Large $\mathcal{N} = 4$ superconformal algebra

The large $\mathcal{N} = 4$ superconformal algebra (SCA), often denoted by “ $\mathcal{A}_\gamma$ ”, is defined as the $SO(4) = SU(2) \times SU(2)$ extension of Virasoro algebra [5] (see also Ref. [6] for a good summary). We have a stress–energy tensor T , four supercurrents G^a ($a = 0, \dots, 3$), two $SU(2)$ currents $A^{+,i}$, $A^{-,i}$ ($i = 1, \dots, 3$) whose levels are k^+ , k^- , one $U(1)$ current U , and four Majorana fermions Q^a ($a = 0, \dots, 3$). The unitarity requires that k^+ , k^- should be positive integers, and the central charge is given as

$$c = \frac{6k^+k^-}{k^+ + k^-}. \quad (2.1)$$

We set $\gamma := \frac{k^-}{k^+ + k^-}$, which parameterizes the “mixing” of two $SU(2)$ currents. The nontrivial part of the large $\mathcal{N} = 4$ algebra is written as follows:¹

$$\begin{aligned} G^a(z)G^b(w) &\sim \frac{2c}{3} \frac{\delta^{ab}}{(z-w)^3} + \frac{8i}{(z-w)^2} \left\{ \gamma \alpha_{ab}^{+,i} A^{+,i}(w) + (1-\gamma) \alpha_{ab}^{-,i} A^{-,i}(w) \right\} \\ &\quad + \frac{4i}{z-w} \left\{ \gamma \alpha_{ab}^{+,i} \partial A^{+,i}(w) + (1-\gamma) \alpha_{ab}^{-,i} \partial A^{-,i}(w) \right\} + \frac{2\delta^{ab}}{z-w} T(w), \\ A^{\pm,i}(z)A^{\pm,j}(w) &\sim \frac{\frac{k^\pm}{2}\delta^{ij}}{(z-w)^2} + \frac{i\epsilon^{ijk}}{z-w} A^{\pm,k}(w), \\ Q^a(z)Q^b(w) &\sim \frac{\frac{k^+ + k^-}{2}\delta^{ab}}{z-w}, \quad U(z)U(w) \sim \frac{\frac{k^+ + k^-}{2}}{(z-w)^2}, \\ A^{\pm,i}(z)G^a(0) &\sim \left(\gamma - \frac{1}{2} \mp \frac{1}{2} \right) \frac{2\alpha_{ab}^{\pm,i}}{(z-w)^2} Q^b(w) + \frac{i\alpha_{ab}^{\pm,i}}{(z-w)} G^b(w), \\ A^{\pm,i}(z)Q^a(w) &\sim \frac{i\alpha_{ab}^{\pm,i}}{z-w} Q^b(w), \\ Q^a(z)G^b(w) &\sim \frac{2}{z-w} \left\{ \alpha_{ab}^{+,i} A^{+,i}(w) - \alpha_{ab}^{-,i} A^{-,i}(w) \right\} + \frac{\delta^{ab}}{z-w} U(w), \\ U(z)G^b(w) &\sim \frac{1}{(z-w)^2} Q^a(w), \end{aligned} \quad (2.2)$$

where we have introduced the 4×4 matrices

$$\alpha_{ab}^{\pm,i} \equiv \frac{1}{2} (\pm \delta_{ia} \delta_{b0} \mp \delta_{ib} \delta_{a0} + \epsilon_{iab}) \quad (2.3)$$

(the third term only contributes if $a, b \neq 0$). They obey the $SO(4)$ commutation relations:

$$[\alpha^{\pm,i}, \alpha^{\pm,j}] = -\epsilon^{ijk} \alpha^{\pm,k}, \quad [\alpha^{\pm,i}, \alpha^{\mp,j}] = 0, \quad \{\alpha^{\pm,i}, \alpha^{\pm,j}\} = -\frac{1}{2} \delta^{ij}. \quad (2.4)$$

¹ We are here using a slightly different convention from that of Refs. [5,6]. Our $A^{\pm,i}$, U , Q^a correspond to $iA^{\pm,i}$, iU , and iQ^a in Refs. [5,6].

2.2. Free-field realization of large $\mathcal{N} = 4$ SCA

Let us consider a conformal system composed of a free boson ϕ , four Majorana fermions ψ^a ($a = 0, \dots, 3$), and an $SU(2)_k$ current j^i ($i = 1, 2, 3$). ϕ and ψ^a are normalized as $\phi(z)\phi(w) \sim -\ln(z-w)$, $\psi^a(z)\psi^b(w) \sim \frac{\delta^{ab}}{z-w}$, and the $SU(2)_k$ current algebra is written in our convention as

$$j^i(z)j^j(w) \sim \frac{\frac{k}{2}\delta^{ij}}{(z-w)^2} + \frac{i\epsilon^{ijk}}{z-w}j^k(w). \quad (2.5)$$

We can combine the fermionic components to the $SU(2)$ currents, and obtain the level $N \equiv k + 2$ “total currents”:

$$J^i(z) := j^i(z) - \frac{i}{2}\epsilon_{ijk}\psi^j(z)\psi^k(z). \quad (2.6)$$

We set

$$\begin{aligned} T &:= -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\psi^a\partial\psi^a + \frac{1}{N}j^ij^i, \\ G^a &:= i\psi^a\partial\phi - 2\sqrt{\frac{2}{N}}\alpha_{ab}^{+,i}j^i\psi^b - \frac{1}{6}\sqrt{\frac{2}{N}}i\epsilon_{abcd}\psi^b\psi^c\psi^d, \\ A^{+,i} &:= -\frac{i}{2}\alpha_{ab}^{+,i}\psi^a\psi^b + j^i, \quad A^{-,i} := -\frac{i}{2}\alpha_{ab}^{-,i}\psi^a\psi^b, \\ U &:= \sqrt{\frac{N}{2}}i\partial\phi, \quad Q^a := \sqrt{\frac{N}{2}}\psi^a. \end{aligned} \quad (2.7)$$

More explicitly, we can rewrite

$$G^0 = i\psi^0\partial\phi + \sqrt{\frac{2}{N}}(\psi^ij^i - i\psi^1\psi^2\psi^3), \quad (2.8)$$

which corresponds to the $\mathcal{N} = 1$ subalgebra (ψ^0 is identified as the superpartner of ϕ), and also

$$G^i = i\psi^i\partial\phi - \sqrt{\frac{2}{N}}\left(\psi^0j^i + \epsilon_{ijk}\psi^j\psi^k - \frac{i}{2}\epsilon_{ijk}\psi^j\psi^k\psi^0\right), \quad (2.9)$$

$$A^{+,i} = -\frac{i}{2}\psi^i\psi^0 - \frac{i}{4}\epsilon^{ijk}\psi^j\psi^k + j^i, \quad A^{-,i} = \frac{i}{2}\psi^i\psi^0 - \frac{i}{4}\epsilon^{ijk}\psi^j\psi^k. \quad (2.10)$$

They generate the large $\mathcal{N} = 4$ algebra with parameters

$$c = \frac{6(N-1)}{N} \left(\equiv \frac{6k^+k^-}{k^+ + k^-} \right), \quad k^+ = N-1, \quad k^- = 1, \quad \gamma = \frac{1}{N}. \quad (2.11)$$

In fact, the total central charge is calculated as

$$c = 1 + 4 \times \frac{1}{2} + \frac{3k}{k+2} = \frac{6(k+1)}{k+2} \equiv \frac{6(N-1)}{N}. \quad (2.12)$$

As is familiar, the “zero-mode subalgebra” of the large $\mathcal{N} = 4$ is the super-Lie algebra $D(2, 1; \alpha)$ with $\alpha \equiv \frac{\gamma}{1-\gamma}$ generated by $L_0, L_{\pm 1}, G_{\pm 1/2}^a, A_0^{\pm, i}, U_0, Q_{\pm 1/2}^a$ (for the Neveu–Schwarz (NS) sector).

2.3. Deformation by linear dilaton

We next consider a deformation of (2.7) by turning on the linear dilaton (background charge) along ϕ . We shall deform it as

$$\begin{aligned} T(z) &\rightarrow \tilde{T}(z) := T(z) - \frac{\mathcal{Q}}{2} \partial^2 \phi, \\ G^a(z) &\rightarrow \tilde{G}^a(z) := G^a(z) + \mathcal{Q} i \partial \psi^a(z). \end{aligned} \quad (2.13)$$

It keeps the $\mathcal{N} = 1$ superconformal symmetry generated by $\tilde{T}(z)$, $\tilde{G}^0(z)$. In other words, $\tilde{G}^a(z)$ behaves as spin-3/2 primary fields with respect to the deformed stress tensor $\tilde{T}(z)$:

$$\tilde{T}(z) \tilde{G}^a(w) \sim \frac{\frac{3}{2}}{(z-w)^2} \tilde{G}^a(w) + \frac{1}{z-w} \partial \tilde{G}^a(w). \quad (2.14)$$

The central charge is shifted as

$$\tilde{c} = c + 3\mathcal{Q}^2 \equiv 6 \left(1 - \frac{1}{N}\right) + 3\mathcal{Q}^2. \quad (2.15)$$

In the end, we obtain the modified large $\mathcal{N} = 4$ SCA generated by $\{\tilde{T}, \tilde{G}, A^{\pm,i}, Q^a, U\}$ as follows:

$$\begin{aligned} \tilde{G}^a(z) \tilde{G}^b(w) &\sim \frac{2\tilde{c}}{3} \frac{\delta^{ab}}{(z-w)^3} + \frac{8i}{(z-w)^2} \left\{ \tilde{\gamma} \alpha_{ab}^{+,i} A^{+,i}(w) + (1 - \tilde{\gamma}) \alpha_{ab}^{-,i} A^{-,i}(w) \right\} \\ &\quad + \frac{4i}{z-w} \left\{ \tilde{\gamma} \alpha_{ab}^{+,i} \partial A^{+,i}(w) + (1 - \tilde{\gamma}) \alpha_{ab}^{-,i} \partial A^{-,i}(w) \right\} + \frac{2\delta^{ab}}{z-w} \tilde{T}(w), \\ A^{\pm,i}(z) A^{\pm,j}(w) &\sim \frac{\frac{k^\pm}{2} \delta^{ij}}{(z-w)^2} + \frac{i\epsilon^{ijk}}{z-w} A^{\pm,k}(w), \\ Q^a(z) Q^b(w) &\sim \frac{\frac{k^+ + k^-}{2}}{z-w} \delta^{ab}, \quad U(z) U(w) \sim \frac{\frac{k^+ + k^-}{2}}{(z-w)^2}, \\ A^{\pm,i}(z) \tilde{G}^a(0) &\sim \left(\tilde{\gamma} - \frac{1}{2} \mp \frac{1}{2} \right) \frac{2\alpha_{ab}^{\pm,i}}{(z-w)^2} Q^b(w) + \frac{i\alpha_{ab}^{\pm,i}}{(z-w)} \tilde{G}^b(w), \\ A^{\pm,i}(z) Q^a(w) &\sim \frac{i\alpha_{ab}^{\pm,i}}{z-w} Q^b(w), \\ \tilde{T}(z) U(w) &\sim \sqrt{\frac{k+2}{2}} \frac{i\mathcal{Q}}{(z-w)^3} + \frac{1}{(z-w)^2} U(w) + \frac{1}{(z-w)} \partial U(w), \\ Q^a(z) \tilde{G}^b(w) &\sim \sqrt{\frac{k+2}{2}} \frac{i\mathcal{Q}\delta_{ab}}{(z-w)^2} + \frac{2}{z-w} \left\{ \alpha_{ab}^{+,i} A^{+,i}(w) - \alpha_{ab}^{-,i} A^{-,i}(w) \right\} + \frac{\delta^{ab}}{(z-w)} U(w), \\ U(z) \tilde{G}^b(w) &\sim \frac{1}{(z-w)^2} Q^a(w), \end{aligned} \quad (2.16)$$

where $\tilde{\gamma}$ is defined as

$$\tilde{\gamma} := \gamma - \frac{\mathcal{Q}}{\sqrt{2(k+2)}} \equiv \frac{1}{N} - \frac{\mathcal{Q}}{\sqrt{2N}}. \quad (2.17)$$

Here we point out a fact that will be crucial in our arguments: even though the central charge (2.15) depends on the background charge $\mathcal{Q}$, the *effective central charge* [7], which counts the net

degrees of freedom, is unchanged under the deformation by linear dilaton. In the relevant system, the effective central charge should be

$$c_{\text{eff}} = c \equiv 6 \left(1 - \frac{1}{N} \right), \quad (2.18)$$

irrespective of the value of Q .

To conclude this section, we note that the whole SCA (2.16) reduces to the small $\mathcal{N} = 4$ SCA, if we choose particular values of Q . In fact, inspecting the operator product expansion (OPE) of $A^{\pm,i}(z)\tilde{G}^a(0)$ given in (2.16), we find that spin-1/2 currents Q^a decouple when we set $\tilde{\gamma} = 0$ or $\tilde{\gamma} = 1$. We thus obtain the next two “small $\mathcal{N} = 4$ points”:

Case 1. $Q = \sqrt{\frac{2}{N}}$: small $\mathcal{N} = 4$ SCA of level $k^- = 1$

This case is just the familiar Callan–Harvey–Strominger (CHS) superconformal system [8]. We have $\tilde{c} = 6$ and $\tilde{\gamma} = 0$. Then, $A^{\pm,i}(z)$, $Q^a(z)$, $U(z)$ are decoupled, and $\{\tilde{T}(z), \tilde{G}^a(z), A^{-,i}(z)\}$ generate the small $\mathcal{N} = 4$ SCA of level 1.

Case 2. $Q = -(N-1)\sqrt{\frac{2}{N}}$: small $\mathcal{N} = 4$ SCA of level $k^+ = N-1$

In this case, we have $\tilde{c} = 6(N-1)$ and $\tilde{\gamma} = 1$. Then, $A^{-,i}(z)$, $Q^a(z)$, $U(z)$ are decoupled, and $\{\tilde{T}(z), \tilde{G}^a(z), A^{+,i}(z)\}$ generate the small $\mathcal{N} = 4$ SCA with level $k^+ = N-1$.

These types of reductions from the large $\mathcal{N} = 4$ to the small $\mathcal{N} = 4$ with level k^+ or k^- have already been discussed in Ref. [5], and also potentially utilized in Refs. [9,10] in order to construct the Feigin–Fuchs representation of $\mathcal{N} = 4$ SCFT. In the context of string theory on the NS5–NS1 background, case 1 is identified with the world-sheet CFT for the “short string” sector (or that describing the “Coulomb branch tube”), while case 2 corresponds to the “long string” sector (or the “Higgs branch tube”) [11]. Therefore, they are expected to be dual to each other from the viewpoints of $\text{AdS}_3/\text{CFT}_2$ duality. In our discussions later, this fact would suggest the existence of two different descriptions of the umbral moonshine [2,3] based on cases 1 and 2. Indeed, these two theories have equal effective central charges, as has already been mentioned, which implies essentially the same asymptotic growth of massive (non-Bogomol’nyi–Prasad–Sommerfield (BPS)) excitations characterizing the moonshine phenomena.

3. Elliptic genera of the $\mathcal{N} = 4$ Liouville model

3.1. Outline sketch

Our main purpose is to evaluate the elliptic genera of the $\mathcal{N} = 4$ Liouville theory with suitable Liouville potentials. However, it seems hard to directly carry out this computation because of the complexity of the $\mathcal{N} = 4$ Liouville potentials.

We shall thus take another route: we regard the relevant $\mathcal{N} = 4$ superconformal system as the $\mathbb{Z}_N$ -orbifold of

$$SU(2)_N/U(1) \otimes SL(2)_N/U(1) \cong SU(2)_N/U(1) \otimes [\mathcal{N} = 2 \text{ Liouville}]_{Q=\sqrt{\frac{2}{N}}},$$

for “case 1” ($\hat{c} = 2$) [12], and then try to deform the system into “case 2” ($\hat{c} = 2(N-1)$). The following statements are crucial in our evaluation of the elliptic genus of case 2.

- (i) Cases 1 and 2 correspond to theories with different central charges. Nevertheless, these two theories should possess equal effective central charges (2.18):

$$c_{\text{eff}} = 6 \left(1 - \frac{1}{N} \right),$$

as we have already emphasized. c_{eff} characterizes the asymptotic growth of the degeneracy of states due to the Cardy formula. In terms of the elliptic genus, since $\mathcal{Z}^{(\text{NS})}(\tau) \equiv \mathcal{Z}^{(\text{NS})}(\tau, 0) := q^{\frac{\hat{c}}{8}} \mathcal{Z}(\tau, \frac{\tau+1}{2})$ is S -invariant, (2.18) implies that the IR behavior of $\mathcal{Z}^{(\text{NS})}(\tau)$ becomes²

$$\lim_{\tau_2 \rightarrow +\infty} e^{-2\pi\tau_2 \frac{c_{\text{eff}}}{24}} \left| \mathcal{Z}^{(\text{NS})}(\tau) \right| \equiv \lim_{\tau_2 \rightarrow +\infty} e^{-2\pi\tau_2 \left(\frac{1}{4} - \frac{1}{4N} \right)} \left| \mathcal{Z}^{(\text{NS})}(\tau) \right| < \infty. \quad (3.1)$$

For case 1, we can easily confirm that this condition is satisfied. We will later discuss how we can refine the constraint (3.1) due to the $SU(2)_{N-2}$ symmetry that is part of the $\mathcal{N} = 4$ superconformal symmetry in case 2. Indeed, the resultant constraint will play a crucial role in determining the elliptic genus of case 2.

- (ii) When evaluating the elliptic genera, the contributing states in these systems are weighted by *different* $U(1)$ currents. Namely, setting

$$\psi^{\pm} := \frac{1}{\sqrt{2}} (\psi^3 \mp i\psi^0), \quad \chi^{\pm} := \frac{1}{\sqrt{2}} (\psi^1 \pm i\psi^2), \quad (3.2)$$

the $U(1)_R$ current for each case is written as follows:

◦ **case 1:**

$$J^{(\hat{c}=2)} := 2A^{-,3} = i(\psi^3\psi^0 - \psi^1\psi^2) = \psi^+\psi^- + \chi^+\chi^-. \quad (3.3)$$

◦ **case 2:**

$$J^{(\hat{c}=2(N-1))} \equiv 2A^{+,3} := i(-\psi^3\psi^0 - \psi^1\psi^2) + 2j^3 = -\psi^+\psi^- + \chi^+\chi^- + 2j^3. \quad (3.4)$$

- (iii) Elliptic genera should be invariant under generic marginal deformations, at least for the holomorphic part that is contributed from the BPS states. Moreover, the nonholomorphic part (“holomorphic anomaly”) only includes the continuous spectrum propagating in the asymptotic region where the relevant Liouville potentials are negligible. These facts imply that both the holomorphic and nonholomorphic terms of the elliptic genus of case 2 do not depend on the details of the Liouville potential.

Based on these considerations, we propose that the elliptic genus of case 2 can be uniquely determined in the following way:

- We first evaluate the elliptic genus of case 1, i.e., $\mathcal{Z}^{\text{case 1}}(\tau, z)$.
- Secondly, we deform the holomorphic anomaly term in $\mathcal{Z}^{\text{case 1}}(\tau, z)$, taking account of the distinction between the $U(1)$ currents of (3.3) and (3.4). The expected nonholomorphic term should be expanded in terms of the $\mathcal{N} = 4$ massive characters of $\hat{c} = 2(N - 1)$.

² Here, we may have a subtlety, since $\mathcal{Z}^{(\text{NS})}(\tau)$ could be nonholomorphic due to the existence of modular completions, the correction terms of which show a continuous spectrum. Thus, the Cardy-type argument in this context truly means that the asymptotic growth of the coefficients of q -expansion of the *holomorphic part* $\mathcal{Z}_{\text{hol}}^{(\text{NS})}(\tau)$, which only includes a discrete spectrum, is governed by the IR behavior of the *total* $\mathcal{Z}^{(\text{NS})}(\tau)$.

- Finally, we determine the holomorphic part of the elliptic genus desired, which is expected to be written in terms of the $\mathcal{N} = 4$ massless characters of $\hat{c} = 2(N - 1)$. It will be crucial that the possible ambiguity by adding general holomorphic Jacobi forms can be removed by examining the IR behavior of the NS-sector elliptic genus, which extends the argument of effective central charge given above.

3.2. Preliminaries

3.2.1. Branching relation for the $\mathcal{N} = 2$ minimal model

As preparation, we start with recalling the coset construction of the $\mathcal{N} = 2$ minimal model.

The $SU(2)_k$ character for the spin- $\ell/2$ -integrable representation is given as

$$\begin{aligned}\chi_\ell^{(k)}(\tau, z) &:= \frac{1}{i\theta_1(\tau, z)} [\Theta_{\ell+1, k+2}(\tau, z) - \Theta_{\ell+1, k+2}(\tau, -z)] \\ &\equiv \frac{2}{i\theta_1(\tau, z)} \Theta_{\ell+1, k+2}^{[-]}(\tau, z), \quad (\ell = 0, 1, \dots, k)\end{aligned}\quad (3.5)$$

and the string function $c_m^\ell(\tau)$ is defined by the expansion

$$\chi_\ell^{(k)}(\tau, z) = \sum_{m \in \mathbb{Z}_{2k}} c_m^\ell(\tau) \Theta_{m, k}(\tau, z). \quad (3.6)$$

The branching relation describing the $\mathcal{N} = 2$ minimal model as the supercoset

$$\frac{SU(2)_k \times SO(2)_1}{U(1)_{k+2}}$$

is written as³

$$\chi_\ell^{(k)}(\tau, w) \Theta_{s, 2}(\tau, w - z) = \sum_{m \in \mathbb{Z}_{2(k+2)}} \chi_m^{\ell, s}(\tau, z) \Theta_{m, k+2}(\tau, w - 2z/(k+2)), \quad (3.7)$$

where the branching function $\chi_m^{\ell, s}(\tau, z)$ is explicitly written as

$$\chi_m^{\ell, s}(\tau, z) = \sum_{r \in \mathbb{Z}_k} c_{m-s+4r}^\ell(\tau) \Theta_{2m+(k+2)(-s+4r), 2k(k+2)}(\tau, z/(k+2)). \quad (3.8)$$

The characters of the $\mathcal{N} = 2$ minimal model are written in terms of the branching functions as follows:

$$\begin{aligned}\text{ch}_{\ell, m}^{(\text{NS})}(\tau, z) &= \chi_m^{\ell, 0}(\tau, z) + \chi_m^{\ell, 2}(\tau, z), \\ \text{ch}_{\ell, m}^{(\widetilde{\text{NS}})}(\tau, z) &= \chi_m^{\ell, 0}(\tau, z) - \chi_m^{\ell, 2}(\tau, z), \\ \text{ch}_{\ell, m}^{(\text{R})}(\tau, z) &= \chi_m^{\ell, 1}(\tau, z) + \chi_m^{\ell, 3}(\tau, z), \\ \text{ch}_{\ell, m}^{(\widetilde{\text{R}})}(\tau, z) &= \chi_m^{\ell, 1}(\tau, z) - \chi_m^{\ell, 3}(\tau, z).\end{aligned}\quad (3.9)$$

Now, the parameter w in (3.7) is interpreted as the deformation parameter $\tilde{\gamma}$ (or $\mathcal{Q}$) in the previous section. It is explicitly identified as

$$w = 2\tilde{\gamma}z, \quad (3.10)$$

³ $s = 0, 2$ (1, 3) describe free fermions in the NS (Ramond (R)) sector.

and the corresponding $U(1)_R$ current is given as

$$J^{\tilde{\gamma}} = 2\tilde{\gamma}(A^{+,3} - A^{-,3}) + 2A^{-,3} \equiv 2\tilde{\gamma}(j^3 - \psi^+\psi^-) + (\psi^+\psi^- + \chi^+\chi^-). \quad (3.11)$$

The relevant branching relations are summarized as

◦ $\tilde{\gamma} = 0$ (**case 1**):

$$\chi_\ell^{(N-2)}(\tau, 0)\Theta_{s,2}(\tau, -z) = \sum_{m \in \mathbb{Z}_{2N}} \chi_m^{\ell,s}(\tau, z)\Theta_{m,N}\left(\tau, -\frac{2z}{N}\right), \quad (3.12)$$

◦ $\tilde{\gamma} = 1$ (**case 2**):

$$\chi_\ell^{(N-2)}(\tau, 2z)\Theta_{s,2}(\tau, z) = \sum_{m \in \mathbb{Z}_{2N}} \chi_m^{\ell,s}(\tau, z)\Theta_{m,N}\left(\tau, \frac{2(N-1)z}{N}\right). \quad (3.13)$$

3.2.2. Modular completions

Let us introduce some notations. For $N \in \mathbb{Z}_{>0}$, we set

$$f^{(N)}(\tau, z) := \sum_{n \in \mathbb{Z}} \frac{y^{2Nn} q^{Nn^2}}{1 - yq^n}, \quad (q \equiv e^{2\pi i \tau}, y \equiv e^{2\pi i z}), \quad (3.14)$$

and, for $N, K \in \mathbb{Z}_{>0}$,

$$\begin{aligned} F^{(N,K)}(v, a; \tau, z) &:= \frac{1}{N} \sum_{b \in \mathbb{Z}_N} e^{-2\pi i \frac{vb}{N}} y^{\frac{2Ka}{N}} q^{\frac{Ka^2}{N}} f^{(NK)}\left(\tau, \frac{z + a\tau + b}{N}\right) \\ &\equiv \sum_{n \in a + N\mathbb{Z}} \frac{(yq^n)^{\frac{v}{N}} y^{\frac{2Kn}{N}} q^{\frac{Kn^2}{N}}}{1 - yq^n}. \end{aligned} \quad (3.15)$$

We often use the abbreviation $F^{(N)}(v, a) \equiv F^{(N,1)}(v, a)$. Note that the extended discrete character of the $SL(2)/U(1)$ supercoset with $\hat{c} = 1 + \frac{2K}{N}$ [13,14] is written as

$$\chi_{\text{dis}}^{(N,K)}(v, a; \tau, z) = \frac{\theta_1(\tau, z)}{i\eta(\tau)^3} F^{(N,K)}(v, a; \tau, z). \quad (3.16)$$

The modular completion of $f^{(N)}(\tau, z)$ is defined as [15]:

$$\widehat{f}^{(N)}(\tau, z) := f^{(N)}(\tau, z) - \frac{1}{2} \sum_{m \in \mathbb{Z}_{2N}} R_{m,N}(\tau) \Theta_{m,N}(\tau, 2z), \quad (3.17)$$

where we set

$$\begin{aligned} R_{m,N}(\tau) &:= \sum_{v \in m + 2N\mathbb{Z}} \text{sgn}(v + 0) \text{Erfc}\left(\sqrt{\frac{\pi \tau_2}{N}} |v|\right) q^{-\frac{v^2}{4N}} \\ &\equiv \frac{1}{i\pi} \sum_{v \in m + 2N\mathbb{Z}} \int_{\mathbb{R} - i0} dp \frac{e^{-\pi \tau_2 \frac{p^2 + v^2}{N}}}{p - iv} q^{-\frac{v^2}{4N}}. \end{aligned} \quad (3.18)$$

Here, we denote $\tau_2 \equiv \text{Im } \tau$ and $\text{Erfc}(x)$ is the error function (A12). $\widehat{f}^{(N)}(\tau, z)$ is a weight 1, index N (real analytic) weak Jacobi form [15].

It is useful to rewrite (3.17) by introducing the antisymmetrization

$$f^{(N)[-]}(\tau, z) := \frac{1}{2} \left[f^{(N)}(\tau, z) - f^{(N)}(\tau, -z) \right].$$

We note that

$$\widehat{f}^{(N)[-]}(\tau, z) := \frac{1}{2} \left[\widehat{f}^{(N)}(\tau, z) - \widehat{f}^{(N)}(\tau, -z) \right] \equiv \widehat{f}^{(N)}(\tau, z),$$

since the completion $\widehat{f}^{(N)}(\tau, z)$ gives an odd function of z (see, e.g., Ref. [14]). Then, we obtain

$$\begin{aligned} \widehat{f}^{(N)}(\tau, z) &= f^{(N)[-]}(\tau, z) - \frac{1}{2} \sum_{m \in \mathbb{Z}_{2N}} R_{m,N}(\tau) \Theta_{m,N}^{[-]}(\tau, 2z) \\ &= f^{(N)[-]}(\tau, z) - \sum_{m=1}^{N-1} R_{m,N}(\tau) \Theta_{m,N}^{[-]}(\tau, 2z). \end{aligned} \quad (3.19)$$

In the second line of (3.19), we made use of the facts that $\Theta_{Nj,N}^{[-]}(\tau, z) \equiv 0$ ($\forall j \in \mathbb{Z}$) and

$$R_{-m,N}(\tau) = -R_{m,N}(\tau) + 2\delta_{m,0}^{(2N)}, \quad R_{m+2N,N}(\tau) = R_{m,N}(\tau).$$

We also introduce the modular completion of $F^{(N,K)}(v, a)$ (3.15) [14]:

$$\begin{aligned} \widehat{F}^{(N,K)}(v, a; \tau, z) &:= \frac{1}{N} \sum_{b \in \mathbb{Z}_N} e^{-2\pi i \frac{vb}{N}} y^{\frac{2Ka}{N}} q^{\frac{Ka^2}{N}} \widehat{f}^{(NK)} \left(\tau, \frac{z + a\tau + b}{N} \right) \\ &\equiv F^{(N,K)}(v, a; \tau, z) - \frac{1}{2} \sum_{j \in \mathbb{Z}_{2K}} R_{v+Nj,NK}(\tau) \Theta_{v+Nj+2Ka,NK} \left(\tau, \frac{2z}{N} \right). \end{aligned} \quad (3.20)$$

In particular, for the cases of $K = 1$, the function $\widehat{F}^{(N)}(v, a) \equiv \widehat{F}^{(N,1)}(v, a)$ becomes

$$\widehat{F}^{(N)}(v, a; \tau, z) = F^{(N)}(v, a; \tau, z) - \frac{1}{2} \sum_{j \in \mathbb{Z}_2} R_{v+Nj,N}(\tau) \Theta_{v+Nj+2a,N} \left(\tau, \frac{2z}{N} \right). \quad (3.21)$$

3.3. Case 1 models

We propose the following elliptic genus for the case 1 models with $\hat{c} = 2$:

$$\begin{aligned} \mathcal{Z}^{\text{case 1}}(\tau, z) &= \frac{1}{N} \sum_{a,b \in \mathbb{Z}_N} \mathcal{Z}_{[a,b]}^{(\min)}(\tau, z) \mathcal{Z}_{[a,b]}^{SL(2)/U(1)}(\tau, z) \\ &= -\frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \sum_{\ell=0}^{N-2} \sum_{a \in \mathbb{Z}_N} \text{ch}_{\ell, \ell+1+2a}^{(\widetilde{R})}(\tau, z) \widehat{F}^{(N)}(\ell+1, a; \tau, -z), \end{aligned} \quad (3.22)$$

where we set

$$\mathcal{Z}_{[a,b]}^{(\min)}(\tau, z) := (-1)^{a+b+ab} q^{\frac{N-2}{2N}a^2} y^{\frac{N-2}{N}a} e^{i\pi \frac{N-2}{N}ab} \mathcal{Z}^{(\min)}(\tau, z + a\tau + b), \quad (3.23)$$

$$\mathcal{Z}^{(\min)}(\tau, z) := \frac{\theta_1\left(\tau, \frac{N-1}{N}z\right)}{\theta_1\left(\tau, \frac{z}{N}\right)} \equiv \sum_{\ell=0}^{N-2} \text{ch}_{\ell, \ell+1}^{(\widetilde{R})}(\tau, z), \quad (3.24)$$

for the minimal model ($SU(2)/U(1)$ sector) [16], and

$$\mathcal{Z}_{[a,b]}^{SL(2)/U(1)}(\tau, z) := (-1)^{a+b+ab} q^{\frac{N+2}{2N}a^2} y^{\frac{N+2}{N}a} e^{i\pi \frac{N+2}{N}ab} \times \mathcal{Z}^{SL(2)/U(1)}(\tau, z + a\tau + b), \quad (3.25)$$

$$\begin{aligned} \mathcal{Z}^{SL(2)/U(1)}(\tau, z) &:= \frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \widehat{f}^{(N)}\left(\tau, \frac{z}{N}\right) \\ &\equiv \frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \sum_{v \in \mathbb{Z}_N} \widehat{F}^{(N)}(v, 0; \tau, z), \end{aligned} \quad (3.26)$$

for the $SL(2)/U(1)$ sector [14,17].

The relevant branching relation is given by (3.12), namely,

$$\begin{aligned} \sum_{m \in \mathbb{Z}_{2N}} \text{ch}_{\ell, m}^{(\widetilde{R})}(\tau, z) \Theta_{m, N} \left(\tau, -\frac{2z}{N} \right) &= -i\theta_1(\tau, z) \chi_{\ell}^{(k)}(\tau, 0) \\ &= -\frac{\theta_1(\tau, z)}{i\pi} \oint_{w=0} \frac{dw}{w} \frac{\Theta_{\ell+1, N}^{[-]}(\tau, 2w)}{\theta_1(\tau, 2w)}. \end{aligned} \quad (3.27)$$

By using this identity, we can show

$$\begin{aligned} &[\text{nonhol. part of } \mathcal{Z}^{\text{case 1}}] \\ &= \frac{1}{2} \frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \sum_{\ell=0}^{N-2} \sum_{a \in \mathbb{Z}_N} \sum_{j \in \mathbb{Z}_2} R_{\ell+1+Nj, N} \Theta_{\ell+1+Nj+2a, N} \left(-\frac{2z}{N} \right) \text{ch}_{\ell, \ell+1+2a}^{(\widetilde{R})}(z) \\ &= -\frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} \frac{1}{i\pi} \oint_{w=0} \frac{dw}{w} \frac{1}{i\theta_1(\tau, 2w)} \sum_{\ell=0}^{N-2} R_{\ell+1, N} \Theta_{\ell+1, N}^{[-]}(\tau, 2w). \end{aligned} \quad (3.28)$$

Therefore, recalling (3.19), we find that

$$\begin{aligned} &\frac{\partial}{\partial \bar{\tau}} \mathcal{Z}^{\text{case 1}}(\tau) \\ &= \frac{\partial}{\partial \bar{\tau}} \left[\frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} \frac{1}{i\pi} \oint_{w=0} \frac{dw}{w} \frac{\widehat{f}^{(N)}(\tau, w)}{i\theta_1(\tau, 2w)} \right] \\ &= \frac{\partial}{\partial \bar{\tau}} \left[\frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} \frac{1}{i\pi} \oint_{w=0} \frac{dw}{w} \frac{\widehat{f}^{(N)}(\tau, w)}{i\theta_1(\tau, 2w)} e^{(N-2)G_2(\tau)w^2} \right], \end{aligned} \quad (3.29)$$

where $G_2(\tau)$ is the (unnormalized) second Eisenstein series (A7). In order to obtain the third line of (3.29), we made use of the fact that the function

$$\frac{\partial \widehat{f}^{(N)}(\tau, w)}{\partial \bar{\tau} i\theta_1(\tau, 2w)} \equiv \frac{\partial \widehat{f}^{(N), [-]}(\tau, w)}{\partial \bar{\tau} i\theta_1(\tau, 2w)},$$

is holomorphic with respect to w . It is important that the integrand

$$g^{(N)}(\tau, w) := \frac{\widehat{f}^{(N)}(\tau, w)}{i\theta_1(\tau, 2w)} e^{(N-2)G_2(\tau)w^2} \quad (3.30)$$

possesses the correct modular property due to the factor $e^{(N-2)G_2(\tau)w^2}$ (see Eq. (A8)), i.e.,

$$g^{(N)}\left(-\frac{1}{\tau}, \frac{w}{\tau}\right) = -\sqrt{-i\tau} g^{(N)}(\tau, w). \quad (3.31)$$

Thus, we conclude that

$$\mathcal{Z}^{\text{case 1}}(\tau, z) = [\text{holomorphic Jacobi form}] + \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} \frac{1}{i\pi} \oint_{w=0} \frac{dw}{w} \frac{\widehat{f}^{(N)}(\tau, w)}{i\theta_1(\tau, 2w)} e^{(N-2)G_2(\tau)w^2}. \quad (3.32)$$

It is easy to identify the first term because of the uniqueness of the weak Jacobi form of weight 0, index 1, i.e.,

$$\phi_{0,1}(\tau, z) \equiv \frac{1}{2} \mathcal{Z}^{\text{K3}}(\tau, z) \equiv 4 \left[\left(\frac{\theta_3(\tau, z)}{\theta_3(\tau, 0)} \right)^2 + \left(\frac{\theta_4(\tau, z)}{\theta_4(\tau, 0)} \right)^2 + \left(\frac{\theta_2(\tau, z)}{\theta_2(\tau, 0)} \right)^2 \right]. \quad (3.33)$$

We also recall that the Witten index should be

$$\mathcal{Z}^{\text{case 1}}(\tau, 0) = N - 1 \quad (3.34)$$

(see, e.g., Ref. [18]), which fixes the normalization of the holomorphic term.

In this way, we finally achieve a simple formula:

$$\mathcal{Z}^{\text{case 1}}(\tau, z) = \frac{N-1}{12} \phi_{0,1}(\tau, z) + \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} \frac{1}{i\pi} \oint_{w=0} \frac{dw}{w} \frac{\widehat{f}^{(N)}(\tau, w)}{i\theta_1(\tau, 2w)} e^{(N-2)G_2(\tau)w^2}. \quad (3.35)$$

3.3.1. Useful facts on Eq. (3.35)

We exhibit some useful computations related to the resultant formula (3.35) and make a few remarks.

For convenience, we define the nonholomorphic modular form $\widehat{H}^{(N)}(\tau)$ of weight 2 by the relation

$$\mathcal{Z}^{\text{case 1}}(\tau, z) = \frac{N-1}{12} \phi_{0,1}(\tau, z) + \frac{\theta_1(\tau, z)^2}{\eta(\tau)^6} \widehat{H}^{(N)}(\tau). \quad (3.36)$$

Namely, we set

$$\widehat{H}^{(N)}(\tau) := \frac{\eta(\tau)^3}{i\pi} \oint_{w=0} \frac{dw}{w} \frac{\widehat{f}^{(N)}(\tau, w)}{i\theta_1(\tau, 2w)} e^{(N-2)G_2(\tau)w^2}. \quad (3.37)$$

Then, by substituting the ‘‘Poincaré series’’ formula [19],

$$\widehat{f}^{(N)}(\tau, z) = \frac{i}{2\pi} \sum_{\lambda \in \Lambda} \frac{e^{-\frac{\pi}{\tau_2} N \{|\lambda|^2 + 2\bar{\lambda}z + z^2\}}}{\lambda + z}, \quad (\Lambda \equiv \mathbb{Z}\tau + \mathbb{Z}), \quad (3.38)$$

into (3.37), we can rewrite it as

$$\begin{aligned} \widehat{H}^{(N)}(\tau) &= \frac{1}{4\pi^2} \left[N\widehat{G}_2(\tau) + \frac{\partial}{\partial w} \sum_{\lambda \in \Lambda'} \frac{e^{-\frac{\pi}{\tau_2} N \{|\lambda|^2 + 2\bar{\lambda}w + w^2\}}}{\lambda + w} \right]_{w=0} \\ &= \frac{1}{4\pi^2} \left[N\widehat{G}_2(\tau) - \sum_{\lambda \in \Lambda'} \frac{e^{-\frac{\pi}{\tau_2} N |\lambda|^2}}{\lambda^2} \left\{ 1 + \frac{2\pi N}{\tau_2} |\lambda|^2 \right\} \right]. \end{aligned} \quad (3.39)$$

Here, the summation is taken over $\lambda \in \Lambda' \equiv \mathbb{Z}\tau + \mathbb{Z} - \{0\}$, and $\widehat{G}_2(\tau) \equiv G_2(\tau) - \frac{\pi}{\tau_2}$ denotes the modular completion of the (unnormalized) second Eisenstein series (A9). We present an explicit derivation of (3.39) in Appendix B.

The above result (3.39) suggests that the holomorphic part $H^{(N)}(\tau)$ would be

$$H^{(N)}(\tau) \sim \frac{N}{4\pi^2} G_2(\tau) + \frac{1}{4\pi^2} \frac{\partial}{\partial w} \sum_{\lambda=m\tau+n \in \Lambda'} \left. \frac{q^{Nm^2} e^{2\pi i(2N)mw}}{\lambda + w} \right|_{w=0}. \quad (3.40)$$

However, the double series appearing in (3.40) does not converge, and thus we have to be more careful. To this end, we introduce the symbol of the “principal value”:

$$\sum_{n \neq 0}^P a_n := \lim_{N \rightarrow \infty} \sum_{n=1}^N (a_n + a_{-n}), \quad \sum_{n \in \mathbb{Z}}^P a_n := a_0 + \sum_{n \neq 0}^P a_n, \quad (3.41)$$

and the correct expression of $H^{(N)}(\tau)$ should be

$$H^{(N)}(\tau) = \frac{N}{4\pi^2} G_2(\tau) + \frac{1}{4\pi^2} \frac{\partial}{\partial w} \left[\sum_{m \neq 0} \sum_{n \in \mathbb{Z}}^P \frac{q^{Nm^2} e^{2\pi i(2N)mw}}{\lambda + w} + \sum_{n \neq 0}^P \frac{1}{w + n} \right] \Bigg|_{w=0}. \quad (3.42)$$

A rigorous derivation of (3.42) is again presented in Appendix B.

One would be interested in the q -expansion of $H^{(N)}(\tau)$. Substituting the familiar formula (A7) as well as

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \equiv \zeta(2) = \frac{\pi^2}{6}, \quad \frac{i}{2\pi} \sum_{n \in \mathbb{Z}}^P \frac{1}{z + n} = \frac{1}{2} + \frac{y}{1-y} = -\frac{1}{2} + \frac{1}{1-y},$$

into (3.42), we readily obtain

$$H^{(N)}(\tau) = \frac{N-1}{12} - 2N \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} + 2 \sum_{m=1}^{\infty} q^{Nm^2} \left[\frac{q^m}{(1-q^m)^2} + Nm \frac{1+q^m}{1-q^m} \right]. \quad (3.43)$$

Let us make the following remarks:

- The constant term $\frac{N-1}{12}$ is anticipated. It precisely cancels the “graviton term” included in the holomorphic Jacobi form $\frac{N-1}{12} \phi_{0,1}(\tau, z)$ appearing in $\mathcal{Z}^{\text{case 1}}(\tau, z)$ (3.35).

In fact, the term $\left(\frac{\theta_2(\tau, z)}{\theta_2(\tau)} \right)^2$ in $\phi_{0,1}(\tau, z)$ yields the leading contribution (“graviton term”), after making the spectral flow $z \mapsto \frac{\tau+1}{2}$. We thus obtain the evaluation

$$[\text{graviton term}] \sim -\frac{N-1}{3} \left(\frac{\theta_4(\tau, z)}{\theta_2(\tau)} \right)^2 \sim -\frac{N-1}{12} q^{-\frac{1}{4}}, \quad (\tau \rightarrow i\infty), \quad (3.44)$$

by using $q^{\frac{1}{8}} y^{\frac{1}{2}} \theta_2(\tau, z + \frac{\tau+1}{2}) = -i\theta_4(\tau, z)$. On the other hand, the first term of (3.43) yields

$$\frac{N-1}{12} \frac{\theta_3(\tau, z)^2}{\eta(\tau)^6} \sim \frac{N-1}{12} q^{-\frac{1}{4}}, \quad (\tau \rightarrow i\infty). \quad (3.45)$$

- It is easily confirmed that the function $\frac{12}{N-1} H^{(N)}(\tau)$ has q -expansion such as

$$\frac{12}{N-1} H^{(N)}(\tau) = 1 - \sum_{n=1}^{\infty} a_n q^n, \quad (3.46)$$

with integer coefficients a_n as long as $N-1$ divides 24. Moreover, since the second term in (3.43) looks more dominant than the third one, we expect that all the coefficients a_n are positive.

Amusingly, for the special case $N = 2$ of Mathieu moonshine, we have an alternative expression for $\widehat{H}^{(2)}(\tau)$ as

$$\begin{aligned}\widehat{H}^{(2)}(\tau) &= \frac{\eta(\tau)^3}{2\pi i} \oint_{w=0} \frac{dw}{w} \frac{\widehat{f}^{(1/2)}(\tau, w)}{i\theta_1(\tau, w)} \\ &= \frac{1}{4\pi^2} \left[G_2(\tau) + \sum_{\lambda=m\tau+n\in\Lambda'} (-1)^{m+n+mn} \frac{2\pi i m e^{-\frac{\pi}{2\tau_2}|\lambda|^2}}{\lambda} \right].\end{aligned}\quad (3.47)$$

The first line follows from the identity⁴

$$\widehat{f}^{(2)}(\tau, z) = \frac{\theta_1(\tau, 2z)}{2\theta_1(\tau, z)} \widehat{f}^{(1/2)}(\tau, z), \quad (3.48)$$

where we set

$$\begin{aligned}\widehat{f}^{(1/2)}(\tau, z) &:= \widehat{F}^{(2)}(1, 0; \tau, z) - \widehat{F}^{(2)}(1, 1; \tau, z) \\ &\equiv \tilde{f}^{(1/2)}(\tau, z) + [\text{nonhol. correction}],\end{aligned}\quad (3.49)$$

$$\tilde{f}^{(1/2)}(\tau, z) := \sum_{n \in \mathbb{Z}} \frac{(-1)^n (yq^n)^{\frac{1}{2}}}{1 - yq^n} y^n q^{\frac{1}{2}n^2}. \quad (3.50)$$

The second line of (3.47)⁵ is obtained by substituting another formula of the nonholomorphic Poincaré series:

$$\widehat{f}^{(1/2)}(\tau, z) = \frac{i}{2\pi} \sum_{\lambda=m\tau+n\in\Lambda} (-1)^{m+n+mn} \frac{e^{-\frac{\pi}{2\tau_2}\{|\lambda|^2+2\bar{\lambda}z+z^2\}}}{\lambda+z}. \quad (3.51)$$

The second line of (3.47) is again derived in Appendix B.

We can also rewrite the first line of (3.47) by replacing $1/w$ in the integrand with an elliptic function $\frac{1}{3}\xi(\tau, w)$ defined by

$$\xi(\tau, w) := \frac{\partial}{\partial w} \ln \left(\frac{\theta_1(\tau, w)^3}{\theta_2(\tau, w)\theta_3(\tau, w)\theta_4(\tau, w)} \right). \quad (3.52)$$

It is easy to see that $\xi(\tau, z)$ is an elliptic function of order 4 that possesses simple poles at $w = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{\tau+1}{2}$ with the residues

$$\text{Res}_{w=0}[\xi(\tau, w)] = 3, \quad \text{Res}_{w=\frac{1}{2}}[\xi(\tau, w)] = \text{Res}_{w=\frac{\tau}{2}}[\xi(\tau, w)] = \text{Res}_{w=\frac{\tau+1}{2}}[\xi(\tau, w)] = -1.$$

⁴ The holomorphic part of the identity (3.48) essentially means the familiar equivalence between the $\mathcal{N} = 4$ massless character of level 1 and the spectral flow sum of the $\mathcal{N} = 2$ massless matter characters with $\hat{c} = 2$ [20,21], which is also presented in Ref. [4]. The identity (3.48) claims that this equivalence still holds *after taking the modular completions*, and it is surely a nontrivial identity. One of its proofs is obtained by setting $N = 2$ in the identity (3.70), which we will prove. Note that the minimal character $\text{ch}_{0,m}^{(\tilde{R})}(\tau, z)$ just reduces to the mod 2 Kronecker delta: $\delta_{m,1}^{(2)} - \delta_{m,-1}^{(2)}$ in the case of $N = 2$ ($\hat{c}_{\min} = 0$).

⁵ This is equivalent to the identity that Zagier gave in his lecture at the Durham workshop, August 2015.

Then, the integrand in (3.47) becomes an elliptic function with a cubic pole $w = 0$ and simple poles $w = \frac{1}{2}, \frac{\tau}{2}, \frac{\tau+1}{2}$. We thus obtain by the contour deformation

$$\begin{aligned}\widehat{H}^{(2)}(\tau) &= \eta(\tau)^3 \operatorname{Res}_{w=0} \left[\frac{1}{3} \xi(\tau, w) \frac{\widehat{f}^{(1/2)}(\tau, w)}{i\theta_1(\tau, w)} \right] \\ &= -\frac{1}{3} \eta(\tau)^3 \sum_{w_0=\frac{1}{2}, \frac{\tau}{2}, \frac{\tau+1}{2}} \operatorname{Res}_{w_0} \left[\xi(\tau, w) \frac{\widehat{f}^{(1/2)}(\tau, w)}{i\theta_1(\tau, w)} \right] \\ &= \frac{1}{3} \eta(\tau)^3 \sum_{w_0=\frac{1}{2}, \frac{\tau}{2}, \frac{\tau+1}{2}} \frac{\widehat{f}^{(1/2)}(\tau, w_0)}{i\theta_1(\tau, w_0)}.\end{aligned}\quad (3.53)$$

The holomorphic part of the R.H.S. of (3.53) is identical to the known expression [1] of Mathieu moonshine, which is derived using the relation among $\mathcal{N} = 4$ character formulas at level 1 [20–22]:

$$\begin{aligned}\operatorname{ch}_0^{(\mathcal{R})}(k=1, \ell=0; \tau, z) &= \frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \widetilde{f}^{(1/2)}(\tau, z) \\ &= \left(\frac{\theta_i(\tau, z)}{\theta_i(\tau, 0)} \right)^2 + \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} \frac{\widetilde{f}^{(1/2)}(\tau, w_i)}{i\theta_1(\tau, w_i)}, \quad (\forall i=2, 3, 4), \\ &\quad \left(w_2 \equiv \frac{1}{2}, w_3 \equiv \frac{\tau+1}{2}, w_4 \equiv \frac{\tau}{2} \right).\end{aligned}\quad (3.54)$$

3.3.2. Comments on the shadow

As a consistency check of (3.39), let us evaluate its “shadow” [15]. After a short calculation, we obtain

$$\sqrt{\tau_2} \frac{\partial}{\partial \bar{\tau}} \widehat{H}^{(N)}(\tau) = \frac{iN}{8\pi \tau_2^{3/2}} \sum_{\lambda \in \Lambda} e^{-\frac{\pi}{\tau_2} N |\lambda|^2} \left(1 - \frac{2\pi N}{\tau_2} |\lambda|^2 \right). \quad (3.55)$$

On the other hand, due to the Poisson resummation, we find

$$\sum_{r \in \mathbb{Z}_{2N}} \Theta_{r,N}(\tau, z_L) \overline{\Theta_{r,N}(\tau, z_R)} = \sqrt{\frac{N}{\tau_2}} e^{-\frac{\pi N}{4\tau_2} (z_L - \bar{z}_R)^2} \sum_{\lambda \in \Lambda} e^{-\frac{\pi N}{\tau_2} \{ |\lambda|^2 + (\bar{\lambda} z_L - \lambda \bar{z}_R) \}}, \quad (3.56)$$

and thus

$$\sum_{r \in \mathbb{Z}_{2N}} \partial_{z_L} \Theta_{r,N}(\tau, z_L) \overline{\partial_{z_R} \Theta_{r,N}(\tau, z_R)} \Big|_{z_L=z_R=0} = \frac{\pi N^{3/2}}{2\tau_2^{3/2}} \sum_{\lambda \in \Lambda} e^{-\frac{\pi}{\tau_2} N |\lambda|^2} \left(1 - \frac{2\pi N}{\tau_2} |\lambda|^2 \right). \quad (3.57)$$

Therefore, introducing the “unary theta function” [15]

$$S_{r,N}(\tau) := \frac{1}{2\pi i} \partial_z \Theta_{r,N}(\tau, 2z) \Big|_{z=0} \equiv \sum_{n \in r+2N\mathbb{Z}} n q^{\frac{n^2}{4N}}, \quad (3.58)$$

we finally obtain

$$\sqrt{\tau_2} \frac{\partial}{\partial \bar{\tau}} \widehat{H}^{(N)}(\tau) = \frac{i}{4\sqrt{N}} \sum_{r \in \mathbb{Z}_{2N}} |S_{r,N}(\tau)|^2. \quad (3.59)$$

This is the expected result. Indeed, by using the familiar property of the “R-function” of Ref. [15],

$$\sqrt{\tau_2} \frac{\partial}{\partial \bar{\tau}} R_{m,N}(\tau) = -\frac{i}{2\sqrt{N}} \overline{S_{m,N}(\tau)}. \quad (3.60)$$

Together with (3.19) and (3.37), we can directly evaluate the shadow of $\widehat{H}^{(N)}(\tau)$ as

$$\begin{aligned} \sqrt{\tau_2} \frac{\partial}{\partial \bar{\tau}} \widehat{H}^{(N)}(\tau) &= -\frac{1}{i\pi} \oint \frac{dw}{w} \frac{\eta(\tau)^3}{i\theta_1(\tau, 2w)} \sum_{v=1}^{N-1} \sqrt{\tau_2} \frac{\partial}{\partial \bar{\tau}} R_{v,N}(\tau) \Theta_{v,N}^{[-]}(\tau, 2w) \\ &= -\frac{1}{2\pi i} \sum_{v=1}^{N-1} \frac{-i}{2\sqrt{N}} \overline{S_{v,N}(\tau)} \partial_w \Theta_{v,N}(\tau, 2w) \Big|_{w=0} \\ &= \frac{i}{4\sqrt{N}} \sum_{r \in \mathbb{Z}_{2N}} |S_{r,N}(\tau)|^2, \end{aligned} \quad (3.61)$$

which coincides with (3.59).

It may also be useful to evaluate the shadow of $\widehat{f}^{(N)}(\tau, z)$ based on the formula of the nonholomorphic Poincaré series (3.38):

$$\widehat{f}^{(N)}(\tau, z) = \frac{i}{2\pi} \sum_{\lambda \in \Lambda} \frac{\rho^{(N)}(\lambda, z)}{\lambda + z} \equiv \frac{i}{2\pi} \sum_{\lambda \in \Lambda} \frac{e^{-\frac{\pi N}{\tau_2} \{|\lambda|^2 + 2\bar{\lambda}z + z^2\}}}{\lambda + z},$$

where we have introduced the notation

$$\begin{aligned} \rho^{(k)}(\lambda, z) &:= e^{-\frac{\pi k}{\tau_2} \{|\lambda|^2 + 2\bar{\lambda}z + z^2\}} \\ &\equiv q^{km^2} y^{2km} e^{2\pi i k m n} e^{-\frac{\pi k}{\tau_2} z^2}. \end{aligned} \quad (3.62)$$

Using

$$\frac{\partial}{\partial \bar{\tau}} \rho^{(N)}(\lambda, z) = \frac{i\pi N}{2\tau_2^2} (\lambda + z)^2 \rho^{(N)}(\lambda, z),$$

we easily obtain

$$\sqrt{\tau_2} \frac{\partial}{\partial \bar{\tau}} \widehat{f}^{(N)}(\tau, z) = -\frac{N}{4\tau_2^{3/2}} \sum_{\lambda \in \Lambda} (\lambda + z) \rho^{(N)}(\lambda, z). \quad (3.63)$$

However, by differentiating with respect to $\bar{z}_R$ on both sides of (3.56), we find

$$\sum_{r \in \mathbb{Z}_{2N}} \Theta_{r,N}(\tau, 2z) \overline{S_{r,N}(\tau)} = \frac{iN^{3/2}}{\tau_2^{3/2}} \sum_{\lambda \in \Lambda} (\lambda + z) \rho^{(N)}(\lambda, z). \quad (3.64)$$

We thus obtain

$$\sqrt{\tau_2} \frac{\partial}{\partial \bar{\tau}} \widehat{f}^{(N)}(\tau, z) = \frac{i}{4\sqrt{N}} \sum_{r \in \mathbb{Z}_{2N}} \Theta_{r,N}(\tau, 2z) \overline{S_{r,N}(\tau)}, \quad (3.65)$$

which is consistent with Eqs. (3.17) and (3.60).

3.4. Elliptic genus of case 2 theory

Now, let us begin our analysis of the system of “case 2”, i.e., the $\mathcal{N} = 4$ theory with $\hat{c} = 2(N - 1)$. According to our strategy addressed in Sect. 3.1, we propose the elliptic genus of case 2 $\mathcal{Z}^{\text{case 2}}(\tau, z)$ by modifying the coupling of the variable z in $\mathcal{Z}^{\text{case 1}}(\tau, z)$. The relevant branching relation is now (3.13), in other words,

$$2 \frac{\theta_1(\tau, z)}{\theta_1(\tau, 2z)} \Theta_{\ell+1, N}^{[-1]}(\tau, 2z) = \sum_{m \in \mathbb{Z}_{2N}} \text{ch}_{\ell, m}^{(\tilde{R})}(\tau, z) \Theta_{m, N} \left(\tau, \frac{2(N-1)}{N} z \right). \quad (3.66)$$

This identity can be interpreted as the $\frac{SU(2)}{U(1)} \times \frac{SL(2)}{U(1)}$ decomposition of the $\mathcal{N} = 4$ massive characters with isospin $\frac{j}{2} \equiv \frac{\ell+1}{2}$, conformal weight $h \equiv \frac{p^2}{2} + \frac{\ell(\ell+2)}{4N} + \frac{(N-1)^2}{4} + \frac{1}{4}$:

$$\begin{aligned} \text{ch}^{(\tilde{R})}(N-1, h, j; \tau, z) &\equiv (-1)^{N-\ell} q^{\frac{p^2}{2}} \frac{2\theta_1(\tau, z)^2}{i\eta(\tau)^3 \theta_1(\tau, 2z)} \Theta_{\ell+1, N}^{[-1]}(\tau, 2z) \\ &= (-1)^{N-\ell} q^{\frac{p^2}{2}} \frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \sum_{m \in \mathbb{Z}_{2N}} \text{ch}_{\ell, m}^{(\tilde{R})}(\tau, z) \Theta_{m, N} \left(\tau, \frac{2(N-1)}{N} z \right). \end{aligned} \quad (3.67)$$

This formula suggests the following elliptic genus of the case 2 theory,

$$\mathcal{Z}^{\text{case 2}}(\tau, z) = \frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \sum_{\ell=0}^{N-2} \sum_{a \in \mathbb{Z}_N} \text{ch}_{\ell, \ell+1+2a}^{(\tilde{R})}(\tau, z) \widehat{F}^{(N)}(\ell+1, a; \tau, (N-1)z), \quad (3.68)$$

up to an overall phase factor. This is obtained by the replacement $-z \mapsto (N-1)z$ in the function $\widehat{F}^{(N)}(*, *)$ appearing in $\mathcal{Z}^{\text{case 1}}$ (3.22), which properly corrects the difference between the $U(1)_R$ charges of cases 1 and 2, and reproduces the expected holomorphic anomaly terms expanded by the $\mathcal{N} = 4$ massive characters. (Recall the branching relations (3.12) and (3.13).)

3.4.1. Proof of an identity

We recall that the function $\widehat{f}^{(N)}(\tau, z)$ is simply related to the $\mathcal{N} = 4$ massless character

$$\widehat{\text{ch}}_0^{(\tilde{R})}(N-1, 0; \tau, z) \equiv (-1)^{N-1} \frac{2\theta_1(\tau, z)^2}{i\eta(\tau)^3 \theta_1(\tau, 2z)} \widehat{f}^{(N)}(\tau, z). \quad (3.69)$$

Here, $\widehat{\text{ch}}_0^{(\tilde{R})}(N-1, 0; \tau, z)$ denotes the modular completion of the $\mathcal{N} = 4$ massless character of level $N-1$, isospin 0 in the $\tilde{R}$ sector. This is actually the unique modular completion of $\mathcal{N} = 4$ massless characters since they are independent of the value of isospin ℓ , as was discussed in Ref. [19]. We shall now prove the following important identity:

$$2 \frac{\theta_1(\tau, z)}{\theta_1(\tau, 2z)} \widehat{f}^{(N)}(\tau, z) = \sum_{\ell=0}^{N-2} \sum_{a \in \mathbb{Z}_N} \text{ch}_{\ell, \ell+1+2a}^{(\tilde{R})}(\tau, z) \widehat{F}^{(N)}(\ell+1, a; \tau, (N-1)z). \quad (3.70)$$

This may be interpreted as a massless counterpart of (3.66), and it implies that (3.68) is written in terms of the modular completion $\widehat{\text{ch}}_0^{(\tilde{R})}(N-1, 0; \tau, z)$.

Proof of (3.70). We set

$$G^{(\tilde{R})}(\tau, z) := 2 \frac{\theta_1(\tau, z)^2}{i\eta(\tau)^3 \theta_1(\tau, 2z)} \widehat{f}^{(N)}(\tau, z) - \frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \sum_{\ell=0}^{N-2} \sum_{a \in \mathbb{Z}_N} \text{ch}_{\ell, \ell+1+2a}^{(\tilde{R})}(\tau, z) \widehat{F}^{(N)}(\ell+1, a; \tau, (N-1)z), \quad (3.71)$$

and prove $G^{(\tilde{R})}(\tau, z) \equiv 0$.

First of all, it is obvious by definition that $G^{(\tilde{R})}(\tau, z)$ possesses the correct modular and spectral flow properties as a weak Jacobi form of weight 0, index $N-1$.

We next discuss more nontrivial properties of the function $G^{(\tilde{R})}(\tau, z)$:

(i) **Holomorphicity with respect to τ :**

By using (3.21), (3.60), and (3.66), we obtain

$$\begin{aligned} & \sqrt{\tau_2} \frac{\partial}{\partial \bar{\tau}} \left[\sum_{\ell=0}^{N-2} \sum_{a \in \mathbb{Z}_N} \text{ch}_{\ell, \ell+1+2a}^{(\tilde{R})}(\tau, z) \widehat{F}^{(N)}(\ell+1, a; \tau, (N-1)z) \right] \\ &= \frac{i}{4\sqrt{N}} \sum_{\ell=0}^{N-2} \sum_{a \in \mathbb{Z}_N} \sum_{j \in \mathbb{Z}_2} \text{ch}_{\ell, \ell+1+2a}^{(\tilde{R})}(\tau, z) \overline{S_{\ell+1+Nj, N}(\tau)} \Theta_{\ell+1+Nj+2a, N} \left(\tau, \frac{2(N-1)}{N} z \right) \\ &= \frac{i}{4\sqrt{N}} \frac{2\theta_1(\tau, z)}{\theta_1(\tau, 2z)} \sum_{\ell=0}^{N-2} \left[\overline{S_{\ell+1, N}(\tau)} \Theta_{\ell+1, N}^{[-1]}(\tau, 2z) + \overline{S_{N-\ell+1, N}(\tau)} \Theta_{N-\ell+1, N}^{[-1]}(\tau, 2z) \right] \\ &= \frac{i}{4\sqrt{N}} \frac{2\theta_1(\tau, z)}{\theta_1(\tau, 2z)} \sum_{r \in \mathbb{Z}_{2N}} \overline{S_{r, N}(\tau)} \Theta_{r, N}(\tau, 2z). \end{aligned} \quad (3.72)$$

In the third line, we have made use of the identities $\text{ch}_{\ell, m}^{(\tilde{R})}(\tau, z) = -\text{ch}_{N-2-\ell, m+N}^{(\tilde{R})}(\tau, z)$ and $S_{r+N, N}(\tau) = -S_{-r, N}(\tau)$. Combining (3.72) with (3.65), we conclude that

$$\frac{\partial}{\partial \bar{\tau}} G^{(\tilde{R})}(\tau, z) = 0. \quad (3.73)$$

(ii) **Holomorphicity with respect to z :**

We next confirm the holomorphicity with respect to z ; in other words, the absence of singularities in the z -variable. To this aim it would be useful to rewrite the function $G^{(\tilde{R})}(\tau, z)$ in the form of a nonholomorphic Poincaré series (3.38):

$$G^{(\tilde{R})}(\tau, z) = \frac{\theta_1(z)^2}{2\pi \eta^3 \theta_1(2z)} \sum_{\lambda \in \Lambda} \frac{\rho^{(N)}(\lambda, z)}{z + \lambda} - \frac{\theta_1(z)}{2\pi \eta^3} \sum_{\lambda \equiv m\tau + n \in \Lambda} (-1)^{m+n} y^{\frac{N-2}{N}m} q^{\frac{N-2}{2N}m^2} e^{-2\pi i \frac{mn}{N}} \frac{\theta_1\left(\frac{N-1}{N}(z + \lambda)\right)}{\theta_1\left(\frac{1}{N}(z + \lambda)\right)} \frac{\rho^{(\frac{1}{N})}(-\lambda, (N-1)z)}{(N-1)z - \lambda}, \quad (3.74)$$

where $\rho^{(k)}(\lambda, z)$ has been defined by (3.62). The first term is obviously holomorphic.

On the other hand, the potential singularities of the second term emerge at the points

$$z = \frac{1}{N-1} \lambda, \quad (\forall \lambda \in \Lambda). \quad (3.75)$$

They are, however, canceled by the simple zeros coming from the factor $\theta_1\left(\frac{N-1}{N}(z+\lambda)\right)$ since

$$\frac{N-1}{N}\left(\frac{\lambda}{N-1}+\lambda\right)=\frac{N-1}{N}\frac{N\lambda}{N-1}=\lambda\in\Lambda.$$

Also, the factor $\theta_1\left(\frac{1}{N}(z+\lambda)\right)$ gives rise to simple poles

$$z=-\lambda+N\nu, \quad (\forall \nu\in\Lambda), \quad (3.76)$$

which are canceled by the remaining $\theta_1(z)$. Therefore we have confirmed the holomorphicity of $G^{(\tilde{R})}(\tau, z)$.

(iii) **IR behavior of $G^{(\text{NS})}(\tau, z)$ around $\tau \sim i\infty$:**

We next examine the IR behavior of $G^{(\text{NS})}(\tau, z)$, which is defined by the half-spectral flow as

$$G^{(\text{NS})}(\tau, z) := q^{\frac{N-1}{4}} y^{N-1} G^{(\tilde{R})}\left(\tau, z + \frac{\tau+1}{2}\right). \quad (3.77)$$

The first term of (3.71) is essentially equal to $\widehat{\text{ch}}_0^{(\tilde{R})}(N-1, \ell=0; \tau, z)$, which is converted into $\widehat{\text{ch}}_0^{(\text{NS})}(N-1, \ell=N-1; \tau, z)$ by the half-spectral flow, yielding the IR behavior $\sim q^{\frac{N-1}{4}}$.

On the other hand, the second term of (3.71) yields (up to phases)

$$\sim \frac{\theta_3(z)}{\eta^3} \sum_{\ell=0}^{N-2} \sum_{a\in\mathbb{Z}_N} \text{ch}_{\ell, \ell+2a}^{(\text{NS})}(z) q^{\frac{(N-1)^2}{4N}} y^{\frac{(N-1)^2}{N}} \widehat{F}^{(N)}\left(\ell+1, a; \tau, (N-1)\left(z + \frac{\tau+1}{2}\right)\right).$$

The leading contribution obviously comes from the term of $\ell=0, a=0$, which gives the IR behavior

$$\sim q^{-\frac{1}{8}-\frac{N-2}{8N}+\frac{(N-1)^2}{4N}+\frac{N-1}{2N}} = q^{\frac{N-1}{4}}.$$

In this way, we find

$$G^{(\text{NS})}(\tau, z) \sim \mathcal{O}(q^{\frac{N-1}{4}}), \quad (\tau \sim i\infty). \quad (3.78)$$

In summary, we have shown that $G^{(\tilde{R})}(\tau, z)$ should be a holomorphic weak Jacobi form with weight 0, index $N-1$, and that it satisfies (3.78). This is enough to conclude that $G^{(\tilde{R})}(\tau, z) \equiv 0$ because of the lemma given in Appendix C. $\square$

3.4.2. Considerations on the effective central charge

The above result for the elliptic genus of case 2 $\mathcal{N}=4$ Liouville theory is summarized as

$$\begin{aligned} \mathcal{Z}^{\text{case 2}}(\tau, z) &= \frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \sum_{\ell=0}^{N-2} \sum_{a\in\mathbb{Z}_N} \text{ch}_{\ell, \ell+1+2a}^{(\tilde{R})}(\tau, z) \widehat{F}^{(N)}(\ell+1, a; \tau, (N-1)z) \\ &= (-1)^{N-1} \widehat{\text{ch}}_0^{(\tilde{R})}(N-1, 0; \tau, z). \end{aligned} \quad (3.79)$$

Namely, we claim that *the elliptic genus of case 2 should be equal to the modular completion of the $\mathcal{N}=4$ massless character $\widehat{\text{ch}}_0^{(\tilde{R})}(N-1, 0; \tau, z)$ itself*. This has been suggested from the consideration of the holomorphic anomaly (or the shadow) given in Sect. 3.1. It is similar to the case of the elliptic

genus of $\mathcal{N} = 2$ Liouville theory ($\cong SL(2)/U(1)$ supercoset), which is given only in terms of modular completions of $\mathcal{N} = 2$ massless characters [14,17].

However, one might still ask: The holomorphic anomaly remains unchanged even if we add any holomorphic Jacobi form with weight 0, index $N - 1$ to (3.79). How can we reject this possibility? In the simplest case of $N = 2$, the absence of the holomorphic Jacobi form ($\propto \phi_{0,1}(\tau, z)$) just means the familiar fact of decoupling gravity in the noncompact models (see, e.g., Ref. [18]). However, for the cases with $N > 2$, we have a number of holomorphic Jacobi forms, and the situation becomes much more nontrivial.

To answer this question and confirm the validity of the above result (3.79), let us present a consideration about the effective central charge mentioned before. We start with refining the c_{eff} condition (3.1) based on the affine $SU(2)$ symmetry as the underlying structure of $\mathcal{N} = 4$ SCA, as we promised. To show this, we expand the elliptic genus in the NS sector in terms of the angular variable $y \equiv e^{2\pi iz}$ as

$$\mathcal{Z}^{(\text{NS})}(\tau, z) \equiv q^{\frac{N-1}{4}} y^{N-1} \mathcal{Z}\left(\tau, z + \frac{\tau+1}{2}\right) = \frac{1}{2} \sum_{s \in \mathbb{Z}_{\geq 0}} (y^s + y^{-s}) \mathcal{Z}_s^{(\text{NS})}(\tau). \quad (3.80)$$

Then, we obtain the constraint

$$\lim_{\tau_2 \rightarrow +\infty} e^{-2\pi\tau_2 \left\{ \frac{c_{\text{eff}}}{24} - \frac{\ell(\ell+2)}{4N} \right\}} \left| \mathcal{Z}_s^{(\text{NS})}(\tau) \right| \equiv \lim_{\tau_2 \rightarrow +\infty} e^{-2\pi\tau_2 \left\{ \frac{1}{4} - \frac{(\ell+1)^2}{4N} \right\}} \left| \mathcal{Z}_s^{(\text{NS})}(\tau) \right| < \infty, \quad (3.81)$$

where $\ell = 0, 1, \dots, N-1$ is defined by

$$\ell := \begin{cases} |s|, & (0 \leq |s| \leq N-2) \\ N-1, & (|s| \geq N-1) \end{cases}. \quad (3.82)$$

In fact,

- $|s| = \ell \leq N-2$:

The leading term contributing to $\mathcal{Z}_s^{(\text{NS})}(\tau)$ is composed of the highest-weight state of spin $\ell/2$ of the bosonic $SU(2)$ current j^a , and the NS vacuum of fermions. Thus, the condition (3.81) is obviously satisfied.

- $|s| \geq N-1$:

The leading term contributing to $\mathcal{Z}_s^{(\text{NS})}(\tau)$ is composed of the highest-weight state of spin $\frac{N-2}{2}$ of j^a , and the level $\geq \frac{1}{2}$ state of fermions. We thus obtain the IR evaluation as

$$\mathcal{Z}_s^{(\text{NS})}(\tau) \sim q^\alpha, \quad \alpha \geq \frac{(N-2)N}{4N} + \frac{1}{2} - \frac{c_{\text{eff}}}{24} \geq \frac{N^2-1}{4N} - \frac{c_{\text{eff}}}{24}.$$

Therefore, the condition (3.81) is still satisfied.

Now, let us expand the holomorphic part of the elliptic genus in the NS sector in terms of $\mathcal{N} = 4$ characters:

$$\begin{aligned} \mathcal{Z}_{\text{hol}}^{(\text{NS})}(\tau, z) &= \sum_{\ell=0}^{N-1} a_\ell \text{ch}_0^{(\text{NS})}(N-1, \ell; \tau, z) \\ &\quad + \sum_{j=0}^{N-2} \sum_{n \in \mathbb{Z}_{\geq 0}} b_{j,n} \text{ch}^{(\text{NS})}\left(N-1, h = n + \frac{N-2}{4}, j; \tau, z\right). \end{aligned} \quad (3.83)$$

Here, the holomorphic part $\mathcal{Z}_{\text{hol}}^{(\text{NS})}(\tau, z)$ has been defined so that it shows the same IR behavior as $\mathcal{Z}^{(\text{NS})}(\tau, z)$ for each term of the y -expansions.⁶

It is easy to confirm that the constraint (3.81) requires that the conformal weight h of the massive representations should satisfy the inequality

$$h \geq h(j) \equiv \frac{j(j+2)}{4N} + \frac{(N-1)^2}{4N}. \quad (3.84)$$

It is notable that the second term $\frac{(N-1)^2}{4N}$ is equal to the mass gap $\frac{Q^2}{8}$ due to the linear dilaton.

On the other hand, the IR behavior of the NS massless character of isospin $\ell/2$ is evaluated as ($\ell = 0, \dots, N-1$):

$$\text{ch}_0^{(\text{NS})}(N-1, \ell; \tau, z) \sim q^{\frac{\ell}{2} - \frac{N-1}{4}}, \quad (\tau \sim i\infty). \quad (3.85)$$

Thus, the constraint (3.81) implies

$$\begin{aligned} \frac{\ell}{2} - \frac{N-1}{4} - \frac{(\ell+1)^2}{4N} + \frac{1}{4} &\geq 0, \\ \iff (\ell+1-N)^2 &\leq 0. \end{aligned} \quad (3.86)$$

That is, only the maximal spin massless representation $\ell = N-1$ is allowed.

Consequently, (3.81) implies that

$$\begin{aligned} \mathcal{Z}_{\text{hol}}^{\text{case 2 (NS)}}(\tau, z) &= a_{N-1} \text{ch}_0^{(\text{NS})}(N-1, \ell = N-1; \tau, z) \\ &+ \sum_{j=0}^{N-2} \sum_{n \in \mathbb{Z}_{\geq 0}, h \geq h(j)} b_{j,n} \text{ch}^{(\text{NS})}\left(N-1, h = n + \frac{N-2}{4}, j; \tau, z\right), \end{aligned} \quad (3.87)$$

or, equivalently,

$$\begin{aligned} \mathcal{Z}_{\text{hol}}^{\text{case 2 (R)}}(\tau, z) &= a'_0 \text{ch}_0^{(\text{R})}(N-1, \ell = 0; \tau, z) \\ &+ \sum_{j=1}^{N-1} \sum_{n \in \mathbb{Z}_{\geq 0}, h \geq h(j-1) + \frac{1}{4}} b'_{j,n} \text{ch}^{(\text{R})}\left(N-1, h = n + \frac{N-1}{4}, j; \tau, z\right). \end{aligned} \quad (3.88)$$

The above result (3.79) is indeed consistent with this character expansion (3.88) (with vanishing coefficients $b'_{j,n}$).

On the other hand, as shown in Ref. [2], any holomorphic Jacobi form can never be written in the form (3.88); i.e., we need additional contributions from the massless characters with $\ell \geq 1$, or the massive characters with conformal weight “below the mass gap” $h < h(j-1) + \frac{1}{4}$ in order to construct a holomorphic Jacobi form. In this way, we conclude that there is no room for adding extra holomorphic Jacobi forms to (3.79).

⁶ In fact, the coefficients of massless characters a_ℓ can be uniquely determined from $\mathcal{Z}^{(\text{NS})}(\tau, z)$ by this assumption, while the massive coefficients $b_{j,n}$ would not be necessarily unique. However, this ambiguity does not affect the following discussions.

3.4.3. Some remarks

We add a few remarks on the analyses in this subsection:

- (i) It should be emphasized that case 2 is *not* equivalent to

$$SU(2)_N/U(1) \otimes [\mathcal{N} = 2 \text{ Liouville}]_{\mathcal{Q} = -(N-1)\sqrt{\frac{2}{N}}} \Big|_{\mathbb{Z}_N\text{-orbifold}}, \quad (3.89)$$

which is a superconformal system of the type studied in Ref. [23]. The same value of linear dilaton $\mathcal{Q} = -(N-1)\sqrt{\frac{2}{N}}$ is expected for our case 2, but the $\mathcal{N} = 2$ Liouville potential does not preserve the $\mathcal{N} = 4$ superconformal symmetry except for the special case $N = 2$ ($\hat{c} = 2$). In fact, we have

$$\widehat{\chi}_{\text{dis}}^{(N, (N-1)^2)}(v, a; \tau, z) \equiv \frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \widehat{F}^{(N, (N-1)^2)}(v, a; \tau, z) \quad (3.90)$$

as suitable building blocks for the $\mathcal{N} = 2$ Liouville theory with $\mathcal{Q} = -(N-1)\sqrt{\frac{2}{N}}$, which is equivalent to the $SL(2)_{k=\frac{N}{(N-1)^2}}/U(1)$ supercoset. The elliptic genus of (3.89) is obtained as

$$\mathcal{Z}(\tau, z) = -\frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \sum_{\ell=0}^{N-2} \sum_{a \in \mathbb{Z}_N} \text{ch}_{\ell, \ell+1+2a}^{(\widetilde{\mathbf{R}})}(\tau, z) \widehat{F}^{(N, (N-1)^2)}(\ell+1, a; \tau, -z), \quad (3.91)$$

which is similar to (3.22). On the other hand, (3.79) can be rewritten as

$$\begin{aligned} \mathcal{Z}^{\text{case 2}}(\tau, z) &= -\frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \sum_{\ell=0}^{N-2} \sum_{a \in \mathbb{Z}_N} \text{ch}_{\ell, \ell+1+2a}^{(\widetilde{\mathbf{R}})}(\tau, z) \\ &\quad \times \frac{1}{N-1} \sum_{\lambda \in \Lambda/(N-1)\Lambda} s_{\lambda}^{(\frac{(N-1)^2}{N})} \cdot \widehat{F}^{(N, (N-1)^2)}(\ell+1, a; \tau, -z), \end{aligned} \quad (3.92)$$

where $s_{\lambda}^{(\kappa)}$ denotes the spectral flow operator defined by (A10). Stated physically, the $\mathbb{Z}_{N-1}$ orbifolding (or the Eichler–Zagier operator [24])

$$\mathcal{W}(N-1) \equiv \frac{1}{N-1} \sum_{\lambda \in \Lambda/(N-1)\Lambda} s_{\lambda}^{(*)}$$

reduces the radius of the asymptotic cylindrical region of the $\mathcal{N} = 2$ Liouville theory so as to be compatible with the $\mathcal{N} = 4$ superconformal symmetry.

- (ii) An important consequence of the “ c_{eff} condition” (3.81), which physically means the normalizability of a spectrum, is the *inevitable* emergence of the holomorphic anomaly in $\mathcal{Z}^{\text{case 2}}(\tau, z)$ as long as we require its good modular property.

It is also worthwhile to note the fact that the formulas of modular S -transformations of any $\mathcal{N} = 4$ superconformal characters in the $\widetilde{\mathbf{R}}$ sector are schematically written as

$$[\mathcal{N} = 4 \text{ character}]|_S = A \text{ch}_0^{(\widetilde{\mathbf{R}})}(\ell = 0) + \sum_j \int_{h \geq h(j)} [\mathcal{N} = 4 \text{ massive character}], \quad (3.93)$$

where the coefficient A on the R.H.S. does not vanish when the character of the L.H.S. is any massless character. Namely, we find the following facts for the representations on the R.H.S. of (3.93):

- Only the $\ell = 0$ massless character can appear.
- Only massive characters with conformal weight satisfying $h \geq h(j)$ (“above the mass gap”) appear.

These representations are precisely identical to those satisfying the condition (3.81). This feature is indeed anticipated, since the modular invariant partition function (and thus the elliptic genus) should only include the normalizable spectrum that contributes to the net degrees of freedom.

4. “Duality” in $\mathcal{N} = 4$ Liouville theory and umbral moonshine

In the previous sections we studied two systems possessing $\mathcal{N} = 4$ superconformal symmetry:

case 1: ($\hat{c} = 2$)

The elliptic genus is given as (3.22):

$$\mathcal{Z}^{\text{case 1}}(\tau, z) = -\frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \sum_{r=1}^{N-1} \sum_{a \in \mathbb{Z}_N} \text{ch}_{r-1, r+2a}^{(\tilde{\mathbf{R}})}(\tau, z) \widehat{F}^{(N)}(r, a; \tau, -z),$$

case 2: ($\hat{c} = 2(N-1)$)

The elliptic genus is given as (3.79):

$$\begin{aligned} \mathcal{Z}^{\text{case 2}}(\tau, z) &= \frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \sum_{r=1}^{N-1} \sum_{a \in \mathbb{Z}_N} \text{ch}_{r-1, r+2a}^{(\tilde{\mathbf{R}})}(\tau, z) \widehat{F}^{(N)}(r, a; \tau, (N-1)z) \\ &= (-1)^{N-1} \widehat{\text{ch}}_0^{(\tilde{\mathbf{R}})}(N-1, 0; \tau, z). \end{aligned}$$

As we have emphasized several times, these two $\mathcal{N} = 4$ systems have the same degrees of freedom even though the central charges differ from each other, and are expected to be dual in the sense of the $\text{AdS}_3/\text{CFT}_2$ correspondence. We can explicitly observe a very simple correspondence:

$$\widehat{F}^{(N)}(v, a; \tau, -z) \text{ for case 1} \longleftrightarrow \widehat{F}^{(N)}(v, a; \tau, (N-1)z) \text{ for case 2.} \quad (4.1)$$

Now, we present some comments on the relation with the analyses on “umbral moonshine” [2–4, 25]. In Ref. [4], the authors studied (the holomorphic part of) the extension of (3.22) with general modular coefficients determined by the simply laced root system X corresponding to each Niemeier lattice. We have $\text{rank } X = 24$ by definition; let N be the Coxeter number of X . A Niemeier lattice is explicitly expressed as

$$X = \coprod_i X_i, \quad \sum_i \text{rank } X_i = 24, \quad (4.2)$$

where each X_i is the irreducible component of the root system possessing the common Coxeter number N .

We can schematically write

$$\begin{aligned}
 \mathcal{Z}_X^{[\hat{c}=2]}(\tau, z) &\equiv \sum_i \mathcal{Z}^{\text{case } 1(X_i)}(\tau, z) \\
 &:= -\frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \sum_i \sum_{r_i, s_i=1}^{N-1} \sum_{a_i \in \mathbb{Z}_N} \mathcal{N}_{r_i, s_i}^{X_i} \text{ch}_{r_i-1, s_i+2a_i}^{(\tilde{R})}(\tau, z) \widehat{F}^{(N)}(s_i, a_i; \tau, -z) \\
 &= -\frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \sum_{r, s=1}^{N-1} \sum_{a \in \mathbb{Z}_N} \mathcal{N}_{r, s}^X \text{ch}_{r-1, s+2a}^{(\tilde{R})}(\tau, z) \widehat{F}^{(N)}(s, a; \tau, -z) \quad (4.3)
 \end{aligned}$$

where $\mathcal{N}_{r, s}^{X_i}$ denotes the modular invariant coefficients of $SU(2)_{N-2}$ associated with the simply laced root system X_i , and we set $\mathcal{N}_{r, s}^X \equiv \sum_i \mathcal{N}_{r, s}^{X_i}$. One may identify $\mathcal{Z}^{\text{case } 1(X_i)}(\tau, z)$ as the elliptic genus of the ALE space associated with the simple singularity of type X_i . In Ref. [4] it was suggested that the root system $X = \coprod_i X_i$ should be identified as the geometrical data of various K3 singularities.

Since we assume $\text{rank } X \equiv \sum_i \text{rank } X_i = 24$, we can rewrite (4.3) as

$$\mathcal{Z}_X^{[\hat{c}=2]}(\tau, z) = 2\phi_{0,1}(\tau, z) - \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} \widehat{h}^X(\tau), \quad (4.4)$$

where $\widehat{h}^X(\tau)$ is the completion of the mock modular form of weight $1/2$ characterized by the shadow

$$\begin{aligned}
 \sqrt{\tau_2} \frac{\partial}{\partial \bar{\tau}} \widehat{h}^X(\tau) &= -\frac{i}{2\sqrt{N}} \sum_{r, s=1}^{N-1} \mathcal{N}_{r, s}^X \chi_{r-1}^{(N-2)}(\tau, 0) \overline{S_{s, N}(\tau)} \\
 &\equiv -\frac{i}{2\sqrt{N}} \frac{1}{\eta(\tau)^3} \sum_{r, s=1}^{N-1} \mathcal{N}_{r, s}^X S_{r, N}(\tau) \overline{S_{s, N}(\tau)}. \quad (4.5)
 \end{aligned}$$

This is derived from (4.3) with the help of the branching relation (3.27). (We recall that the space of the weight 0, index 1 holomorphic Jacobi form is 1D and is spanned by $\phi_{0,1}$).

Let us next consider the type- X generalization of (3.79), which is related to (4.3) via the “duality correspondence” (4.1):

$$\mathcal{Z}_X^{[\hat{c}=2(N-1)]}(\tau, z) := \frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \sum_{r, s=1}^{N-1} \sum_{a \in \mathbb{Z}_N} \mathcal{N}_{r, s}^X \text{ch}_{r-1, s+2a}^{(\tilde{R})}(\tau, z) \widehat{F}^{(N)}(s, a; \tau, (N-1)z). \quad (4.6)$$

Another useful realization of the duality correspondence is given as the natural extension of the identity (3.35):

$$\begin{aligned}
 \mathcal{Z}_X^{[\hat{c}=2]}(\tau, z) &= 2\phi_{0,1}(\tau, z) + \theta_1(\tau, z)^2 \frac{1}{2\pi i} \oint_{w=0} \frac{dw}{w} \frac{e^{(N-2)G_2(\tau)w^2}}{\theta_1(\tau, w)^2} \mathcal{Z}_X^{[\hat{c}=2(N-1)]}(\tau, w) \\
 &\equiv 2\phi_{0,1}(\tau, z) + \frac{\theta_1(\tau, z)^2}{\eta(\tau)^6} \frac{1}{8\pi^3 i} \oint_{w=0} \frac{dw}{w} \frac{e^{(N-1)G_2(\tau)w^2}}{\sigma(\tau, w)^2} \mathcal{Z}_X^{[\hat{c}=2(N-1)]}(\tau, w), \quad (4.7)
 \end{aligned}$$

where we have introduced the Weierstrass σ -function (A6) in the second line. In fact, one can straightforwardly confirm that the second term possesses the correct modular property and reproduces the expected shadow (4.5) in a manner similar to the derivation of (3.35).

Similarly to (4.4), the R.H.S. of (4.6) can be decomposed as

$$\begin{aligned}\mathcal{Z}_X^{[\hat{c}=2(N-1)]}(\tau, z) &= \Phi_{0,N-1}^X(\tau, z) - \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} \sum_{r=1}^{N-1} \widehat{h}_r^X(\tau) \chi_{r-1}^{(N-2)}(\tau, 2z) \\ &\equiv \Phi_{0,N-1}^X(\tau, z) - \frac{2\theta_1(\tau, z)^2}{i\eta(\tau)^3 \theta_1(\tau, 2z)} \sum_{r=1}^{N-1} \widehat{h}_r^X(\tau) \Theta_{r,N}^{[-]}(\tau, 2z).\end{aligned}\quad (4.8)$$

In the above expression, $\Phi_{0,N-1}^X(\tau, z)$ is a weak Jacobi form of weight 0, index $N-1$, which is holomorphic with respect to τ , but generically *meromorphic with respect to* z . $\chi_\ell^{(k)}(\tau, z)$ is the affine $SU(2)$ character of level k , isospin $\ell/2$, and $\widehat{h}_r^X(\tau)$ are the completions of vector-valued mock modular forms of weight $1/2$, whose shadow is given as

$$\sqrt{\tau_2} \frac{\partial}{\partial \bar{\tau}} \widehat{h}_r^X(\tau) = -\frac{i}{2\sqrt{N}} \sum_{s=1}^{N-1} \mathcal{N}_{r,s}^X \overline{S_{s,N}(\tau)}. \quad (4.9)$$

Equation (4.9) again follows from the definition (4.8) and the identity (3.66).

We note a subtlety in the decomposition (4.8). $\widehat{h}_r^X(\tau)$ is not necessarily determined only from the shadow (4.9). This is in contrast to $\widehat{h}^X(\tau)$ in (4.4), which is indeed determined uniquely. For a sufficiently large N , there exist nontrivial holomorphic weak Jacobi forms of weight 1, index N , which we denote here by, say, $\psi_{1,N}(\tau, z)$. Then, we may add $\frac{2\theta_1(\tau, z)^2}{i\eta(\tau)^3 \theta_1(\tau, 2z)} \psi_{1,N}^{[-]}(\tau, z)$ to the first term of (4.8), while subtracting the same function from the second term. We can thus modify $\widehat{h}_r^X(\tau)$ in (4.8) while keeping the shadow (4.9) unchanged. To avoid this ambiguity, we should impose the *optimal growth condition*:

$$\lim_{\tau \rightarrow i\infty} q^{\frac{1}{4N}} \left| \widehat{h}_r^X(\tau) \right| < \infty, \quad (\forall r = 1, \dots, N-1), \quad (4.10)$$

according to Ref. [3]. In fact, the above $\psi_{1,N}^{[-]}(\tau, z)$ can be expanded by theta functions as

$$\psi_{1,N}^{[-]}(\tau, z) = \sum_{r=1}^{N-1} \alpha_r(\tau) \Theta_{r,N}^{[-]}(\tau, 2z), \quad (4.11)$$

and the holomorphic coefficients $\alpha_r(\tau)$ cannot satisfy the condition (4.10) due to Theorem 9.7 of Ref. [26]. Consequently, we can remove the ambiguity in the decomposition (4.8), and can determine $\widehat{h}_r^X(\tau)$ as well as $\Phi_{0,N-1}^X(\tau, z)$ uniquely.

Now, substituting the decompositions (4.4), (4.8) into Eq. (4.7), we find

$$\widehat{h}^X(\tau) = \sum_{r=1}^{N-1} \widehat{h}_r^X(\tau) \chi_{r-1}^{(N-2)}(\tau, 0) \equiv \frac{1}{\eta(\tau)^3} \sum_{r=1}^{N-1} \widehat{h}_r^X(\tau) S_{r,N}(\tau), \quad (4.12)$$

since the contour integral $\oint \frac{dw}{w} \frac{e^{(N-1)G_2(\tau)w^2}}{\sigma(\tau, w)^2} \Phi_{0,N-1}^X(\tau, w)$ should be a holomorphic modular form of weight 2, and thus vanishes.⁷ This is the duality relation between the expansion coefficients of

⁷ We remark that the optimal growth condition is not necessary for the purpose of proving the identity (4.12). In other words, the ambiguity of $\widehat{h}_r^X(\tau)$ mentioned above does not spoil this relation; $\sum_{r=1}^{N-1} \alpha_r(\tau) \chi_{r-1}^{(N-2)}(\tau, 0) = 0$ always holds for the coefficients $\alpha_r(\tau)$ appearing in (4.11).

massive representations of $\mathcal{Z}_X^{[\hat{c}=2]}(\tau, z)$ and $\mathcal{Z}_X^{[\hat{c}=2(N-1)]}(\tau, z)$. In the case of Mathieu moonshine ($N = 2, X = A_1^{24}$), one has the self-dual situation $\widehat{h}_1^{A_1^{24}}(\tau) = \widehat{h}_{r=1}^{A_1^{24}}(\tau)$. In general, holomorphic parts of $\widehat{h}_r^X(\tau)$ should reproduce the mock modular form of umbral moonshine on which the umbral group G_X should act [2].

Let us discuss the relation of the present analysis to a closely related consideration given in Ref. [4]. For this purpose it is convenient to introduce the meromorphic Jacobi form $\Psi_{1,N}^X(\tau, z)$ with weight 1, index N defined by

$$\Psi_{1,N}^X(\tau, z) := \frac{i\eta(\tau)^3 \theta_1(\tau, 2z)}{2\theta_1(\tau, z)^2} \Phi_{0,N-1}^X(\tau, z) \equiv \widehat{f}^{(1)}(\tau, z) \Phi_{0,N-1}^X(\tau, z), \quad (4.13)$$

following Refs. [2,3]. Then, (4.8) can be rewritten as

$$\Psi_{1,N}^X(\tau, z) = \widehat{f}^{(1)}(\tau, z) \mathcal{Z}_X^{[\hat{c}=2(N-1)]}(\tau, z) + \sum_{r=1}^{N-1} \widehat{h}_r^X(\tau) \Theta_{r,N}^{[-]}(\tau, 2z), \quad (4.14)$$

where the first and second terms on the R.H.S. correspond to the *polar* and *finite* parts in the terminology of Refs. [2,3].

If X includes only the A -type components, our results are manifestly consistent with those given in Ref. [4]. Namely, both the polar and finite parts in (4.14) (the “case 2 theory” with $\hat{c} = 2(N-1)$) separately correspond to the first and second terms in (4.4) (the “case 1 theory” with $\hat{c} = 2$) under the transformation given in Ref. [4]:

$$\begin{aligned} \varphi_{1,N}(\tau, z) &\mapsto \varphi'_{0,N^2}(\tau, z) := \frac{\theta_1(\tau, (N-1)z) \theta_1(\tau, Nz)}{i\eta(\tau)^3 \theta_1(\tau, z)} \varphi_{1,N}(\tau, z) \\ &\mapsto \varphi''_{0,1}(\tau, z) := \frac{1}{N} \sum_{\lambda \in \Lambda/N\Lambda} s_\lambda^{(N^2)} \cdot \varphi'_{0,N^2}(\tau, z/N). \end{aligned} \quad (4.15)$$

Here, we have denoted a (holomorphic or nonholomorphic) weak Jacobi form of weight w and index d by the symbol “ $\varphi_{w,d}(\tau, z)$ ”, and the second line is identified as the Eichler–Zagier operator $\mathcal{W}(N)$ ($\mathbb{Z}_N$ orbifolding). In fact, the first term on the R.H.S. of (4.14) just becomes

$$\widehat{f}^{(1)}(\tau, z) \mathcal{Z}_X^{[\hat{c}=2(N-1)]}(\tau, z) = \frac{24}{N-1} \widehat{f}^{(N)}(\tau, z), \quad (4.16)$$

by substituting the identity (3.70) into (4.6), and the map (4.15) converts (4.14) into the definition of $\mathcal{Z}_X^{[\hat{c}=2]}(\tau, z)$ itself. (Recall the formula for the elliptic genus of the $\mathcal{N} = 2$ minimal model (3.24).) Furthermore, the correspondence of finite parts by (4.15) is found to be equivalent to the relation (4.12) due to the branching relation (3.27). The fact (4.16) also implies that $\Psi_{1,N}^X(\tau, z)$ here coincides with the “umbral Jacobi form” constructed in Ref. [3]. It is also obvious that (4.16) is consistent with our duality relation (4.7) because of Eq. (3.35).

In the cases when X includes D - or E -type components, however, $\Psi_{1,N}^X(\tau, z)$ generally differs from the umbral Jacobi form. Indeed, according to the explicit construction of the umbral Jacobi form given in Ref. [3], it would contain n -torsion points with $n|N$ created by the Eichler–Zagier operator written schematically as

$$\mathcal{W}_X \equiv \sum_{n_i|N} c_i \mathcal{W}(n_i).$$

On the other hand, $\Psi_{1,N}^X(\tau, z)$, given above, possesses n' -torsion points with $n'|(N-1)$, which is inherited from the functions $\widehat{F}^{(N)}(r, a; \tau, (N-1)z)$ in the expression (4.6).⁸

Nevertheless, if we replace $\Psi_{1,N}^X(\tau, z)$ with the umbral Jacobi form of Ref. [3] in the “elliptic genus” $\mathcal{Z}_X^{[\hat{c}=2(N-1)]}(\tau, z)$, the duality relation (4.7) is still satisfied, because the difference between these meromorphic Jacobi forms just yields a vanishing contribution to the contour integral.

How about the correspondence (4.15)? Indeed, one can show that the claim of Ref. [4] still holds even when we replace the umbral Jacobi form with $\Psi_{1,N}^X(\tau, z)$. Namely, $\widehat{f}^{(1)}(\tau, z)\mathcal{Z}_X^{[\hat{c}=2(N-1)]}(\tau, z)$ and $\Psi_{1,N}^X(\tau, z)$ still correspond to $\mathcal{Z}_X^{[\hat{c}=2]}(\tau, z)$ and $\mathcal{Z}^{K3}(\tau, z) \equiv 2\phi_{0,1}(\tau, z)$ by the map (4.15), as shown by the following arguments:

- (i) It turns out that the finite part of (4.14) is mapped to that of $\mathcal{Z}_X^{[\hat{c}=2]}(\tau, z)$, and the relation (4.12) still holds.
- (ii) When considering the map (4.15), all the n' -torsion points included in $\Psi_{1,N}^X(\tau, z)$ are canceled out with the zeros of the factor $\theta_1(\tau, (N-1)z)$.⁹ We thus find that $\Psi_{1,N}^X(\tau, z)$ is always mapped to the unique holomorphic Jacobi form $\alpha\phi_{0,1}(\tau, z)$ with some constant coefficient α .
- (iii) This coefficient α is, however, just determined from the finite part so that the contribution of “graviton representation” should be canceled out. In fact, it is straightforward to confirm that (4.6) cannot include such a term after converting it into the $\hat{c} = 2$ system by the transformation (4.15). This is quite similar to the argument given in Sect. 3.3.1 (see the discussions around (3.44), (3.45)). On the other hand, the finite part of (4.6) is mapped to $\widehat{h}^X(\tau) \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3}$ as noted above, yielding the $h = 0$ behavior $\sim -\frac{N-1}{12}q^{-\frac{1}{4}}$ in the NS sector as in (3.45). Therefore, the cancellation of the graviton term implies that

$$\alpha = \frac{24}{N-1} \cdot \frac{N-1}{12} = 2,$$

which completes the proof of our statement.

5. Discussions

We conclude that our ansatz (4.6) for $\mathcal{Z}_X^{[\hat{c}=2(N-1)]}(\tau, z)$ should reproduce the expected expansion coefficients $\widehat{h}_r^X(\tau)$ whose holomorphic parts give the coefficients of massive representations of Mathieu and umbral moonshine. Therefore, we naturally consider that there are two world-sheet descriptions of the moonshine phenomena based on the theories $\mathcal{Z}_X^{[\hat{c}=2]}(\tau, z)$ and $\mathcal{Z}_X^{[\hat{c}=2(N-1)]}(\tau, z)$, depending on the choice of $SU(2)_R$ symmetry. Mathieu moonshine sits at the self-dual point.

Our observation is related to Ref. [4] (and also to Ref. [25]). Without invoking the idea of duality, the authors of Ref. [4] introduced a transformation rule between the corresponding Jacobi forms of the two theories that seem to fit our results very well. A possible geometrical interpretation of the umbral moonshine based on particular types of singular K3 is also discussed. As we briefly discussed in Sect. 4, the correspondence (4.15) proposed in Ref. [4] is likely to be consistent with our “duality relation” (4.7).

Some of the results in this paper appear correct by symmetry arguments (modularity, etc.) but are not verified explicitly: We have not computed the holomorphic parts of functions $\widehat{h}_r^X$ except for

⁸ Such torsion points are canceled out in the cases when X is made up only of A -type components, as we illustrated in Sect. 3.4.1.

⁹ On the other hand, the n -torsion points appearing in the umbral Jacobi form are canceled out by the factor $\theta_1(\tau, Nz)$.

in the case of Mathieu moonshine. In subsequent work we want to fill these gaps. We also mainly discussed the A -type modular invariant and not the D and E types.

One should keep in mind that the Jacobi form $\Phi_{0,N-1}^X(\tau, z)$ given here is at most meromorphic except for the purely A -type models, since $\Psi_{1,N}^X(\tau, z)$ generally includes torsion points, as we have mentioned. Consequently, it would be subtle whether one can strictly interpret $\mathcal{Z}_X^{[\hat{c}=2(N-1)]}(\tau, z)$ as the elliptic genus of a well-defined superconformal system. Of course, we have no such subtlety for the “dual” $\hat{c} = 2$ -realization (4.3) for an arbitrary X . We would like to further discuss this point in future work.

Acknowledgements

We thank the organizers of the LMS–EPSRC Durham Symposium on “New Moonshines, Mock Modular Forms and String Theory” at Durham University, 3–12 August 2015, where part of this work was done, for their kind hospitality. The research of T.E. is supported in part by JSPS KAKENHI grant nos. 25400273, 22224001 and 23340115. The research of Y.S. is supported in part by JSPS KAKENHI grant no. 23540322.

Funding

Open Access funding: SCOAP³.

Appendix A. Notations and useful formulas

In this appendix we summarize the notations adopted in this paper and useful formulas related to them. We assume that $\tau \equiv \tau_1 + i\tau_2$, $\tau_2 > 0$ and set $q := e^{2\pi i\tau}$, $y := e^{2\pi iz}$:

Theta functions:

$$\begin{aligned}\theta_1(\tau, z) &= i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n-1/2)^2/2} y^{n-1/2} \equiv 2 \sin(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)(1 - yq^m)(1 - y^{-1}q^m), \\ \theta_2(\tau, z) &= \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2} y^{n-1/2} \equiv 2 \cos(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)(1 + yq^m)(1 + y^{-1}q^m), \\ \theta_3(\tau, z) &= \sum_{n=-\infty}^{\infty} q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1 - q^m)(1 + yq^{m-1/2})(1 + y^{-1}q^{m-1/2}), \\ \theta_4(\tau, z) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1 - q^m)(1 - yq^{m-1/2})(1 - y^{-1}q^{m-1/2}).\end{aligned}\tag{A1}$$

$$\Theta_{m,k}(\tau, z) = \sum_{n=-\infty}^{\infty} q^{k(n+\frac{m}{2k})^2} y^{k(n+\frac{m}{2k})},\tag{A2}$$

$$\Theta_{m,k}^{[-]}(\tau, z) = \frac{1}{2} [\Theta_{m,k}(\tau, z) - \Theta_{m,k}(\tau, -z)].\tag{A3}$$

We use the abbreviations $\theta_i(\tau) \equiv \theta_i(\tau, 0)$ ($\theta_1(\tau) \equiv 0$), $\Theta_{m,k}(\tau) \equiv \Theta_{m,k}(\tau, 0)$. We also set

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).\tag{A4}$$

The spectral flow properties of the theta functions are summarized as follows:

$$\theta_1(\tau, z + m\tau + n) = (-1)^{m+n} q^{-\frac{m^2}{2}} y^{-m} \theta_1(\tau, z),$$

$$\begin{aligned}
\theta_2(\tau, z + m\tau + n) &= (-1)^n q^{-\frac{m^2}{2}} y^{-m} \theta_2(\tau, z), \\
\theta_3(\tau, z + m\tau + n) &= q^{-\frac{m^2}{2}} y^{-m} \theta_3(\tau, z), \\
\theta_4(\tau, z + m\tau + n) &= (-1)^m q^{-\frac{m^2}{2}} y^{-m} \theta_4(\tau, z), \\
\Theta_{a,k}(\tau, 2(z + m\tau + n)) &= e^{2\pi i n a} q^{-k m^2} y^{-2k m} \Theta_{a+2k m, k}(\tau, 2z). \quad (A5)
\end{aligned}$$

We introduce the Weierstrass σ -function:

$$\begin{aligned}
\sigma(\tau, z) &:= e^{\frac{1}{2} G_2(\tau) z^2} \frac{\theta_1(\tau, z)}{2\pi \eta(\tau)^3} \\
&\equiv z \prod_{\omega \in \Lambda'} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2} \left(\frac{z}{\omega}\right)^2}. \quad (\Lambda' \equiv \Lambda - \{0\}), \quad (A6)
\end{aligned}$$

where $G_2(\tau)$ is the (unnormalized) second Eisenstein series:

$$\begin{aligned}
G_2(\tau) &:= \sum_{n \in \mathbb{Z} - \{0\}} \frac{1}{n^2} + \sum_{m \in \mathbb{Z} - \{0\}} \sum_{n \in \mathbb{Z}} \frac{1}{(m\tau + n)^2} \\
&\equiv \frac{\pi^2}{3} \left[1 - 24 \sum_{n=1}^{\infty} \frac{n q^n}{1 - q^n} \right]. \quad (A7)
\end{aligned}$$

It is useful to note the anomalous S -transformation formula of $G_2(\tau)$:

$$G_2\left(-\frac{1}{\tau}\right) = \tau^2 G_2(\tau) - 2\pi i \tau. \quad (A8)$$

We also set

$$\widehat{G}_2(\tau) := G_2(\tau) - \frac{\pi}{\tau_2}, \quad (A9)$$

which is a nonholomorphic modular form of weight 2.

Spectral flow operator (see also Ref. [24]):

$$\begin{aligned}
s_{\lambda}^{(\kappa)} \cdot f(\tau, z) &:= e^{2\pi i \frac{\kappa}{\tau_2} \lambda_2 (\lambda + 2z)} f(\tau, z + \lambda) \\
&\equiv q^{\kappa \alpha^2} y^{2\kappa \alpha} e^{2\pi i \kappa \alpha \beta} f(\tau, z + \alpha \tau + \beta), \\
&\quad (\lambda \equiv \alpha \tau + \beta, \forall \alpha, \beta \in \mathbb{R}). \quad (A10)
\end{aligned}$$

An important property of the spectral flow operator $s_{\lambda}^{(\kappa)}$ is the modular covariance, which precisely means the following:

Assume that $f(\tau, z)$ is an arbitrary function with the modular property

$$f(\tau + 1, z) = f(\tau, z), \quad f\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = e^{2\pi i \frac{\kappa}{\tau} z^2} \tau^{\alpha} f(\tau, z);$$

then, we obtain for $\forall \lambda \in \mathbb{C}$

$$s_{\lambda}^{(\kappa)} \cdot f(\tau + 1, z) = s_{\lambda}^{(\kappa)} \cdot f(\tau, z), \quad s_{\frac{\lambda}{\tau}}^{(\kappa)} \cdot f\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = e^{2\pi i \frac{\kappa}{\tau} z^2} \tau^{\alpha} s_{\lambda}^{(\kappa)} \cdot f(\tau, z).$$

Error functions:

$$\text{Erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \quad (x \in \mathbb{R}), \quad (\text{A11})$$

$$\text{Erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \equiv 1 - \text{Erf}(x) \quad (x > 0). \quad (\text{A12})$$

The next identity is elementary but useful:

$$\text{sgn}(v+0) - \text{Erf}(v) = \text{sgn}(v+0) \text{Erfc}(|v|) = \frac{1}{i\pi} \int_{\mathbb{R}-i0} dp \frac{e^{-(p^2+v^2)}}{p-iv} \quad (v \in \mathbb{R}). \quad (\text{A13})$$

Weak Jacobi forms:

The weak Jacobi form [24] for the full modular group $\Gamma(1) \equiv SL(2, \mathbb{Z})$ with weight $k (\in \mathbb{Z}_{\geq 0})$ and index $r (\in \frac{1}{2}\mathbb{Z}_{\geq 0})$ is defined by the following conditions:

(i) modularity:

$$\Phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = e^{2\pi i r \frac{cz^2}{c\tau+d}} (c\tau+d)^k \Phi(\tau, z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1). \quad (\text{A14})$$

(ii) double quasiperiodicity:

$$\Phi(\tau, z + m\tau + n) = (-1)^{2r(m+n)} q^{-rm^2} y^{-2rm} \Phi(\tau, z). \quad (\text{A15})$$

In this paper, we shall use this terminology in a broader sense. We allow a half-integral index r , and, more crucially, allow nonholomorphic dependence on τ , while we keep the holomorphicity with respect to z .¹⁰

Appendix B. Derivation of Eqs. (3.39), (3.42), and (3.47)

In this appendix, we derive Eqs. (3.39), (3.42), and (3.47).

Derivation of (3.39):

We start with the definition of $\widehat{H}^{(N)}(\tau)$ (3.37). Substituting (3.38) into (3.37), we obtain ($\Lambda \equiv \mathbb{Z}\tau + \mathbb{Z}$)

$$\widehat{H}^{(N)}(\tau) = \frac{1}{2\pi^2} \oint_{w=0} \frac{dw}{w^2} \left[\frac{w\eta(\tau)^3}{i\theta_1(\tau, 2w)} e^{(N-2)G_2(\tau)w^2} \sum_{\lambda \in \Lambda} \frac{e^{-\frac{\pi}{\tau_2} N \{|\lambda|^2 + 2\bar{\lambda}w + w^2\}}}{\lambda + w} \right]. \quad (\text{B1})$$

It would be easiest to carry out this residue integral by introducing the Weierstrass σ -function defined in (A6). Namely,

$$\widehat{H}^{(N)}(\tau) = \frac{1}{8\pi^3 i} \oint_{w=0} \frac{dw}{w^2} \left[\frac{2w}{\sigma(\tau, 2w)} e^{NG_2(\tau)w^2} \sum_{\lambda \in \Lambda} \frac{e^{-\frac{\pi}{\tau_2} N \{|\lambda|^2 + 2\bar{\lambda}w + w^2\}}}{\lambda + w} \right]$$

¹⁰ According to the original terminology of Ref. [24], the “weak Jacobi form” of weight k and index r ($k, r \in \mathbb{Z}_{\geq 0}$) means that $\Phi(\tau, z)$ should be Fourier expanded as $\Phi(\tau, z) = \sum_{n \in \mathbb{Z}_{\geq 0}} \sum_{\ell \in \mathbb{Z}} c(n, \ell) q^n y^\ell$, in addition to the conditions (A14) and (A15). It is called the “Jacobi form” if it further satisfies the condition $c(n, \ell) = 0$ for $\forall n, \ell$ s.t. $4nr - \ell^2 < 0$.

$$\begin{aligned}
&= \frac{1}{8\pi^3 i} \oint_{w=0} \frac{dw}{w^2} \left[\frac{2w}{\sigma(\tau, 2w)} e^{NG_2(\tau)w^2} \sum_{\lambda \in \Lambda'} \frac{e^{-\frac{\pi}{\tau_2} N \{|\lambda|^2 + 2\bar{\lambda}w + w^2\}}}{\lambda + w} \right] \\
&\quad + \frac{1}{8\pi^3 i} \oint_{w=0} \frac{dw}{w^3} \frac{2w}{\sigma(\tau, 2w)} e^{N\widehat{G}_2(\tau)w^2}. \quad (\Lambda' \equiv \Lambda - \{0\}). \quad (\text{B2})
\end{aligned}$$

Moreover, since we have

$$\frac{2w}{\sigma(\tau, 2w)} = 1 + \mathcal{O}(w^4),$$

due to (A6), (B2) just becomes

$$\begin{aligned}
\widehat{H}^{(N)}(\tau) &= \frac{1}{8\pi^3 i} \oint_{w=0} \frac{dw}{w^2} \sum_{\lambda \in \Lambda'} \frac{e^{-\frac{\pi}{\tau_2} N \{|\lambda|^2 + 2\bar{\lambda}w + w^2\}}}{\lambda + w} + \frac{1}{8\pi^3 i} \oint_{w=0} \frac{dw}{w^3} e^{N\widehat{G}_2(\tau)w^2} \\
&= \frac{1}{4\pi^2} \left[\frac{\partial}{\partial w} \sum_{\lambda \in \Lambda'} \frac{e^{-\frac{\pi}{\tau_2} N \{|\lambda|^2 + 2\bar{\lambda}w + w^2\}}}{\lambda + w} \Big|_{w=0} + N\widehat{G}_2(\tau) \right] \\
&= \frac{1}{4\pi^2} \left[- \sum_{\lambda \in \Lambda'} \frac{e^{-\frac{\pi}{\tau_2} N |\lambda|^2}}{\lambda^2} \left\{ 1 + \frac{2\pi N}{\tau_2} |\lambda|^2 \right\} + N\widehat{G}_2(\tau) \right]. \quad (\text{B3})
\end{aligned}$$

We have thus obtained (3.39).

Derivation of (3.42):

We next evaluate the holomorphic part of (3.39) or (B3). Let us derive the expression of the holomorphic function $H^{(N)}(\tau)$ given in (3.42).

Recalling (3.28), the nonholomorphic part of $\mathcal{Z}^{\text{case } 1}(\tau, z)$ can be rewritten as

$$\begin{aligned}
\Delta \mathcal{Z}^{\text{case } 1}(\tau, z) &\equiv \frac{\theta_1(\tau, z)^2}{\eta(\tau)^6} \Delta \widehat{H}^{(N)}(\tau), \\
\Delta \widehat{H}^{(N)}(\tau) &= 2\eta(\tau)^3 \lim_{w \rightarrow 0} \left[\frac{\widehat{f}^{(N)}(w) - f^{(N)}[-]}{i\theta_1(2w)} \right] \\
&= \frac{1}{2\pi i} \frac{\partial}{\partial w} \left[\widehat{f}^{(N)}(w) - f^{(N)}[-] \right] \Big|_{w=0}. \quad (\text{B4})
\end{aligned}$$

Moreover, we note the formula

$$\widehat{f}^{(N)}(w) - f^{(N)}[-](w) = \frac{i}{2\pi} \sum_{\lambda \in \Lambda} s_{\lambda}^{(N)} \cdot \left[\frac{e^{-\frac{\pi}{\tau_2} N w^2} - 1}{w} \right], \quad (\text{B5})$$

where $s_{\lambda}^{(N)}$ denotes the spectral flow operator (A10).

Substituting (B5) into (B4), we obtain

$$\begin{aligned}
\Delta \widehat{H}^{(N)}(\tau) &= \frac{1}{4\pi^2} \lim_{\epsilon \rightarrow +0} \left[\frac{\partial}{\partial w} \sum_{\lambda \in \Lambda'} s_{\lambda}^{(N)} \cdot \left\{ \frac{e^{-\frac{\pi}{\tau_2} N w^2} - e^{-\frac{\pi}{\tau_2} \epsilon w^2}}{w} \right\} \right]_{w=0} \\
&\quad + \frac{\partial}{\partial w} \left\{ \frac{e^{-\frac{\pi}{\tau_2} N w^2} - 1}{w} \right\} \Big|_{w=0} \Big] \\
&= \frac{1}{4\pi^2} \left[\frac{\partial}{\partial w} \left\{ \sum_{\lambda \in \Lambda'} s_{\lambda}^{(N)} \cdot \frac{e^{-\frac{\pi}{\tau_2} N w^2}}{w} \right\} \right]_{w=0} - \frac{\pi N}{\tau_2} \\
&\quad - \lim_{\epsilon \rightarrow +0} \frac{1}{4\pi^2} \frac{\partial}{\partial w} \left\{ \sum_{\lambda \in \Lambda'} s_{\lambda}^{(N)} \cdot \frac{e^{-\frac{\pi}{\tau_2} \epsilon w^2}}{w} \right\} \Big|_{w=0} \\
&= \frac{1}{4\pi^2} \left[\frac{\partial}{\partial w} \left\{ \sum_{\lambda \in \Lambda'} s_{\lambda}^{(N)} \cdot \frac{e^{-\frac{\pi}{\tau_2} N w^2}}{w} \right\} \right]_{w=0} - \frac{\pi N}{\tau_2} \\
&\quad - \frac{1}{4\pi^2} \frac{\partial}{\partial w} \left[\sum_{m \neq 0} \sum_{n \in \mathbb{Z}}^P s_{m\tau+n}^{(N)} \cdot \frac{1}{w} + \sum_{n \neq 0}^P \frac{1}{w+n} \right] \Big|_{w=0}. \quad (\text{B6})
\end{aligned}$$

Here, we have included the convergence factor $e^{-\frac{\pi}{\tau_2} \epsilon w^2}$ to make the second line well-defined, and the symbol “ $\sum_n^P$ ” denotes the principal value defined in (3.41):

$$\sum_{n \neq 0}^P a_n := \lim_{N \rightarrow \infty} \sum_{n=1}^N (a_n + a_{-n}), \quad \sum_n^P a_n := a_0 + \sum_{n \neq 0}^P a_n.$$

Then, by comparing (B6) with (B3), and recalling $\widehat{G}_2(\tau) = G_2(\tau) - \frac{\pi}{\tau_2}$, we find that the holomorphic part $H^{(N)}(\tau) \equiv \widehat{H}^{(N)}(\tau) - \Delta \widehat{H}^{(N)}(\tau)$ is indeed given by Eq. (3.42).

Derivation of (3.47):

Finally, let us derive the formula given in the second line of (3.47) for the $N = 2$ case. To this end, it is again convenient to make use of the σ -function (A6). Substituting the identity (3.51) into the first line of (3.47), we obtain

$$\begin{aligned}
\widehat{H}^{(2)}(\tau) &\equiv \frac{1}{2\pi i} \oint_{w=0} \frac{dw}{w} \frac{\eta(\tau)^3}{i\theta_1(\tau, w)} \widehat{f}^{(1/2)}(\tau, w) \\
&= \frac{1}{8\pi^3 i} \oint_{w=0} \frac{dw}{w^2} \frac{w}{\sigma(\tau, w)} e^{\frac{1}{2} G_2(\tau) w^2} \sum_{\lambda \in \Lambda} (-1)^{m+n+mn} \frac{e^{-\frac{\pi}{2\tau_2} \{|\lambda|^2 + 2\bar{\lambda}w + w^2\}}}{\lambda + w} \\
&= \frac{1}{8\pi^3 i} \oint_{w=0} \frac{dw}{w^2} e^{\frac{1}{2} G_2(\tau) w^2} \sum_{\lambda = m\tau + n \in \Lambda'} (-1)^{m+n+mn} \frac{e^{-\frac{\pi}{2\tau_2} \{|\lambda|^2 + 2\bar{\lambda}w + w^2\}}}{\lambda + w} \\
&\quad + \frac{1}{8\pi^3 i} \oint_{w=0} \frac{dw}{w^3} e^{\frac{1}{2} \widehat{G}_2(\tau) w^2}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4\pi^2} \left[\frac{\partial}{\partial w} \sum_{\lambda \in \Lambda'} (-1)^{m+n+mn} \frac{e^{-\frac{\pi}{2\tau_2} \{|\lambda|^2 + 2\bar{\lambda}w + w^2\}}}{\lambda + w} \right]_{w=0} + \frac{1}{2} \widehat{G}_2(\tau) \\
&= \frac{1}{4\pi^2} \left[\sum_{\lambda \in \Lambda'} (-1)^{m+n+mn} e^{-\frac{\pi}{2\tau_2} |\lambda|^2} \left(-\frac{1}{\lambda^2} + \frac{2\pi i m}{\lambda} - \frac{\pi}{\tau_2} \right) + \frac{1}{2} \widehat{G}_2(\tau) \right] \\
&= \frac{1}{4\pi^2} \left[\sum_{\lambda \in \Lambda'} (-1)^{m+n+mn} e^{-\frac{\pi}{2\tau_2} |\lambda|^2} \left(-\frac{1}{\lambda^2} + \frac{2\pi i m}{\lambda} \right) + \frac{\pi}{\tau_2} + \frac{1}{2} \widehat{G}_2(\tau) \right]. \quad (\text{B7})
\end{aligned}$$

In the last line, we made use of the identity

$$\sum_{\lambda=m\tau+n\in\Lambda} (-1)^{m+n+mn} e^{-\frac{\pi}{2\tau_2} |\lambda|^2} = 0. \quad (\text{B8})$$

In fact,

$$\begin{aligned}
\sum_{\lambda \in \Lambda} (-1)^{m+n+mn} e^{-\frac{\pi}{2\tau_2} |\lambda|^2} &= \left(\sum_{m,n \in 2\mathbb{Z}} - \sum_{m \in 2\mathbb{Z}+1 \text{ or } n \in 2\mathbb{Z}+1} \right) e^{-\frac{\pi}{2\tau_2} |\lambda|^2} \\
&= \sum_{\lambda \in \Lambda} e^{-\frac{2\pi}{\tau_2} |\lambda|^2} - \left\{ \sum_{\lambda \in \Lambda} e^{-\frac{\pi}{2\tau_2} |\lambda|^2} - \sum_{\lambda \in \Lambda} e^{-\frac{2\pi}{\tau_2} |\lambda|^2} \right\} \\
&= 2 \sum_{\lambda \in \Lambda} e^{-\frac{2\pi}{\tau_2} |\lambda|^2} - \sum_{\lambda \in \Lambda} e^{-\frac{\pi}{2\tau_2} |\lambda|^2};
\end{aligned}$$

the last line vanishes due to the Poisson resummation with respect to both m and n .

To proceed further, we set

$$g(\tau) := - \sum_{\lambda \in \Lambda'} (-1)^{m+n+mn} \frac{e^{-\frac{\pi}{2\tau_2} |\lambda|^2}}{\lambda^2} \equiv - \sum_{\lambda \in \Lambda'} (-1)^{m+n} \frac{q^{\frac{1}{2}m^2} e^{-\frac{\pi}{2\tau_2} \lambda^2}}{\lambda^2}. \quad (\text{B9})$$

Then, we obtain

$$\frac{\partial}{\partial \bar{\tau}} g(\tau) = - \sum_{\lambda \in \Lambda'} (-1)^{m+n+mn} e^{-\frac{\pi}{2\tau_2} |\lambda|^2} \frac{i\pi}{4\tau_2^2} = \frac{i\pi}{4\tau_2^2} = -\frac{1}{2} \frac{\partial}{\partial \bar{\tau}} \left(\frac{\pi}{\tau_2} \right). \quad (\text{B10})$$

Here we have again used (B8). By the definition (B9), $g(\tau)$ should be a nonholomorphic modular form of weight 2 for $\Gamma(1)$. Thus $g(\tau)$ is uniquely determined by (B10) as

$$g(\tau) = \frac{1}{2} \widehat{G}_2(\tau). \quad (\text{B11})$$

Combining (B9), (B11) with (B7), we finally obtain

$$\begin{aligned}
\widehat{H}^{(2)}(\tau) &= \frac{1}{4\pi^2} \left[\sum_{\lambda \in \Lambda'} (-1)^{m+n+mn} \frac{2\pi i m e^{-\frac{\pi}{2\tau_2} |\lambda|^2}}{\lambda} + \widehat{G}_2(\tau) + \frac{\pi}{\tau_2} \right] \\
&= \frac{1}{4\pi^2} \left[\sum_{\lambda \in \Lambda'} (-1)^{m+n+mn} \frac{2\pi i m e^{-\frac{\pi}{2\tau_2} |\lambda|^2}}{\lambda} + G_2(\tau) \right], \quad (\text{B12})
\end{aligned}$$

which is the second line of (3.47).

Appendix C. A lemma for the weak Jacobi form

In this appendix, we present a simple lemma that is necessary for the proof of (3.70).

LEMMA 1 Let $\Phi(\tau, z)$ be any holomorphic weak Jacobi form of weight 0 and index $d(\in \mathbb{Z}_{>0})$. Then, the “NS counterpart” $\Phi^{(\text{NS})}(\tau, z) \equiv q^{\frac{d}{4}} y^d \Phi\left(\tau, z + \frac{\tau+1}{2}\right)$ has the following behavior around $\tau \sim i\infty$ (for a generic value of z):

$$\Phi^{(\text{NS})}(\tau, z) \sim q^\alpha, \quad -\frac{d}{4} \leq \alpha \leq \frac{d}{12}, \quad (\tau \sim i\infty). \quad (\text{C1})$$

Proof. The proof is straightforward. Let us recall that bases of holomorphic weak Jacobi forms of weight 0 and index $d(\in \mathbb{Z}_{>0})$ consist of functions defined by

$$\begin{aligned} \phi_{(n_1, n_2, n_3)}(\tau, z) &\propto \sum_{\sigma \in S_3} \left(\frac{\theta_2(\tau, z)}{\theta_2(\tau)} \right)^{2n_{\sigma(1)}} \left(\frac{\theta_3(\tau, z)}{\theta_3(\tau)} \right)^{2n_{\sigma(2)}} \left(\frac{\theta_4(\tau, z)}{\theta_4(\tau)} \right)^{2n_{\sigma(3)}}, \\ \phi_{(n_1, n_2, n_3)}(\tau, 0) &= 1, \quad \forall (n_1, n_2, n_3) \in \mathcal{S}^{(d)}, \end{aligned} \quad (\text{C2})$$

with

$$\mathcal{S}^{(d)} := \{(n_1, n_2, n_3) \in \mathbb{Z}_{\geq 0}^3; n_1 \geq n_2 \geq n_3 \geq 0, n_1 + n_2 + n_3 = d\}. \quad (\text{C3})$$

Now, we define the NS counterpart of the function $\phi_{(n_1, n_2, n_3)}(\tau, z)$ by the spectral flow $z \mapsto z + \frac{\tau+1}{2}$:

$$\phi_{(n_1, n_2, n_3)}^{(\text{NS})}(\tau, z) := q^{\frac{d}{4}} y^d \phi_{(n_1, n_2, n_3)}\left(\tau, z + \frac{\tau+1}{2}\right).$$

Then, by using the evaluations around $\tau \sim i\infty$,

$$\left(\frac{\theta_4(\tau, z)}{\theta_2(\tau)} \right)^2 \sim q^{-\frac{1}{4}}, \quad \left(\frac{\theta_1(\tau, z)}{\theta_3(\tau)} \right)^2 \sim q^{\frac{1}{4}}, \quad \left(\frac{\theta_2(\tau, z)}{\theta_4(\tau)} \right)^2 \sim q^{\frac{1}{4}},$$

we can easily find that the behavior of $\phi_{(n_1, n_2, n_3)}^{(\text{NS})}(\tau, z)$ becomes

$$\phi_{(n_1, n_2, n_3)}^{(\text{NS})}(\tau, z) \sim q^{-\frac{1}{4}n_1 + \frac{1}{4}(n_2+n_3)} = q^{-\frac{1}{2}(n_1 - \frac{d}{2})}, \quad (\tau \sim i\infty). \quad (\text{C4})$$

Since we are assuming that $n_1 + n_2 + n_3 = d$ and $n_1 \geq n_2 \geq n_3 \geq 0$, we find

$$\frac{d}{3} \leq n_1 \leq d.$$

Therefore, we obtain

$$\phi_{(n_1, n_2, n_3)}^{(\text{NS})}(\tau, z) \sim q^\alpha, \quad -\frac{d}{4} \leq \alpha \leq \frac{d}{12}, \quad (\tau \sim i\infty), \quad (\text{C5})$$

for $\forall (n_1, n_2, n_3) \in \mathcal{S}^{(d)}$. □

References

- [1] T. Eguchi, H. Ooguri, and Y. Tachikawa, *Exper. Math.* **20**, 91 (2011) [[arXiv:1004.0956](#) [hep-th]] [[Search INSPIRE](#)].
- [2] M. C. N. Cheng, J. F. R. Duncan, and J. A. Harvey, *Commun. Num. Theor. Phys.* **08**, 101 (2014) [[arXiv:1204.2779](#) [math.RT]] [[Search INSPIRE](#)].
- [3] M. C. N. Cheng, J. F. R. Duncan, and J. A. Harvey, [arXiv:1307.5793](#) [math.RT] [[Search INSPIRE](#)].
- [4] M. C. N. Cheng and S. Harrison, *Commun. Math. Phys.* **339**, 221 (2015) [[arXiv:1406.0619](#) [hep-th]] [[Search INSPIRE](#)].
- [5] A. Sevrin, W. Troost, and A. Van Proeyen, *Phys. Lett. B* **208**, 447 (1988).
- [6] S. Gukov, E. Martinec, G. W. Moore, and A. Strominger, [arXiv:hep-th/0404023](#).
- [7] D. Kutasov and N. Seiberg, *Nucl. Phys. B* **358**, 600 (1991).
- [8] C. G. Callan, J. A. Harvey, and A. Strominger, *Nucl. Phys. B* **359**, 611 (1991) [[arXiv:hep-th/9112030](#)] [[Search INSPIRE](#)].
- [9] S. Matsuda and Y. Ishimoto, *Mod. Phys. Lett. A* **11**, 2611 (1996) [[arXiv:hep-th/9609184](#)] [[Search INSPIRE](#)].
- [10] S. Matsuda, *Int. J. Mod. Phys. A* **12**, 4707 (1997) [[arXiv:hep-th/9610212](#)] [[Search INSPIRE](#)].
- [11] N. Seiberg and E. Witten, *J. High Energy Phys.* **9904**, 017 (1999) [[arXiv:hep-th/9903224](#)] [[Search INSPIRE](#)].
- [12] H. Ooguri and C. Vafa, *Nucl. Phys. B* **463**, 55 (1996) [[arXiv:hep-th/9511164](#)] [[Search INSPIRE](#)].
- [13] T. Eguchi and Y. Sugawara, *J. High Energy Phys.* **0401**, 025 (2004) [[arXiv:hep-th/0311141](#)] [[Search INSPIRE](#)].
- [14] T. Eguchi and Y. Sugawara, *J. High Energy Phys.* **1103**, 107 (2011) [[arXiv:1012.5721](#) [hep-th]] [[Search INSPIRE](#)].
- [15] S. Zweegers, Mock theta functions, Ph.D. Thesis, Utrecht University (2002).
- [16] E. Witten, *Int. J. Mod. Phys. A* **9**, 4783 (1994) [[arXiv:hep-th/9304026](#)] [[Search INSPIRE](#)].
- [17] J. Troost, *J. High Energy Phys.* **1006**, 104 (2010) [[arXiv:1004.3649](#) [hep-th]] [[Search INSPIRE](#)].
- [18] T. Eguchi and Y. Sugawara, *J. High Energy Phys.* **0405**, 014 (2004) [[arXiv:hep-th/0403193](#)] [[Search INSPIRE](#)].
- [19] T. Eguchi and Y. Sugawara, *J. High Energy Phys.* **1411**, 156 (2014) [[arXiv:1407.7721](#) [hep-th]] [[Search INSPIRE](#)].
- [20] T. Eguchi and A. Taormina, *Phys. Lett. B* **200**, 315 (1988).
- [21] T. Eguchi and A. Taormina, *Phys. Lett. B* **210**, 125 (1988).
- [22] T. Eguchi, H. Ooguri, A. Taormina, and S. K. Yang, *Nucl. Phys. B* **315**, 193 (1989).
- [23] A. Giveon, D. Kutasov, and O. Pele, *J. High Energy Phys.* **9910**, 035 (1999) [[arXiv:hep-th/9907178](#)] [[Search INSPIRE](#)].
- [24] M. Eichler and D. Zagier, *The Theory of Jacobi Forms: Progress in Mathematics Vol. 55* (Springer, Berlin, 1985).
- [25] J. A. Harvey, S. Murthy, and C. Nazaroglu, *J. High Energy Phys.* **1505**, 126 (2015) [[arXiv:1410.6174](#) [hep-th]] [[Search INSPIRE](#)].
- [26] A. Dabholkar, S. Murthy, and D. Zagier, [arXiv:1208.4074](#) [hep-th] [[Search INSPIRE](#)].