

Heat kernel approach to the relations between covariant and consistent currents in chiral gauge theories

Masaharu Takeuchi^{1,*} and Ryusuke Endo²

¹*Graduate School of Science and Engineering, Yamagata University, Yamagata 990-8560, Japan*

²*Department of Physics, Yamagata University, Yamagata 990-8560, Japan*

*E-mail: m.takeuchi.258@gmail.com

Received January 25, 2017; Accepted January 29, 2017; Published March 4, 2017

.....
We apply the heat kernel method to relations between covariant and consistent currents in anomalous chiral gauge theories. Banerjee et al. have shown that the relation between these currents is expressed by a “functional curl” of the covariant current. Using the heat kernel method, we evaluate the functional curl explicitly in arbitrary even dimensions. We also apply the heat kernel method to evaluate Osabe and Suzuki’s results of the difference between covariant and consistent currents in two and four dimensions. Applying the arguments of Banerjee et al. to gravitational anomalies, we investigate the relationship between the covariant and consistent energy–momentum tensors. The relation is found to be expressed by a functional curl of the covariant energy–momentum tensor.
.....

Subject Index B06, B31, B33

1. Introduction

Chiral gauge anomalies can be viewed in one of two ways, namely, covariant and consistent. Covariant anomalies are defined as covariant divergences of the covariant current, i.e., a covariant divergence of the covariantly regularized expectation value of the current. Consistent anomalies can be considered as gauge transformations of a regularized effective action. From this definition, consistent anomalies satisfy the Wess–Zumino consistency condition (Ref. [1]).

The covariant and consistent anomalies are known to be equivalent in the sense that they lead to the same anomaly-cancellation condition. Bardeen and Zumino (Ref. [2]) have given a general proof for this equivalence of the anomalies using algebraic prescriptions. Their approach does not need any explicit form for the Lagrangians, thus giving model-independent results. Lagrangian-based field-theoretical approaches to the equivalence of the gauge anomalies have been given by various authors (Refs. [3–8]). In particular, Banerjee et al. (Ref. [3]) have shown equivalence by introducing a regularized effective action defined through covariant current.

To prove the equivalence of covariant and consistent gauge anomalies, Banerjee et al. (Ref. [3]) gave a relationship between the covariant and consistent currents. The consistent current was derived as a functional derivative of a regularized effective action, which was defined using the covariant current (Ref. [3]). As a result, the relationship between the covariant and consistent currents is

expressed by a “functional curl” of the covariant current.¹ The authors of Ref. [3] argued that the functional curl of the covariant current is proportional to the delta function. With the help of the delta-function-type behavior of the functional curl, they have derived the relationship between the covariant and consistent gauge anomalies. Although their result agrees with Bardeen and Zumino (Ref. [2]), the delta-function-type behavior of the functional curl is not clearly explained in their arguments. Thus, it is desirable to prove the behavior of the functional curl more explicitly.

The functional curl of the covariant current has been discussed by various authors (Refs. [6,7,9–12]). Fujikawa and Suzuki (Ref. [6]) gave a formal proof of the relationship between the functional curl and the covariant anomaly; this relation was derived by Banerjee et al. (Ref. [3]) using the delta-function-type behavior of the functional curl. Ohshima et al. (Ref. [7]) evaluated the functional curl of the covariant current in supersymmetric chiral gauge theory. This curl was evaluated explicitly by using the Fourier transformation in four dimensions. Based on their motivation, which differed from that of Banerjee et al., Qiu and Ren (Ref. [12]) evaluated the functional curl explicitly by using the point-splitting method in two and four dimensions. All of these results are consistent with the curl’s delta-function-type behavior.

Other studies concerning the relationship between the covariant and consistent currents have been reported in Refs. [4,5,8], where the functional curl does not appear in the arguments. The difference between the covariant and consistent currents has been directly calculated using Pauli–Villars regularization (Ref. [4]) and the point-splitting method (Ref. [5]). Osabe and Suzuki (Ref. [8]) also discussed the difference between covariant and consistent currents, which they defined by invoking different types of exponential regulators. These regulators were then used to obtain a formal expression of the difference between the covariant and consistent currents.

In this paper, by using the heat kernel method (Ref. [13]), we evaluate the functional curl of the covariant current explicitly. The curl that we derive agrees with that of Refs. [3,6]. Our result presents another direct proof of the delta-function-type behavior of the functional curl in arbitrary even dimensions. We also apply the heat kernel method to evaluate Osabe and Suzuki’s formal expression of the difference between the covariant and consistent currents (Ref. [8]). This difference, which we calculate in two and four dimensions, agrees with previous results (Refs. [2,3]). The arguments of Banerjee et al. (Ref. [3]) are also applied to gravitational anomalies (Ref. [14]).² We investigate the relationship between the covariant and consistent energy–momentum tensors, which is found to be expressed by a functional curl of the covariant energy–momentum tensor.

The rest of this paper is outlined as follows. In Sect. 2, we review the arguments of Banerjee et al. (Ref. [3]) concerning covariant and consistent gauge anomalies. In Sect. 3, we evaluate the functional curl of the covariant current explicitly by using the heat kernel method in arbitrary even dimensions. In Sect. 4, we apply the heat kernel method to Osabe and Suzuki’s difference of the covariant and consistent currents (Ref. [8]) in two and four dimensions. In Sect. 5, by applying the arguments of Banerjee et al. (Ref. [3]) to the gravitational anomalies, we investigate the relationship between the covariant and consistent energy–momentum tensors. Section 6 is devoted to a summary and discussion.

¹ The functional curl of the covariant current appears also in the covariant commutator anomaly (Refs. [10, 11]).

² The equivalence of the covariant and consistent gravitational anomalies is also shown in Ref. [2] by the algebraic approach. We are interested here in the field-theoretical approach to the equivalence.

2. Functional curl of the covariant current

We consider a chiral gauge theory given by the following $2n$ -dimensional Euclidean Lagrangian:

$$\mathcal{L} = \bar{\psi} \gamma^\mu \left(\partial_\mu + i A_\mu^a T^a \frac{1 - \gamma_5}{2} \right) \psi, \quad (2.1)$$

where ψ and $\bar{\psi}$ are the Dirac spinors, and A_μ^a are the gauge fields. The metric we use is $\eta^{\mu\nu} = -\delta^{\mu\nu}$. The Dirac gamma matrices γ^μ are antihermitian, and $\gamma_5 = i^n \gamma^1 \gamma^2 \cdots \gamma^{2n}$ is hermitian. The matrices γ^μ and the hermitian generators T^a satisfy

$$\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu}, \quad (2.2)$$

$$[T^a, T^b] = i f^{abc} T^c, \quad (2.3)$$

$$\text{tr } T^a T^b = \frac{1}{2} \delta^{ab}, \quad (2.4)$$

where f^{abc} are the structure constants of the gauge group. The Lagrangian $\mathcal{L}$ is invariant under these gauge transformations:

$$\delta_\alpha \psi(x) = i \alpha^a(x) T^a \frac{1 - \gamma_5}{2} \psi(x), \quad (2.5)$$

$$\delta_\alpha \bar{\psi}(x) = -i \bar{\psi}(x) \alpha^a(x) T^a \frac{1 + \gamma_5}{2}, \quad (2.6)$$

$$\delta_\alpha A_\mu^a(x) = -D_\mu \alpha^a(x) = -\partial_\mu \alpha^a(x) + f^{abc} A_\mu^b(x) \alpha^c(x). \quad (2.7)$$

2.1. Covariant and consistent currents

Although the Lagrangian (2.1) is invariant under gauge transformations, the effective action is not. The effective action $W[A_\mu^a]$ transforms as

$$\begin{aligned} \delta_\alpha W[A_\mu^a] &= \int dx \frac{\delta W}{\delta A_\mu^a(x)} \delta_\alpha A_\mu^a(x) \\ &= i \int dx \alpha^a(x) G^a(x), \end{aligned} \quad (2.8)$$

where the gauge anomaly $G^a(x)$ is defined by

$$G^a(x) = D_\mu \langle J^{\mu a}(x) \rangle, \quad (2.9)$$

with the vacuum expectation value of the current $\langle J^{\mu a}(x) \rangle$ given by

$$\langle J^{\mu a}(x) \rangle = \frac{1}{i} \frac{\delta}{\delta A_\mu^a(x)} W[A_\nu^b]. \quad (2.10)$$

These expressions have only formal meanings, i.e., $\langle J^{\mu a}(x) \rangle$ is divergent since $W[A_\nu^b]$ is divergent. To treat current (2.10) meaningfully, we should adopt an appropriate regularization.

We usually adopt either the consistent or covariant regularization. The consistently regularized current $\langle J^{\mu a}(x) \rangle_{\text{cons}}$ is defined through the regularization of the effective action $W[A_\nu^b]$. Using the regularized effective action $W_{\text{reg}}[A_\nu^b]$, we define a regularized current

$$\langle J^{\mu a}(x) \rangle_{\text{cons}} = \frac{1}{i} \frac{\delta}{\delta A_\mu^a(x)} W_{\text{reg}}[A_\nu^b]. \quad (2.11)$$

We note that the consistent current $\langle J^{\mu a}(x) \rangle_{\text{cons}}$ given by Eq. (2.11) satisfies the integrability condition

$$\frac{\delta}{\delta A_\mu^a(x)} \langle J^{\nu b}(x') \rangle_{\text{cons}} - \frac{\delta}{\delta A_\nu^b(x')} \langle J^{\mu a}(x) \rangle_{\text{cons}} = 0. \quad (2.12)$$

If $W_{\text{reg}}[A_\nu^b]$ is gauge invariant, the current $\langle J^{\mu a}(x) \rangle_{\text{cons}}$ transforms covariantly under gauge transformation. In the anomalous gauge theory, however, $W_{\text{reg}}[A_\nu^b]$ is not gauge invariant and thus the current $\langle J^{\mu a}(x) \rangle_{\text{cons}}$ does not transform covariantly.

The covariant current $\langle J^{\mu a}(x) \rangle_{\text{cov}}$ is the expectation value of current regularized covariantly with respect to gauge transformation. In contrast with the current $\langle J^{\mu a}(x) \rangle_{\text{cons}}$, the $\langle J^{\mu a}(x) \rangle_{\text{cov}}$ transforms covariantly under the gauge transformation (2.7). Consequently, $\langle J^{\mu a}(x) \rangle_{\text{cov}}$ cannot be expressed in the form of Eq. (2.11) in the anomalous theory. In particular, the covariant current does not satisfy the integrability condition (2.12). These expectation values are functionals of A_ν^b . When we need to pay attention to the functional property, we use a symbol such as $\langle J^{\mu a}(x) \rangle_{\text{cov}}[A_\nu^b]$.

Substituting these regularized currents into Eq. (2.9), we obtain the following gauge anomalies:

$$G_{\text{cov}}^a(x) = D_\mu \langle J^{\mu a}(x) \rangle_{\text{cov}} \quad (2.13)$$

and

$$G_{\text{cons}}^a(x) = D_\mu \langle J^{\mu a}(x) \rangle_{\text{cons}}, \quad (2.14)$$

where $G_{\text{cov}}^a(x)$ and $G_{\text{cons}}^a(x)$ are called covariant and consistent, respectively. The consistent anomaly $G_{\text{cons}}^a(x)$ satisfies the Wess–Zumino consistency condition (Ref. [1]), which is ascribed to the integrability condition (2.12).

The covariant anomaly $G_{\text{cov}}^a(x)$ can be expressed as (see, e.g., Ref. [6])

$$G_{\text{cov}}^a(x) = \lim_{s \rightarrow 0} \lim_{x' \rightarrow x} \text{tr} T^a \gamma_5 e^{-s \mathcal{D}^2} \delta(x - x'), \quad (2.15)$$

where s is the cut-off parameter and $\mathcal{D} = \gamma^\mu (\partial_\mu + i A_\mu^a T^a)$. This quantity can be calculated (Ref. [6]) as

$$G_{\text{cov}}^a(x) = \frac{(-1)^n}{(4\pi)^n n!} \varepsilon^{\mu_1 \nu_1 \cdots \mu_n \nu_n} \text{tr} T^a F_{\mu_1 \nu_1}(x) \cdots F_{\mu_n \nu_n}(x), \quad (2.16)$$

where $\varepsilon^{\mu_1 \nu_1 \cdots \mu_n \nu_n}$ is the totally antisymmetric tensor with $\varepsilon^{12 \cdots 2n} = 1$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ is the field strength of the gauge fields $A_\mu = A_\mu^a T^a$. The covariant anomaly $G_{\text{cov}}^a(x)$ is a finite local polynomial of field strength $F_{\mu\nu}(x)$.

2.2. Relationship between the covariant and consistent currents

We follow Banerjee et al. (Ref. [3]) in deriving the relationship between the covariant and consistent currents. We introduce a parameter g and define

$$W_g = W[gA_\mu^a]. \quad (2.17)$$

If we put $g = 1$, then W_g reduces to the original effective action $W[A_\mu^a]$. We can express $W[A_\mu^a]$ using W_g as

$$W[A_\mu^a] = \int_0^1 dg \frac{\partial W_g}{\partial g} + W_{g=0}. \quad (2.18)$$

Note that the g -dependence arises only through the combination gA_μ^a ; we obtain

$$W[A_\mu^a] = \int_0^1 dg \int dx \frac{\delta W_g}{\delta (gA_\mu^a(x))} A_\mu^a(x), \quad (2.19)$$

where we have dropped the $W_{g=0}$ term since it is an A_μ^a -independent constant. From definition (2.10), we rewrite Eq. (2.19) as

$$W[A_\mu^a] = i \int_0^1 dg \int dx A_\mu^a(x) \langle J^{\mu a}(x) \rangle^g, \quad (2.20)$$

where we have used the notation

$$\langle J^{\mu a}(x) \rangle^g = \langle J^{\mu a}(x) \rangle [gA_\nu^b]. \quad (2.21)$$

Expression (2.20) has only a formal meaning because the current $\langle J^{\mu a}(x) \rangle^g$ is not yet regularized. The crucial point of the prescription of Ref. [3] is to substitute covariant current $\langle J^{\mu a}(x) \rangle_{\text{cov}}^g = \langle J^{\mu a}(x) \rangle_{\text{cov}} [gA_\nu^b]$ for $\langle J^{\mu a}(x) \rangle^g$ in Eq. (2.20) to construct a regularized effective action $W[A_\mu^a]_{\text{reg}}$:

$$W[A_\mu^a]_{\text{reg}} = i \int_0^1 dg \int dx A_\mu^a(x) \langle J^{\mu a}(x) \rangle_{\text{cov}}^g. \quad (2.22)$$

We can obtain a consistent current from the regularized effective action (2.22). Taking the functional derivative of Eq. (2.22) with respect to $A_\mu^a(x)$, we obtain the relationship between the covariant and consistent currents (Ref. [3]):

$$\langle J^{\mu a}(x) \rangle_{\text{cons}} = \langle J^{\mu a}(x) \rangle_{\text{cov}} + \int_0^1 dg \int dx' g A_\nu^b(x') \left\{ \frac{\delta \langle J^{\nu b}(x') \rangle_{\text{cov}}^g}{\delta (gA_\mu^a(x))} - \frac{\delta \langle J^{\mu a}(x) \rangle_{\text{cov}}^g}{\delta (gA_\nu^b(x'))} \right\}. \quad (2.23)$$

Note that the “functional curl” of the covariant current appears in the integrand of the second term on the right-hand side. The functional curl in Eq. (2.23) is obtained by substituting gA_μ^a into A_μ^a in the functional curl

$$\frac{\delta \langle J^{\nu b}(x') \rangle_{\text{cov}}}{\delta A_\mu^a(x)} - \frac{\delta \langle J^{\mu a}(x) \rangle_{\text{cov}}}{\delta A_\nu^b(x')}, \quad (2.24)$$

which does not vanish since the covariant current does not satisfy the integrability condition (2.12) in the anomalous theory.³ Taking the covariant divergence of Eq. (2.23), we obtain the relationship between the covariant and consistent gauge anomalies:

$$G_{\text{cons}}^a(x) = G_{\text{cov}}^a(x) + D_\mu^{ac} \int_0^1 dg \int dx' g A_\nu^b(x') \left\{ \frac{\delta \langle J^{\nu b}(x') \rangle_{\text{cov}}^g}{\delta (gA_\mu^c(x))} - \frac{\delta \langle J^{\mu c}(x) \rangle_{\text{cov}}^g}{\delta (gA_\nu^b(x'))} \right\}, \quad (2.25)$$

where $D_\mu^{ac} = \delta^{ac} \partial_\mu - f^{adc} A_\mu^d$.

³ It can be seen that the parity-conserving part of the functional curl vanishes.

Banerjee et al. (Ref. [3]) have evaluated the functional curl of the covariant current by using the fact that this curl has delta-function-type behavior at $x = x'$:

$$\frac{\delta \langle J^{vb}(x') \rangle_{\text{cov}}}{\delta A_\mu^a(x)} - \frac{\delta \langle J^{\mu a}(x) \rangle_{\text{cov}}}{\delta A_\nu^b(x')} \propto \delta(x - x'). \quad (2.26)$$

Using expression (2.26), they showed that the functional curl can be expressed by the covariant gauge anomaly,

$$\frac{\delta \langle J^{vb}(x') \rangle_{\text{cov}}}{\delta A_\mu^a(x)} - \frac{\delta \langle J^{\mu a}(x) \rangle_{\text{cov}}}{\delta A_\nu^b(x')} = -2 \frac{\delta G_{\text{cov}}^b(x')}{\delta F_{\mu\nu}^a(x)}. \quad (2.27)$$

Substituting this equation into Eq. (2.25), they derived an expression for the consistent gauge anomaly that agrees with the result of Ref. [2]. In the arguments of Ref. [3] given above, it is crucial for expression (2.26) to actually hold. In Ref. [3], however, a detailed proof of expression (2.26) is not shown. Considering this point, we evaluate the functional curl explicitly in the next section.

3. Explicit evaluation of the functional curl of the covariant current

The expectation value of the current can be expressed by

$$\begin{aligned} \langle J^{\mu a}(x) \rangle &= \left\langle \bar{\psi}(x) \gamma^\mu T^a \frac{1 - \gamma_5}{2} \psi(x) \right\rangle \\ &= \lim_{x' \rightarrow x} \text{tr} \frac{1 + \gamma_5}{2} \gamma^\mu T^a \frac{1}{\mathcal{D}} \delta(x - x'), \end{aligned} \quad (3.1)$$

where $\mathcal{D} = \gamma^\mu (\partial_\mu + iA_\mu^a T^a)$. To regularize Eq. (3.1), we employ the Gaussian regulator to define a covariant current (Ref. [6]),

$$\langle J^{\mu a}(x) \rangle_{\text{cov}} = \lim_{s \rightarrow 0} \lim_{x' \rightarrow x} \text{tr} \frac{1 + \gamma_5}{2} \gamma^\mu T^a \frac{1}{\mathcal{D}} e^{-s\mathcal{D}^2} \delta(x - x'), \quad (3.2)$$

where s is the cut-off parameter. Because the regulator $e^{-s\mathcal{D}^2}$ is covariant, the current $\langle J^{\mu a}(x) \rangle_{\text{cov}}$ transforms covariantly. Taking the functional curl of Eq. (3.2) and using trace properties, we have (Ref. [6])

$$\begin{aligned} &\frac{\delta \langle J^{vb}(x') \rangle_{\text{cov}}}{\delta A_\mu^a(x)} - \frac{\delta \langle J^{\mu a}(x) \rangle_{\text{cov}}}{\delta A_\nu^b(x')} \\ &= -i \lim_{s \rightarrow 0} s \text{tr} \gamma_5 \gamma^\nu T^b \int_0^1 d\alpha \left(e^{-(1-\alpha)s\mathcal{D}'^2} \delta(x' - x) \right) \gamma^\mu T^a e^{-\alpha s\mathcal{D}'^2} \delta(x - x'), \end{aligned} \quad (3.3)$$

where $\mathcal{D}' = \gamma^\mu (\partial'_\mu + iA_\mu^a(x') T^a)$. Here Fujikawa and Suzuki have shown that the right-hand side of Eq. (3.3) is equal to the functional derivative of the expression for the covariant anomaly (2.15) with respect to the field strength (Ref. [6]), which gives a formal proof of Eq. (2.27) and thus gives the proof of expression (2.26).

In the following, we evaluate the functional curl (3.3) explicitly by using the heat kernel method (Ref. [13]). The functional curl (3.3) can be expressed by

$$\begin{aligned} &\frac{\delta \langle J^{vb}(x') \rangle_{\text{cov}}}{\delta A_\mu^a(x)} - \frac{\delta \langle J^{\mu a}(x) \rangle_{\text{cov}}}{\delta A_\nu^b(x')} \\ &= -i \lim_{s \rightarrow 0} s \text{tr} \gamma_5 \gamma^\nu T^b \int_0^1 d\alpha K(x', x; (1 - \alpha)s) \gamma^\mu T^a K(x, x'; \alpha s), \end{aligned} \quad (3.4)$$

where $K(x, x'; s)$ is the heat kernel defined by

$$K(x, x'; s) = e^{-s\mathcal{D}^2} \delta(x - x'). \quad (3.5)$$

Substituting the heat-kernel expansion

$$K(x, x'; s) = \frac{1}{(4\pi s)^n} \exp((x - x')^2/4s) \sum_{k=0}^{\infty} a_k(x, x') s^k \quad (3.6)$$

into Eq. (3.4), we have

$$\begin{aligned} \text{Eq. (3.4)} = & -i \frac{1}{(4\pi)^{2n}} \int_0^1 d\alpha \sum_{k,l} (1-\alpha)^{k-n} \alpha^{l-n} s^{1+k+l-2n} \exp((x-x')^2/4\alpha(1-\alpha)s) \\ & \times \text{tr } \gamma_5 \gamma^\nu T^b a_k(x', x) \gamma^\mu T^a a_l(x, x'), \end{aligned} \quad (3.7)$$

where we have suppressed the symbol $\lim_{s \rightarrow 0}$. The exponential function appearing on the right-hand side can be understood as the heat kernel of the free theory. That is, if we define

$$K_0(x, x'; s) = \frac{1}{(4\pi s)^n} \exp((x - x')^2/4s), \quad (3.8)$$

then $K_0(x, x'; s)$ satisfies

$$\frac{\partial}{\partial s} K_0(x, x'; s) = -\square K_0(x, x'; s), \quad K_0(x, x'; s=0) = \delta(x - x'), \quad (3.9)$$

where $\square = \partial_\mu \partial^\mu$. A formal solution to Eq. (3.9) can be written as

$$K_0(x, x'; s) = e^{-s\square} \delta(x - x'). \quad (3.10)$$

Taking the Taylor expansion of $e^{-s\square}$ with respect to s , we have⁴

$$\exp((x - x')^2/4s) = (4\pi s)^n \sum_{k=0}^{\infty} \frac{(-s\square)^k}{k!} \delta(x - x'). \quad (3.11)$$

With this formula and the integration formula

$$\int_0^1 d\alpha (1-\alpha)^{k+m} \alpha^{l+m} = \frac{(k+m)!(l+m)!}{(k+l+2m+1)!}, \quad (3.12)$$

Eq. (3.7) can be written as

$$\begin{aligned} \text{Eq. (3.7)} = & -i \frac{1}{(4\pi)^n} \sum_{k,l,m} \frac{1}{m!} \frac{(k+m)!(l+m)!}{(k+l+2m+1)!} s^{1+k+l+m-n} \\ & \times \text{tr } \gamma_5 \gamma^\nu T^b a_k(x', x) \gamma^\mu T^a a_l(x, x') (-\square)^m \delta(x - x'). \end{aligned} \quad (3.13)$$

Considering that the terms higher than zeroth order in s vanish in the limit $s \rightarrow 0$, we find that the indices k , l , and m of the surviving terms (3.13) satisfy the condition

$$1 + k + l + m - n \leq 0. \quad (3.14)$$

⁴ A proof of Eq. (3.11) using test functions is given in Appendix A.

In addition, the surviving terms must contain at least $2n$ factors of gamma matrices γ^μ , because of the existence of γ_5 in the trace over spinor indices. As shown in Appendix B, $a_k(x, x')$ contains at most $2k$ factors of γ^μ . Consequently, indices k and l of the surviving terms satisfy the condition

$$2 + 2k + 2l \geq 2n. \quad (3.15)$$

Conditions (3.14) and (3.15) lead to

$$m = 0, \quad (3.16)$$

$$l = n - k - 1. \quad (3.17)$$

Then, Eq. (3.13) becomes

Eq. (3.13)

$$= -i \frac{1}{(4\pi)^n} \sum_{k=0}^{n-1} \frac{k!(n-k-1)!}{n!} \text{tr} \gamma_5 \gamma^\nu T^b a_k(x, x) \gamma^\mu T^a a_{n-k-1}(x, x) \delta(x - x'). \quad (3.18)$$

Here $a_k(x, x)$ is given by Eq. (B.7), starting with the term containing $2k$ factors of γ^μ :

$$a_k(x, x) = \frac{(-1)^k}{k!} \left(\frac{i}{2} \gamma^\mu \gamma^\nu F_{\mu\nu} \right)^k + \dots, \quad (B.7)$$

where the dots on the right-hand side express terms with a lower power of γ^μ . Substituting Eq. (B.7) into Eq. (3.18), we obtain the final expression for the functional curl,

$$\begin{aligned} & \frac{\delta \langle J^{vb}(x') \rangle_{\text{cov}}}{\delta A_\mu^a(x)} - \frac{\delta \langle J^{\mu a}(x) \rangle_{\text{cov}}}{\delta A_\nu^b(x')} \\ &= -2 \frac{(-1)^n}{(4\pi)^n (n-1)!} \varepsilon^{\mu\nu\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}} \text{Str} T^a T^b F_{\mu_1\nu_1} \cdots F_{\mu_{n-1}\nu_{n-1}} \delta(x - x'), \end{aligned} \quad (3.19)$$

where the symbol “Str” denotes the symmetrized trace (Ref. [15]) indicating that the factors in the trace are to be totally symmetrized. We notice here that our evaluation gives a direct proof of expression (2.26). Comparing this expression with the final expression of the covariant anomaly (2.16), we again obtain Eq. (2.27).

4. Explicit evaluation of Osabe and Suzuki’s expression for the current difference

Osabe and Suzuki (Ref. [8]) have also discussed the difference between the consistent and covariant currents. Their consistent current $\langle J^{\mu a}(x) \rangle_{\text{cons}}$ can be written as

$$\langle J^{\mu a}(x) \rangle_{\text{cons}} = \lim_{s \rightarrow 0} \text{tr} \left\langle x \left| \frac{1 + \gamma_5}{2} \gamma^\mu T^a \frac{1}{\mathcal{D}} e^{-s\mathcal{D}\not{\partial}} \right| x \right\rangle \quad (4.1)$$

in our notation, while the covariant current $\langle J^{\mu a}(x) \rangle_{\text{cov}}$ is given by Eq. (3.2), i.e.,

$$\langle J^{\mu a}(x) \rangle_{\text{cov}} = \lim_{s \rightarrow 0} \text{tr} \left\langle x \left| \frac{1 + \gamma_5}{2} \gamma^\mu T^a \frac{1}{\mathcal{D}} e^{-s\mathcal{D}^2} \right| x \right\rangle. \quad (4.2)$$

From these definitions, they derived an expression for the difference between currents. Their derivation can be explained essentially as follows: Introducing $\mathcal{D}_g = \gamma^\mu D_\mu^g = \gamma^\mu (\partial_\mu + igA_\mu)$ and noticing the equality

$$e^{-s\mathcal{D}^2} - e^{-s\mathcal{D}\not{\partial}} = \int_0^1 dg \frac{d}{dg} e^{-s\mathcal{D}\mathcal{D}_g}, \quad (4.3)$$

we obtain

$$\begin{aligned}
& \langle J^{\mu a}(x) \rangle_{\text{cons}} - \langle J^{\mu a}(x) \rangle_{\text{cov}} \\
&= - \lim_{s \rightarrow 0} \int_0^1 dg \frac{d}{dg} \text{tr} \left\langle x \left| \frac{1 + \gamma_5}{2} \gamma^\mu T^a \frac{1}{\mathcal{D}} e^{-s \mathcal{D} \mathcal{D}_g} \right| x \right\rangle \\
&= - \lim_{s \rightarrow 0} \int_0^1 dg \int_0^1 d\alpha \text{tr} \left\langle x \left| \frac{1 + \gamma_5}{2} \gamma^\mu T^a \frac{1}{\mathcal{D}} e^{-(1-\alpha)s \mathcal{D} \mathcal{D}_g} \left(-s \mathcal{D} \frac{d \mathcal{D}_g}{dg} \right) e^{-\alpha s \mathcal{D} \mathcal{D}_g} \right| x \right\rangle \\
&= \lim_{s \rightarrow 0} s \int_0^1 dg \int_0^1 d\alpha \text{tr} \frac{1 + \gamma_5}{2} \gamma^\mu T^a \left\langle x \left| e^{-(1-\alpha)s \mathcal{D}_g \mathcal{D}} \frac{d \mathcal{D}_g}{dg} e^{-\alpha s \mathcal{D} \mathcal{D}_g} \right| x \right\rangle. \tag{4.4}
\end{aligned}$$

In the third line, we have used the identity

$$e^{-s \mathcal{D} \mathcal{D}_g} \mathcal{D} = \mathcal{D} e^{-s \mathcal{D}_g \mathcal{D}}. \tag{4.5}$$

Equation (4.4) is equivalent to Osabe and Suzuki's expression for the current difference (Ref. [8]).

Now, we calculate current difference (4.4) by applying the heat kernel method. Introducing heat kernels

$$K_g(x, x'; s) = \langle x | e^{-s \mathcal{D} \mathcal{D}_g} | x' \rangle, \tag{4.6}$$

$$\tilde{K}_g(x, x'; s) = \langle x | e^{-s \mathcal{D}_g \mathcal{D}} | x' \rangle, \tag{4.7}$$

we express Eq. (4.4) as

$$\begin{aligned}
& \langle J^{\mu a}(x) \rangle_{\text{cons}} - \langle J^{\mu a}(x) \rangle_{\text{cov}} \\
&= \frac{i}{2} \lim_{s \rightarrow 0} s \int_0^1 dg \int_0^1 d\alpha \int dx' \text{tr} \gamma_5 \gamma^\mu T^a \tilde{K}_g(x, x'; (1-\alpha)s) \mathcal{A}' K_g(x', x; \alpha s), \tag{4.8}
\end{aligned}$$

where $\mathcal{A}' = \gamma^\nu A_\nu(x')$ and we have omitted the parity-conserving terms since only parity-violating terms contribute to the anomalies. These kernels K_g and $\tilde{K}_g$ are not independent of each other. In fact, owing to the relation

$$\mathcal{D}_g \mathcal{D} = (\mathcal{D} \mathcal{D}_g)^\dagger, \tag{4.9}$$

they satisfy

$$\tilde{K}_g(x, x'; s) = K_g(x', x; s)^\dagger. \tag{4.10}$$

We expand $K_g(x, x'; s)$ and $\tilde{K}_g(x, x'; s)$ in $2n$ dimensions as

$$K_g(x, x'; s) = \frac{1}{(4\pi s)^n} \exp((x - x')^2/4s) \sum_{k=0}^{\infty} b_k(x, x') s^k, \tag{4.11}$$

$$\tilde{K}_g(x, x'; s) = \frac{1}{(4\pi s)^n} \exp((x - x')^2/4s) \sum_{k=0}^{\infty} \tilde{b}_k(x, x') s^k. \tag{4.12}$$

Note here that Eq. (4.10) indicates

$$\tilde{b}_k(x, x') = b_k(x', x)^\dagger. \tag{4.13}$$

Substituting expansions (4.11) and (4.12) into Eq. (4.8), we have

$$\begin{aligned} \text{Eq. (4.8)} &= \frac{i}{2} \frac{1}{(4\pi)^{2n}} \int_0^1 dg \int_0^1 d\alpha \int dx' \sum_{k,l} (1-\alpha)^{k-n} \alpha^{l-n} s^{1+k+l-2n} \\ &\quad \times \exp\left((x-x')^2/4\alpha(1-\alpha)s\right) \text{tr} \gamma_5 \gamma^\mu T^a \tilde{b}_k(x, x') A' b_l(x', x), \end{aligned} \quad (4.14)$$

where we have suppressed the symbol $\lim_{s \rightarrow 0}$. With the help of Eqs. (3.11) and (3.12), Eq. (4.14) becomes

$$\begin{aligned} \text{Eq. (4.14)} &= \frac{i}{2} \frac{1}{(4\pi)^n} \int_0^1 dg \int dx' \sum_{k,l,m} \frac{1}{m!} \frac{(k+m)!(l+m)!}{(k+l+2m+1)!} s^{1+k+l+m-n} \\ &\quad \times \text{tr} \gamma_5 \gamma^\mu T^a \tilde{b}_k(x, x') A' b_l(x', x) (-\square)^m \delta(x-x'). \end{aligned} \quad (4.15)$$

Note that the terms higher than zeroth order in s vanish in the limit $s \rightarrow 0$; we find that the indices k , l , and m of the surviving terms on the right-hand side satisfy the condition

$$1 + k + l + m - n \leq 0. \quad (4.16)$$

Below, we work in two and four dimensions.

In two dimensions ($n = 1$), condition (4.16) becomes

$$k + l + m \leq 0, \quad (4.17)$$

which means that $k = l = m = 0$. Thus, Eq. (4.15) reads

$$\begin{aligned} \langle J^{\mu a}(x) \rangle_{\text{cons}} - \langle J^{\mu a}(x) \rangle_{\text{cov}} &= \frac{i}{8\pi} \int_0^1 dg \int dx' \text{tr} \gamma_5 \gamma^\mu T^a \tilde{b}_0(x, x') A' b_0(x', x) \delta(x-x') \\ &= \frac{1}{4\pi} \varepsilon^{\mu\nu} \text{tr} T^a A_\nu(x), \end{aligned} \quad (4.18)$$

where we have used the coincidence limits $b_0(x, x) = \tilde{b}_0(x, x) = \mathbf{1}$ (Eqs. (C.4) and (4.13)). This agrees with the previous results (Refs. [2,3]).

In four dimensions ($n = 2$), condition (4.16) becomes

$$k + l + m - 1 \leq 0. \quad (4.19)$$

The solutions of this condition are $(k, l, m) = (0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. Calculating the four terms corresponding to these solutions, we obtain

$$\begin{aligned} &\langle J^{\mu a}(x) \rangle_{\text{cons}} - \langle J^{\mu a}(x) \rangle_{\text{cov}} \\ &= \frac{i}{2} \frac{1}{(4\pi)^2} \int_0^1 dg \int dx' \text{tr} \gamma_5 \gamma^\mu T^a \left(\frac{1}{s} \tilde{b}_0(x, x') A' b_0(x', x) \right. \\ &\quad \left. + \frac{1}{2} \tilde{b}_1(x, x') A' b_0(x', x) + \frac{1}{2} \tilde{b}_0(x, x') A' b_1(x', x) - \frac{1}{3!} \tilde{b}_0(x, x') A' b_0(x', x) \square \right) \delta(x-x') \\ &= \frac{i}{2} \frac{1}{(4\pi)^2} \int_0^1 dg \left(\frac{1}{s} \text{tr} \gamma_5 \gamma^\mu T^a [\tilde{b}_0] A [b_0] \right. \\ &\quad \left. + \frac{1}{2} \text{tr} \gamma_5 \gamma^\mu T^a [\tilde{b}_1] A [b_0] + \frac{1}{2} \text{tr} \gamma_5 \gamma^\mu T^a [\tilde{b}_0] A [b_1] - \frac{1}{3!} [\square' \text{tr} \gamma_5 \gamma^\mu T^a \tilde{b}_0 A' b_0] \right), \end{aligned} \quad (4.20)$$

where we have used Sygne's symbol (Ref. [16]) to denote coincidence limits such as $[\partial_\alpha b_0] = \lim_{x' \rightarrow x} \partial_\alpha b_0(x, x')$. Since $[b_0] = [\tilde{b}_0] = \mathbf{1}$, the first term in the integrand of Eq. (4.20) vanishes after taking the trace over the spinor indices. With the help of coincidence limits (C.4), (C.16), and (4.13), the second and third terms become

$$\frac{1}{2} \text{tr} \gamma_5 \gamma^\mu T^a [\tilde{b}_1] A[b_0] = i(1+g) \varepsilon^{\mu\alpha\beta\gamma} \text{tr} T^a A_\gamma \partial_\beta A_\alpha + (1+g^2) \varepsilon^{\mu\alpha\beta\gamma} \text{tr} T^a A_\alpha A_\beta A_\gamma, \quad (4.21)$$

$$\frac{1}{2} \text{tr} \gamma_5 \gamma^\mu T^a [\tilde{b}_0] A[b_1] = i(1+g) \varepsilon^{\mu\alpha\beta\gamma} \text{tr} A_\gamma T^a \partial_\beta A_\alpha + (1+g^2) \varepsilon^{\mu\alpha\beta\gamma} \text{tr} T^a A_\alpha A_\beta A_\gamma. \quad (4.22)$$

The last term in Eq. (4.20) can be calculated as follows. Note that

$$\begin{aligned} \square'(\tilde{b}_0 A' b_0) &= (\square' \tilde{b}_0) A' b_0 + \tilde{b}_0 A' \square' b_0 + \tilde{b}_0 (\square' A') b_0 \\ &\quad + 2(\partial'_\alpha \tilde{b}_0)(\partial'^\alpha A') b_0 + 2\tilde{b}_0 (\partial'_\alpha A') \partial'^\alpha b_0 + 2(\partial'_\alpha \tilde{b}_0) A' \partial'^\alpha b_0. \end{aligned} \quad (4.23)$$

The coincidence limit of Eq. (4.23) can be evaluated by using Eqs. (C.4), (C.14), (C.15), and (4.13); thus, we obtain

$$[\square' \text{tr} \gamma_5 \gamma^\mu T^a \tilde{b}_0 A' b_0] = 2i(1-g) \varepsilon^{\mu\alpha\beta\gamma} \text{tr}\{T^a, A_\gamma\} \partial_\beta A_\alpha + 8(1-g) \varepsilon^{\mu\alpha\beta\gamma} \text{tr} T^a A_\alpha A_\beta A_\gamma. \quad (4.24)$$

From these results, we finally obtain

$$\begin{aligned} \langle J^{\mu a}(x) \rangle_{\text{cons}} - \langle J^{\mu a}(x) \rangle_{\text{cov}} \\ = \frac{1}{24\pi^2} \varepsilon^{\mu\alpha\beta\gamma} \text{tr}\{T^a, A_\gamma\} \partial_\beta A_\alpha - \frac{i}{16\pi^2} \varepsilon^{\mu\alpha\beta\gamma} \text{tr} T^a A_\alpha A_\beta A_\gamma, \end{aligned} \quad (4.25)$$

which agrees with the previous results (Refs. [2,3]).

5. Relationship between the covariant and consistent energy–momentum tensors

In this section, we apply the arguments of Banerjee et al. (Ref. [3]), as explained in Sect. 2, to gravitational anomalies (Ref. [14]). The vacuum expectation value of the energy–momentum tensor density $\langle eT^a_\mu(x) \rangle$ is expressed by the effective action $W[e_b^\nu]$:

$$\langle eT^a_\mu(x) \rangle = \frac{\delta}{\delta e_a^\mu(x)} W[e_b^\nu], \quad (5.1)$$

where e_a^μ is the vielbein field, $e = \det e_a^\mu$, and e_a^μ is the inverse matrix of e_a^μ . Gravitational anomalies appear as nonzero values of $D^\mu \langle T^a_\mu(x) \rangle$ (Einstein anomaly) and/or $\langle T_{[ab]}(x) \rangle = \frac{1}{2} \langle T_{ab}(x) - T_{ba}(x) \rangle$ (Lorentz anomaly).

In Eq. (5.1), $\langle eT^a_\mu(x) \rangle$ is ill defined because $W[e_b^\nu]$ is a divergent quantity. To treat the energy–momentum tensor $\langle eT^a_\mu(x) \rangle$ meaningfully, we should adopt an appropriate regularization, either consistent or covariant. The consistently regularized energy–momentum tensor $\langle eT^a_\mu(x) \rangle_{\text{cons}}$ is defined by the regularized effective action $W[e_b^\nu]_{\text{reg}}$ as

$$\langle eT^a_\mu(x) \rangle_{\text{cons}} = \frac{\delta}{\delta e_a^\mu(x)} W[e_b^\nu]_{\text{reg}}. \quad (5.2)$$

The covariant energy–momentum tensor $\langle eT^a_\mu(x) \rangle_{\text{cov}}$ is the expectation value of the energy–momentum tensor regularized covariantly with respect to both the general coordinate and local Lorentz transformations. These expectation values are functionals of $e_a^\mu(x)$. When we need to pay attention to the functional property, we use a symbol such as $\langle eT^a_\mu(x) \rangle_{\text{cov}}[e_b^\nu]$.

Now, we introduce a vielbein field $e_a^\mu(t) = e_a^\mu(x, t)$, with one parameter t , which connects the original vielbein $e_a^\mu(x)$ to the flat space-time vielbein δ_a^μ such that

$$e_a^\mu(x, 0) = \delta_a^\mu, \quad (5.3)$$

$$e_a^\mu(x, 1) = e_a^\mu(x). \quad (5.4)$$

For example, we may adopt $e_a^\mu(t) = \delta_a^\mu + t(e_a^\mu(x) - \delta_a^\mu)$ or $e_a^\mu(t) = (e^{tH})_a^\mu$ with the matrix $H_a^\mu = (\ln e)_a^\mu$. We define a t -parametrized effective action W_t by

$$W_t = W[e_a^\mu(t)], \quad (5.5)$$

which reduces to the original effective action $W[e_a^\mu]$ if $t = 1$. We can express the effective action $W[e_a^\mu]$ by using W_t as

$$W[e_a^\mu] = \int_0^1 dt \frac{\partial W_t}{\partial t} + W_{t=0}. \quad (5.6)$$

Note that the t -dependence of W_t arises only through $e_a^\mu(t)$; we obtain

$$W[e_a^\mu] = \int_0^1 dt \int dx \frac{\delta W_t}{\delta e_a^\mu(x, t)} \frac{\partial e_a^\mu(x, t)}{\partial t}, \quad (5.7)$$

where we have dropped the $W_{t=0}$ term, since it is an e_a^μ -independent constant. From definition (5.1), we rewrite this equation as

$$W[e_a^\mu] = \int_0^1 dt \int dx \langle eT_\mu^a \rangle^t \frac{\partial e_a^\mu(t)}{\partial t}, \quad (5.8)$$

where we have used the notation

$$\langle eT_\mu^a \rangle^t = \langle eT_\mu^a \rangle[e_b^\nu(t)]. \quad (5.9)$$

To construct a regularized effective action $W_{\text{reg}}[e_a^\mu]$, we substitute the covariant energy–momentum tensor $\langle eT_\mu^a \rangle_{\text{cov}}^t = \langle eT_\mu^a \rangle_{\text{cov}}[e_b^\nu(t)]$ for $\langle eT_\mu^a \rangle^t$ on the right-hand side of Eq. (5.8):

$$W_{\text{reg}}[e_a^\mu] = \int_0^1 dt \int dx \langle eT_\mu^a \rangle_{\text{cov}}^t \frac{\partial e_a^\mu(t)}{\partial t}. \quad (5.10)$$

We can obtain a consistent energy–momentum tensor from the regularized effective action (5.10). Taking the variation of Eq. (5.10) with respect to e_a^μ , we obtain the following relationship between the covariant and consistent energy–momentum tensors:

$$\begin{aligned} & \int dx \langle eT_\mu^a \rangle_{\text{cons}} \delta e_a^\mu \\ &= \int_0^1 dt \int dx \langle eT_\mu^a \rangle_{\text{cov}}^t \frac{\partial}{\partial t} \delta e_a^\mu(t) + \int_0^1 dt \int dx \int dx' \frac{\delta \langle eT_\mu^a \rangle_{\text{cov}}^t}{\delta e_{b'}^{\nu'}(t)} \delta e_{b'}^{\nu'}(t) \frac{\partial e_a^\mu(t)}{\partial t} \\ &= \int dx \langle eT_\mu^a \rangle_{\text{cov}} \delta e_a^\mu \\ &+ \int_0^1 dt \int dx \int dx' \left\{ \frac{\delta \langle eT_\mu^a \rangle_{\text{cov}}^t}{\delta e_{b'}^{\nu'}(t)} - \frac{\delta \langle e'T_{\nu'}^{b'} \rangle_{\text{cov}}^t}{\delta e_a^\mu(t)} \right\} \delta e_{b'}^{\nu'}(t) \frac{\partial e_a^\mu(t)}{\partial t}, \end{aligned} \quad (5.11)$$

where we have applied integration by parts to the first term in the second line and used the fact that the t -dependence of $\langle eT_\mu^a \rangle_{\text{cov}}^t$ arises only through $e_a^\mu(t)$. In Eq. (5.11), primed indices denote those attached at the point x' such as

$$\langle e'T_{v'}^{b'} \rangle = \langle e(x')T_v^b(x') \rangle, \quad e_{b'}^{v'}(t) = e_b^v(x', t). \quad (5.12)$$

We emphasize that the “functional curl” of the covariant energy–momentum tensor appears in Eq. (5.11). This curl vanishes only when the theory is not anomalous. In fact, if the theory is anomaly free, the regularized effective action is invariant under the general coordinate and local Lorentz transformations. In this case, the consistent energy–momentum tensor becomes covariant, and thus the covariant energy–momentum tensor satisfies the integrability condition, i.e., the condition of vanishing functional curl. Conversely, if the functional curl of the covariant energy–momentum tensor is zero, the consistent energy–momentum tensor coincides with the covariant one, as seen from Eq. (5.11). In this case, the covariant and consistent gravitational anomalies coincide with each other. The diagrammatic approach to the anomaly, however, tells us that the leading terms of these anomalies differ by the Bose-symmetrization factor $1/(n+1)$ in $2n$ dimensions. This is true only when both anomalies are zero. Thus, the vanishing functional curl indicates an anomaly-free theory.

The relationships between the covariant and consistent gravitational anomalies are derived immediately from Eq. (5.11). For example, if we adopt the parametrization $e_a^\mu(t) = \delta_a^\mu + t(e_a^\mu(x) - \delta_a^\mu)$, Eq. (5.11) becomes

$$\langle eT_\mu^a \rangle_{\text{cons}} = \langle eT_\mu^a \rangle_{\text{cov}} + \int_0^1 dt \int dx' t(e_{b'}^{v'} - \delta_{b'}^{v'}) \left\{ \frac{\delta \langle e'T_{v'}^{b'} \rangle_{\text{cov}}^t}{\delta e_a^\mu(t)} - \frac{\delta \langle eT_\mu^a \rangle_{\text{cov}}^t}{\delta e_{b'}^{v'}(t)} \right\}. \quad (5.13)$$

Taking the covariant divergence of both sides, we obtain a relationship between the covariant and consistent Einstein anomalies:

$$D^\mu \langle eT_\mu^a \rangle_{\text{cons}} = D^\mu \langle eT_\mu^a \rangle_{\text{cov}} + D^\mu \int_0^1 dt \int dx' t(e_{b'}^{v'} - \delta_{b'}^{v'}) \left\{ \frac{\delta \langle e'T_{v'}^{b'} \rangle_{\text{cov}}^t}{\delta e_a^\mu(t)} - \frac{\delta \langle eT_\mu^a \rangle_{\text{cov}}^t}{\delta e_{b'}^{v'}(t)} \right\}. \quad (5.14)$$

The relationship between the Lorentz anomalies can be similarly obtained.

6. Summary and discussion

In Sect. 3, we evaluated the functional curl of the covariant current explicitly using the heat kernel method in arbitrary even dimensions. The result gives a direct proof of the delta-function-type behavior of the functional curl. Our explicit form of this curl leads immediately to the relationship between the covariant and consistent currents presented by Bardeen and Zumino (Refs. [2,3]). In Sect. 4, we applied the heat kernel method to evaluate Osabe and Suzuki’s results of the difference between the covariant and consistent currents (Ref. [8]) in two and four dimensions. The results are the same as previously reported (Refs. [2,3]). In Sect. 5, applying the arguments of Banerjee et al. (Ref. [3]) to gravitational anomalies, we have investigated the relationship between the covariant and consistent energy–momentum tensors. The relation is found to be expressed by the functional curl of the covariant energy–momentum tensor.

The energy–momentum tensors considered in Sect. 5 have both Einstein and Lorentz anomalies in general. As shown in Ref. [2], these anomalies are not independent of each other. Moreover, using the regularization ambiguity, we can always choose the energy–momentum tensor to have either a vanishing Lorentz anomaly or a vanishing Einstein anomaly. From the covariant regularization viewpoint, this is explained below.

Given a covariantly regularized energy–momentum tensor $\langle T_{\mu\nu} \rangle_{\text{cov}}$, we have in general both the Einstein anomaly $D^\mu \langle T_{\mu\nu} \rangle_{\text{cov}}$ and the Lorentz anomaly $\langle T_{[\mu\nu]} \rangle_{\text{cov}}$. Note that these covariant gravitational anomalies are local polynomials of the Riemann curvature (and its derivative for the Einstein anomaly). Because of the regularization ambiguity, we can add a finite, local, and covariant counterterm to $\langle T_{\mu\nu} \rangle_{\text{cov}}$ to obtain another covariantly regularized energy–momentum tensor. Adopting the Lorentz anomaly as a counterterm, we can obtain a Lorentz-anomaly-free energy–momentum tensor,

$$\langle T_{\mu\nu} \rangle_{\text{cov}}^{\text{pE}} = \langle T_{\mu\nu} \rangle_{\text{cov}} - \langle T_{[\mu\nu]} \rangle_{\text{cov}}, \quad (6.1)$$

which gives the pure covariant Einstein anomaly $D^\mu \langle T_{\mu\nu} \rangle_{\text{cov}}^{\text{pE}}$. Since the energy–momentum tensor $\langle T_{\mu\nu} \rangle_{\text{cov}}^{\text{pE}}$, given above, is nothing but the symmetric part of $\langle T_{\mu\nu} \rangle_{\text{cov}}$, we can say that the pure covariant Einstein anomaly is the covariant divergence of the symmetric part of the covariant energy–momentum tensor (Refs. [6,14,17,18]).

We can also define a covariant energy–momentum tensor with a vanishing Einstein anomaly. It is known from Refs. [6,14,17–20] that the pure covariant Einstein anomaly has the form

$$D^\mu \langle T_{\mu\nu} \rangle_{\text{cov}}^{\text{pE}} = -D^\mu L_{\mu\nu}, \quad (6.2)$$

where $L_{\mu\nu}$ is a local polynomial of Riemann curvature⁵ and is antisymmetric with respect to the indices μ and ν . To obtain an Einstein-anomaly-free energy–momentum tensor $\langle T_{\mu\nu} \rangle_{\text{cov}}^{\text{pL}}$, we may adopt $L_{\mu\nu}$ as a local counterterm to $\langle T_{\mu\nu} \rangle_{\text{cov}}^{\text{pE}}$:

$$\langle T_{\mu\nu} \rangle_{\text{cov}}^{\text{pL}} = \langle T_{\mu\nu} \rangle_{\text{cov}}^{\text{pE}} + L_{\mu\nu}, \quad (6.3)$$

which has vanishing Einstein anomaly: $D^\mu \langle T_{\mu\nu} \rangle_{\text{cov}}^{\text{pL}} = 0$. Thus, the pure covariant Lorentz anomaly is given by

$$\langle T_{[\mu\nu]} \rangle_{\text{cov}}^{\text{pL}} = L_{\mu\nu}. \quad (6.4)$$

In Sect. 5, we defined a regularized effective action using the covariant energy–momentum tensor (Eq. (5.10)). Since the covariant energy–momentum tensor retains the ambiguity of adding covariant local curvature and vielbein polynomials, corresponding ambiguity arises in the effective action (5.10). Then, one might wonder what kind of covariant energy–momentum tensor leads to the Lorentz-anomaly-free effective action, which is local Lorentz invariant but which does not have general coordinate invariance. It can be seen that the Lorentz-anomaly-free covariant energy–momentum tensor does not necessarily lead to a Lorentz-anomaly-free effective action. In fact, for spin-1/2 chiral fermions in two-dimensional space-time, an explicit calculation with the use of a symmetric covariant energy–momentum tensor $\langle eT_{\mu\nu} \rangle_{\text{cov}}^{\text{pE}}[e_b^\lambda(t)]$ to define the effective action (5.10) shows that the second term on the right-hand side of Eq. (5.13) contributes to the consistent Lorentz anomaly. Thus, obtaining a Lorentz-anomaly-free (or Einstein-anomaly-free) consistent energy–momentum tensor is not yet straightforward in the context of Eq. (5.11). Future work will aim to clarify these points.

Funding

Open Access funding: SCOAP³.

⁵ The quantity $L_{\mu\nu}$ is related to axial U(1) anomalies in $d + 2$ dimensions (Refs. [6,14,17–20]).

Appendix A. Proof of Eq. (3.11) using test functions

In this appendix, we prove Eq. (3.11) using test functions. Namely, we give a proof of equality

$$\int d^{2n}x \exp(x^2/4s) f(x) = (4\pi s)^n \sum_{k=0}^{\infty} \int d^{2n}x f(x) \frac{(-s\Box)^k}{k!} \delta(x), \quad (\text{A.1})$$

where $f(x)$ is an arbitrary test function. Changing the integration variables from x^μ to ξ^μ ,

$$\xi^\mu = \frac{x^\mu}{\sqrt{4s}}, \quad (\text{A.2})$$

we express the left-hand side of Eq. (A.1) as

$$\begin{aligned} \text{LHS of Eq. (A.1)} &= (4s)^n \int d^{2n}\xi \exp(\xi^2) f(\sqrt{4s}\xi) \\ &= (4s)^n \sum_{l=0}^{\infty} \frac{(4s)^{l/2}}{l!} \int d^{2n}\xi \xi^{\mu_1} \xi^{\mu_2} \dots \xi^{\mu_l} \exp(\xi^2) f_{,\mu_1\mu_2\dots\mu_l}(0), \end{aligned} \quad (\text{A.3})$$

where we have taken the Taylor expansion of $f(\sqrt{4s}\xi)$ with respect to ξ and $f_{,\mu_1\mu_2\dots\mu_l} = \partial^l f / \partial x^{\mu_1} \dots \partial x^{\mu_l}$. Owing to formula

$$\begin{aligned} &\int d^{2n}\xi \xi^{\mu_1} \xi^{\mu_2} \dots \xi^{\mu_l} \exp(\xi^2) \\ &= \begin{cases} \frac{\pi^n}{2^k} (\delta^{\mu_1\mu_2} \delta^{\mu_3\mu_4} \dots \delta^{\mu_{2k-1}\mu_{2k}} + \text{permutations}) & (l = 2k), \\ 0 & (l = 2k + 1), \end{cases} \end{aligned} \quad (\text{A.4})$$

Eq. (A.3) becomes

$$\begin{aligned} \text{Eq. (A.3)} &= (4s)^n \sum_{k=0}^{\infty} \frac{(4s)^k}{(2k)!} \frac{\pi^n}{2^k} (2k-1)!! \left((-\Box)^k f \right)(0) \\ &= (4\pi s)^n \sum_{k=0}^{\infty} \frac{((-s\Box)^k f)(0)}{k!}, \end{aligned} \quad (\text{A.5})$$

which is equal to the right-hand side of Eq. (A.1).

Appendix B. The largest number of gamma matrices included in $a_k(x, x')$

The heat kernel $K(x, x'; s) = e^{-s\mathcal{D}^2} \delta(x - x')$ satisfies the differential equation

$$-\frac{\partial K(x, x'; s)}{\partial s} = \mathcal{D}^2 K(x, x'; s) \quad (\text{B.1})$$

and the boundary condition

$$K(x, x'; 0) = \delta(x - x'). \quad (\text{B.2})$$

We assume the following expansion of $K(x, x'; s)$:

$$K(x, x'; s) = \frac{1}{(4\pi s)^n} \exp((x - x')^2/4s) \sum_{k=0}^{\infty} a_k(x, x') s^k. \quad (\text{B.3})$$

From the boundary condition (B.2), we have

$$a_0(x, x) = \mathbf{1}. \quad (\text{B.4})$$

Equation (B.1) leads to the following recurrence formulas for a_k :

$$(x - x')^\mu D_\mu a_0 = 0, \quad (\text{B.5})$$

$$(k + 1)a_{k+1} + (x - x')^\mu D_\mu a_{k+1} + D_\mu D^\mu a_k + \frac{i}{2} \gamma^\mu \gamma^\nu F_{\mu\nu} a_k = 0 \quad (k \geq 0). \quad (\text{B.6})$$

From Eqs. (B.4) and (B.5), we confirm that $a_0(x, x')$ is the parallel-displacement matrix of the gauge group (Ref. [13]). Then, it is obvious that $a_0(x, x')$ does not contain any gamma matrices γ^μ . Equation (B.6) shows that $a_{k+1}(x, x')$ has two more gamma matrices than $a_k(x, x')$, since D_μ has none. From these observations, we find that the largest number of gamma matrices included in $a_k(x, x')$ is equal to $2k$.

In the coincidence limit $x' \rightarrow x$, then $a_k(x, x')$ still has at most $2k$ gamma matrices. In fact, Eqs. (B.4), (B.5), and (B.6) lead us to

$$a_k(x, x) = \frac{(-1)^k}{k!} \left(\frac{i}{2} \gamma^\mu \gamma^\nu F_{\mu\nu} \right)^k + \cdots, \quad (\text{B.7})$$

where the dots on the right-hand side express terms with a lower power of gamma matrices.

Appendix C. Heat kernel appearing in Osabe and Suzuki's currents

The heat kernel $K_g(x, x'; s) = \langle x | e^{-s \mathcal{D} \mathcal{D}_g} | x' \rangle$ satisfies the differential equation

$$-\frac{\partial K_g(x, x'; s)}{\partial s} = \mathcal{D} \mathcal{D}_g K_g(x, x'; s) \quad (\text{C.1})$$

and the boundary condition

$$K_g(x, x'; 0) = \delta(x - x'). \quad (\text{C.2})$$

We assume the following expansion of $K_g(x, x'; s)$:

$$K_g(x, x'; s) = \frac{1}{(4\pi s)^n} \exp((x - x')^2 / 4s) \sum_{k=0}^{\infty} b_k(x, x') s^k. \quad (\text{C.3})$$

From the boundary condition (C.2), we have

$$[b_0] = \mathbf{1}, \quad (\text{C.4})$$

where we have used Synge's symbol to denote the coincidence limit $[f] = f(x, x)$. Equation (C.1) leads to the following recurrence formulas for b_k :

$$(x - x')^\mu \mathcal{D}_\mu b_0 = 0, \quad (\text{C.5})$$

$$(k + 1)b_{k+1} + (x - x')^\mu \mathcal{D}_\mu b_{k+1} + (\mathcal{D}_\mu \mathcal{D}^\mu + P)b_k = 0 \quad (k \geq 0), \quad (\text{C.6})$$

where we have used the equation

$$\mathcal{D} \mathcal{D}_g = \mathcal{D}_\mu \mathcal{D}^\mu + P, \quad (\text{C.7})$$

with

$$\mathcal{D}_\mu = \partial_\mu + \mathcal{A}_\mu, \quad (\text{C.8})$$

$$\mathcal{A}_\mu = \frac{i}{2}(1+g)A_\mu + \frac{i}{2}(1-g)\gamma_{\alpha\mu}A^\alpha, \quad (\text{C.9})$$

$$\begin{aligned} P = & -\frac{i}{2}(1-g)\partial_\alpha A^\alpha - \frac{1}{2}(n-1)(1-g)^2 A_\alpha A^\alpha \\ & + \frac{i}{2}\gamma^{\alpha\beta} \left\{ (1+g)\partial_\alpha A_\beta + i \left((n-1)g^2 - 2(n-2)g + n-1 \right) A_\alpha A_\beta \right\} \end{aligned} \quad (\text{C.10})$$

and $\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$. From recurrence formulas (C.5) and (C.6), together with Eq. (C.4), we obtain the following coincidence limits (Ref. [13]):

$$[\mathcal{D}_\mu b_0] = 0, \quad (\text{C.11})$$

$$[\mathcal{D}_\mu \mathcal{D}_\nu b_0] = \frac{1}{2} \mathcal{F}_{\mu\nu}, \quad (\text{C.12})$$

$$[b_1] = -P, \quad (\text{C.13})$$

where $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]$. From these equations, we obtain

$$[\partial_\alpha b_0] = -\frac{i}{2}(1+g)A_\alpha - \frac{i}{2}(1-g)\gamma_{\beta\alpha}A^\beta, \quad (\text{C.14})$$

$$\begin{aligned} [\square b_0] = & -\frac{i}{2}(1+g)\partial_\alpha A^\alpha + \frac{1}{2} \left((n-1)g^2 - 2ng + n-1 \right) A_\alpha A^\alpha \\ & + \frac{i}{2}\gamma^{\alpha\beta} \left((1-g)\partial_\alpha A_\beta - i(n-1)(1-g)^2 A_\alpha A_\beta \right), \end{aligned} \quad (\text{C.15})$$

$$\begin{aligned} [b_1] = & \frac{i}{2}(1-g)\partial_\alpha A^\alpha + \frac{1}{2}(n-1)(1-g)^2 A_\alpha A^\alpha \\ & - \frac{i}{2}\gamma^{\alpha\beta} \left\{ (1+g)\partial_\alpha A_\beta + i \left((n-1)g^2 - 2(n-2)g + n-1 \right) A_\alpha A_\beta \right\}. \end{aligned} \quad (\text{C.16})$$

References

- [1] J. Wess and B. Zumino, Phys. Lett. B **37**, 95 (1971).
- [2] W. A. Bardeen and B. Zumino, Nucl. Phys. B **244**, 421 (1984).
- [3] H. Banerjee, R. Banerjee, and P. Mitra, Z. Phys. C **32**, 445 (1986).
- [4] H. Banerjee and R. Banerjee, Phys. Lett. B **174**, 313 (1986).
- [5] R. Banerjee and H. Banerjee, Z. Phys. C **39**, 89 (1988).
- [6] K. Fujikawa and H. Suzuki, *Path Integrals and Quantum Anomalies* (Oxford University Press, New York, 2004).
- [7] Y. Ohshima, K. Okuyama, H. Suzuki, and H. Yasuta, Phys. Lett. B **457**, 291 (1999).
- [8] S. Osabe and H. Suzuki, Int. J. Mod. Phys. A **09**, 3377 (1994).
- [9] M. T. Grisaru and S. Penati, Phys. Lett. B **504**, 89 (2001).
- [10] G. V. Dunne and C. A. Trugenberger, Ann. Phys. **204**, 281 (1990).
- [11] G. Kelnhofer, Z. Phys. C **52**, 89 (1991).
- [12] Z. Qiu and H. Ren, Phys. Rev. D **38**, 2530 (1988).
- [13] B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, London, 1965).
- [14] L. Alvarez-Gaumé and E. Witten, Nucl. Phys. B **234**, 269 (1984).
- [15] B. Zumino, Y.-S. Wu, and A. Zee, Nucl. Phys. B **239**, 477 (1984).
- [16] J. L. Synge, *Relativity: The General Theory* (North-Holland Publishing, Amsterdam, 1960).

- [17] K. Fujikawa, M. Tomiya, and O. Yasuda, Z. Phys. C **28**, 289 (1985).
- [18] R. Endo and M. Takao, Prog. Theor. Phys. **78**, 440 (1987).
- [19] T. Matsuki, Prog. Theor. Phys. **75**, 461 (1986).
- [20] F. Bastianelli and P. van Nieuwenhuizen, *Path Integrals and Anomalies in Curved Space* (Cambridge University Press, Cambridge, 2006).