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Resurgence structure to all orders of multi-bions in deformed SUSY quantum mechanics

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We investigate the resurgence structure in quantum mechanical models originating in 2d nonlinear sigma models with emphasis on nearly supersymmetric and quasi-exactly solvable parameter regimes. By expanding the ground state energy in powers of a supersymmetry-breaking deformation parameter $\delta\epsilon$, we derive exact results for the expansion coefficients. In the class of models described by real multiplets, the $\mathcal{O}(\delta\epsilon)$ ground state energy has a non-Borel summable asymptotic series, which gives rise to imaginary ambiguities leading to rich resurgence structure. We discuss sine-Gordon quantum mechanics (QM) as an example and show that the semiclassical contributions from complex multi-bion solutions correctly reproduce the corresponding part in the exact result including the imaginary ambiguities. As a typical model described by chiral multiplets, we discuss $\mathbb{C}P^{N-1}$ QM and show that the exact $\mathcal{O}(\delta\epsilon)$ ground state energy can be completely reconstructed from the semiclassical multi-bion contributions. Although the $\mathcal{O}(\delta\epsilon)$ ground state energy has trivial resurgence structure, a simple but rich resurgence structure appears at $\mathcal{O}(\delta\epsilon^2)$. We show the complete cancelation between the $\mathcal{O}(\delta\epsilon^2)$ imaginary ambiguities arising from the non-Borel summable perturbation series and those in the semiclassical contributions of $N-1$ complex bion solutions. We also discuss the resurgence structure of a squashed $\mathbb{C}P^1$ QM.

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1. Introduction

Resurgence theory and trans-series formalism (Ref. [1–8]) in quantum theories have shed new light on nonperturbative analysis and the definition of the path integral formalism in quantum mechanics (QM) (Refs. [9–42]), 2d quantum field theories (QFT) (Refs. [43–56]), 4d (or 3d) QFT (Refs. [57–64]), matrix models and topological string theories (Refs. [65–89]), and localization-applicable supersymmetric Yang–Mills (SYM) theories (Refs. [90–92]). They have also been discussed in terms of the exact WKB analysis of Schrödinger-type ordinary differential equations (Refs. [93–105]). In resurgence theory, the Borel resummations of perturbation series around all nontrivial backgrounds are taken into account, and it is expected that such a full semiclassical expansion (resurgent trans-series) leads to an unambiguous definition of quantum theories (Refs. [106–108]).

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In the original argument of the resurgent expansion (Refs. [11–21]), one needs to take account of configurations composed of instanton–antiinstanton pairs called "bions" (Refs. [109–115]). Imaginary ambiguities emerging from such bion contributions cancel out those arising in the non-Borel-summable perturbation series. Recent studies have manifested the true nature of the bion configurations from the viewpoint of the complexified path integral, where each bion configuration emerges as a complex saddle point (Refs. [28,29,32,38]). In the framework of the complexified theory, the original integration contour of the path integral is decomposed into the so-called Lefschetz thimbles (Refs. [116–128]), each of which is associated with one of the saddle points. The contribution from each bion background in the resurgent trans-series is given by the path integral along the associated Lefschetz thimble. Those deformed contours vary depending on the complexified coupling constant ($\arg g^2 \neq 0$) and some of the integration cycles discontinuously jump at $\arg g^2 = 0$. Such a discontinuity of a thimble gives rise to an imaginary ambiguity reflecting the "Stokes phenomenon" in the corresponding sector of the trans-series. Those ambiguities are expected to cancel among themselves and hence there is no ambiguity in the entire trans-series, which corresponds to the path integral along the original contour.

In Refs. [28,29] and our previous work Refs. [32,38], exact solutions of the holomorphic equations of motion were investigated in the double-well, sine-Gordon, and $\mathbb{C}P^1$ QM with fermionic degrees of freedom (incorporated as a parameter ϵ), and it was shown that the bions appear as the exact solutions of the complexified equation of motions. In Ref. [38], an infinite tower of exact multi-bion solutions were found, and the exact resurgent trans-series was obtained to all orders in the perturbative and nonperturbative expansion. It was shown that the response of the exact ground state energy under a deformation from the supersymmetric (SUSY) point ($\epsilon = 1$) can be expressed as a trans-series with nonperturbative terms corresponding to multi-bion saddle points, together with perturbation series around them. In the trans-series, the imaginary ambiguity associated with the non-Borel summable perturbation series around the p-bion background is canceled by that arising from the semiclassical contribution of a $(p + 1)$-bion saddle point. By applying the Lefschetz thimble method, all the semiclassical contributions from the multi-bion solutions are shown to agree with the corresponding parts in the exact result.

In this work, we investigate resurgence structure in a broader class of quantum mechanical models from the viewpoint of the complexified path integral and its complex multi-bion saddle points. We focus on SUSY and quasi-exactly solvable (QES) QM, where we can take advantage of exact results to probe the resurgence structure of those models. By introducing a SUSY-breaking (or QES-breaking) deformation parameter $\delta\epsilon$ and expanding the ground state energy around the SUSY (or QES) point in powers of $\delta\epsilon$, we reveal all order multi-bion contributions with nontrivial resurgence structure.

We classify the models into two classes: (i) quantum mechanics on a Riemannian manifold described by real multiplets and (ii) quantum mechanics on a Kähler manifold described by chiral multiplets. In both classes, the ground state energy at the SUSY and QES points ($\delta\epsilon = 0$) does not receive any nonperturbative correction due to cancelation among various (real and complex) multi-bion contributions. In the first class, the $\mathcal{O}(\delta\epsilon)$ ground state energy has a non-Borel-summable asymptotic series in each sector of the trans-series, which gives rise to an imaginary ambiguity leading to a rich resurgence structure. In the second class, a localization method (Refs. [129,130]) is (partially) available to determine the $\mathcal{O}(\delta\epsilon)$ ground state energy, which leads to a simpler (sometimes trivial) resurgence structure than that in the first class.

As typical examples in the two classes, we consider sine-Gordon QM (the first class) and the (squashed) $\mathbb{C}P^{N-1}$ QM (the second class). In sine-Gordon QM, we obtain the exact result for the

$\mathcal{O}(\delta\epsilon)$ ground state energy, which is composed of a perturbation series and an infinite tower of nonperturbative terms, each of which has a non-Borel-summable asymptotic series. Based on the complexified path integral, we show that the semiclassical multi-bion contributions reproduce the corresponding parts in the exact result including the imaginary ambiguities which cancel those in the other sectors. This supports the resurgence to all orders in the nonperturbative exponential. In $\mathbb{C}P^{N-1}$ QM, we find $(N-1)$ types of (real and complex) bion solutions and show that the exact result for the $\mathcal{O}(\delta\epsilon)$ ground state energy can be completely reconstructed from the semiclassical multi-bion contributions. We determine the non-Borel summable perturbation series of the $\mathcal{O}(\delta\epsilon^2)$ ground state energy to all orders in g^2 and confirm that its imaginary ambiguities are canceled by those in the single-bion contributions. We also show by deforming the target space that a nontrivial resurgence structure can be seen in the $\mathcal{O}(\delta\epsilon)$ ground state energy for a generic target manifold.

The organization of this paper is as follows. In Sect. 2, we discuss generic properties of the ground state energy in the two classes of SUSY models. By expanding in terms of a SUSY-breaking parameter $\delta\epsilon$, we show that the "generating function" $\langle 0|0\rangle$ plays an important role in determining the resurgence structure of the ground state energy around the SUSY point in the parameter space. In Sect. 3, we investigate the resurgence structure in sine-Gordon QM. The $\mathcal{O}(\delta\epsilon)$ ground state energy is derived exactly and compared with the semiclassical multi-bion contributions. In Sect. 4, we investigate the resurgence structure in $\mathbb{C}P^{N-1}$ QM, with emphasis on the exact complex solutions and their contributions to the ground state energy around the SUSY and QES points. In Sect. 5, we investigate the resurgence structure in the squashed $\mathbb{C}P^1$ QM. Section 6 is devoted to summary and discussion. Appendices A, B, C, D, E, F, and G are devoted to the supersymmetric QM, the localization method, the perturbative part in $\mathbb{C}P^{N-1}$ QM, the quasi-moduli space, the kink–antikink effective potential, the Lefschetz thimble analysis, and the one-loop determinant, respectively.

2. SUSY QM with deformation parameter

2.1. Quantum mechanics on Riemannian manifolds

In this paper, we discuss quantum mechanics of a particle on a manifold $\mathcal{M}$ with a potential V. The Schrödinger equation takes the form

$$H\Psi = \left[-g^2\Delta + V\right]\Psi, \qquad \Delta\Psi \equiv \frac{1}{\sqrt{\det G}}\partial_i\left(G^{ij}\sqrt{\det G}\right)\partial_j\Psi, \tag{2.1}$$

where G_{ij} is the metric of the target manifold $\mathcal{M}$, Δ is the Laplacian, and $\partial_i \equiv \frac{\partial}{\partial\varphi^i}$ stands for the partial derivatives with respect to the real bosonic variables φ^i ($i = 1, \ldots, n = \dim\mathcal{M}$) corresponding to the coordinates on $\mathcal{M}$. The focus of this paper is on the class of models that can be obtained from supersymmetric quantum mechanical models by a SUSY-breaking deformation. In particular, we put a special emphasis on the expansion around SUSY and quasi-exactly solvable (QES) points. In such a model, the Hamiltonian projected onto the lowest fermion number eigenspace takes the form of Eq. (2.1) with a bosonic potential of the form (see Appendix A for details)

$$V = \frac{1}{g^2}G^{ij}\partial_i W\partial_j W - \epsilon\Delta W, \tag{2.2}$$

where W is a real function on $\mathcal{M}$, which we call the superpotential, and ϵ is the SUSY-breaking deformation parameter.

The model with $\epsilon = 1$ corresponds to the SUSY case, where the exact wave function for the SUSY ground state $|0\rangle$ is obtained as

$$\Psi^{(0)} \equiv \langle\varphi|0\rangle = \exp\left(-\frac{W}{g^2}\right). \tag{2.3}$$

This SUSY invariant state is well defined only when it is normalizable:

$$\langle 0|0\rangle = \int_{\mathcal{M}} dv \exp\left(-\frac{2W}{g^2}\right) < \infty, \tag{2.4}$$

where dv is the volume form on $\mathcal{M}$: $dv = \sqrt{\det G}\, d\varphi^1 \wedge \cdots \wedge d\varphi^n$. For example, in a single-variable case, a polynomial superpotential $W(\varphi)$ of even degree satisfies the normalizability condition while that of odd degree does not. This means that the present setup includes triple-well, 5-well, …, $(2n+1)$-well potentials while it does not include double-well, 4-well, …, $2n$-well potentials. As is well known, SUSY is spontaneously broken in the latter cases.

We will investigate the ground state energy and the wave function by expanding them in powers of the SUSY-deformation parameter $\delta\epsilon = \epsilon - 1$ as $E = \delta\epsilon E^{(1)} + \delta\epsilon^2 E^{(2)} + \cdots$ and $\Psi = \Psi^{(0)} + \delta\epsilon \Psi^{(1)} + \cdots$. In other words, we consider their responses under the SUSY-breaking deformation

$$E^{(n)} = \frac{1}{n!}\frac{\partial^n}{\partial\epsilon^n}E\bigg|_{\epsilon=1}, \qquad \Psi^{(n)} = \frac{1}{n!}\frac{\partial^n}{\partial\epsilon^n}\Psi\bigg|_{\epsilon=1}. \tag{2.5}$$

These expansion coefficients can be determined by the standard Rayleigh–Schrödinger perturbation theory,

$$E^{(1)} = \frac{\langle 0|\delta H|0\rangle}{\langle 0|0\rangle}, \qquad E^{(2)} = -\frac{\langle \Psi^{(1)}|H_{\epsilon=1}|\Psi^{(1)}\rangle}{\langle 0|0\rangle}, \qquad \ldots, \tag{2.6}$$

where $\delta H = -\Delta W$ is the SUSY-breaking perturbation Hamiltonian.

For the purpose of understanding the resurgence structure at each order of $\delta\epsilon$, the property of the denominator $\langle 0|0\rangle$ in Eq. (2.6) is of great importance. Since this gives the vacuum expectation value (VEV) of W when differentiated with respect to $1/g^2$, we call this quantity the generating function of W in the following. Applying the saddle point method, we obtain the following schematic form of the generating function

$$\langle 0|0\rangle = \int_{\mathcal{M}} dv\, \exp\left(-\frac{2W}{g^2}\right) = \sum_{s\in\mathfrak{S}} F_s(g^2)\exp\left(-\frac{2W_s}{g^2}\right), \tag{2.7}$$

where $\mathfrak{S}$ denotes the set of the saddle points of W, W_s is the value of W at the saddle point s, and $F_s(g^2)$ is the perturbation series around the saddle point s.

Depending on the type of SUSY models from which we obtain the bosonic potential, the generating function $\langle 0|0\rangle$ has different properties. If the target space $\mathcal{M}$ is a Riemannian manifold whose coordinates are parametrized by the bosonic components of "real multiplets", the functions $F_s(g^2)$ have nontrivial asymptotic expansions, giving a complicated resurgent structure to $E^{(n)}$. This class includes sine-Gordon QM and the $(2n+1)$-well models ($n \in \mathbb{Z}$), while if the target space is a Kähler manifold whose complex coordinates constitute "chiral multiplets", the functions $F_s(g^2)$ are monomials of g^2, which leads to a simplified (sometimes trivial) resurgent trans-series in $E^{(n)}$. Examples of this type are $\mathbb{C}P^{N-1}$ QM, Grassmannian QM, and their deformed models.

2.2. Real multiplet: generating function and Lefschetz thimble

Let us first see some generic properties of the generating function $\langle 0|0\rangle$ in the SUSY QM described by real multiplets. After the projection onto the lowest fermion number states, the action takes the form (see Appendix A for details)

$$S = \int dt \left(\frac{1}{4g^2} G_{ij} \dot{\varphi}^i \dot{\varphi}^j - \frac{1}{g^2} G^{ij} \partial_i W \partial_j W + \Delta W \right). \tag{2.8}$$

This class of models includes the SUSY $(2n+1)$-well model and the SUSY sine-Gordon model.

As we have discussed, the generating function for W can be obtained by evaluating the integral

$$\langle 0|0\rangle = \int_{\mathcal{M}} dv \, \exp\left(-\frac{2W}{g^2}\right). \tag{2.9}$$

Let us apply the Lefschetz thimble method to this integral.

We assume that the target manifold $\mathcal{M}$ has a suitable complexification $\mathcal{M}^{\mathbb{C}}$ parametrized by the complexified coordinates φ^i and the function W can be analytically continued to $\mathcal{M}^{\mathbb{C}}$ as a holomorphic function of φ^i. The thimble $\mathcal{J}_s$ associated with a saddle point s of W is a middle-dimensional subspace of $\mathcal{M}^{\mathbb{C}}$ that can be reached by upward flows from the saddle point s:

$$\mathcal{G}_{i\bar{j}} \frac{d\varphi^i}{dt} = \overline{\frac{1}{g^2} \frac{\partial W}{\partial \varphi^j}}, \quad \lim_{t \to -\infty} \varphi^i = \varphi^i_s, \tag{2.10}$$

where t is a formal flow parameter and $\mathcal{G}_{i\bar{j}}$ is a suitable positive definite metric on $\mathcal{M}^{\mathbb{C}}$ such that $\mathcal{G}_{i\bar{j}} = G_{ij}$ on $\mathcal{M}$. The dual thimble $\mathcal{K}_s$ is defined as a middle-dimensional subspace of $\mathcal{M}^{\mathbb{C}}$ ($\dim \mathcal{J}_s = \dim \mathcal{K}_s$) that flows to the saddle point s:

$$\mathcal{G}_{i\bar{j}} \frac{d\varphi^i}{dt} = \overline{\frac{1}{g^2} \frac{\partial W}{\partial \varphi^j}}, \quad \lim_{t \to \infty} \varphi^i = \varphi^i_s. \tag{2.11}$$

The definition of the thimble $\mathcal{J}_s$ and its dual $\mathcal{K}_s$ implies that the real and imaginary parts of the complexified superpotential satisfy

$$\operatorname{Re} W|_{\mathcal{J}_s} \geq \operatorname{Re} W_s \geq \operatorname{Re} W|_{\mathcal{K}_s}, \qquad \operatorname{Im} W|_{\mathcal{J}_s} = \operatorname{Im} W_s = \operatorname{Im} W|_{\mathcal{K}_s}. \tag{2.12}$$

It follows from these properties that $\mathcal{J}_s$ and $\mathcal{K}_s$ can intersect only at the saddle point s. Since the imaginary parts are generically different ($\operatorname{Im} W_s \neq \operatorname{Im} W_{s'}$) at two different saddle points $s \neq s'$ for a generic function W, there is no intersection between $\mathcal{J}_s$ and $\mathcal{K}_{s'}$ ($s' \neq s$). Provided Stokes phenomena do not arise, we can define the intersection pairing of thimbles and their duals as

$$\langle \mathcal{J}_s, \mathcal{K}_{s'} \rangle = \delta_{ss'}. \tag{2.13}$$

By using this pairing, the original integration cycle $\mathcal{M}$ can be decomposed as

$$\mathcal{M} = \sum_{s \in \mathfrak{S}} n_s \mathcal{J}_s, \quad n_s = \langle \mathcal{M}, \mathcal{K}_s \rangle, \tag{2.14}$$

where n_s are the intersection numbers between $\mathcal{M}$ and the dual thimble $\mathcal{K}_s$. Correspondingly, the generating function can be decomposed as

$$\langle 0|0\rangle = \sum_{s \in \mathfrak{S}} F_s(g^2) \exp\left(-\frac{2W_s}{g^2}\right), \tag{2.15}$$

with

$$F_s(g^2) = n_s \int_{\mathcal{J}_s} dv \, \exp\left[-\frac{2(W - W_s)}{g^2}\right]. \tag{2.16}$$

However, this decomposition becomes ambiguous if there exists a kink solution described by the BPS equation in the original SUSY model

$$G_{ij}\frac{\partial \varphi^j}{\partial \tau} = \frac{\partial W}{\partial \varphi^i}, \tag{2.17}$$

where τ is interpreted as the Euclidean time. Since the BPS kink solution is a flow connecting 2 different saddle points ($s \neq s'$), its existence implies that $\mathcal{J}_s$ coincides with $\mathcal{K}_{s'}$ and hence the intersection pairing is ill defined. We can make it well defined by giving a small imaginary part to g^2 as a regularization parameter. Although such a complexified coupling constant gives a well-defined intersection pairing, it can give different decompositions of $\mathcal{M}$ depending on the sign of $\arg g^2 = 0$. This is because the thimbles can have discontinuity at $\arg g^2 = 0$,

$$\mathcal{J}_s^+ \neq \mathcal{J}_s^-, \qquad \mathcal{K}_s^+ \neq \mathcal{K}_s^-, \tag{2.18}$$

where thimbles with $\pm$ denote those for positive and negative $\arg g^2$. Therefore, the generating function can also have different decompositions depending on $\arg g^2$:

$$\langle 0|0\rangle = \sum_{s\in\mathfrak{S}} F_s^{\pm}(g^2)\exp\left(-\frac{2W_s}{g^2}\right), \tag{2.19}$$

with

$$F_s^{\pm}(g^2) = n_s^{\pm} \int_{\mathcal{J}_s^{\pm}} dv \, \exp\left[-\frac{2(W - W_s)}{g^2}\right], \quad n_s^{\pm} \equiv \langle \mathcal{M}, \mathcal{K}_s^{\pm}\rangle. \tag{2.20}$$

This means that a Stokes phenomenon occurs on the line $\arg g^2 = 0$, and consequently the asymptotic series of $\langle 0|0\rangle$ at each saddle point has an ambiguity for real g. As we will see below, the Stokes phenomenon for the generating function $\langle 0|0\rangle$ gives a nontrivial resurgence structure to $E^{(1)} = \langle 0|\delta H|0\rangle/\langle 0|0\rangle$.

It is worth noting that the generating function can be written as

$$\langle 0|0\rangle = \exp\left(-\frac{2W_0}{g^2}\right)\sum_{s\in\mathfrak{S}} F_s^{\pm}(g^2)\exp\left(-2S_{\mathrm{kink},s}\right), \tag{2.21}$$

where $W_0 \equiv \min_{s\in\mathfrak{S}} W_s$ is the value of W at the global minimum of the potential and $S_{\mathrm{kink},s} \equiv (W_s - W_0)/g^2$ are the on-shell values of the Euclidean action for the BPS kink solutions. Equation (2.21) implies that pairs of kinks, i.e., bions, give the nonperturbative contribution to $\langle 0|0\rangle$.

2.3. *Chiral multiplet: generating function and Lefschetz thimble*

Next let us consider the case of a Kähler target manifold parametrized by the bosonic components of chiral multiplets, which includes the $\mathbb{C}P^{N-1}$ QM and Grassmannian QM models. In the previous subsection, we have seen that the Stokes phenomenon for $\langle 0|0\rangle$ can give a nontrivial resurgence structure of $E^{(1)}$. Here we see that in the case of a Kähler target manifold, no Stokes phenomenon occurs for $\langle 0|0\rangle$ and hence the expansion coefficients $E^{(n)}$ have relatively simple resurgence structure.

The Kähler metric can be written in terms of a Kähler potential K on each coordinate patch as

$$G_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K, \tag{2.22}$$

where $i, \bar{j} = 1, \ldots, n$. For example, the model with $\mathcal{M} \cong \mathbb{C}P^{N-1}$ is given by $K = \log(1 + \varphi^i \bar{\varphi}^{\bar{i}})$ with the inhomogeneous coordinates φ^i $(i = 1, \ldots N-1)$. We also see that there is a holomorphic isometry, whose holomorphic Killing vector $\Xi = \xi^i \partial_i + \bar{\xi}^{\bar{i}} \bar{\partial}_{\bar{i}}$ satisfies

$$\partial_i \xi_{\bar{j}} + \bar{\partial}_{\bar{j}} \bar{\xi}_i = \bar{\partial}_{\bar{j}} \xi^i = \partial_i \bar{\xi}^{\bar{j}} = 0. \tag{2.23}$$

We here define the moment map μ for the holomorphic isometry as $d\mu = i_\Xi \omega$, where ω is the Kähler form $\omega = iG_{i\bar{j}} d\varphi^i \wedge d\bar{\varphi}^{\bar{j}}$ and i_Ξ denotes the interior product (contraction) with respect to the Killing vector Ξ. In terms of the components, the equation for μ can be rewritten as

$$\partial_i \mu = -iG_{i\bar{j}} \bar{\xi}^{\bar{j}}, \qquad \bar{\partial}_{\bar{i}} \mu = iG_{j\bar{i}} \xi^j. \tag{2.24}$$

The SUSY QM of chiral multiplets we discuss in this paper can be obtained from the 2d $\mathcal{N} = (2, 0)$ nonlinear sigma model,

$$S_{2d} = \frac{1}{g_{2d}^2} \int d^2x \, G_{i\bar{j}} \Big[-\partial_\mu \varphi^i \partial^\mu \bar{\varphi}^{\bar{j}} + i\bar{\psi}^{\bar{j}} (\mathcal{D}_t - \mathcal{D}_x) \psi^i \Big]. \tag{2.25}$$

By imposing the periodic boundary condition twisted by the isometry and reducing the spatial direction, we obtain the following 1d action:

$$S = \frac{1}{g^2} \int dt \, G_{i\bar{j}} \Big[\dot{\varphi}^i \dot{\bar{\varphi}}^{\bar{j}} - \xi^i \bar{\xi}^{\bar{j}} + i\bar{\psi}^{\bar{j}} \mathcal{D}_t \psi^i - i\nabla_k \xi^i \bar{\psi}^{\bar{j}} \psi^k \Big], \tag{2.26}$$

where $\mathcal{D}_i \psi^i = \partial_i \psi^i + \Gamma^i_{jk} \partial_i \varphi^j \psi^k$, $\nabla_k \xi^i = \partial_k \xi^i + \Gamma^i_{jk} \xi^j$, $\Gamma^i_{jk} = \partial_j G_{k\bar{l}} G^{\bar{l}i}$, and

$$\frac{1}{g^2} = \frac{1}{g_{2d}^2} \times 2\pi \{\text{compactification radius}\}. \tag{2.27}$$

Under the compactification with the twisted boundary condition, sigma model instantons in the Euclidean 2d nonlinear sigma model (Ref. [131]) decompose into a set of “fractional instantons” (Refs. [132–134]; see also Refs. [135,136]). From the viewpoint of quantum mechanics, such fractional instantons appear as the BPS kinks (see, e.g., Refs. [137–140]) carrying a fractional instanton number

$$S_{E,\,\text{inst}} = \frac{1}{g_{2d}^2} \int iG_{i\bar{j}} \, d\varphi^i \wedge d\bar{\varphi}^{\bar{j}} \quad \to \quad S_{E,\,\text{kink}} = \frac{1}{g^2} \int \Big(\partial_i \mu \, d\varphi^i + \bar{\partial}_{\bar{i}} \mu \, d\bar{\varphi}^{\bar{i}} \Big). \tag{2.28}$$

Thus it could be speculated that there is a close relationship between the 2d and 1d nonperturbative effects induced by those objects.

As discussed in Appendix A, the projection onto the lowest fermion number eigenstates gives a bosonic potential of the form (2.2) with the identification

$$W = \mu. \tag{2.29}$$

Thus the Schrödinger equation takes form

$$H\Psi = G^{\bar{j}i} \left[-g^2 \partial_i \bar{\partial}_{\bar{j}} + \frac{1}{g^2} \partial_i \mu \bar{\partial}_{\bar{j}} \mu - \epsilon \partial_i \bar{\partial}_{\bar{j}} \mu \right] \Psi. \tag{2.30}$$

As in the case of real multiplets, we can find the exact SUSY ground state wave function as

$$\Psi^{(0)} \equiv \langle \varphi | 0 \rangle = \exp\left(-\frac{\mu}{g^2}\right). \tag{2.31}$$

The normalization factor $\langle 0|0\rangle$ can be regarded as the generating function of the vacuum expectation value of μ

$$\langle 0|0\rangle = \int_{\mathcal{M}} dv \, \exp\left(-\frac{2\mu}{g^2}\right), \quad dv = \frac{\omega^n}{n!}. \tag{2.32}$$

Again, we can use the saddle point method to decompose the generating function,

$$\langle 0|0\rangle = \sum_{s\in\mathfrak{S}} F_s(g^2)\exp\left(-\frac{2\mu_s}{g^2}\right) = \exp\left(-\frac{2\mu_0}{g^2}\right)\sum_{s\in\mathfrak{S}} F_s(g^2)\exp\left(-2S_{\mathrm{kink},s}\right), \tag{2.33}$$

where $\mu_0 \equiv \min_{s\in\mathfrak{S}}\mu_s$ is the value of the moment map at the global minimum of the potential and $S_{\mathrm{kink},s}$ is the on-shell value of the action for the BPS kink satisfying

$$G_{i\bar{j}}\frac{d\varphi^i}{d\tau} = \frac{\partial\mu}{\partial\bar{\varphi}^{\bar{j}}}, \quad \lim_{\tau\to-\infty}\varphi^i = \varphi_0^i, \quad \lim_{\tau\to\infty}\varphi^i = \varphi_s^i. \tag{2.34}$$

Equation (2.33) implies that the nonperturbative contributions to $\langle 0|0\rangle$, which are proportional to $\exp\left(-2S_{\mathrm{kink},s}\right)$, can be regarded as bion contributions. The most important property of the generating function in the Kähler case is that $F_s(g^2)$ can be computed by the Duistermaat–Heckman localization formula (Refs. [129,130])

$$F_s(g^2) = \left(\frac{\pi g^2}{2}\right)^n \frac{1}{\det M_s}, \tag{2.35}$$

where M_s is the n-by-n matrix that represents the action of the Killing vector $-i\xi$ on the tangent space at the saddle point s (see Appendix B for details). Since $F_s(g^2)$ is not a divergent asymptotic series, there is no ambiguity and hence the decomposition (2.33) is unambiguous. This property ensures that the expansion coefficients of the ground state energy (2.6) have relatively simple resurgence structure. In particular, when the perturbation Hamiltonian is a polynomial of μ,

$$\delta H = -\Delta\mu = P(\mu), \tag{2.36}$$

the first expansion coefficient $E^{(1)}$ can be calculated from the generating function $\langle 0|0\rangle = \int dv \, \exp\left(-\frac{2\mu}{g^2}\right)$ as

$$E^{(1)} = -\frac{\langle 0|\Delta\mu|0\rangle}{\langle 0|0\rangle} = \frac{P(\hat{\mu})\langle 0|0\rangle}{\langle 0|0\rangle}, \quad \hat{\mu} \equiv \frac{1}{2}g^4\frac{\partial}{\partial g^2}. \tag{2.37}$$

This implies that $E^{(1)}$ has a finite-order g^2 asymptotic series in each bion sector and hence there is no ambiguity. This indicates that the resurgence structure is trivial at $\mathcal{O}(\delta\epsilon)$ in these types of models. We will see these properties in detail in $\mathbb{C}P^{N-1}$ QM in Sect. 4.

On the other hand, when $P(\mu)$ is not a polynomial of μ, g^2 asymptotic series can be infinite order (non-Borel summable) and give an ambiguous contribution to $E^{(1)}$ at the Stokes line $\arg g^2 = 0$. Therefore, the resurgence structure can be nontrivial at $\mathcal{O}(\delta\epsilon)$. We note that it is still relatively simple since there is no Stokes phenomenon in the denominator $\langle 0|0\rangle$, so that imaginary ambiguities cancel between adjacent nonperturbative sectors. We will see details of these properties in the squashed $\mathbb{C}P^{N-1}$ QM in Sect. 5.

It is notable that $E^{(2)}$ has a richer resurgence structure in any type of model with a Kähler target manifold. Nevertheless, since the denominator $\langle 0|0\rangle$ is free from ambiguity, the resurgence structure is relatively "clean", where imaginary ambiguities arising from perturbation series around p-bion backgrounds are completely canceled by those in semiclassical $(p+1)$-bion contributions. We will see these properties in $\mathbb{C}P^{N-1}$ QM in detail in Sect. 4.

3. Resurgence structure in sine-Gordon QM

In this section, we consider sine-Gordon QM as an example of a model described by real multiplets discussed in Sect. 2. We first derive some exact results for the expansion coefficients of the ground state energy around the SUSY and QES solvable points. Then we look into the multi-bion solution and show by applying the Lefschetz thimble method to the quasi-moduli integral that the semiclassical bion contributions with imaginary ambiguities reproduce the corresponding parts in the exact results.

3.1. Sine-Gordon QM

Throughout this section, we use θ instead of φ as the periodic coordinate of S^1. The superpotential for sine-Gordon QM is given by

$$W(\theta) = -\frac{m}{2}\cos\theta. \tag{3.1}$$

After the projection onto the lowest fermion number eigenspace, the Hamiltonian with the SUSY-breaking deformation takes the form

$$H\Psi = \left[-g^2\partial_\theta^2 + \frac{m^2}{4g^2}\sin^2\theta - \frac{m\epsilon}{2}\cos\theta\right]\Psi. \tag{3.2}$$

This sine-Gordon model becomes supersymmetric at $\epsilon = 1$.

Since the potential of sine-Gordon QM is a periodic function, its energy spectrum has a band structure. Energy eigenstates within each band are characterized by the Bloch angle $2\pi\alpha$ defined as a twisted angle of boundary conditions

$$\Psi(\theta + 2\pi) = \exp(2\pi i\alpha)\,\Psi(\theta). \tag{3.3}$$

The Bloch angle takes values $0 \le \alpha < 1$ since α and $\alpha + \mathbb{Z}$ are equivalent. The factor $2\pi\alpha$ can be eliminated from the boundary condition by the redefinition $\Psi(\theta) \to \tilde{\Psi}(\theta) \equiv \exp(i\alpha\theta)\,\Psi(\theta)$. Then the Schrödinger equation (3.2) for the periodic wave function $\tilde{\Psi}$ becomes

$$H\tilde{\Psi} = \left[-g^2(\partial_\theta - i\alpha)^2 + \frac{m^2}{4g^2}\sin^2\theta - \frac{m\epsilon}{2}\cos\theta\right]\tilde{\Psi} = E\tilde{\Psi}. \tag{3.4}$$

In the following, we discuss the ground state of the system described by this Schrödinger equation. The ground state energy can be read off from the low temperature limit ($\beta \to \infty$) of the partition function, which can be defined for each α by the following Euclidean path integral over periodic configuration $\theta(\tau + \beta) = \theta(\tau) \bmod 2\pi$,

$$Z(\alpha) = \mathrm{Tr}[\exp(-\beta H)] = \int \mathcal{D}\theta\,\exp(-S_E), \tag{3.5}$$

where the Euclidean action is given by

$$S_E = \int_0^\beta d\tau \left[\frac{1}{4g^2} \left(\dot{\theta}^2 + m^2 \sin^2 \theta \right) - \frac{m\epsilon}{2} \cos\theta - i\alpha\dot{\theta} \right]. \tag{3.6}$$

The last term is the topological term related to the Bloch angle, which gives $\mp\pi i\alpha$ for each (anti)kink corresponding to the tunneling process between the local and global minima of the potential ($\theta = \pi$ and $\theta = 0$). It is notable that the Hamiltonian can be rewritten as

$$H = \bar{Q}Q - \frac{m}{2}\delta\epsilon \cos\theta, \tag{3.7}$$

where $\delta\epsilon = \epsilon - 1$ and

$$Q = ig\left(\partial_\theta - i\alpha + \frac{m}{2g^2}\sin\theta \right), \qquad \bar{Q} = ig\left(\partial_\theta - i\alpha - \frac{m}{2g^2}\sin\theta \right). \tag{3.8}$$

When $\delta\epsilon = 0$, the Hamiltonian describes the $F = 0$ (zero fermion) sector of the SUSY sine-Gordon QM. Furthermore, for $\alpha = 0$, the ground state preserves SUSY and its wave function can be exactly determined as

$$\Psi^{(0)} = \exp\left(\frac{m}{2g^2}\cos\theta \right), \qquad H\Psi^{(0)} = 0. \tag{3.9}$$

For $\alpha \neq 0$, SUSY is spontaneously broken due to the topological term.

3.2. *Exact results around the SUSY and QES points*

Before discussing nonperturbative contributions in the path integral formalism, we derive some exact results for the expansion coefficients of the ground state energy around the SUSY and QES points.

3.2.1. *Expansion around the SUSY point*

Small $\delta\epsilon$ expansion. First, we consider the expansion around the SUSY point ($\epsilon = 1$, $\alpha = 0$). The generating function is given by the modified Bessel function of the first kind

$$\langle 0|0\rangle = \int_0^{2\pi} d\theta \, \exp\left(z\cos\theta \right) = 2\pi I_0(z), \quad z \equiv \frac{m}{g^2}. \tag{3.10}$$

The leading-order Rayleigh–Schrödinger perturbation theory gives the first expansion coefficient of the ground state energy as

$$E^{(1)} = \frac{\partial}{\partial\epsilon}E\bigg|_{\epsilon=1} = -\frac{m}{2}\frac{\langle 0|\cos\theta|0\rangle}{\langle 0|0\rangle} = -\frac{m}{2}\frac{\partial}{\partial z}\log I_0(z). \tag{3.11}$$

This is the exact result that will be compared with the semiclassical multi-bion contributions discussed below. To extract the perturbative contribution, let us consider the weak coupling (large z) asymptotic expansion of $I_0(z)$:

$$I_0(z) = \frac{e^z}{\sqrt{2\pi z}}\left[1 + \cdots + \frac{1}{\pi}\frac{\Gamma(n+\frac{1}{2})^2}{\Gamma(n+1)n}\left(\frac{1}{2z} \right)^n + \cdots \right] + \mathcal{O}(e^{-z}). \tag{3.12}$$

This is the divergent power series that gives the asymptotic expansion of the generating function $\langle 0|0\rangle$. Now let us consider the Borel resummation and see how the imaginary ambiguity arises

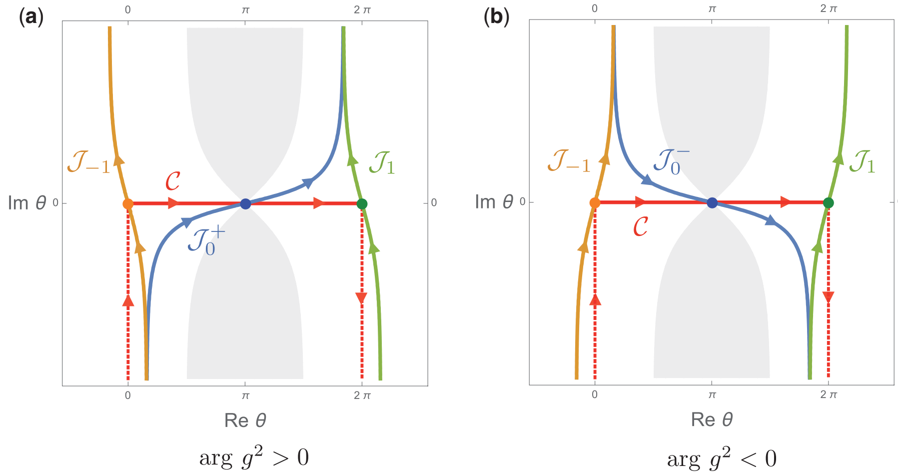

Fig. 1. The integration contour C for the generating function $\langle 0|0\rangle$ and the Lefschetz thimbles $\mathcal{J}_n$ associated with the saddle points $\theta = (n+1)\pi$. The thimble with $n = 0$ jumps at $\arg g^2 = 0$ due to the Stokes phenomenon. The original integration contour $\mathcal{C}$ can be deformed and decomposed as $\mathcal{J}_0^+ - \mathcal{J}_1$ or $\mathcal{J}_0^- + \mathcal{J}_{-1}$ depending on $\arg g^2$. The ambiguous Borel resummation $I_0(z) \pm \frac{i}{\pi}K_0(z)$ corresponds to integration along $\mathcal{J}_0^\pm$.

from the perturbation series. First, replacing $\Gamma(n+\frac{1}{2})$ with its integral representation $\Gamma\left(n+\frac{1}{2}\right) = \int_0^\infty \frac{dt}{t} e^{-t} t^{n+\frac{1}{2}}$, we can rewrite the series as

$$\frac{e^z}{\sqrt{2\pi z}} \sum_{n=0}^{\infty} \frac{1}{\pi} \frac{\Gamma(n+\frac{1}{2})^2}{\Gamma(n+1)} \left(\frac{1}{2z}\right)^n \to \frac{e^z}{\sqrt{2\pi z}} \int_0^\infty dt\, ds\, \exp(-t-s) \sum_{n=0}^{\infty} \frac{(ts)^{n+\frac{1}{2}}}{\pi \Gamma(n+1)} \left(\frac{1}{2z}\right)^n . \tag{3.13}$$

Then summing over n before the integration, we obtain the Borel resummation of the divergent series in I_0:

$$(3.13) = \frac{1}{\pi} \int \frac{dt\, ds}{\sqrt{2\pi z t s}} \exp\left(z - t - s + \frac{st}{2z}\right) = \frac{1}{\pi} \int_0^\infty dt\, \frac{\exp(z-t)}{\sqrt{t(2z-t)}}. \tag{3.14}$$

Corresponding to the Stokes phenomenon at $\arg g^2 = 0$ shown in Fig. 1, this integral representation has an imaginary ambiguity associated with the branch cut starting from $t = 2z$ to infinity:

$$\frac{1}{\pi} \int_0^\infty dt \frac{\exp(z-t)}{\sqrt{t(2z-t)}} = I_0(z) \pm \frac{i}{\pi} K_0(z), \tag{3.15}$$

where $+$ is for $\arg z < 0$ and $-$ is for $\arg z > 0$. Thus, we obtain the ambiguous perturbative part of $E^{(1)}$ as

$$E_0^{(1)} = -\frac{m}{2}\frac{\partial}{\partial z} \log\left[\frac{1}{\pi} \int_0^\infty dt \frac{\exp(z-t)}{\sqrt{t(2z-t)}}\right] = -\frac{m}{2}\frac{\partial}{\partial z} \log\left[I_0(z) \pm \frac{i}{\pi} K_0(z)\right]. \tag{3.16}$$

The remaining nonperturbative part, which cancels the imaginary ambiguity of the perturbative part, can be expressed as the convergent power series in the nonperturbative exponential e^{-2z} as

$$E^{(1)} - E_0^{(1)} = -\frac{m}{2}\frac{\partial}{\partial z} \log\left[1 - \frac{\pm \frac{i}{\pi} K_0(z)}{I_0(z) \pm \frac{i}{\pi} K_0(z)}\right] = \sum_{p=1}^{\infty} E_p^{(1)}, \tag{3.17}$$

where $E_p^{(1)}$ denotes the contribution with pth power of nonperturbative exponential e^{-2z},

$$E_p^{(1)} = \frac{m}{2}\frac{\partial}{\partial z}\left[\frac{1}{p}\left(\frac{\pm\frac{i}{\pi}K_0(z)}{I_0(z)\pm\frac{i}{\pi}K_0(z)}\right)^p\right] \sim \mathcal{O}(e^{-2pz}). \tag{3.18}$$

This can be further expanded in powers of g^2 and the leading-order term is given by

$$E_p^{(1)} = -m\left(\pm i\exp\left(-\frac{2m}{g^2}\right)\right)^p + \cdots, \tag{3.19}$$

where we have used $K_0(z)/I_0(z) \approx \pi e^{-2z}$ for large z. This nonperturbative contribution with the imaginary ambiguity is expected to correspond to the semiclassical part of the p-bion contribution. We will compare this exact result with the multi-bion contributions later.

Small α expansion. It is also possible to obtain exact results by expanding the ground state energy in powers of α around the SUSY point $\epsilon = 1, \alpha = 0$. The leading-order correction $\Psi^{(1)}$ to the wave function can be determined from the expanded Schrödinger equation $\bar{Q}_0 Q_0 \Psi^{(1)} = -2ig^2\partial_\theta\Psi^{(0)}$ with $Q_0 = Q(\alpha = 0)$ and $\bar{Q}_0 = \bar{Q}(\alpha = 0)$, which can be solved as

$$\Psi^{(1)} = i\Psi^{(0)}\int_0^\theta d\theta'\left(1 - \frac{\exp\left(-z\cos\theta'\right)}{I_0(z)}\right). \tag{3.20}$$

Since the Hamiltonian takes the form $H = \bar{Q}_0 Q_0 + 2i\alpha g^2\partial_\theta + \alpha^2 g^2$, the ground state energy can be expanded as

$$E = \alpha\frac{\langle 0|2ig^2\partial_\theta|0\rangle}{\langle 0|0\rangle} + \alpha^2\left[\frac{\langle 0|g^2|0\rangle}{\langle 0|0\rangle} - \frac{\langle\Psi^{(1)}|\bar{Q}_0 Q_0|\Psi^{(1)}\rangle}{\langle 0|0\rangle}\right] + \cdots. \tag{3.21}$$

Therefore, the first- and the second-order expansion coefficients are given by

$$\frac{\partial}{\partial\alpha}E\bigg|_{\alpha=0} = 0, \qquad \frac{1}{2}\frac{\partial^2}{\partial\alpha^2}E\bigg|_{\alpha=0} = g^2 I_0(z)^{-2}. \tag{3.22}$$

The second-order expansion coefficient can be decomposed into the multi-bion contributions whose p-bion part is given by

$$\frac{1}{2}\frac{\partial^2}{\partial\alpha^2}E_p\bigg|_{\alpha=0} = g^2 p\frac{\left[\pm\frac{i}{\pi}K_0(z)\right]^{p-1}}{\left[I_0(z)\pm\frac{i}{\pi}K_0(z)\right]^{p+1}} = \mp 2\pi ipm\left(\pm i\exp\left(-\frac{2m}{g^2}\right)\right)^p + \cdots. \tag{3.23}$$

This shows that the nontrivial structure appears from the second-order in the small α expansion. We will later discuss these results in comparison with the multi-bion contributions.

3.2.2. *Expansion around the QES points*

In addition to the SUSY case, we can also obtain exact results in the QES case. To investigate the QES case, we rewrite the Hamiltonian as

$$H = \Psi^{(0)}\left[g^2 J_3^2 - \frac{m}{2}(J_+ + J_-)\right](\Psi^{(0)})^{-1}, \tag{3.24}$$

where J_3 and $J_\pm$ are differential operators defined by

$$J_3 = i(\partial_\theta - i\alpha), \qquad J_\pm = \mp \exp(\mp i\theta)\left[\partial_\theta - i(\alpha \mp j)\right], \tag{3.25}$$

with $j \equiv \frac{\epsilon-1}{2}$. These operators satisfy the $\mathfrak{sl}(2)$ algebra

$$[J_+, J_-] = 2J_3, \quad [J_3, J_\pm] = \pm J_\pm, \quad \text{with } J_3^2 + \tfrac{1}{2}(J_+J_- + J_-J_+) = j(j+1). \tag{3.26}$$

This $\mathfrak{sl}(2)$ algebra have a finite-dimensional irreducible representation when j is a nonnegative half integer. Note that eigenfunctions in such an irreducible representation satisfy the periodic boundary condition $\Psi(\theta + 2\pi) = \Psi(\theta)$ only when $j \pm \alpha$ is an integer (e.g., the highest weight state $\Psi(\theta) = \exp(i(\alpha - j)\theta)\,\Psi^{(0)}$). Since the action of the Hamiltonian is closed within the irreducible representation, we can find its eigenfunctions using the ansatz

$$\Psi = \Psi^{(0)}\left(a_0 + a_1 J_- + \cdots + a_{2j} J_-^{2j}\right)\exp(i(\alpha - j)\theta). \tag{3.27}$$

Thus, by diagonalizing the Hamiltonian in this finite-dimensional sector of the Hilbert space, we can obtain $2j+1$ exact eigenstates with the Bloch angle $2\pi\alpha = 0$ (periodic Ψ) for $j \in \mathbb{Z}$ and $2\pi\alpha = \pi$ (antiperiodic Ψ) for $j \in \mathbb{Z} + 1/2$. For the singlet case $j = \alpha = 0$ ($\epsilon = 1$), the eigenfunction of the Hamiltonian (3.24) is given by

$$\Psi = \Psi_0, \qquad E = 0, \tag{3.28}$$

which is consistent with the results we have already shown in the SUSY case.

For $j = \frac{1}{2}$ ($\epsilon = 2$), $\alpha = \frac{1}{2}$, we find two eigenstates corresponding to the doublet representation

$$\Psi_\pm = \Psi_0(1 \mp e^{i\theta}), \qquad E_\pm = \tfrac{1}{4}(g^2 \pm 2m). \tag{3.29}$$

This indicates that the energy eigenvalues of the ground state E_- and the first excited state E_+ do not receive any nonperturbative correction (Ref. [35]). The small $\delta\epsilon = \epsilon - 1$ expansion gives the first-order expansion coefficient of the ground state energy as

$$E^{(1)} = \frac{\partial}{\partial\epsilon}E_-\bigg|_{j=\alpha=\frac{1}{2}} = -\frac{m}{2}\frac{\partial}{\partial z}\log\left[I_0(z) + I_1(z)\right]. \tag{3.30}$$

As in the SUSY case, we can extract the perturbative part using the Borel resummation as

$$E_0^{(1)} = -\frac{m}{2}\frac{\partial}{\partial z}\log\left[I_0(z) + I_1(z) \pm \frac{i}{\pi}(K_0(z) - K_1(z))\right]. \tag{3.31}$$

The remaining nonperturbative part can be decomposed into the p-bion contribution, whose leading-order part in the small g^2 expansion is given by

$$\begin{aligned} E_p^{(1)} &= \frac{m}{2}\frac{\partial}{\partial z}\left[\frac{1}{p}\left(\frac{\pm\frac{i}{\pi}\left[K_0(z) - K_1(z)\right]}{I_0(z) + I_1(z) \pm \frac{i}{\pi}\left[K_0(z) - K_1(z)\right]}\right)^p\right] \\ &= -m\left(\mp\frac{ig^2}{4m}\exp\left(-\frac{2m}{g^2}\right)\right)^p + \cdots, \end{aligned} \tag{3.32}$$

where we have used $[K_0(z) - K_1(z)]/[I_0(z) + I_1(z)] \approx -\frac{\pi}{4z}e^{-2z}$ for large z. We will discuss this result in comparison with the multi-bion contributions later.

3.3. Multi-bion solutions and semiclassical contributions

In this subsection, we discuss the multi-bion contributions to the ground state energy and show that they have imaginary ambiguities that are necessary for consistent resurgence structure of sine-Gordon QM. We derive the semiclassical contributions from multi-bion saddle points by applying the Lefschetz thimble method. We will see that their expansion coefficients around the SUSY and QES points consistently reproduce those obtained from the exact results discussed in the previous subsection.

3.3.1. Multi-bion solutions

We first identify the multi-bion solutions corresponding to the complex saddle points of the Euclidean action of sine-Gordon QM (3.6). They can be easily obtained from those in $\mathbb{C}P^1$ QM (Ref. [38]) by ignoring the azimuthal angle variable. The solutions of the Euclidean equation of motion that have nontrivial contributions in the $\beta \to \infty$ limit take the form

$$\tan \frac{\theta(\tau)}{2} = \frac{f(\tau - \tau_c)}{\sin \alpha}, \tag{3.33}$$

where τ_c is the complexified position modulus and $f(\tau)$ is the elliptic function

$$f(\tau) = \mathrm{cs}(\Omega\tau, k) \equiv \mathrm{cn}(\Omega\tau, k)/\,\mathrm{sn}(\Omega\tau, k), \tag{3.34}$$

which satisfies the differential equation $(\partial_\tau f)^2 = \Omega^2(f^2 + 1)(f^2 + 1 - k^2)$. Since the periods of the doubly periodic function $\mathrm{cs}(z, k)$ are given by the elliptic integrals $2K(k)$ and $4iK'(k) \equiv 4iK(\sqrt{1-k^2})$, the periodic boundary condition $\theta(\tau + \beta) = \theta(\tau)$ is satisfied when the parameters (Ω, k) are related to two integers (p, q) as

$$\beta = \frac{2pK + 4iqK'}{\Omega}, \quad 0 \leq q < p. \tag{3.35}$$

The parameters (α, Ω, k) are given in terms of the period β and the pair of integers (p, q), and their asymptotic forms for large β are given by (see Appendix B of Ref. [38] for details)

$$\begin{gathered} k \approx 1 - 8\exp\left(-\frac{\omega\beta - 2\pi i q}{p}\right), \qquad \Omega \approx \omega\left(1 + 8\frac{\omega^2 + m^2}{\omega^2 - m^2}\exp\left(-\frac{\omega\beta - 2\pi i q}{p}\right)\right), \\ \cos\alpha \approx \frac{m}{\omega}\left(1 - \frac{8m^2}{\omega^2 - m^2}\exp\left(-\frac{\omega\beta - 2\pi i q}{p}\right)\right), \end{gathered} \tag{3.36}$$

where $\omega = m\left(1 + \epsilon g^2/m\right)^{1/2}$. The asymptotic value of the action for the (p, q) solution takes the form

$$S \approx pS_1 + \pi i \epsilon l \quad \text{for large } \beta, \tag{3.37}$$

where S_1 denotes the on-shell value of the action for the single bion configuration

$$S_1 = \frac{2m}{g^2} + \epsilon \log \frac{\omega + m}{\omega - m}, \tag{3.38}$$

and we have ignored the vacuum value of the action. The imaginary part $\pi i \epsilon l$ is related to the so-called hidden topological angle (Ref. [52]) and the integer l is zero or the greatest common divisor of p and $2q$ depending on the value of $\mathrm{Im}\,\tau_c$. The real part of the action is p times the single bion

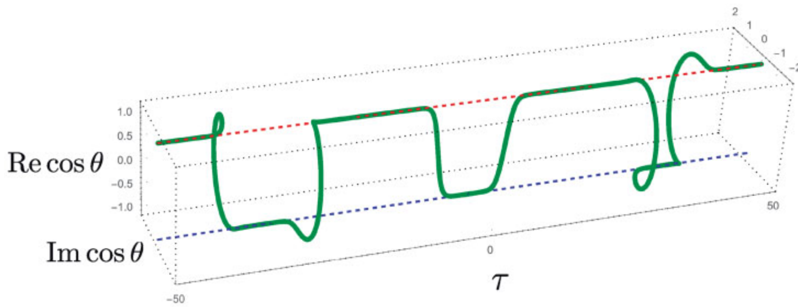

Fig. 2. Multi-bion solution: $p = 3$, $q = 2$, $m = 1$, $\epsilon = 1$, $g^2 = 1/20\,000$, $\beta = 100$, $\tau_c = 0$.

action S_1, which shows that the integer p is the number of bions. We can see from the solution (3.33) that the bions are equally spaced and the nth kink and antikink are located at τ_n^+ and τ_n^- given by

$$\tau_n^{\pm} = \tau_c + \frac{n-1}{\omega p}(\omega\beta - 2\pi i q) \pm \frac{1}{2\omega}\log\frac{4\omega^2}{\omega^2 - m^2}. \tag{3.39}$$

Figure 2 shows an example of a multi-bion solution with $(p, q) = (3, 2)$.

3.3.2. Multi-bion contributions and quasi-moduli integral

Next, we compute the semiclassical contributions from the multi-bion saddle points in the weak coupling limit $g \to 0$. As can be seen from Eq. (3.39), each constituent (anti)kink is almost isolated for large β and small g and hence the interkink binding force becomes small. Therefore, the positions of the constituent (anti)kinks can be regarded as quasi-moduli parameters that parametrize the nearly flat directions around the multi-bion saddle points. As explained in Appendix D, the semiclassical contribution from the saddle points can be obtained by reducing the path integral to a finite-dimensional integral over the quasi-moduli parameters.

Let us consider the semiclassical contributions from the p-bion saddle points. There are $2p$ constituent kinks and each of them can be either instanton or antiinstanton. Assigning $s_i = +1$ (-1) if ith kink is an (anti)instanton, we can write the p-bion contribution to the partition function Z_p as

$$\frac{Z_p}{Z_0} = \sum_{s_1=\pm 1} \cdots \sum_{s_{2p}=\pm 1} X_{(s_1,\dots,s_{2p})}, \tag{3.40}$$

where Z_0 is the perturbative part of the partition function and $X_{(s_1,\dots,s_{2p})}$ denotes the quasi-moduli integral for a fixed set of s_i:

$$X_{(s_1,\dots,s_{2p})} = \frac{1}{p}\exp\left(-\frac{2pm}{g^2}\right)\int \prod_{i=1}^{2p}\left[m d\tau_i \left(\frac{2m}{\pi g^2}\right)^{1/2} \exp\left(-V_i\right)\right]. \tag{3.41}$$

The factor $\left(\frac{2m}{\pi g^2}\right)^{1/2}$ is the one-loop determinant around each kink and $1/p$ is inserted since the kinks are indistinguishable. The nearest-neighbor interaction potential V_i between the ith and $(i+1)$th kinks, which is discussed in Appendix E, is given by

$$V_i = \frac{4m}{g^2} s_i s_{i-1} e^{-y_i} + \epsilon_i y_i - \pi i \alpha s_i, \quad y_i = m(\tau_i - \tau_{i-1}), \tag{3.42}$$

where $\tau_0 = \tau_{2p}$, $s_0 = s_{2p}$, and

$$\epsilon_i = \begin{cases} \epsilon & \text{for even } i, \\ 0 & \text{for odd } i. \end{cases} \tag{3.43}$$

We can check that the saddle points of the effective potential $\sum_{i=1}^{2p} V_i$ give complexified kink positions that are consistent with those read off from the multi-bion solution (3.39).

To evaluate the quasi-moduli integral, it is convenient to introduce the Lagrange multiplier for the constraint

$$\delta\left(\sum_{i=1}^{2p}\tau_i - \beta\right) = m\int_{-\infty}^{\infty}\frac{d\sigma}{2\pi}\exp\left[im\sigma\left(\sum_{i=1}^{2p}\tau_i - \beta\right)\right]. \tag{3.44}$$

Then, we can rewrite $X_{(s_1,\ldots,s_{2p})}$ as

$$X_{(s_1,\ldots,s_{2p})} = \frac{m\beta}{p}\exp\left(-\frac{2pm}{g^2}\right)\int\frac{d\sigma}{2\pi}\exp\left(-im\beta\sigma\right)\prod_{i=1}^{2p} I_i, \tag{3.45}$$

where I_i is given by the following integral with respect to the single variable τ_i:

$$I_i = \left(\frac{2m}{\pi g^2}\right)^{1/2}\int m\, d\tau_i \exp\left(-\frac{4m}{g^2}s_i s_{i-1}e^{-m\tau_i} - (m\epsilon_i - i\sigma)\tau_i + \pi i\alpha s_i\right), \tag{3.46}$$

where the integration contour is determined by the Lefschetz thimble method. Since the details of the thimble calculation are parallel to those for the quasi-moduli integral in $\mathbb{C}P^1$ QM in Ref. [38], we show only the essential parts of the calculation below.

As shown in Appendix F, we can evaluate I_i by means of the Lefschetz thimble method as

$$I_i = \left(\frac{2m}{\pi g^2}\right)^{1/2}\left(\frac{4m}{g^2}\right)^{i\sigma-\epsilon_i}\Gamma\left(\epsilon_i - i\sigma\right)\exp\left[\pm\frac{\pi i}{2}(i\sigma - \epsilon_i)(1 - s_i s_{i-1}) + \pi i\alpha s_i\right], \tag{3.47}$$

where the ambiguous sign comes from the Stokes phenomenon: the sign $\pm$ corresponds to the limit $\arg g^2 \to \pm 0$. To obtain this expression, we have shifted $\mathrm{Im}\,\sigma$ so that $\mathrm{Re}(\epsilon_i - i\sigma) > 0$ for all i. By closing the integration contour for σ as shown in Fig. 3, $X_{(s_1,\ldots,s_{2p})}$ can be evaluated by picking up the poles of the integrand located at

$$\sigma = -ik \quad \text{and} \quad -i(\epsilon + k), \quad k \in \mathbb{Z}_{\geq 0}. \tag{3.48}$$

Since the residues at $\sigma \neq 0$ vanish in the $\beta \to \infty$ limit, the pole at $\sigma = 0$ gives the leading-order term for large β,

$$X_{(s_1,\ldots,s_{2p})} = -\frac{im\beta}{p}\exp\left(-\frac{2pm}{g^2}\right)\operatorname*{Res}_{\sigma=0}\left[\exp\left(-im\beta\sigma\right)\prod_{i=1}^{2p} I_i\right] + \mathcal{O}(\exp\left(-m\epsilon\beta\right), e^{-m\beta}). \tag{3.49}$$

Thus we obtain the following semiclassical contribution of the p-bion solutions:

$$\frac{Z_p}{Z_0} \approx -\frac{im\beta}{p}\exp\left(-\frac{2pm}{g^2}\right)\operatorname*{Res}_{\sigma=0}\left[\exp\left(-im\beta\sigma\right)\left\{\frac{1}{2\pi}\left(\frac{4m}{g^2}\right)^{2i\sigma-\epsilon+1}\Gamma(-i\sigma)\Gamma(\epsilon - i\sigma)\right\}^p \mathcal{Z}_{\pm}\right], \tag{3.50}$$

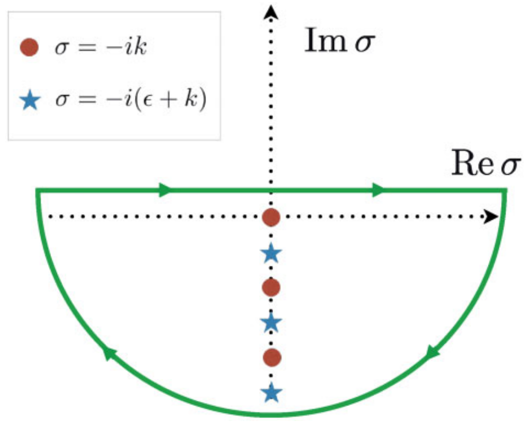

Fig. 3. Integration contour for σ. The poles of the integrand are located at $\sigma = -ik$ and $\sigma = -i(\epsilon + k)$ ($k \in \mathbb{Z}_{\geq 0}$).

where $\mathcal{Z}_\pm$ is given by

$$\mathcal{Z}_\pm = \sum_{s_1=\pm 1} \cdots \sum_{s_{2p}=\pm 1} \prod_{i=1}^{2p} \exp\left[\pm\frac{\pi i}{2}(i\sigma - \epsilon_i)(1 - s_i s_{i-1}) + \pi i \alpha s_i\right]. \tag{3.51}$$

By using the transfer matrix

$$T_\pm = \begin{pmatrix} \exp(\pi i \alpha) & \exp(\pm\pi i(i\sigma - \epsilon)) \\ \exp(\pm\pi i(i\sigma - \epsilon)) & \exp(-\pi i \alpha) \end{pmatrix} \begin{pmatrix} \exp(\pi i \alpha) & \exp(\pm\pi i(i\sigma)) \\ \exp(\pm\pi i(i\sigma)) & \exp(-\pi i \alpha) \end{pmatrix}, \tag{3.52}$$

$\mathcal{Z}_\pm$ can also be written as

$$\mathcal{Z}_\pm = \mathrm{Tr}(T_\pm^p). \tag{3.53}$$

By calculating $\mathrm{Tr}(T_\pm^p)$ and evaluating the residue of Eq. (3.50) at $\sigma = 0$, we can obtain the explicit p-bion contribution to the partition function.

3.3.3. *Comparison with the exact results*

We are now ready to compare the semiclassical contributions from the multi-bion solutions with the exact results (3.19), (3.23), and (3.32). We show below that the semiclassical multi-bion contributions obtained above precisely agree with the exact results for the expansion coefficients of the ground state energy around the SUSY and QES points.

Small $\delta\epsilon$ expansion around $j \in \mathbb{Z}$, $\alpha = 0$. The trace $\mathrm{Tr}(T_\pm^p)$ can be expanded around the QES point $\epsilon = 2j + 1$ ($j \in \mathbb{Z}$), $\alpha = 0$ as

$$\mathrm{Tr}(T_\pm^p) = 2^p \exp(\mp p\pi\sigma)\left[2\sinh^p(\pm\pi\sigma) \pm ip\pi\exp(\mp\pi\sigma)\sinh^{p-1}(\pm\pi\sigma)\delta\epsilon + \mathcal{O}(\delta\epsilon^2)\right], \tag{3.54}$$

where $\delta\epsilon = \epsilon - 2j - 1$ (for all $j \in \mathbb{Z}$). The first term indicates that the trace $\mathrm{Tr}(T_\pm^p)$ has a pth-order zero at $\sigma = 0$ for $\delta\epsilon = 0$, so that the p-bion contribution (3.50) vanishes at the QES point

$$E_p^{(0)} = \lim_{\beta\to\infty}\lim_{\delta\epsilon\to 0}\left(-\frac{1}{\beta}\frac{Z_p}{Z_0}\right) = 0. \tag{3.55}$$

The second term in Eq. (3.54) gives the following contribution to the first expansion coefficient of the ground state energy:

$$E_p^{(1)} = \lim_{\beta\to\infty}\lim_{\delta\epsilon\to 0}\frac{\partial}{\partial\delta\epsilon}\left(-\frac{1}{\beta}\frac{Z_p}{Z_0}\right) = -m\left[\pm i\frac{\Gamma(2j+1)}{(4z)^{2j}}\exp\left(-\frac{2m}{g^2}\right)\right]^p + \cdots . \tag{3.56}$$

This is consistent with the exact result for $j = 0$ (3.19), which indicates that the imaginary ambiguity from the perturbation series is canceled by those from the semiclassical bion contributions.

Small α expansion around $j \in \mathbb{Z}$, $\alpha = 0$. Next, let us consider the expansion with respect to α. For $\epsilon = 2j+1$, the trace $\mathrm{Tr}(T_\pm^p)$ can be expanded as

$$\mathrm{Tr}(T_\pm^p) = 2^p\exp(\mp p\pi\sigma)\left[2\sinh^p(\pm\pi\sigma)+2p^2\pi^2\exp(\mp\pi\sigma)\sinh^{p-1}(\pm\pi\sigma)\alpha^2+\mathcal{O}(\alpha^3)\right]. \tag{3.57}$$

The absence of an $\mathcal{O}(\alpha)$ term implies that the leading-order expansion coefficient vanishes:

$$\left.\frac{\partial E_p}{\partial\alpha}\right|_{\alpha=0} = 0. \tag{3.58}$$

The first nontrivial contribution appears from the second-order coefficient

$$\left.\frac{1}{2}\frac{\partial^2 E_p}{\partial\alpha^2}\right|_{\alpha=0} = \lim_{\beta\to\infty}\lim_{\alpha\to 0}\frac{1}{2}\frac{\partial^2}{\partial\alpha^2}\left(-\frac{1}{\beta}\frac{Z_p}{Z_0}\right) \tag{3.59}$$

$$= \mp 2\pi ipm\left[\pm i\frac{\Gamma(2j+1)}{(4z)^{2j}}\exp\left(-\frac{2m}{g^2}\right)\right]^p + \cdots . \tag{3.60}$$

These results are consistent with those obtained from the exact expressions for $j = 0$ given in Eqs. (3.22) and (3.23).

Small $\delta\epsilon$ expansion around $j \in \mathbb{Z}+\frac{1}{2}$, $\alpha = \frac{1}{2}$. Finally, let us look into the expansion around $j \in \mathbb{Z}+\frac{1}{2}$, $\alpha = \frac{1}{2}$. The trace $\mathrm{Tr}(T_\pm^p)$ can be expanded as

$$\mathrm{Tr}(T_\pm^p) = 2^p\exp(\mp p\pi\sigma)\left[2\sinh^p(\mp\pi\sigma)\mp ip\pi\exp(\mp\pi\sigma)\sinh^{p-1}(\mp\pi\sigma)\delta\epsilon+\mathcal{O}(\delta\epsilon)\right], \tag{3.61}$$

where $\delta\epsilon = \epsilon - 2j - 1$ (for all $j \in \mathbb{Z}+\frac{1}{2}$). As above, we can obtain the following semiclassical p-bion contribution to the expansion coefficients, which is consistent with the exact result (3.32) for $j = \frac{1}{2}$:

$$E_p^{(0)} = 0, \qquad E_p^{(1)} = -m\left[\mp i\frac{\Gamma(2j+1)}{(4z)^{2j}}\exp\left(-\frac{2m}{g^2}\right)\right]^p + \cdots . \tag{3.62}$$

We have shown in sine-Gordon QM that the multi-bion semiclassical contributions correctly reproduce the leading (semiclassical) terms of the exact results for the ground state energy around the SUSY and QES points in the parameter space. We emphasize that this agreement reveals the resurgence structure to all orders of the nonperturbative exponential, even though the resurgence structure in models such as the sine-Gordon model is more complicated than that of Kähler QM, such as $\mathbb{C}P^{N-1}$ QM discussed in the next section.

4. Resurgence structure in $\mathbb{C}P^{N-1}$ QM

In this section, we consider quantum mechanics on the complex projective space $\mathbb{C}P^{N-1}$ as an example of a model on a Kähler target space discussed in Sect. 2.

4.1. $\mathbb{C}P^{N-1}$ QM

The Kähler potential K and Kähler metric $G_{i\bar{j}}$ of $\mathbb{C}P^{N-1}$ are given by

$$K = \log(1 + |\varphi^i|^2), \qquad G_{i\bar{j}} = \frac{\partial^2 K}{\partial\varphi^i \partial\bar{\varphi}^{\bar{j}}} = \frac{(1 + |\varphi^i|^2)\delta_{i\bar{j}} - \varphi^j \bar{\varphi}^{\bar{i}}}{(1 + |\varphi^i|^2)^2}. \tag{4.1}$$

This Kähler metric has an SU(N) holomorphic isometry and we use the following specific linear combination of the Killing vectors in its Cartan subalgebra:

$$\Xi = -\sum_{i=1}^{N-1} i m_i (\varphi^i \partial_i - \bar{\varphi}^{\bar{i}} \bar{\partial}_{\bar{i}}), \qquad \mu = \sum_{j=1}^{N-1} \frac{m_j |\varphi^j|^2}{1 + |\varphi^i|^2}. \tag{4.2}$$

The coefficients m_i, which parametrize the Killing vector Ξ and the moment map μ, determine the potential in the Schrödinger equation of $\mathbb{C}P^{N-1}$ QM,

$$H\Psi = G^{\bar{j}i} \left[-g^2 \partial_i \bar{\partial}_{\bar{j}} + \frac{1}{g^2} \partial_i \mu \, \bar{\partial}_{\bar{j}} \mu - \epsilon \partial_i \bar{\partial}_{\bar{j}} \mu \right] \Psi = E\Psi. \tag{4.3}$$

For $\epsilon = 1$, this equation describes the SUSY $\mathbb{C}P^{N-1}$ QM projected to the sector with the lowest fermion number F. The SUSY ground state wave function and its energy eigenvalue are given by

$$\Psi^{(0)} = \langle \varphi | 0 \rangle = \exp\left(-\frac{\mu}{g^2} \right), \qquad E^{(0)} = 0. \tag{4.4}$$

For $\epsilon \approx 1$, we can solve the Schrödinger equation by expanding the wave function and the ground state energy with respect to $\delta\epsilon = \epsilon - 1$ as

$$\Psi = \Psi^{(0)} + \delta\epsilon \Psi^{(1)} + \cdots, \qquad E = \delta\epsilon E^{(1)} + \delta\epsilon^2 E^{(2)} + \cdots. \tag{4.5}$$

These expansion coefficients are determined by the standard Rayleigh–Schrödinger perturbation theory,

$$E^{(1)} = -\frac{\langle 0 | \Delta\mu | 0 \rangle}{\langle 0 | 0 \rangle}, \qquad E^{(2)} = -\frac{\langle \Psi^{(1)} | H_{\epsilon=1} | \Psi^{(1)} \rangle}{\langle 0 | 0 \rangle}, \qquad \cdots, \tag{4.6}$$

where we have used $\delta H = -\Delta\mu$. Here $\langle 0|0 \rangle$ is the normalization factor of the SUSY ground state wave function in Eq. (4.4), which can also be viewed as the generating function for μ:

$$\langle 0 | 0 \rangle = \int dv \, \exp\left(-\frac{2\mu}{g^2} \right), \qquad dv = \frac{1}{(N-1)!} \left(\frac{i}{2} G_{i\bar{j}} d\varphi^i \wedge d\bar{\varphi}^{\bar{j}} \right)^{N-1}. \tag{4.7}$$

For $N = 2$ (the $\mathbb{C}P^1$ case), the resurgence structure to all orders with complex multi-bion solutions have been investigated in our previous work Ref. [38]. Here we discuss the case with general $N \geq 2$.

4.2. Near-SUSY exact results

First let us derive some exact results for the expansion coefficients $E^{(n)}$ around the SUSY point $\epsilon = 1$. Since the perturbation Hamiltonian $\delta H = -\Delta\mu$ in this case is simply given by the linear function of μ,

$$-\Delta\mu = G^{\bar{j}i}\partial_{\bar{j}}\partial_i\mu = -\sum_{i=1}^{N-1} m_i + N\mu, \tag{4.8}$$

then the first expansion coefficient can be rewritten as

$$E^{(1)} = -\frac{\langle 0|\Delta\mu|0\rangle}{\langle 0|0\rangle} = -\sum_{i=1}^{N-1} m_i + \frac{N}{2}g^4\frac{\partial}{\partial g^2}\log\langle 0|0\rangle. \tag{4.9}$$

As shown in Appendix B, the generating function $\langle 0|0\rangle$ can be calculated by the Duistermaat–Heckman localization formula (Refs. [129,130]) as

$$\langle 0|0\rangle = \left(\prod_{i=1}^{N-1}\frac{\pi g^2}{2m_i}\right)\left(1 - \sum_{i=1}^{N-1} A_i \exp\left(-\frac{2m_i}{g^2}\right)\right), \tag{4.10}$$

where the coefficients A_i are given by

$$A_i = \prod_{j=1, j\neq i}^{N-1}\frac{m_j}{m_j - m_i}. \tag{4.11}$$

Combining Eqs. (4.9)–(4.11), we obtain the following exact expression for the first-order expansion coefficient $E^{(1)}$:

$$E^{(1)} = \frac{N(N-1)}{2}g^2 - \sum_{i=1}^{N-1} m_i\left(1 + \frac{NA_i\exp\left(-\frac{2m_i}{g^2}\right)}{1 - \sum_{j=1}^{N-1}A_j\exp\left(-\frac{2m_j}{g^2}\right)}\right). \tag{4.12}$$

As mentioned in Sect. 2, no divergent asymptotic series appears in each sector of the trans-series. For example, the perturbative contribution is given by

$$E_0^{(1)} = \frac{N(N-1)}{2}g^2 - \sum_{i=1}^{N-1} m_i. \tag{4.13}$$

The absence of divergent asymptotic series indicates that the first-order expansion coefficient $E^{(1)}$ has no nontrivial resurgence structure among the sectors with different orders of nonperturbative exponentials.

Next, let us move on to the second-order expansion coefficient of the ground state energy $E^{(2)}$. To obtain $E^{(2)}$ from Eq. (4.6), we need to solve the $\mathcal{O}(\delta\epsilon)$ Schrödinger equation for $\Psi^{(1)}$. Although it is difficult to obtain the exact solution for $\Psi^{(1)}$, the Bender–Wu method (Ref. [141]) can be used to determine $\Psi^{(1)}$ in a perturbative way. Resumming the perturbative series of $\Psi^{(1)}$, we obtain the following leading-order part of $\Psi^{(1)}$ in the weak coupling limit $g \to 0$:

$$\Psi^{(1)} = \frac{N}{2}\exp\left(-\frac{\mu}{g^2}\right)\log\frac{1}{1+\sum_{k=1}^{N-1}|\varphi^k|^2} + \cdots, \tag{4.14}$$

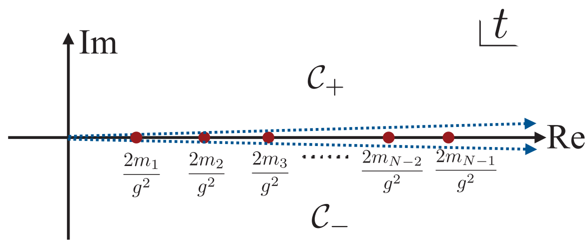

Fig. 4. Singularities on the Borel plane for the perturbative Borel transform of $\mathbb{C}P^{N-1}$ QM.

where $\cdots$ denotes nonperturbative corrections. We can also directly check that Eq. (4.14) is the correct perturbative part of $\Psi^{(1)}$ by substituting it into the $\mathcal{O}(\delta\epsilon)$ Schrödinger equation.

Then, from the relation $E^{(2)} = -\langle\Psi^{(1)}|H_{\epsilon=1}|\Psi^{(1)}\rangle/\langle 0|0\rangle$, we obtain the perturbative part of the second-order expansion coefficient as (see Appendix C)

$$E_0^{(2)} = \frac{N^2}{4}\left[g^2 + \sum_{i=1}^{N-1} 2m_i A_i \int_0^\infty dt \frac{e^{-t}}{t - \frac{2m_i}{g^2 \pm i0}}\right]. \tag{4.15}$$

This perturbative part has the following imaginary ambiguity due to the regularization $g^2 \to g^2 \pm i0$:

$$\operatorname{Im} E_0^{(2)} = \mp\frac{\pi i}{2} N^2 \sum_{i=1}^{N-1} m_i A_i \exp\left(-\frac{2m_i}{g^2}\right). \tag{4.16}$$

It is notable that the number of singularities on the Borel plane is $N-1$, which means that there are multiple singularities for $N > 2$ as shown in Fig. 4. We will show that the imaginary ambiguities that originate from these $N-1$ singularities are canceled by the $(N-1)$-type single (real and complex) bions.

4.3. *Bion solutions and semiclassical contributions*

In the previous section, we derived the imaginary ambiguities arising from the perturbation series of $E^{(2)}$. In this section, we show that they are canceled by the semiclassical contributions from single-bion solutions. We also show that the contributions from multi-bion solutions correctly reproduce $E^{(1)}$ to all orders in the nonperturbative exponentials.

4.3.1. *Embedding $\mathbb{C}P^1$ bion solutions*

First, we investigate exact (complex) saddle point solutions in $\mathbb{C}P^{N-1}$ QM. Here we focus on the $\beta \to \infty$ limit for simplicity. The Euclidean action for the projected $\mathbb{C}P^{N-1}$ QM takes the form

$$S_E = \frac{1}{g^2}\int_{-\infty}^{\infty} d\tau \left[G_{i\bar{j}}\partial_\tau\varphi^i\partial_\tau\bar{\varphi}^{\bar{j}} + G^{\bar{j}i}\left(\partial_i\mu\bar{\partial}_{\bar{j}}\mu - g^2\epsilon\partial_i\bar{\partial}_{\bar{j}}\mu\right)\right]. \tag{4.17}$$

To find the simplest saddle point solution, let us consider the ansatz

$$\varphi^j = 0 \quad \text{for } j \neq 1. \tag{4.18}$$

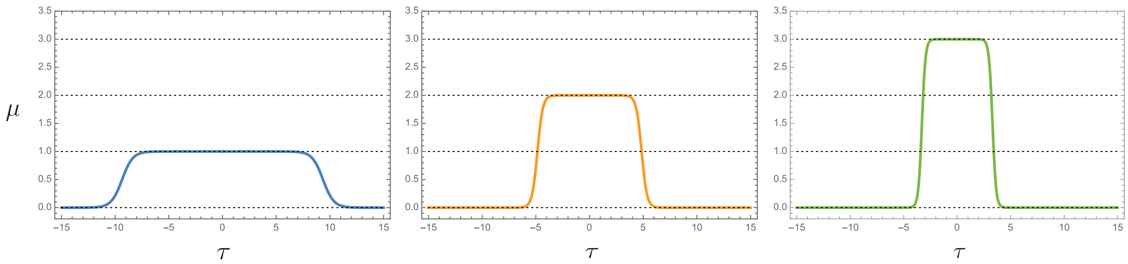

Fig. 5. Examples of $(N-1)$ bion solutions for $N=4$: $m_1=1, m_2=2, m_3=3, g=10^{-4}, \epsilon=1$.

Then the equation of motion reduces to that in $\mathbb{C}P^1$ QM (Ref. [38]) with $2\epsilon \to N\epsilon$, and hence we can embed the $\mathbb{C}P^1$ bion solutions into φ^1 as

$$\varphi^1 = \left(\frac{\omega_1^2}{\omega_1^2 - m_1^2}\right)^{1/2} \frac{e^{i\phi_0}}{\sinh \omega_1(\tau-\tau_0)}, \quad \varphi^j = 0 \ (j \neq 1), \tag{4.19}$$

where $\omega_1 \equiv m_1\left(1+N\epsilon g^2/m_1\right)^{1/2}$ and (τ_0, ϕ_0) are moduli parameters. This is the single real bion solution, whose action is given by

$$S_{\rm rb}^1 = \frac{2\omega_1}{g^2} + N\epsilon \log \frac{\omega_1 + m_1}{\omega_1 - m_1}. \tag{4.20}$$

The corresponding complex bion solution can be obtained by shifting $\tau_0 \to \tau_0 + \pi i/2\omega_1$ and its action has an imaginary ambiguity related to the hidden topological angle

$$S_{\rm cb}^1 = S_{\rm rb}^1 \pm N\epsilon\pi i \quad \text{for } \arg g^2 \to \pm 0. \tag{4.21}$$

Similarly, we can also embed the real and complex bion solution into φ^i $(i=1,\dots,N-1)$:

$$\varphi^i = \left(\frac{\omega_i^2}{\omega_i^2 - m_i^2}\right)^{1/2} \frac{e^{i\phi_0}}{\sinh \omega_i(\tau-\tau_0)}, \quad \varphi^j = 0 \ (j \neq i), \tag{4.22}$$

where $\omega_i \equiv m_i\left(1+N\epsilon g^2/m_i\right)^{1/2}$. Therefore, $\mathbb{C}P^{N-1}$ QM has $(N-1)$ types of real and complex single bion solutions as shown in Fig. 5. We next look into the semiclassical contributions from these solutions.

4.3.2. *Semiclassical contribution of bion saddle points*

Next, let us calculate nonperturbative contributions to the ground state energy from the viewpoint of the saddle point method. We first note that the bion configuration can be rewritten into the kink–antikink form

$$\varphi^i_{k\bar{k}} = \left(\exp\left(\omega_i(\tau-\tau_+) - i\phi_+\right) - \exp\left(-\omega_i(\tau-\tau_-) - i\phi_-\right)\right)^{-1} \tag{4.23}$$

$$\text{with } \exp\left(\pm\omega_i(\tau_\pm - \tau_0) + i(\phi_\pm - \phi_0)\right) = \left(\frac{4\omega_i^2}{\omega_i^2 - m_i^2}\right)^{1/2}.$$

In the weak coupling limit $g \to 0$, the kink and antikink are well separated ($|\tau_+ - \tau_-| \approx \log(4m_i/N\epsilon g^2)$), and the binding force between them becomes small. For such a configuration, the relative position $\tau_r = \tau_+ - \tau_-$ and phase $\phi_r = \phi_+ - \phi_-$ can be regarded as quasi-moduli parameters corresponding to the nearly flat directions around the saddle point configuration.

As discussed in Appendix D, we can decompose the degrees of freedom into the quasi-moduli parameters and orthogonal massive modes $\delta\varphi^j$:

$$S = V_{\rm eff} + \sum_{j=1}^{N-1} \int d\tau\, \delta\bar{\varphi}^{\bar{j}}\, \Delta_j \delta\varphi^j + \mathcal{O}(g^4), \qquad \Delta_j = -\partial_\tau^2 + \mathcal{V}_{i,j}(\tau), \tag{4.24}$$

where $V_{\rm eff}$ is the effective potential between the well-separated kink and antikink and Δ_j is the differential operator with the potential $\mathcal{V}_{i,j}(\tau)$ for the jth fluctuation $\delta\varphi^j$ on the ith bion background (see Appendix E for details)

$$V_{\rm eff} = \frac{2m_i}{g^2} - \frac{4m_i}{g^2}\cos\phi_r \exp(-m_i\tau_r) + N m_i \epsilon\tau_r. \tag{4.25}$$

We can easily check that $V_{\rm eff}$ has the correct saddle points corresponding to the real and complex bion solutions (4.23). The nonperturbative contribution from each single bion saddle point can be obtained from the quasi-moduli integral

$$\lim_{\beta\to\infty}\left(-\frac{1}{\beta}\frac{Z_1}{Z_0}\right)_{i,\rm bion} = -\frac{8m_i^4}{\pi g^2}\int d\tau_r d\phi_r \left(\prod_{j=1,j\neq i}^{N-1} \det\Delta_j^{-1}\right)\exp(-V_{\rm eff}), \tag{4.26}$$

where $\det\Delta_j$ denotes the one-loop determinant for the fluctuation $\delta\varphi^j$. The overall factor $8m_i^4/\pi g^2$, which includes the one-loop determinant for $\delta\varphi^i$, can be obtained in the same way as in the $\mathbb{C}P^1$ case (Ref. [32]). As shown in Appendix G, the determinants $\det\Delta_j$ are given by

$$\prod_{j=1,j\neq i}^{N-1} \det\Delta_j^{-1} = A_i \exp\left((N-2)m_i\tau_r\right). \tag{4.27}$$

Hence the one-bion contribution can be rewritten as

$$-\lim_{\beta\to\infty}\frac{1}{\beta}\frac{Z_1}{Z_0} = -\sum_{i=1}^{N-1}\frac{8A_i m_i^4}{\pi g^2}\int d\tau_r\, d\phi_r\, \exp\left(-V'_{\rm eff}\right), \tag{4.28}$$

with the modified effective potential

$$V'_{\rm eff} = \frac{2m_i}{g^2} - \frac{4m_i}{g^2}\cos\phi_r \exp(-m_i\tau_r) + 2\epsilon' m_i\tau_r, \tag{4.29}$$

where ϵ' is the "renormalized" parameter related to ϵ as

$$\epsilon' = 1 - \frac{N}{2}(1-\epsilon). \tag{4.30}$$

Since this modified kink–antikink potential is the same as that of $\mathbb{C}P^1$ QM with $m \to m_i$ and $\epsilon \to \epsilon'$, the quasi-moduli integral can be performed as in $\mathbb{C}P^1$ QM (Ref. [32]). Summing all the single bion contributions, we obtain the following semiclassical contribution with an imaginary ambiguity:

$$E_1 = -\lim_{\beta\to\infty}\frac{1}{\beta}\frac{Z_1}{Z_0} = -\sum_{i=1}^{N-1} 2m_i A_i \left(\frac{2m_i}{g^2}\right)^{2(1-\epsilon')}\frac{\Gamma(\epsilon')}{\Gamma(1-\epsilon')}\exp\left(-\frac{2m_i}{g^2} \mp \pi i\epsilon'\right). \tag{4.31}$$

Let us consider the expansion around the SUSY point $\epsilon = 1$. From Eq. (4.31), we obtain the following first-order expansion coefficient of the ground state energy:

$$E_1^{(1)} = \lim_{\epsilon\to 1} \partial_\epsilon \left(-\lim_{\beta\to\infty} \frac{1}{\beta}\frac{Z_1}{Z_0} \right) = -\sum_{i=1}^{N-1} N m_i A_i \exp\left(-\frac{2m_i}{g^2}\right). \tag{4.32}$$

This precisely agrees with the leading-order nonperturbative corrections that can be extracted from the exact result (4.12). The second-order coefficient is given by

$$\begin{aligned} E_1^{(2)} &= \frac{1}{2}\lim_{\epsilon\to 1} \partial_\epsilon^2 \left(-\lim_{\beta\to\infty} \frac{1}{\beta}\frac{Z_1}{Z_0} \right) \\ &= N^2 \sum_{i=1}^{N-1} m_i A_i \exp\left(-\frac{2m_i}{g^2}\right) \left[\gamma + \log\frac{2m_i}{g^2} \pm \frac{\pi i}{2} + \mathcal{O}(g^2) \right], \end{aligned} \tag{4.33}$$

where γ is the Euler–Mascheroni constant. Here we find complete cancelation between the imaginary ambiguities of the $(N-1)$ types of the semiclassical single-bion contributions and those of the perturbative part (4.16) arising from the $(N-1)$ Borel singularities.

4.3.3. *Multi-bion contributions*

Next, let us consider the multi-bion contributions to the partition function. Although we have not found multi-bion solutions except for the embedded ones, we assume that they consist of well-separated kink–antikink pairs, each of which is one of the $N-1$ types of bions. Then the semiclassical contribution to the partition function can be schematically written as

$$\frac{Z_p}{Z_0} = \sum_{i_1=1}^{N-1} \cdots \sum_{i_p=1}^{N-1} \int \prod_{n=1}^{2p} \Big[d\tau_n d\phi_n\, B_n \exp\big(-V_{n,n-1}\big) \Big], \tag{4.34}$$

where (τ_n, ϕ_n) are quasi-moduli parameters corresponding to the position and phase of the nth constituent kink, B_n are constants related to the integration measure and the one-loop determinant, and $(i_1, \ldots, i_p)$ denote the types of bions. Since the nearest-neighbor asymptotic interaction potential $V_{n,n-1}$ is a function of the relative quasi-moduli parameters

$$\tau_n^r \equiv \tau_n - \tau_{n-1}, \qquad \phi_n^r \equiv \phi_n - \phi_{n-1}, \tag{4.35}$$

the integral can be factorized by introducing the Lagrange multipliers (σ, s) for the constraints

$$\sum_{n=1}^{2p} \tau_n^r = \beta, \qquad \sum_{n=1}^{2p} \phi_n^r = 0 + \mathrm{mod} 2\pi, \tag{4.36}$$

and changing the integration variables as $(\tau_n, \phi_n) \to (\tau_n^r, \phi_n^r)$. Then, the summation over $(1, \ldots, i_p)$ can be recast into the matrix form

$$\frac{Z_p}{Z_0} = \frac{2\pi\beta}{p} \sum_{s=-\infty}^{\infty} \frac{1}{2\pi} \int \frac{d\sigma}{2\pi} \exp(-i\beta\sigma)\, \mathrm{Tr}(\mathcal{T}^p), \tag{4.37}$$

where the volume factor of the overall moduli $2\pi\beta$ is divided by p since the bions are indistinguishable. The matrix $\mathcal{T}$ takes the form

$$(\mathcal{T})_{ij} = \left(\frac{2m_i^2}{\pi g^2}\right)^2 A_i \mathcal{I}_i \mathcal{J}_{ij} \exp\left(-\frac{2m_i}{g^2}\right), \tag{4.38}$$

with

$$\mathcal{I}_i = \int d\tau\, d\phi\, \exp\left(-V_i + i\sigma\tau + is\phi\right), \qquad \mathcal{J}_{ij} = \int d\tau\, d\phi\, \exp\left(-U_{ij} + i\sigma\tau + is\phi\right), \tag{4.39}$$

where V_i is the interaction potential between kink and antikink of the same type,

$$V_i = \frac{4m_i}{g^2}\exp\left(-m_i\tau\right)\cos\phi + 2m_i\epsilon'\tau, \quad \epsilon' = 1 + \frac{N}{2}(\epsilon - 1), \tag{4.40}$$

and U_{ij} is that between a kink of ith type and an antikink of jth type. As in the $\mathbb{C}P^1$ case (Ref. [38]), the leading contribution in the large-β limit is given by the residue at $\sigma = 0$ of the term with $s = 0$,

$$\frac{Z_p}{Z_0} \approx -\frac{i\beta}{p}\operatorname*{Res}_{\sigma=0}\left[\exp\left(-i\beta\sigma\right)\mathrm{Tr}(\mathcal{T}^p)\right]_{s=0}. \tag{4.41}$$

In the following, we focus on the term with $s = 0$ only.

As in the single-bion case, we can show by using the Lefschetz thimble method that

$$\mathcal{I}_i = \frac{\pi}{m_i}\left(\frac{2m_i}{g^2}\exp\left(\pm\frac{\pi i}{2}\right)\right)^{i\frac{\sigma}{m_i} - 2\epsilon'} \frac{\Gamma\left(\epsilon' - \frac{i\sigma}{2m_i}\right)}{\Gamma\left(1 - \epsilon' + \frac{i\sigma}{2m_i}\right)}. \tag{4.42}$$

Although the explicit form of U_{ij} is not known, we can show from the fact that U_{ij} vanishes for large τ that $\mathcal{J}_{ij}$ has a pole at $\sigma = 0$:

$$\mathcal{J}_{ij} = \int d\tau\, d\phi\, \theta(\tau)\exp\left(i\sigma\tau\right) + \int d\tau\, d\phi\left[e^{-U_{ij}} - \theta(\tau)\right]\exp\left(i\sigma\tau\right) = \frac{2\pi i}{\sigma} + \mathcal{O}(1). \tag{4.43}$$

From Eqs. (4.42) and (4.43), we find that $\mathcal{T}$ and its derivative at $\epsilon = 1$ can be expanded around $\sigma = 0$ as

$$(\mathcal{T})_{ij}\Big|_{\epsilon=1} = A_i\exp\left(-\frac{2m_i}{g^2}\right) + \mathcal{O}(\sigma), \qquad \partial_\epsilon(\mathcal{T})_{ij}\Big|_{\epsilon=1} = \frac{iNm_i}{\sigma}A_i\exp\left(-\frac{2m_i}{g^2}\right) + \mathcal{O}(1). \tag{4.44}$$

Since $\mathcal{T}$ has no pole at $\sigma = 0$ for $\epsilon = 1$, the bion contribution to the partition function (4.41) vanishes and hence there is no bion correction to the ground state energy

$$\frac{Z_p}{Z_0}\bigg|_{\epsilon=1} = 0 \quad \Longrightarrow \quad E_p^{(0)} = 0. \tag{4.45}$$

Due to the pole of $\partial_\epsilon(\mathcal{T})_{ij}|_{\epsilon=1}$ at $\sigma = 0$, there is nontrivial bion contribution to the first expansion coefficient

$$E_p^{(1)} = -\lim_{\beta\to\infty}\frac{1}{\beta}\partial_\epsilon\frac{Z_p}{Z_0}\bigg|_{\epsilon=1} = -N\left(\sum_{i=1}^{N-1} m_iA_i\exp\left(-\frac{2m_i}{g^2}\right)\right)\left(\sum_{i=1}^{N-1} A_i\exp\left(-\frac{2m_i}{g^2}\right)\right)^{p-1}. \tag{4.46}$$

Summing the perturbative part and all the bion contributions, we obtain

$$E^{(1)} = E_0^{(1)} + \sum_{p=1}^{\infty}E_p^{(1)} = \frac{N(N-1)}{2}g^2 - \sum_{i=1}^{N-1} m_i\left(1 + \frac{NA_i\exp\left(-\frac{2m_i}{g^2}\right)}{1 - \sum_{j=1}^{N-1}A_j\exp\left(-\frac{2m_j}{g^2}\right)}\right). \tag{4.47}$$

Thus, the first expansion coefficient $E^{(1)}$ is completely reproduced by the semiclassical bion contributions.

We note that as opposed to $E_p^{(1)}$, the p-bion contribution to the nth expansion coefficient $E_p^{(n)}$ with $n \geq 2$ is an asymptotic series of g^2,

$$E_p^{(n)} = E_{p,0}^{(n)} + E_{p,2}^{(n)} g^2 + E_{p,4}^{(n)} g^4 + \cdots = E_{p,0}^{(n)} + \int_0^\infty dt \, \exp\left(-t/g^2\right) \tilde{E}_p^{(n)}(t). \tag{4.48}$$

In the $\mathbb{C}P^1$ case (Ref. [38]), the Borel transform $\tilde{E}_p^{(2)}(t)$ of the perturbative corrections to the second-order expansion coefficient around the p-bion background has a pole that gives an imaginary ambiguity canceled by that of the semiclassical $(p+1)$-bion contribution

$$\mathrm{Im}\left(\int_0^\infty dt \, \exp\left(-t/g^2\right) \tilde{E}_p^{(2)}(t)\right) \quad \underset{\text{cancelation}}{\longleftrightarrow} \quad \mathrm{Im}\, E_{p+1,0}^{(2)}. \tag{4.49}$$

Although it is difficult to obtain the perturbative corrections directly from the p-bion background, we can determine $E^{(2)}$ by summing all the semiclassical contributions,

$$\sum_{p=1}^{\infty} E_{p,0}^{(2)} = m \frac{\cosh \frac{m}{g^2}}{\sinh^3 \frac{m}{g^2}} \left(\gamma + \log \frac{2m}{g^2} \pm \frac{\pi i}{2}\right), \tag{4.50}$$

using the dispersion relation, and imposing the symmetry $m \to -m$ as[1]

$$\begin{aligned} E^{(2)} &= g^2 - 2m \coth \frac{m}{g^2} \int_0^m \frac{d\mu}{\mu} \frac{\sinh^2 \frac{\mu}{g^2}}{\sinh^2 \frac{m}{g^2}} \\ &= g^2 - m \frac{\cosh \frac{m}{g^2}}{\sinh^3 \frac{m}{g^2}} \left[\mathrm{Chi}\left(\frac{2m}{g^2}\right) - \gamma - \log \frac{2m}{g^2}\right]. \end{aligned} \tag{4.51}$$

Thus, taking advantage of the resurgence structure, we can completely reconstruct $E^{(2)}$ from the semiclassical bion contributions.

In the $\mathbb{C}P^{N-1}$ case, the total semiclassical bion contributions can be formally written as

$$\begin{aligned} \sum_{p=1}^{\infty} E_{p,0}^{(2)} = N^2 \sum_{i=1}^{N-1} & m_i A_i \exp\left(-\frac{2m_i}{g^2}\right) \\ & \times \left[\left(\gamma + \log \frac{2m_i}{g^2} \pm \frac{\pi i}{2}\right) Y_{ii} - \sum_{j=1}^{N-1} m_j A_j \exp\left(-\frac{2m_j}{g^2}\right) Y_{ij} X_{ij}\right], \end{aligned} \tag{4.52}$$

where X_{ij} and Y_{ij} are defined by

$$X_{ij} \equiv \underset{\sigma=0}{\mathrm{Res}} \left(\frac{\mathcal{J}_{ij}}{2\pi\sigma}\right), \qquad Y_{ij} \equiv \frac{R_i R_j}{1 - \sum_k A_k \exp\left(-\frac{2m_k}{g^2}\right)}, \tag{4.53}$$

$$R_i \equiv \frac{1 - \sum_k \frac{m_i - m_k}{m_i} A_k \exp\left(-\frac{2m_k}{g^2}\right)}{1 - \sum_k A_k \exp\left(-\frac{2m_k}{g^2}\right)}.$$

[1] The first term g^2 is added so that the perturbative part agrees with the Bender–Wu analysis. The argument here does not fix the degree of freedom to add any convergent series that is an even function of m.

In the previous subsection, we saw that the imaginary ambiguity of the single-bion contribution in Eq. (4.52) is canceled by that of the non-Borel-summable perturbation series (4.15). Since the generating function $\langle 0|0\rangle$ does not have a divergent asymptotic series, it is natural to expect that the higher bion sectors in $E^{(2)} = -\langle \Psi^{(1)}|H_{\epsilon=1}|\Psi^{(1)}\rangle/\langle 0|0\rangle$ have the same cancelation structure: the imaginary ambiguity of the semiclassical $(p+1)$-bion contribution is canceled by that of the non-Borel-summable perturbation series around the p-bion background. Thus we expect that it is also possible to recover $E^{(2)}$ from the semiclassical bion contributions (4.52) in a parallel manner to $\mathbb{C}P^1$ QM. It would be interesting to check whether there is such a resurgence structure in $\mathbb{C}P^{N-1}$ QM by explicitly determining $E^{(2)}$ and X_{ij} in Eqs. (4.53) and (4.52).

4.4. *Quasi-exact solvability of* $\mathbb{C}P^{N-1}$ *QM*

As in the case of sine-Gordon QM discussed in the previous section, $\mathbb{C}P^{N-1}$ QM becomes quasi-exactly solvable at some specific points in the parameter space. By introducing a deformation parameter around those QES points, we obtain exact results for the expansion coefficients of the ground state energy, which show a nontrivial resurgence structure around the QES points.

Here we focus on the sector with vanishing conserved charges, where wave functions are independent of $\arg \varphi^i$. For later convenience, we define the following new variables: $x_i = \frac{|\varphi^i|^2}{1+|\varphi^k|^2}$. Redefining the wave function as

$$\Psi = \Psi_0 u(x_i), \qquad \Psi_0 = \exp\left(-\frac{\mu}{g^2}\right), \tag{4.54}$$

we can rewrite the Schrödinger equation as $\tilde{H}u = Eu$ with

$$\tilde{H} = g^2 \sum_{i=1}^{N-1} \left[\left(T_i{}^N - \frac{2m_i}{g^2} \right) \left(T_i{}^i - T_N{}^i \right) + \frac{1-\epsilon}{2} \left(N(T_i{}^i - 1) + \frac{2m_i}{g^2} \right) \right], \tag{4.55}$$

where $T_I{}^J$ are the following differential operators:

$$T_i{}^j = -x_j \frac{\partial}{\partial x_i}, \quad T_i{}^N = \frac{\partial}{\partial x_i}, \quad T_N{}^i = -x_i T_N{}^N, \quad T_N{}^N = \sum_{i=1}^{N-1} x_i \frac{\partial}{\partial x_i} - \frac{N}{2}(\epsilon - 1). \tag{4.56}$$

The operators $T_I{}^J$ $(I, J = 1, \ldots, N)$ satisfy the $\mathfrak{gl}(N, \mathbb{C})$ algebra

$$[T_I{}^J, T_K{}^L] = \delta_K{}^J T_I{}^L - \delta_I{}^L T_K{}^J. \tag{4.57}$$

The quadratic Casimir invariant is given by

$$\sum_{I,J=1}^{N} T_I{}^J T_J{}^I = (\epsilon' - 1)(\epsilon' + N - 2), \quad \epsilon' = 1 + \frac{N}{2}(\epsilon - 1). \tag{4.58}$$

If ϵ' is an integer, the action of the operators $T_I{}^J$ is closed on the set of polynomials of x_i of degree $\epsilon' - 1$. Therefore, we can find eigenfunctions of $\tilde{H}$ by using the polynomial ansatz for $u(x_i)$ (corresponding to the symmetric representation of $\mathfrak{gl}(N, \mathbb{C})$ with $\epsilon' - 1$ indices).

Since $\epsilon' = 1$ is equivalent to $\epsilon = 1$, the deformation from $\epsilon' = 1$ is nothing but the SUSY-breaking deformation whose resurgence structure has been already discussed in the previous three subsections. Now, we consider the case of $\epsilon' = 2$. We can find eigenfunctions by using the ansatz $u = a_N + \sum_{i=1}^{N-1} a_i x_i$. Substituting into the Schrödinger equation, we find that

$$u = 1 + \sum_{i=1}^{N-1} \frac{m_i x_i}{M - m_i}, \qquad E = g^2 N + \frac{2}{N}\left(M + \sum_{i=1}^{N-1}(M - m_i)\right), \tag{4.59}$$

where M is one of the solutions of the equation

$$\sum_{i=1}^{N-1} \frac{1}{m_i - M} - \frac{1}{M} = \frac{2}{g^2}. \tag{4.60}$$

There are N solutions corresponding to the N-dimensional fundamental representation of $\mathfrak{gl}(N, \mathbb{C})$. Let us consider the small ϵ expansion of the smallest eigenvalue corresponding to the ground state energy

$$E^{(1)} = \lim_{\epsilon' \to 2} \frac{\partial}{\partial \epsilon} E = \frac{\langle \Psi | \delta H | \Psi \rangle}{\langle \Psi | \Psi \rangle}. \tag{4.61}$$

Here, we again note that $\delta H = \partial_\epsilon H$ is given by $\delta H = -\Delta\mu = -\sum_{i=1}^{N-1} m_i(1 - N x_i)$. Evaluating the integral in Eq. (4.61), we obtain

$$E^{(1)} = -\sum_{i=1}^{N-1} m_i \left[(1 + \mathcal{O}(g^2)) + \frac{N\tilde{A}_i \exp\left(-\frac{2m_i}{g^2}\right)}{1 - \sum_i \tilde{A}_i \exp\left(-\frac{2m_i}{g^2}\right)} \right], \tag{4.62}$$

where $\tilde{A}_i$ are constants that have the following weak coupling forms:

$$\tilde{A}_i = \left(\frac{g^2}{2m_i}\right)^2 \left[A_i + \mathcal{O}(g^2)\right] = \left(\frac{g^2}{2m_i}\right)^2 \left[\prod_{i \neq j} \frac{m_j}{m_j - m_i} + \mathcal{O}(g^2)\right]. \tag{4.63}$$

Thus we find that the pth-order nonperturbative correction takes the form

$$E_p^{(1)} = -N \left(\sum_{i=1}^{N-1} m_i \tilde{A}_i \exp\left(-\frac{2m_i}{g^2}\right)\right) \left(\sum_{i=1}^{N-1} \tilde{A}_i \exp\left(-\frac{2m_i}{g^2}\right)\right)^{p-1}. \tag{4.64}$$

We can check the agreement between the leading-order part of Eq. (4.64) for small g and the semiclassical multi-bion contribution (4.41) expanded around $\epsilon' = 2$. Thus, in the weak coupling limit, the nonperturbative corrections in the exact result are correctly reproduced by semiclassical multi-bion contributions not only around the SUSY regime but also around the near-QES regime of $\mathbb{C}P^{N-1}$ QM.

5. Resurgence structure in squashed $\mathbb{C}P^1$ QM

In this section, we briefly discuss another type of model belonging to the class described by chiral multiplets in which the $\mathcal{O}(\delta\epsilon)$ ground state energy has a nontrivial resurgence structure.

We here focus on the model described by the Kähler potential with a parameter $a \geq 0$,

$$K = \log\left(\frac{1}{1 - x}\right) + a x^2, \tag{5.1}$$

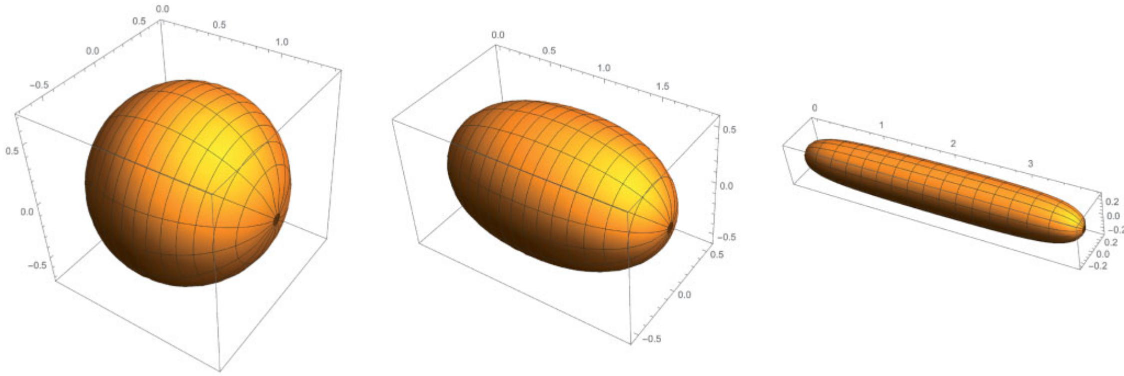

Fig. 6. The squashed $\mathbb{C}P^1$ with $a = 0$ (left), $a = 1$ (center), $a = 10$ (right).

where x is the function of φ determined by

$$|\varphi| = e^{ax}\left(\frac{x}{1-x}\right)^{1/2}. \tag{5.2}$$

The target space is the squashed $\mathbb{C}P^1$ (see Fig. 6) whose metric is given by

$$ds^2 = \partial_\varphi \partial_{\bar{\varphi}} K \, d\varphi \, d\bar{\varphi} = \frac{1}{2}(U \, dx^2 + U^{-1} \, d \arg \varphi^2), \quad U \equiv a + \frac{1}{2}\left(\frac{1}{x} + \frac{1}{1-x}\right). \tag{5.3}$$

This model reduces to the standard $\mathbb{C}P^1$ QM for $a = 0$.

Now let us consider the small $\delta\epsilon = \epsilon - 1$ expansion of the ground state energy,

$$E^{(1)} = \frac{\langle 0|\delta H|0\rangle}{\langle 0|0\rangle}, \qquad \langle \varphi|0\rangle = \exp\left(-\frac{\mu}{g^2}\right), \qquad \mu = mx. \tag{5.4}$$

It is quite notable that the perturbation Hamiltonian $\delta H = -\Delta\mu = -G^{-1}\partial\bar{\partial}\mu$ is not a polynomial but the following rational function:

$$\delta H = -\frac{m}{2}\partial_x U^{-1} = m\frac{2x-1}{(1+2a(1-x)x)^2}. \tag{5.5}$$

This implies that the trans-series expression for the first expansion coefficient $E^{(1)}$ has non-Borel-summable g^2 series at each order of the nonperturbative exponential. This is a crucial difference from the standard $\mathbb{C}P^{N-1}$ QM.

Again the generating function $\langle 0|0\rangle$ can be calculated by the localization formula. It does not depend on a and hence it is identical to that in the standard $\mathbb{C}P^1$ QM,

$$\langle 0|0\rangle = \frac{\pi g^2}{2m}\left(1 - \exp\left(-\frac{2m}{g^2}\right)\right). \tag{5.6}$$

This is finite order in terms of g^2 at each order of nonperturbative exponential. Evaluating the integral

$$\langle 0|\delta H|0\rangle = -\frac{\pi m}{2}\int_0^1 dx \, \exp\left(-\frac{2\mu}{g^2}\right)\partial_x U^{-1}, \tag{5.7}$$

we obtain the first-order coefficient as

$$E^{(1)} = -\frac{m^2}{ag^2}\left(1 + \frac{2m}{g^2}\frac{X}{1-\exp\left(-\frac{2m}{g^2}\right)}\right), \qquad X = \int_0^1 dx \frac{\exp\left(-\frac{2mx}{g^2}\right)}{1+2ax(1-x)}. \tag{5.8}$$

The quantity X is a linear combination of the exponential integral function whose asymptotic series in powers of g^2 can be obtained by changing variable to $t = 2mx/g^2$ and taking the upper limit of integration $2m/g^2 \to \infty$,

$$X = \frac{1}{2a(2b-1)} \int_0^{2m/g^2} dt\, e^{-t} \left(\frac{1}{t + \frac{2m(b-1)}{g^2}} - \frac{1}{t - \frac{2mb}{g^2}} \right)$$

$$\approx \frac{1}{2a(2b-1)} \left[\sum_{n=0}^{\infty} n! \left(\frac{-g^2}{2m(b-1)} \right)^{n+1} + \sum_{n=0}^{\infty} n! \left(\frac{g^2}{2mb} \right)^{n+1} \right], \tag{5.9}$$

where $b = \frac{1}{2} + \left(\frac{a+2}{4a}\right)^{1/2}$. Since the perturbation series of X is a sum of Borel-summable and Borel-non-summable series, the Borel resummation of the perturbation series gives the perturbative contribution X_{pert} containing the imaginary ambiguities as

$$X_{\text{pert}} = \frac{1}{2a(2b-1)} f\left(\frac{2m}{g^2} \mp i0 \right), \quad f(z) = \int_0^{\infty} dt\, e^{-t} \left(\frac{1}{t + (b-1)z} - \frac{1}{t - bz} \right). \tag{5.10}$$

The remaining part of X is multiplied by a single power of the nonperturbative exponential

$$X - X_{\text{pert}} = \frac{\exp\left(-\frac{2m}{g^2}\right)}{2a(2b-1)} g\left(\frac{2m}{g^2} \mp i0 \right), \quad g(z) = \int_0^{\infty} dt\, e^{-t} \left(-\frac{1}{t + bz} + \frac{1}{t - (b-1)z} \right), \tag{5.11}$$

which corresponds to the Borel resummation of a sum of Borel-summable and Borel-non-summable asymptotic series g^2 on the single-bion background. Expanding with respect to $\exp\left(-\frac{2m}{g^2}\right)$, we obtain the trans-series expression $E^{(1)} = \sum_{p=0}^{\infty} E_p^{(1)}$ with

$$E_0^{(1)} = -\frac{m^2}{ag^2} - \frac{m^3}{a^2 g^4 (2b-1)} f\left(\frac{2m}{g^2} \mp i0 \right), \tag{5.12}$$

$$E_p^{(1)} = -\frac{m^3}{a^2 g^4 (2b-1)} \left[f\left(\frac{2m}{g^2} \mp i0 \right) + g\left(\frac{2m}{g^2} \mp i0 \right) \right] \exp\left(-\frac{2pm}{g^2} \right). \tag{5.13}$$

Here, the $\pm$ signs correspond to $\arg g^2 \to \pm 0$. At each order of the nonperturbative exponential $\exp\left(-\frac{2m}{g^2}\right)$, $E_p^{(1)}$ has the following imaginary ambiguities due to the poles at $t = 2mb/g^2$ and $t = 2m(b-1)/g^2$:

$$\operatorname{Im} E_0^{(1)} = \pm \frac{\pi m^3}{a^2 g^4} \frac{1}{2b-1} \exp\left(-\frac{2bm}{g^2} \right). \tag{5.14}$$

$$\operatorname{Im} E_p^{(1)} = \pm \frac{\pi m^3}{a^2 g^4} \frac{1}{2b-1} \left[\exp\left(-\frac{2(p+b)m}{g^2} \right) - \exp\left(-\frac{2(p-1+b)m}{g^2} \right) \right] \quad (p \geq 1). \tag{5.15}$$

These imaginary ambiguities cancel out between the adjacent sectors. Thus, we conclude that the resurgence structure of the $\mathcal{O}(\delta\epsilon)$ ground state energy is nontrivial. Nevertheless, it has a simple structure where imaginary ambiguities cancel between the adjacent sectors of the p and $p+1$ power of the nonperturbative exponentials. It is worth noting that the cancelation mechanism of the imaginary ambiguities seems different from that in the standard $\mathbb{C}P^1$ QM. For example, the ambiguities have noninteger powers of $\exp\left(-\frac{2m}{g^2}\right)$ and all of them are from the Borel resummation of the asymptotic series. It is interesting to interpret these cancelation mechanisms from the viewpoint of the complexified path integral.

6. Summary and discussion

In this paper, we investigated the resurgence structure in SUSY QM with emphasis on the expansion around the SUSY and quasi-exactly solvable (QES) parameter regimes. First, we generically showed that bions play a vital role in nonperturbative contributions based on the Lefschetz thimble decomposition. We discussed two different classes of SUSY models: (i) quantum mechanics on a Riemannian manifold described by real multiplets and (ii) quantum mechanics on a Kähler manifold described by chiral multiplets.

In the models belonging to class (i), the generating function $\langle 0|0\rangle$ has a non-Borel-summable asymptotic series that gives rise to an imaginary ambiguity at each order of nonperturbative exponentials in trans-series. This property of the generating function $\langle 0|0\rangle$ provides the $\mathcal{O}(\delta\epsilon)$ ground state energy with a rich resurgence structure. As an example of a model in this class, we discussed the sine-Gordon model in Sect. 3. Using the Rayleigh–Schrödinger perturbation theory, we obtained the exact $\mathcal{O}(\delta\epsilon)$ ground state energy, which was expressed as a trans-series of non-Borel-summable series with nonperturbative exponentials corresponding to multi-bions. We showed that the semiclassical contributions from the complex multi-bion solutions are in agreement with those in the exact result including the imaginary ambiguities which cancel those in the other sectors.

In the models belonging to class (ii), the generating function $\langle 0|0\rangle$ can be exactly calculated by the Duistermaat–Heckman localization formula. Since it has a convergent (finite) power series of g^2 in each sector of nonperturbative exponentials, the $\mathcal{O}(\delta\epsilon)$ ground state energy has a relatively simple resurgence structure. As an example of a model in this class, $\mathbb{C}P^{N-1}$ QM was discussed in Sect. 4. We determined the exact $\mathcal{O}(\delta\epsilon)$ ground state energy and showed that it has a trivial resurgence structure with no imaginary ambiguities in the trans-series. This property enabled us to completely reconstruct the $\mathcal{O}(\delta\epsilon)$ ground state energy from the semiclassical multi-bion contributions. The ground state energy has a nontrivial resurgence structure at $\mathcal{O}(\delta\epsilon^2)$ and higher orders. We found $N-1$ types of real and complex bion solutions and showed the resurgence structure with cancelation between imaginary ambiguities arising from the Borel resummation of the perturbation series around the perturbative vacuum (zero-bion background) and the semiclassical contributions of the single-bion solutions. As shown in the example of the squashed $\mathbb{C}P^1$ QM, the $\mathcal{O}(\delta\epsilon)$ ground state energy of a generic model in class (ii) has a richer resurgence structure with a nontrivial cancelation of imaginary ambiguities in trans-series.

This work reveals that a broad class of quantum mechanical models have a nontrivial resurgent structure, where exact results for physical quantities are expressed as resurgent trans-series that consist of perturbative Borel resummations and complex multi-bion contributions. The cancelation of imaginary ambiguities enables us to reproduce the contribution of one sector from another by use of the dispersion relation of Cauchy's theorem. However, in some special cases such as the $\mathcal{O}(\delta\epsilon)$ ground state energy in $\mathbb{C}P^{N-1}$ QM, each order term of nonperturbative exponentials produces no imaginary ambiguities and the resurgence structure is trivial. This situation is similar to the case of the partition function in the $\mathcal{N}=2$ SYM (Refs. [90,91]) obtained by the localization method, where each sector of the trans-series does not talk to other sector. One of our next plans is to apply our analysis to solvable field-theoretical models where the localization method is applicable to find nontrivial resurgent structures in the expansion with respect to deformation parameters. It has been observed that the information on the level number of the perturbation series on a zero-bion background gives all p-bion contributions (Ref. [34]). It is an interesting future task to obtain resurgent trans-series for states other than the ground state.

One of the goals is to discuss the roles of complexified solutions in Yang–Mills or QCD in 4 dimensions. In the 1970s and 80s, complex instanton solutions were discussed in gauge theories with complexified gauge groups (Refs. [142,143]). It will be interesting to discuss contributions from these complex solutions in terms of resurgence theory. Another way of studying complexified solutions in Yang–Mills theory is to consider $U(N)$ Yang–Mills theory coupled with Higgs fields in the fundamental representation. In the Higgs phase, there exists a non-Abelian vortex whose low-energy dynamics is effectively described by the $\mathbb{C}P^{N-1}$ model localized around the vortex (Refs. [144–147]). By introducing appropriate fermions coupled in the original bulk theory, we can localize fermion quasi-zero modes around the vortex and the $\mathbb{C}P^{N-1}$ model is coupled to the fermions. For the case of the SUSY bulk theory, the vortex can be BPS and the SUSY $\mathbb{C}P^{N-1}$ model is obtained as the vortex theory. It means that our complexified solutions are able to be embedded into it. The $\mathbb{C}P^{N-1}$ model instantons in the vortex theory correspond to Yang–Mills instantons in the bulk theory (Ref. [132]). Therefore, complexified bions can be interpreted as those in (complexified) Yang–Mills theory in the bulk. A decoupling limit of the Higgs phase leads to isolation of complexified solutions in Yang–Mills theory. We hope that these are useful to reveal the resurgence structure of Yang–Mills theory or QCD in 4 dimensions.

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Appendix A. Supersymmetric QM

In this appendix, we review the models of SUSY QM discussed in this paper. Consider the SUSY algebra

$$\{Q, \bar{Q}\} = H - P, \tag{A.1}$$

where H is the Hamiltonian and P denotes the central charge, which may exist when there is an internal symmetry that commutes with Q and $\bar{Q}$. We consider two types of supermultiplets, namely real and chiral multiplets, each of which has a bosonic degree of freedom in its lowest component.

A.1. Real multiplets

Let us first consider the SUSY QM described by real multiplets. For simplicity, we assume that there is no central charge appearing in the superalgebra ($P = 0$). The SUSY transformation of a real multiplet is given by

$$\delta\varphi = \frac{1}{2}\left(\varepsilon\psi + \bar{\varepsilon}\bar{\psi}\right), \qquad \delta\psi = \bar{\varepsilon}(i\partial_t\varphi + F), \tag{A.2}$$

$$\delta F = \frac{i}{2}(\varepsilon\partial_t\psi - \bar{\varepsilon}\partial_t\bar{\psi}), \qquad \delta\bar{\psi} = \varepsilon(i\partial_t\varphi - F), \tag{A.3}$$

where ε and $\bar{\varepsilon}$ are transformation parameters for Q and $\bar{Q}$. Let us consider a Riemannian manifold $\mathcal{M}$ parametrized by the scalar components φ^i. After integrating out the auxiliary fields F^i, the SUSY Lagrangian takes the form

$$L = \frac{1}{g^2}\left[\frac{1}{4}G_{ij}\left(\dot{\varphi}^i\dot{\varphi}^j + i\bar{\psi}^i\mathcal{D}_t\psi^j\right) - \frac{1}{32}R_{ijkl}\psi^i\psi^j\bar{\psi}^k\bar{\psi}^l - G^{ij}\partial_i W\partial_j W + \frac{1}{2}\nabla_i\partial_j W\psi^i\bar{\psi}^j\right], \tag{A.4}$$

where G_{ij} is the Riemannian metric, $\mathcal{D}_t\psi^i = \partial_t\psi^i + \Gamma^i_{jk}\dot{\varphi}^j\psi^j$, Γ^i_{ij} is the Christoffel symbol, R_{ijkl} is the Riemannian curvature tensor, and W is the superpotential, which can be an arbitrary function on $\mathcal{M}$.

BPS kink solution. Next let us discuss the BPS kink configuration in this model (see, e.g., Refs. [137–140] for BPS kinks). After the Wick rotation, the bosonic part of the Euclidean action can be rewritten as

$$S_E = \frac{1}{g^2}\int d\tau\, G_{ij}\left(\frac{1}{2}\partial_\tau\varphi^i \pm G^{ik}\partial_k W\right)\left(\frac{1}{2}\partial_\tau\varphi^j \pm G^{jl}\partial_l W\right) \mp \frac{1}{g^2}\int dW. \tag{A.5}$$

This form of the Euclidean action implies that there exist BPS kink solutions obeying the flow equation

$$\frac{1}{2}\partial_t\varphi^i = \mp G^{ik}\partial_k W. \tag{A.6}$$

Such BPS kinks correspond to tunneling processes between two minima of the potential, i.e., saddle points of W. For a BPS kink interpolating two saddle points s and s', the on-shell value of the Euclidean action is given by

$$S_{E,(s,s')} = \left|\frac{W_s - W_{s'}}{g^2}\right|, \tag{A.7}$$

where W_s and $W_{s'}$ are the values of the superpotential at the corresponding saddle points. Typical nonperturbative effects for the ground state are of order $\exp\left(-2S_{E,(s,s')}\right)$, which implies that they are given by bound states of kink and antikink, i.e., bion configurations.

Hamiltonian and ground state wave function. Let us quantize the system by introducing the commutation relation between the canonical coordinates, whose nontrivial part is given by

$$[\varphi^i, p_j] = i\delta^i{}_j, \qquad \{\psi^i, \pi_{\psi^j}\} = i\delta^i{}_j, \tag{A.8}$$

where p_i and π_{ψ^i} are the conjugate momenta

$$p_i = \frac{\partial L}{\partial\dot{\varphi}^i} = \frac{1}{2g^2}G_{ij}\left[\dot{\varphi}^j + \frac{i}{2}\Gamma^j_{kl}\bar{\psi}^k\psi^l\right], \qquad \pi_{\psi^i} = \frac{\partial L}{\partial\dot{\psi}^i} = \frac{i}{4g^2}G_{ij}\bar{\psi}^j. \tag{A.9}$$

Let us project the Hilbert space onto the subspace with the lowest fermion number $F = G_{ij}\psi^i\bar{\psi}^j$:

$$0 = \langle\Psi|F|\Psi\rangle = \sum_{a=1}^{n} \|e^a{}_i\bar{\psi}^i|\Psi\rangle\|^2 \quad \Longrightarrow \quad \bar{\psi}^i|\Psi\rangle = 0, \tag{A.10}$$

where $e^a{}_i$ are the vielbein defined by $G_{ij} = \sum_{a=1}^{n} e^a{}_i e^a{}_j$. On this subspace, the Hamiltonian $H = \{Q, \bar{Q}\}$ reduces to

$$H|\Psi\rangle = \bar{Q}Q|\Psi\rangle \quad \Longleftrightarrow \quad -g^2 G^{ij}\left(\nabla_i - \frac{1}{g^2}\partial_i W\right)\left(\partial_j + \frac{1}{g^2}\partial_j W\right)\langle\varphi|\Psi\rangle, \tag{A.11}$$

where we have used the following explicit form of the supercharges written in terms of the canonical coordinate:

$$Q = \frac{1}{2}\psi^i\left(p_i - \frac{i}{g^2}\partial_i W\right), \qquad \bar{Q} = \frac{1}{2}\bar{\psi}^i\left(p_i - \frac{i}{4g^2}G_{il}\Gamma^l_{jk}\psi^k\bar{\psi}^j + \frac{i}{g^2}\partial_i W\right). \tag{A.12}$$

Therefore, the SUSY ground state, which has the lowest energy $H|\Psi\rangle = 0$, is the one annihilated by $p_i - \frac{i}{g^2}\partial_i W$, i.e.,

$$\langle\varphi|\Psi\rangle = \exp\left(-\frac{W}{g^2}\right). \tag{A.13}$$

A.2. Chiral multiplets

Next let us consider the SUSY QM described by chiral multiplets. Such a model can be obtained from the corresponding 2d nonlinear sigma model with $\mathcal{N} = (2, 0)$ SUSY

$$\{Q, \bar{Q}\} = H - P, \tag{A.14}$$

where P is the spatial momentum. Introducing the Grassmannian coordinates θ and $\bar{\theta}$, we can associate the supercharges Q and $\bar{Q}$ with the differential operators

$$Q = \frac{\partial}{\partial\theta} + i\bar{\theta}\partial_+, \qquad \bar{Q} = -\frac{\partial}{\partial\bar{\theta}} - i\theta\partial_-, \tag{A.15}$$

with $\partial_\pm = \frac{1}{2}(\partial_t \pm \partial_x)$. The chiral and antichiral superfields are respectively defined as those annihilated by the differential operators,

$$D = \frac{\partial}{\partial\theta} - i\bar{\theta}\partial_+, \qquad \bar{D} = -\frac{\partial}{\partial\bar{\theta}} + i\theta\partial_+, \tag{A.16}$$

which anticommute with Q and $\bar{Q}$. The explicit forms of the chiral and antichiral superfields are given by

$$\bar{D}\Phi = 0 \quad \Longrightarrow \quad \Phi = \varphi + \theta\psi + i\bar{\theta}\theta\partial_+\varphi, \tag{A.17}$$

$$D\bar{\Phi} = 0 \quad \Longrightarrow \quad \bar{\Phi} = \bar{\varphi} - \bar{\theta}\bar{\psi} - i\bar{\theta}\theta\partial_+\bar{\varphi}. \tag{A.18}$$

The Lagrangian of the nonlinear sigma model with Kähler potential K and Kähler metric $G_{i\bar{j}} = \partial_i\bar{\partial}_{\bar{j}}K$ can be written as

$$\mathcal{L} = \frac{1}{g_{2d}^2}\int d\theta\, d\bar{\theta}(-2i\partial_-\Phi^i\partial_i K) = \frac{1}{g_{2d}^2}G_{i\bar{j}}\left(-\partial_\mu\varphi^i\partial^\mu\bar{\varphi}^{\bar{j}} + 2i\bar{\psi}^{\bar{j}}\mathcal{D}_-\psi^i\right) + \cdots, \tag{A.19}$$

where $\mathcal{D}_-\psi^i = \partial_-\psi^i + \Gamma^i_{jk}\partial_-\varphi^j\psi^k$ and $\cdots$ denotes total derivative terms.

Let us assume that the target manifold has a holomorphic isometry with moment map μ. The corresponding holomorphic Killing vector $\Xi \equiv \xi^i \partial_i + \bar{\xi}^i \bar{\partial}_i$ satisfies

$$\xi^i = iG^{\bar{j}i}\bar{\partial}_{\bar{j}}\mu, \qquad \bar{\xi}^{\bar{j}} = -iG^{\bar{j}i}\partial_i\mu, \qquad \bar{\partial}_{\bar{j}}\xi^i = \partial_i\bar{\xi}^{\bar{j}} = 0. \tag{A.20}$$

The SUSY QM of chiral multiplets discussed in this paper can be obtained from Eq. (A.19) by dimensional reduction twisted by the isometry

$$L = \frac{1}{g^2} G_{i\bar{j}} \left[\dot{\varphi}^i \dot{\bar{\varphi}}^{\bar{j}} - \xi^i \bar{\xi}^{\bar{j}} + i\bar{\psi}^{\bar{j}} \mathcal{D}_t \psi^i - i\nabla_k \xi^i \bar{\psi}^{\bar{j}} \psi^k \right], \tag{A.21}$$

where $\mathcal{D}_t\psi^i = \dot{\psi}^i + \Gamma^i_{jk}\dot{\varphi}^j\psi^k$, $\nabla_k\xi^i = \partial_k\xi^i + \Gamma^i_{jk}\xi^j$, $\Gamma^i_{jk} = \partial_j G_{k\bar{l}} G^{\bar{l}i}$, and

$$\frac{1}{g^2} = \frac{1}{g_{2d}^2} \times 2\pi\{\text{compactification radius}\}. \tag{A.22}$$

Symmetry. The SUSY transformations for the components of a chiral multiplet are given by

$$\delta\varphi^i = \varepsilon\psi^i, \qquad \delta\psi^i = i\bar{\varepsilon}(\dot{\varphi}^i + \xi^i), \tag{A.23}$$

$$\delta\bar{\varphi}^{\bar{i}} = \bar{\varepsilon}\bar{\psi}^{\bar{i}}, \qquad \delta\bar{\psi}^{\bar{i}} = i\varepsilon(\dot{\bar{\varphi}}^{\bar{i}} + \bar{\xi}^{\bar{i}}), \tag{A.24}$$

where ε and $\bar{\varepsilon}$ are transformation parameters for Q and $\bar{Q}$. We can see from these SUSY transformations that the SUSY algebra takes the form

$$\{Q, \bar{Q}\} = H + P, \tag{A.25}$$

where P is given by the Noether charge q for the holomorphic isometry $\delta_q\varphi^i = \xi^i$, $\delta_q\psi^i = \psi^j\partial_j\xi^i$:

$$q = \frac{1}{g^2} G_{i\bar{j}} \left(\xi^i \dot{\bar{\varphi}}^{\bar{j}} + \dot{\varphi}^i \bar{\xi}^{\bar{j}} + i\bar{\psi}^{\bar{j}}\psi^k \nabla_k \xi^i \right). \tag{A.26}$$

BPS kink. The original 2d system (A.19) in the Euclidean spacetime has instanton solutions characterized by the topological charge (Ref. [131])

$$Q = \frac{1}{2\pi} \int dx_1\, dx_2\, i\epsilon^{\mu\nu} G_{i\bar{j}} \partial_\mu \varphi^i \partial_\nu \bar{\varphi}^{\bar{j}}. \tag{A.27}$$

After the twisted dimensional reduction, such an instanton decomposes into fractional instantons characterized by the topological charge (Refs. [132–134]; see also Refs. [135,136])

$$Q_{\text{fractional}} = \frac{1}{2\pi} \int d\mu. \tag{A.28}$$

We can see that there exist such BPS solutions in the 1d system by rewriting the bosonic part of the Euclidean model as

$$S_E = \frac{1}{g^2} \int d\tau\, G_{i\bar{j}} \left(\dot{\varphi}^i \pm i\xi^i \right) \left(\dot{\bar{\varphi}}^{\bar{j}} \mp i\bar{\xi}^{\bar{j}} \right) \pm \frac{1}{g^2} \int d\mu. \tag{A.29}$$

For a given boundary condition, this Euclidean action is minimized when the following flow equation is satisfied (see, e.g., Refs. [137–140]):

$$\partial_\tau \varphi^i = \pm G^{\bar{j}i} \frac{\partial \mu}{\partial \bar{\varphi}^{\bar{j}}}, \tag{A.30}$$

where we have used $\xi^i = iG^{i\bar{j}}\bar{\partial}_{\bar{j}}\mu$. This equation describes kink solutions connecting saddle points of μ. The on-shell value of the Euclidean action for a kink interpolating two saddle points s and s' of μ is given by

$$S_{E,(s,s')} = \left|\frac{\mu_s - \mu_{s'}}{g^2}\right|. \tag{A.31}$$

This implies that bion configurations, i.e., bound states of kink and antikink, give typical nonperturbative effects of order $\exp\left(-2S_{E,(s,s')}\right)$.

Hamiltonian and ground state wave function. Let us quantize the system by introducing the commutation relation, whose nontrivial part is given by

$$[\varphi^i, p_j] = i\delta^i{}_j, \qquad [\bar{\varphi}^{\bar{i}}, \bar{p}_{\bar{j}}] = i\delta^{\bar{i}}{}_{\bar{j}}, \qquad \{\psi^i, \pi_{\psi^j}\} = i\delta^i{}_j, \tag{A.32}$$

where $p_i, \bar{p}_i$, and π_{ψ^i} are the conjugate momenta, which can be read off from the Lagrangian as

$$p_i = \frac{1}{g^2}\left[G_{i\bar{j}}\dot{\bar{\varphi}}^{\bar{j}} + iG_{k\bar{j}}\Gamma^k_{il}\bar{\psi}^{\bar{j}}\psi^l\right], \qquad \bar{p}_{\bar{i}} = \frac{1}{g^2}G_{j\bar{i}}\dot{\varphi}^j, \qquad \pi_{\psi^i} = \frac{i}{g^2}G_{i\bar{j}}\bar{\psi}^{\bar{j}}. \tag{A.33}$$

In terms of these canonical coordinates, the supercharges can be written as

$$Q = \psi^i\left[p_i + \frac{1}{g^2}G_{j\bar{l}}\left(\delta^j{}_i\bar{\xi}^{\bar{l}} - i\Gamma^j_{ik}\{\psi^k, \bar{\psi}^{\bar{l}}\}\right)\right], \qquad \bar{Q} = \bar{\psi}^{\bar{i}}\left(\bar{p}_{\bar{i}} + \frac{1}{g^2}G_{j\bar{i}}\xi^j\right). \tag{A.34}$$

From the superalgebra $H = \{Q, \bar{Q}\} - q$, we can find the explicit form of the Hamiltonian as

$$H = g^2G^{\bar{j}i}\left(p_i - \frac{1}{g^2}iG_{l\bar{m}}\bar{\psi}^{\bar{m}}\psi^k\Gamma^l_{ik}\right)\bar{p}_{\bar{j}} + g^2\bar{\psi}^{\bar{j}}\psi^iR_{i\bar{j}} + \frac{1}{g^2}G_{i\bar{j}}\left(\xi^i\bar{\xi}^{\bar{j}} - i\nabla_k\xi^i\psi^k\bar{\psi}^{\bar{j}}\right), \tag{A.35}$$

where $R_{i\bar{j}}$ is the Ricci curvature $R_{i\bar{j}} = -\bar{\partial}_{\bar{j}}\Gamma^k_{ik}$.

The fermion number operator $F = \frac{1}{g^2}G_{i\bar{j}}\bar{\psi}^{\bar{j}}\psi^i$ commutes with the Hamiltonian, we thus consider the eigenvalue problem of H within each sector with a fixed fermion number F. In the zero fermion sector $F = 0$, any state vector satisfies

$$\langle\Psi|F|\Psi\rangle = \frac{1}{g^2}\sum_{a=1}^{n}\left\|e^a{}_i\psi^i|\Psi\rangle\right\|^2 = 0 \quad\Longrightarrow\quad \psi^i|\Psi\rangle = 0, \tag{A.36}$$

where $e^a{}_i$ denote the vielbein of the target space

$$\sum_{a=1}^{n}e^a{}_i\overline{e^a{}_j} = G_{i\bar{j}}. \tag{A.37}$$

In this sector, the Hamiltonian reduces to

$$H = G^{\bar{j}i}\left[g^2p_i\bar{p}_{\bar{j}} + \frac{1}{g^2}\partial_i\mu\bar{\partial}_{\bar{j}}\mu - \partial_i\bar{\partial}_{\bar{j}}\mu\right]. \tag{A.38}$$

We also note that the conserved charge in this sector becomes $q = \xi^ip_i + \bar{\xi}^{\bar{j}}\bar{p}_{\bar{j}}$. The SUSY ground state, which satisfies $H|\Psi\rangle = 0$, is the one annihilated by $\bar{p}_i - \frac{i}{g^2}\bar{\partial}_{\bar{i}}\mu$, and hence the wave function is given by

$$\langle\varphi|\Psi\rangle = \exp\left(-\frac{\mu}{g^2}\right). \tag{A.39}$$

Appendix B. Localization method for $\langle 0|0\rangle$ in Kähler QM

In this appendix, we calculate $\langle 0|0\rangle$, i.e., the generating function for $\langle \mu\rangle$ in Kähler QM by means of SUSY localization (Duistermaat–Heckman formula). By introducing the fermionic coordinates, $\langle 0|0\rangle$ can be rewritten as

$$\langle 0|0\rangle = \left(\frac{i}{2}g^2\right)^n \int d^{2n}\varphi\, d^n\psi\, d^n\bar{\psi}\ \exp(-X), \quad X = \frac{2}{g^2}\left(\mu + iG_{i\bar{j}}\bar{\psi}^{\bar{j}}\psi^i\right). \tag{B.1}$$

The integrand is invariant under the following SUSY transformation:

$$\delta\varphi^i = \psi^i, \qquad \delta\psi^i = \xi^i. \tag{B.2}$$

Note that the square of this SUSY transformation is the holomorphic isometry $\delta_q\varphi^i = \xi^i$, $\delta_q\psi^i = \psi^j\partial_j\xi^i$. To calculate $\langle 0|0\rangle$, let us consider the following integral:

$$\langle \exp(-t\,\delta V)\rangle = \left(\frac{i}{2}g^2\right)^n \int d^{2n}\varphi\, d^n\psi\, d^n\bar{\psi}\ \exp(-X - t\,\delta V), \tag{B.3}$$

where t is a formal parameter and V is the following operator, which is invariant under the holomorphic isometry $\delta^2 V = \delta_q V = 0$:

$$V = G_{i\bar{j}}\xi^i\bar{\psi}^{\bar{j}} + \text{(c.c.)}. \tag{B.4}$$

The invariance under the SUSY transformation implies that $\langle \exp(-t\,\delta V)\rangle$ does not depend on t, i.e.,

$$\frac{d}{dt}\langle \exp(-t\,\delta V)\rangle = -\left(\frac{i}{2}g^2\right)^n \int d^{2n}\varphi\, d^n\psi\, d^n\bar{\psi}\ \delta\Big[V\exp(-X - t\,\delta V)\Big] = 0. \tag{B.5}$$

Since $\langle \exp(-t\,\delta V)\rangle$ reduces to $\langle 0|0\rangle$ for $t = 0$, its t-independence implies that

$$\langle 0|0\rangle = \langle \exp(-t\,\delta V)\rangle \quad \text{for arbitrary } t. \tag{B.6}$$

In the $t \to \infty$ limit, the saddle point approximation becomes exact and hence the integral can be computed only from the data around the saddle point of δV, whose explicit form is given by

$$\delta V = G_{i\bar{j}}\left(\xi^i\bar{\xi}^{\bar{j}} + \nabla_k\xi^i\psi^k\bar{\psi}^{\bar{j}}\right). \tag{B.7}$$

This implies that the saddle points are the zeros of ξ, i.e., the fixed points of the isometry. Around each saddle point, δV can be expanded by rescaling the fluctuation $(\delta\varphi^i, \psi^i) \to t^{-1}(\delta\varphi^i, \psi^i)$ and taking $t \to \infty$:

$$t\,\delta V = G^s_{i\bar{j}}\Big[(M_s\delta\varphi)^i(M_s^\dagger\delta\bar{\varphi})^{\bar{j}} + i(M_s\psi)^i\bar{\psi}^{\bar{j}}\Big] + \mathcal{O}(t^{-1}), \tag{B.8}$$

where we have assumed $\xi^i = i(M_s)^i{}_j\delta\varphi^j$ around the saddle point s. Since $\langle \exp(-t\,\delta V)\rangle$ is independent of t, we can ignore the subleading terms depending on t. By performing the Gaussian integration at each saddle point and collecting the contributions from all the saddle points, we obtain the generating function $\langle 0|0\rangle$ as

$$\begin{aligned}\langle 0|0\rangle &= \sum_{s\in\mathfrak{S}}\left(\frac{i}{2}g^2\right)^n \int d^{2n}\varphi\, d^n\psi\, d^n\bar{\psi}\ \exp\left\{-\frac{2\mu_s}{g^2} - G^s_{i\bar{j}}\Big[(M_s\delta\varphi)^i(M_s^\dagger\delta\bar{\varphi})^{\bar{j}} + i(M_s\psi)^i\bar{\psi}^{\bar{j}}\Big]\right\}\\ &= \left(\frac{\pi g^2}{2}\right)^n \sum_{s\in\mathfrak{S}}\frac{1}{\det M_s}\exp\left(-\frac{2\mu_s}{g^2}\right).\end{aligned} \tag{B.9}$$

Appendix C. Perturbative part of $E^{(2)}$ in $\mathbb{C}P^{N-1}$ QM

In this appendix, we calculate the perturbative part of $E^{(2)}$ in the $\mathbb{C}P^{N-1}$ model. The leading-order correction to the wave function can be obtained by solving the Schrödinger equation expanded around $\epsilon = 1$. Its asymptotic behavior in the weak coupling limit is given by

$$\Psi^{(1)} = \frac{N}{2} \exp\left(-\frac{\mu}{g^2}\right) \log \frac{1}{1+\sum_{k=1}^{N-1} |\varphi^k|^2} + \cdots , \tag{C.1}$$

where $\cdots$ denotes nonperturbative corrections. From the relation $E^{(2)} = -\langle \Psi^{(1)} | H_{\epsilon=1} | \Psi^{(1)} \rangle / \langle 0|0 \rangle$, we find that

$$E^{(2)} = -\frac{N^2 \pi^{N-1} g^{2N}}{4\langle 0|0\rangle} \int_{\Delta_{N-1}} dx_1 \cdots dx_{N-1} \, \frac{g^2 \sum_{i=1}^{N-1} x_i}{1 - g^2 \sum_{i=1}^{N-1} x_i} \exp\left(-2 \sum_{i=1}^{N-1} m_i x_i\right) + \cdots , \tag{C.2}$$

where we have changed the integration variables from $|\varphi^i|$ to $x_i = \frac{1}{g^2} \frac{|\varphi^i|^2}{1+\sum_{k=1}^{N-1} |\varphi^k|^2}$. The integration domain is the $(N-1)$-simplex

$$\Delta_{N-1} = \left\{ (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1} \,\middle|\, x_i > 0,\, x_1 + \cdots + x_{N-1} \leq \frac{1}{g^2} \right\}. \tag{C.3}$$

Expanding the integrand with respect to g, we obtain

$$\begin{aligned} E^{(2)} = &-\frac{N^2 \pi^{N-1} g^{2N}}{4\langle 0|0\rangle} \sum_{n=1}^{\infty} \left(-\frac{g^2}{2} \sum_{i=1}^{N-1} \frac{\partial}{\partial m_i} \right)^n \\ &\times \int_{\Delta_{N-1}} dx_1 \cdots dx_{N-1} \, \exp\left(-2 \sum_{i=1}^{N-1} m_i x_i\right) + \cdots . \end{aligned} \tag{C.4}$$

Since the condition $x_1 + \cdots + x_{N-1} \leq 1/g^2$ can be ignored in the weak coupling limit, the perturbation series can be obtained by integrating over the region with $x_i > 0$. Then we obtain the following perturbation series:

$$E_0^{(2)} = -\frac{N^2}{4} g^2 \left(\prod_{i=1}^{N-1} \frac{g^2}{m_i} \right)^{-1} \sum_{n=1}^{\infty} \left(-\frac{g^2}{2} \sum_{i=1}^{N-1} \frac{\partial}{\partial m_i} \right)^n \left(\prod_{i=1}^{N-1} \frac{g^2}{m_i} \right), \tag{C.5}$$

where we have used $\langle 0|0\rangle = \prod_{i=1}^{N-1} \frac{\pi g^2}{2 m_i} + \cdots$. This is a divergent series since the coefficient of g^n is of order $n!$. By the Borel resummation, $E_0^{(2)}$ can be rewritten as

$$E_0^{(2)} = \frac{N^2}{4} g^2 \left(\prod_{i=1}^{N-1} \frac{g^2}{m_i} \right)^{-1} \int_0^{\infty} dt\, e^{-t} \left[\prod_{i=1}^{N-1} \frac{g^2}{m_i} - \prod_{i=1}^{N-1} \frac{g^2}{m_i - \frac{tg^2}{2}} \right], \tag{C.6}$$

where we have used $\sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{tg^2}{2} \sum_{i=1}^{N-1} \frac{\partial}{\partial m_i} \right)^n f(m_i) = f(m_i - \frac{tg^2}{2}) - f(m_i)$. Using the partial fraction decomposition

$$\prod_{i=1}^{N-1} \frac{1}{m_i - \frac{tg^2}{2}} = \sum_{j=1}^{N-1} \frac{1}{m_j - \frac{tg^2}{2}} \prod_{i=1, i\neq j}^{N-1} \frac{1}{m_j - m_i}, \tag{C.7}$$

we obtain the perturbative part of the second-order expansion coefficient as

$$E_0^{(2)} = \frac{N^2}{4}\left[g^2 + \sum_{i=1}^{N-1} 2m_i A_i \int_0^\infty dt \frac{e^{-t}}{t - \frac{2m_i}{g^2 \pm i0}}\right]. \tag{C.8}$$

Appendix D. Quasi-moduli space

When a saddle point has a nearly flat direction, which corresponds to an eigenmode whose mass (frequency) vanishes in the weak coupling limit $g \to 0$, the Gaussian approximation is not applicable for evaluating the contribution of that saddle point to the path integral. In such a case, we need to integrate all the way along the nearly flat directions, which are parametrized by the quasi-moduli parameters η^α. Let φ_η^i be the configuration along the nearly flat direction, which we define as the configuration such that $\frac{\partial \varphi_\eta^i}{\partial \eta^\alpha}$ is proportional to the quasi-zero modes at the saddle point and the equation of motion is satisfied up to terms proportional to $\frac{\partial \varphi_\eta^i}{\partial \eta^\alpha}$:

$$\left.\frac{\delta S}{\delta \varphi^i}\right|_{\varphi^i = \varphi_\eta^i} = \sum_\alpha c_\alpha(\eta) \frac{\partial \varphi_\eta^i}{\partial \eta^\alpha}. \tag{D.1}$$

By using this ansatz on the quasi-moduli space, we decompose the field as

$$\varphi^i = \varphi_\eta^i + g\delta\varphi_\perp^i, \tag{D.2}$$

where $\delta\varphi_\perp^i$ denotes the fluctuation containing all the massive modes, which are orthogonal to the quasi-zero modes $\frac{\partial \varphi_s^i}{\partial \eta^\alpha}$. Then the action can be schematically expanded as

$$S[\varphi] = S[\varphi_\eta] + g^2 \delta\bar{\varphi}^i \Delta_{ij} \delta\varphi^j + \mathcal{O}(g^4), \tag{D.3}$$

where Δ_{ij} is the differential operator appearing in the linearized equation of motion. The absence of the linear term indicates that the quasi-moduli parametrize the bottom of the valley of the action. In the weak coupling limit, the path integral for the partition function reduces to the quasi-moduli integral

$$Z \approx \int dv \, \frac{1}{\det \Delta_{ij}} \exp\left(-\frac{V_{\mathrm{eff}}}{g^2}\right), \tag{D.4}$$

where dv is the volume form of the quasi-moduli space, $\det \Delta$ is the one-loop determinant, and V_{eff} is the kink–antikink effective potential which can be obtained by substituting the kink–antikink ansatz into the original action:

$$V_{\mathrm{eff}}(\eta) = S[\varphi_\eta]. \tag{D.5}$$

Appendix E. Kink–antikink effective potential

In this appendix, we derive a general formula for the effective potential between a well-separated kink–antikink pair. To find the effective potential, it is necessary to find an appropriate ansatz for the kink–antikink configuration φ_η^i parametrized by the quasi-moduli parameters η^α. Since the kink and antikink are well-separated in the bion configuration for small g, their positions τ_k and $\tau_{\bar{k}}$ become almost free parameters and hence can be interpreted as the quasi-moduli parameters ($\eta^\alpha = \tau_k, \tau_{\bar{k}}$, and other possible internal degrees of freedom).

When the kink and antikink are well separated so that $|\tau_k - \tau_{\bar{k}}|$ is much larger than any length scale in the model, then the kink–antikink configuration can be schematically written as

$$\varphi^i_\eta = \begin{cases} \varphi^i_k + \delta\varphi^i_{\bar{k}} + \cdots, & \tau \approx \tau_k, \\ v^i + \delta\varphi^i_k + \delta\varphi^i_{\bar{k}} + \cdots, & \tau \approx \tau_0, \\ \varphi^i_{\bar{k}} + \delta\varphi^i_k + \cdots, & \tau \approx \tau_{\bar{k}}, \end{cases} \tag{E.1}$$

where v^i are the vacuum expectation values and τ_0 is a point in between the kink and antikink ($\tau_k \ll \tau_0 \ll \tau_{\bar{k}}$) such that the tail from the kink (antikink) can be approximated by a small perturbation $\delta\varphi^i_k$ ($\delta\varphi^i_{\bar{k}}$) for $\tau > \tau_0$ ($\tau < \tau_0$).

Let us decompose V_{eff} into the contributions from $\tau > \tau_0$ and $\tau < \tau_0$:

$$V_{\text{eff}} = S[\varphi^i_{k\bar{k}}] = S_{\tau>\tau_0} + S_{\tau<\tau_0}. \tag{E.2}$$

Since the kink contribution can be treated as the small perturbation $\delta\varphi^i_k$ in the region $\tau > \tau_0$, $S_{\tau<\tau_0}$ is approximately given by

$$\begin{aligned} S_{\tau>\tau_0} &= S[\varphi^i_{\bar{k}}] + \int d\tau \left(\delta\varphi^i_k \frac{\partial \mathcal{L}}{\partial \varphi^i} + \delta\dot{\varphi}^i_k \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^i} \right)\bigg|_{\varphi^i = \varphi^i_{\bar{k}}} + \cdots \\ &= S[\varphi^i_{\bar{k}}] - \left(\delta\varphi^i_k \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^i}\bigg|_{\varphi^i=\varphi^i_{\bar{k}}} \right)_{\tau=\tau_0} + \cdots, \end{aligned} \tag{E.3}$$

where we have used the fact that $\varphi^i_{\bar{k}}$ satisfies the Euler–Lagrange equation and the contribution from $\tau = \infty$ is trivial. In a similar way, we can calculate the contribution from the region $\tau < \tau_0$. Adding two contributions, we obtain

$$V_{\text{eff}} = S[\varphi^i_k] + S[\varphi^i_{\bar{k}}] + \left(\delta\varphi^i_{\bar{k}} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^i}\bigg|_{\varphi^i=\varphi^i_k} - \delta\varphi^i_k \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^i}\bigg|_{\varphi^i=\varphi^i_{\bar{k}}} \right)_{\tau=\tau_0} + \cdots. \tag{E.4}$$

Let $\delta\varphi^i$ be the fluctuations around the VEVs v^i orthonormalized so that the expanded Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}(\delta\dot{\varphi}^i)^2 + \frac{m_i^2}{2}(\delta\varphi^i)^2 + \cdots. \tag{E.5}$$

Then, for $\tau \approx \tau_0$, the small deviations $\delta\varphi^i_k$ and $\delta\varphi^i_{\bar{k}}$ in Eq. (E.1) take the following forms:

$$\delta\varphi^i_k = \mathcal{A}^i \exp(-m_i\tau), \qquad \delta\varphi^i_{\bar{k}} = \mathcal{B}^i \exp(m_i\tau), \tag{E.6}$$

where $\mathcal{A}^i$ and $\mathcal{B}^i$ are constants depending on the quasi-moduli parameters. Substituting into Eq. (E.4), we obtain the effective potential as

$$V_{\text{eff}} \approx S[\varphi^i_k] + S[\varphi^i_{\bar{k}}] + \sum_i 2m_i \mathcal{A}^i \mathcal{B}^i. \tag{E.7}$$

Therefore, the effective kink–antikink potential can be obtained by determining the coefficients $\mathcal{A}^i$ and $\mathcal{B}^i$ in Eq. (E.6).

In the presence of a potential term induced by the fermion projection, the vacuum $\varphi^i = v_i$ between kink and antikink may not be the true minimum but can be lifted, so that the following confining potential term is induced in the effective potential:

$$V_{\text{eff}} \approx S[\varphi^i_k] + S[\varphi^i_{\bar{k}}] + \sum_i 2m_i \mathcal{A}^i \mathcal{B}^i + |\tau_k - \tau_{\bar{k}}| \delta S, \tag{E.8}$$

where δS is the difference of the action between $\varphi^i = v^i$ and the true minimum:

$$\delta S \equiv S[\varphi^i = v^i] - S[\varphi^i = \text{true minimum}] > 0. \tag{E.9}$$

All the effective kink–antikink potential used in this paper can be obtained by substituting the explicit forms of $\mathcal{A}^i$, $\mathcal{B}^i$, and δS in each model.

Appendix F. Lefschetz thimble analysis of a quasi-moduli integral

In this appendix, we calculate the following quasi-moduli integral of the form (3.46):

$$I = \int_{\mathcal{C}} dy \, \exp\left[-V(y)\right], \quad V(y) \equiv a e^{-y} + b y, \ \operatorname{Re} b > 0, \tag{F.1}$$

where $\mathcal{C}$ denotes the integration contour along the real axis. The parameter a is positive for kinks of the same type and negative for a kink–antikink pair. Note that the divergence of V for $\operatorname{Re} y \to -\infty$ is an artifact of the approximation and the form of V around its saddle points is relevant for the asymptotic behavior of the integral for small g (large a).

The flow equation, which determines the Lefschetz thimbles, is given by

$$\frac{\partial y}{\partial t} = \overline{\frac{\partial V}{\partial y}} = -\bar{a} e^{-\bar{y}} + \bar{b}. \tag{F.2}$$

The Lefschetz thimbles $\mathcal{J}_q$ and their duals $\mathcal{K}_q$ for the saddle points

$$y = \log \frac{a}{b} + 2\pi i q, \quad q \in \mathbb{Z} \tag{F.3}$$

are depicted in Fig. F.1 ($a > 0$) and Fig. F.2 ($a < 0$).

For $a > 0$, the real axis $\mathcal{C}$ intersects with the dual thimble of the saddle point $q = 0$, so that the integral is given by

$$I = \int_{\mathcal{J}_0} dy \, \exp\left[-V(y)\right] = a^{-b} \Gamma(b). \tag{F.4}$$

For $a < 0$, the Stokes phenomena occurs when we vary $\arg a$ around $\arg a = -\pi$ as shown in Fig. F.2. Thus we obtain the following ambiguous result for $a < 0$:

$$I = \begin{cases} \int_{\mathcal{J}_1} dy \, \exp\left[-V(y)\right] \\ \int_{\mathcal{J}_0} dy \, \exp\left[-V(y)\right] \end{cases} = |a|^{-b} \exp(\mp \pi i b) \Gamma(b) \quad \text{for } \theta \equiv -\pi - \arg a = \pm 0. \tag{F.5}$$

Combining these results, we obtain Eq. (3.47).

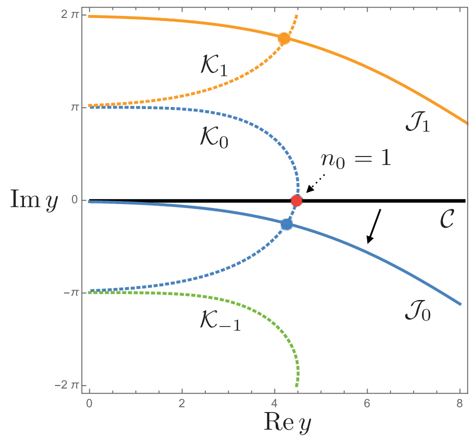
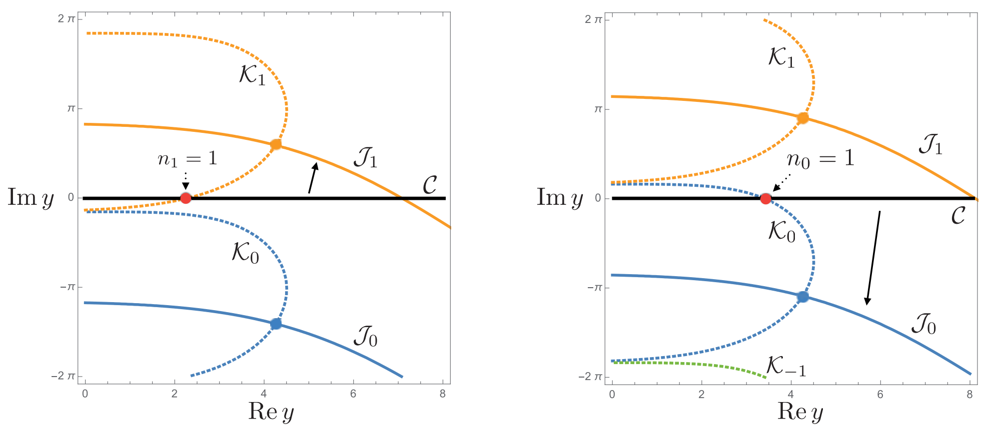


Fig. F.1. The Lefschetz thimbles $\mathcal{J}_q$ and their duals $\mathcal{K}_q$.



(b) $\theta > 0$ (b) $\theta < 0$

Fig. F.2. Stokes phenomenon at $\arg a = -\pi$ ($\theta = -\pi - \arg a = 0$). The original integration contour $\mathcal{C}$ intersects with $\mathcal{K}_1$ ($\mathcal{K}_0$) for $\theta > 0$ ($\theta < 0$) and hence $\mathcal{C}$ is deformed to $\mathcal{J}_1$ ($\mathcal{J}_0$).

Appendix G. One-loop determinant in $\mathbb{C}P^{N-1}$ QM

In this appendix, we calculate the one-loop determinant around the ith single-bion background φ^i_η in the $\mathbb{C}P^{N-1}$ model. It is convenient to normalize the fluctuations in the background of the ith bion as

$$\varphi^i = \varphi^i_\eta + g(1 + |\varphi^i_\eta|^2)\delta\varphi^i, \qquad \varphi^j = g\left(1 + |\varphi^i_\eta|^2\right)^{1/2} \delta\varphi^j \quad (j \neq i). \tag{G.1}$$

It was shown in Ref. [32] that the contribution from $\delta\varphi^i$ to the one-loop determinant gives the overall factor $8m_i^4/\pi g^2$ in Eq. (4.26) . To calculate the contribution from $\delta\varphi^j$, let us consider the linearized equation for $\delta\varphi^j$ in the background φ^i_η:

$$\Delta_j \delta\varphi^j = (-\partial_\tau^2 + \mathcal{V}_{i,j})\delta\varphi^j = 0, \tag{G.2}$$

where $\Delta_j = -\partial_\tau^2 + \mathcal{V}_{i,j}$ is the differential operator appearing in Eq. (4.24). The determinant Δ_j can be read off from the following asymptotic behavior of the solution of the linearized equation (see, e.g., Appendix B of Ref. [32])

$$1 = \lim_{\tau\to-\infty} \exp\left(-m_j\tau\right)\delta\varphi^j, \qquad \det\Delta_j = \lim_{\tau\to\infty} \exp\left(-m_j\tau\right)\delta\varphi^j. \tag{G.3}$$

In the weak coupling limit $g \to 0$, the single bion can be viewed as a well-separated kink–antikink pair, for which $\mathcal{V}_{i,j}$ can be approximated as

$$\mathcal{V}_{i,j} \approx \begin{cases} m_j^2, & \tau \ll \tau_-, \\ \mathcal{U}_{i,j}(\tau - \tau_-), & \tau \sim \tau_-, \\ (m_j - m_i)^2, & \tau_- \ll \tau \ll \tau_+, \\ \bar{\mathcal{U}}_{i,j}(\tau - \tau_+), & \tau \sim \tau_+, \\ m_j^2, & \tau_+ \ll \tau, \end{cases} \tag{G.4}$$

where $\mathcal{U}_{i,j}$ and $\bar{\mathcal{U}}_{i,j}$ are the following background potential felt by $\delta\varphi^j$ due to the BPS kink $\varphi_k^i = \exp(m_i(\tau - \tau_-))$ and the anti-BPS kink $\varphi_{\bar{k}}^i = \exp(-m_i(\tau - \tau_+))$:

$$\begin{aligned} \mathcal{U}_{i,j}(\tau-\tau_-) &= \left(m_j - \frac{m_i}{1+\exp(-2m_i(\tau-\tau_-))}\right)^2 + \partial_\tau \left(m_j - \frac{m_i}{1+\exp(-2m_i(\tau-\tau_-))}\right), \\ \bar{\mathcal{U}}_{i,j}(\tau-\tau_+) &= \left(m_j - \frac{m_i}{1+\exp(2m_i(\tau-\tau_-))}\right)^2 - \partial_\tau \left(m_j - \frac{m_i}{1+\exp(2m_i(\tau-\tau_-))}\right). \end{aligned} \tag{G.5}$$

In each region, the linearized equation (G.2) can be solved as

$$\delta\varphi^j \approx \begin{cases} \exp(m_j\tau), & \tau \ll \tau_-, \\ f_-, & \tau \sim \tau_-, \\ a_j' \exp\left((m_j - m_i)\tau\right) + b_j' \exp\left(-(m_j - m_i)\tau\right), & \tau_- \ll \tau \ll \tau_+, \\ f_+, & \tau \sim \tau_+, \\ \det \Delta_j \exp(m_j\tau) + b_j'' \exp(-m_j\tau), & \tau_+ \ll \tau, \end{cases} \tag{G.6}$$

with

$$\begin{aligned} f_- &= \frac{c_j \exp(m_j\tau)}{(1+\exp(2m_i(\tau-\tau_-)))^{1/2}} \\ &\quad + \frac{d_j \exp(-m_j\tau)}{(1+\exp(2m_i(\tau-\tau_-)))^{1/2}} \left(1 + \frac{m_j}{m_j - m_i}\exp(2m_i(\tau-\tau_-))\right), \end{aligned} \tag{G.7}$$

$$\begin{aligned} f_+ &= \frac{c_j' \exp(-m_j\tau)}{(1+\exp(-2m_i(\tau-\tau_+)))^{1/2}} \\ &\quad + \frac{d_j' \exp(m_j\tau)}{(1+\exp(-2m_i(\tau-\tau_+)))^{1/2}} \left(1 + \frac{m_j}{m_j - m_i}\exp(-2m_i(\tau-\tau_+))\right). \end{aligned} \tag{G.8}$$

Connecting these solutions, we find that the coefficients are related as

$$c_j = 1, \qquad a_j' = \exp(m_j\tau_-), \qquad d_j' = \frac{m_j - m_i}{m_j}\exp(m_i(\tau_- - \tau_+)) = \det \Delta_j. \tag{G.9}$$

Therefore, the one-loop determinant is given by

$$\prod_{j=1, j\neq i}^{N-1} \det \Delta_j^{-1} = A_i \exp((N-2)m_i\tau_r), \tag{G.10}$$

where $\tau_r = \tau_+ - \tau_-$ is the relative position and the constant A_i is defined in Eq. (4.11).

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