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Condensate $\langle A_\mu^+ A_\mu^- \rangle$ and massive magnetic potential in Euclidean gauge theories

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Euclidean SU(2) gauge theory is studied in a nonlinear gauge. In this theory, ghost condensation happens and gauge fields acquire tachyonic masses. It is shown that these tachyonic masses are removed by a gauge field condensate $\langle A_\mu^+ A_\mu^- \rangle$. Because of the ghost condensation, monopole solutions are included naturally. We find that the condensate $\langle A_\mu^+ A_\mu^- \rangle$ makes the magnetic potential massive.

Subject Index B00, B03, B06

1. Introduction

In non-Abelian gauge theories, a magnetic monopole may play an important role. In the dual superconductor model of color confinement (see, e.g., Ref. [1]), monopole condensation is expected. As a result, a dual gauge field [2] acquires a mass, and color confinement may happen.

How can monopoles appear without Higgs fields? One way is the Abelian projection by 't Hooft [3]. Let us consider a field $X(x)$, which belongs to the adjoint representation, and perform the Abelian gauge fixing. When we diagonalize $X(x)$, there may be points at which the eigenvalues of $X(x)$ degenerate. Monopoles appear at these points. Another way is to introduce a unit color vector $\hat{n}^A(x)$ in the internal space [4,5]. A non-Abelian magnetic potential is defined by

$$C_\mu^A = \frac{-1}{g}(\hat{n} \times \partial_\mu \hat{n})^A, \tag{1.1}$$

and the gauge field $A_\mu^A(x)$ is decomposed by using $\hat{n}^A$. This model is called extended quantum chromodynamics (QCD) [6].

What is the origin of $\hat{n}^A$? Is it possible to make the magnetic potential massive? In this paper, we propose a scenario that produces $\hat{n}^A$ and massive C_μ^A. For simplicity, we consider the SU(2) gauge theory in Euclidean space-time, and sometimes call the gauge field a gluon. In the next section, we briefly review ghost condensation and tachyonic gluon masses. In Sect. 3, it is shown that these tachyonic masses are removed by the vacuum expectation value (VEV) $\langle A_\mu^+ A_\mu^- \rangle$. Under the ghost condensation, we can include a magnetic monopole as a classical solution. In Sect. 4, we introduce it in the Abelian gauge. In the presence of the VEV $\langle A_\mu^+ A_\mu^- \rangle$, a magnetic potential is expected to become massive. By using the background covariant gauge, we confirm this expectation in Sect. 5. In Sect. 6, to remove the Dirac string, we perform a singular gauge transformation, and derive the extended QCD with massive C_μ^A. In Sect. 7, the case of 3D space-time is discussed briefly. Section 8 is devoted to a summary and comments. In Appendix A, it is shown that $\langle (A_\mu^3)^2 \rangle$ vanishes. The

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tachyonic gluon mass is derived in the background covariant gauge in Appendix B. The singular gauge transformation used in Sect. 6 is summarized in Appendix C. The Becchi–Rouet–Stora (BRS) symmetry and the breakdown of the global gauge symmetry are explained in Appendix D.

2. Ghost condensation and tachyonic gluon mass

We consider the SU(2) gauge theory with structure constants f_{ABC}. Using the notations

$$F\cdot G = F^A G^A, \quad (F\times)^{AB} = f_{ACB}F^C, \quad (F\times G)^A = f_{ABC}F^B G^C, \quad A=1,2,3,$$

the Lagrangian in the nonlinear gauge is given by [7]

$$\mathcal{L} = \mathcal{L}_{\rm inv} + \mathcal{L}_{\rm NL}, \quad \mathcal{L}_{\rm inv} = \frac{1}{4}F_{\mu\nu}^2,$$
$$\mathcal{L}_{\rm NL} = B\cdot\partial_\mu A_\mu + i\bar{c}\cdot\partial_\mu D_\mu c - \frac{\alpha_1}{2}B^2 - \frac{\alpha_2}{2}\bar{B}^2 - B\cdot w, \tag{2.1}$$

where $\bar{B} = -B + ig\bar{c}\times c$, α_1 and α_2 are gauge parameters, and w is a constant. Introducing the auxiliary field φ, which represents $\alpha_2\bar{B}$, $\mathcal{L}_{\rm NL}$ is rewritten as

$$\mathcal{L}_\varphi = -\frac{\alpha_1}{2}B^2 + B\cdot(\partial_\mu A_\mu + \varphi - w) + i\bar{c}\cdot(\partial_\mu D_\mu + g\varphi\times)c + \frac{\varphi^2}{2\alpha_2}. \tag{2.2}$$

In Ref. [8], we have shown that $g\varphi$ acquires the VEV $v = g\varphi_0$ under an energy scale μ_0. Dividing the quantum fluctuation $\tilde{\varphi}(x)$ from φ_0, we substitute $\varphi(x) = \varphi_0 + \tilde{\varphi}(x)$ into Eq. (2.2), and choose the constant $w = \varphi_0$. This choice is necessary to maintain the BRS symmetry [9].

Then Eq. (2.2) becomes

$$\mathcal{L}'_\varphi = -\frac{\alpha_1}{2}B^2 + B\cdot(\partial_\mu A_\mu + \tilde{\varphi}) + i\bar{c}\cdot(\partial_\mu D_\mu + g\tilde{\varphi}\times + g\varphi_0\times)c. \tag{2.3}$$

Because of the VEV φ_0, the global SU(2) symmetry breaks down to U(1). The relation between this breaking and BRS-invariant Green functions is explained in Ref. [10].[1]

Under the ghost condensation, the ghost-loop contribution to the gluon propagator was calculated. When the gluon momentum becomes small, it is found that the gluon acquires tachyonic mass [7,11]. The term $g\varphi_0\times$ in the ghost propagator $(\partial_\mu^2 + g\varphi_0\times)^{-1}$ gives rise to the tachyonic mass. Now we choose the VEV in the third direction as $\langle\varphi^A\rangle = \varphi_0\delta^{A3}$, and use the notation $(A_\mu^a)^2 = (A_\mu^1)^2 + (A_\mu^2)^2$. Then, the scale μ_0, the value of the VEV $v = g\varphi_0$, and the tachyonic gluon mass term in 4D Euclidean space ($D=4$) are summarized as

$$\mu_0 = \Lambda e^{-4\pi^2/(\alpha_2 g^2)},\ v = \left\{\frac{\mu_0^4 - \mu^4}{1 - e^{-16\pi^2/(\alpha_2 g^2)}}\right\}^{1/2},\ \frac{1}{2}\left(\frac{-g^2 v}{64\pi}\right)\left\{(A_\mu^a)^2 + 2(A_\mu^3)^2\right\}, \tag{2.4}$$

where Λ is a momentum cut-off, and μ is the momentum scale satisfying $\mu < \mu_0$. The coupling constant g and the gauge parameter α_2 in Eq. (2.4) are the quantities at the scale Λ. In the same way, the corresponding quantities in 3D Euclidean space ($D=3$) are

$$\mu_0 = \frac{\alpha_2 g_3^2\Lambda}{\pi^2\Lambda + \alpha_2 g_3^2},\ v_3 = \frac{2}{16\pi^2}(\alpha_2 g_3^2)^2,\ \frac{1}{2}\left(\frac{-\alpha_2}{24\pi^2}g_3^4\right)\left\{(A_\mu^a)^2 + \frac{3}{2}(A_\mu^3)^2\right\}, \tag{2.5}$$

where g_3 and v_3 are the coupling constant and the VEV $g\varphi_0$ in $D=3$, respectively.

[1] In Appendix D, the necessity of w is explained. The BRS symmetry and the broken global gauge symmetry are also discussed.

3. Condensate $\langle A^+_\mu A^-_\mu \rangle$

From now on, we concentrate on the $D = 4$ case, and try to remove the tachyonic gluon mass. We write Eq. (2.4) as

$$-m_4^2\left[A^+_\mu A^-_\mu + \frac{\kappa_4}{2}(A^3_\mu)^2\right],\ m_4^2 = \frac{g^2 v}{64\pi},\ \kappa_4 = 2, \tag{3.1}$$

where $A^\pm_\mu = (A^1_\mu \pm iA^2_\mu)/\sqrt{2}$.

3.1. The effective potential for $\langle A^+_\mu A^-_\mu \rangle$

When the tachyonic mass term (3.1) exists, the gluon propagator blows up as $p^2 \to m_4^2$. To avoid this, we return to the Lagrangian $\mathcal{L}_{\rm inv} + \mathcal{L}'_\varphi$, and introduce the source term $KA^+_\mu A^-_\mu$ for the local composite operator (LCO) $A^+_\mu(x)A^-_\mu(x)$.[2] The partition function is

$$Z(J,K) = \int D\mu \exp\left[-\frac{1}{\hbar}\left\{S + \int dxJ\Psi + \int dxKA^+_\mu A^-_\mu\right\}\right],$$

where the usual source terms are

$$J\Psi = \bar{c}\eta + \bar{\eta}c + A^3_\mu J^3_\mu + A^+_\mu J^-_\mu + A^-_\mu J^+_\mu.$$

Dividing the action S into free and interaction parts as $S = S_{\rm free} + S_{\rm int}$, and performing the Gaussian integration, we obtain

$$Z(K) = \exp\left[-\frac{1}{\hbar}W(K)\right], \quad W(K) = W_1(K) + W_2(K) + \cdots,$$

where W_n is $O(\hbar^n)$, and we set $J = 0$ after constructing $W(J,K)$. From $Z(K)$, we find

$$\frac{\delta W}{\delta K(x)} = \langle A^+_\mu(x)A^-_\mu(x)\rangle = \hbar\Phi_4(x), \tag{3.2}$$

and we define the effective action as

$$\Gamma(\Phi_4) = W(K) - \hbar\int dxK\Phi_4. \tag{3.3}$$

If we write the $A^\pm_\mu$-related part in $S_{\rm free}$ as $-A^+_\mu\Delta_{\mu\nu}A^-_\nu$, the K-dependent term in W_1 becomes

$$W_1(K) = \hbar{\rm Tr}\ln(-\Delta + K).$$

Thus, at $O(\hbar)$, Eq. (3.2) gives

$$\Phi_4(x) = \langle x|\,(-\Delta + K)^{-1}_{\mu\mu}\,|x\rangle. \tag{3.4}$$

From $S_{\rm int}$, the diagrams depicted in Fig. 1 contribute to W_2. Since the kinetic term $(F^A_{\mu\nu})^2/4$ contains the terms

$$\begin{aligned}\frac{g^2}{4}(f^{ABC}A^B_\mu A^C_\nu)^2 = &\frac{g^2}{2}(A^+_\mu A^-_\mu)^2 + g^2(A^+_\mu A^-_\mu)(A^3_\nu)^2 \\ &-\frac{g^2}{2}(A^+_\mu)^2(A^-_\nu)^2 - g^2(A^+_\mu A^3_\mu)(A^-_\nu A^3_\nu),\end{aligned} \tag{3.5}$$

[2] In this subsection, we write $\hbar$ explicitly. For scalar fields $\phi(x)$, the effective potential for the LCO ϕ^2 is studied in Refs. [12,13].

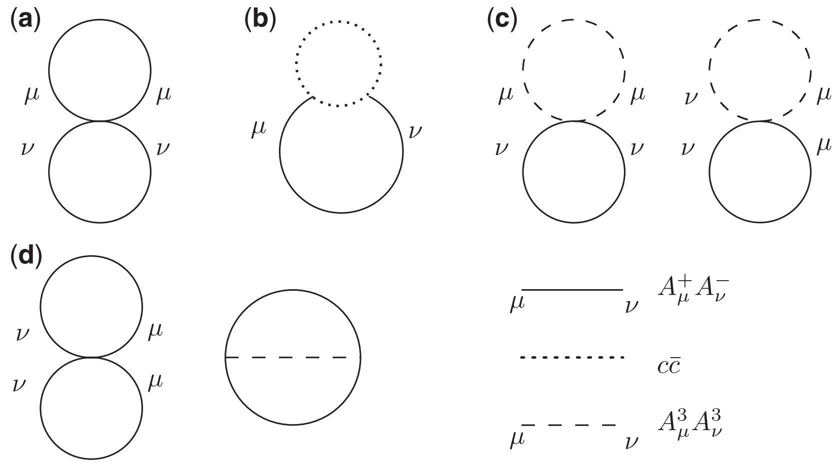

Fig. 1. The diagrams that contribute to the effective potential $\Gamma(\Phi_4)$ at $O(\hbar^2)$. The solid line represents $A^+_\mu A^-_\nu$, the dotted line is $c\bar{c}$, and the dashed line represents $A^3_\mu A^3_\nu$.

the diagram in Fig. 1(a) comes from the vertex $\frac{g^2}{2}(A^+_\mu A^-_\mu)^2$ in Eq. (3.5). Applying Eq. (3.4), it gives $\hbar^2 \frac{g^2}{2}\Phi_4^2$. The ghost loop in Fig. 1(b) yields the term $-m_4^2\hbar\delta_{\mu\nu}$, which makes $A^\pm_\mu$ tachyonic. Using Eq. (3.4) again, Fig. 1(b) gives $-m_4^2\hbar^2\Phi_4$. The diagrams in Fig. 1(c) contain the loops with A^3_μ. In the dimensional regularization, as $\int d^4k(1/k^2) = 0$, these diagrams vanish. If the source K is constant, it plays the role of mass-squared M_4^2 for the $A^\pm_\mu$. So the diagrams in Fig. 1(d) can produce the term $z_1 K\hbar^2\Phi_4$, where z_1 is a divergent constant. This divergence should be removed by the renormalization of K. By summing up these results, W_2 becomes $\Gamma_2(\Phi_4) = \hbar^2 V(\Phi_4)$, where

$$V(\Phi_4) = \frac{g^2}{2}\Phi_4^2 - m_4^2\Phi_4. \tag{3.6}$$

Thus, up to $O(\hbar^2)$, we obtain

$$\Gamma(\Phi_4) = W_1(K) + \hbar^2 V(\Phi_4) - \hbar\int dx K\Phi_4. \tag{3.7}$$

This Γ satisfies

$$\frac{\delta\Gamma}{\delta\Phi_4} = \left(\frac{\delta W_1}{\delta K} - \hbar\Phi_4\right)\frac{\delta K}{\delta\Phi_4} + \hbar^2\frac{dV}{d\Phi_4} - \hbar K = \hbar^2\frac{dV}{d\Phi_4} - \hbar K, \tag{3.8}$$

where Eq. (3.4) has been used. Since Γ satisfies $\delta\Gamma/\delta\Phi_4 = -\hbar K$, from Eq. (3.8), we obtain $dV/d\Phi_4 = 0$.

In Fig. 2(a), the diagrams that contribute to $V(\Phi_4)$ are depicted. The closed loop with a solid line represents $\Phi_4(x)$. As we show in Fig. 2(b), by removing one closed loop, the diagrams that correspond to $dV/d\Phi_4$ are obtained. If we cut one closed loop as in Fig. 2(c), the diagrams that contribute to the mass term for $A^+_\mu(x)A^-_\mu(x)$ are obtained. Therefore, when the condition $dV/d\Phi_4 = 0$ holds, the two diagrams in Fig. 2(c) cancel out. In the same way, in $D = 4$, the diagrams depicted in Fig. 1(d) cancel out. Namely, the diagrams in Figs. 2(c) and (d) do not contribute to the masses for $A^\pm_\mu$ and A^3_μ, respectively.

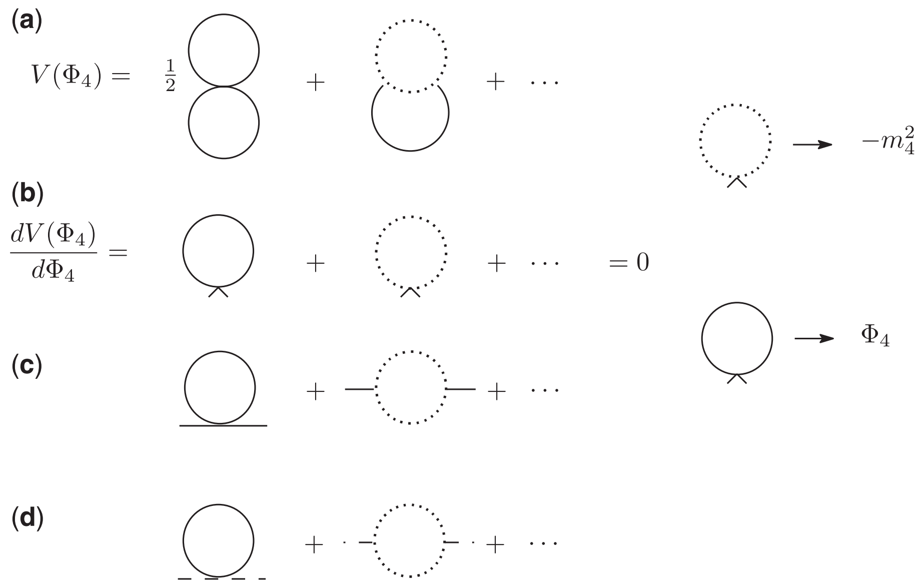

Fig. 2. The diagrams that contribute to $V(\Phi)$, $dV/d\Phi_4 = 0$, and the mass terms for $A_\mu^\pm$ and A_μ^3.

We summarize the results. Up to $O(\hbar^2)$, the effective potential Γ is Eq. (3.7). As it satisfies $\delta\Gamma/\delta\Phi_4 = -\hbar K$, we obtain Eq. (3.4) and $dV/d\Phi_4 = 0$. Using Eq. (3.6), $dV/d\Phi_4 = 0$ gives

$$\Phi_4 = \langle A_\mu^+ A_\mu^- \rangle = \frac{m_4^2}{g^2} = \frac{v}{64\pi}, \tag{3.9}$$

and Eqs. (3.4) and (3.9) give

$$\frac{m_4^2}{g^2} = \langle x| \left(-\Delta + M_4^2\right)^{-1}_{\mu\mu} |x\rangle, \tag{3.10}$$

where we set $K = M_4^2$. Thus, when $m_4 \neq 0$, the condensate Φ_4 exists, and the mass M_4 for $A_\mu^\pm$ is determined by Eq. (3.10).

As $\Delta_{\mu\nu}$ contains a gauge parameter, Eq. (3.10) is gauge-dependent and requires regularization. So, although Eq. (3.10) implies that $M_4 \neq 0$, the determination of this must be done carefully. We do not try to determine M_4 in this paper, and make it a future task.

3.2. *Expedient procedure*

The above results are obtained simply by the following procedure. Add the tachyonic mass term (3.1) to the Lagrangian. Then replace $A_\mu^+ A_\mu^-$ as

$$A_\mu^+ A_\mu^- \rightarrow \Phi_4 + A_\mu^+ A_\mu^-, \quad \Phi_4 = \langle A_\mu^+ A_\mu^- \rangle, \tag{3.11}$$

and add the source term $M_4^2 A_\mu^+ A_\mu^-$. Then we obtain the following terms that contain m_4^2, M_4^2, or Φ_4:

$$V(\Phi_4) + (g^2\Phi_4 - m_4^2)[A_\mu^+ A_\mu^- + (A_\mu^3)^2] + M_4^2 A_\mu^+ A_\mu^-, \tag{3.12}$$

where Eq. (3.5) has been used. From $dV/d\Phi_4 = 0$, Eq. (3.9) holds and Eq. (3.12) becomes $M_4^2 A_\mu^+ A_\mu^-$. Namely, the tachyonic masses are removed and, whereas A_μ^3 is massless, $A_\mu^\pm$ acquire the mass M_4. In the following sections, we apply this procedure for simplicity.

We note, although the component A^3_μ has the tachyonic mass in Eq. (3.1) as well, we did not take the source term for $(A^3_\mu)^2$ and the VEV $\langle (A^3_\mu)^2 \rangle$ into account. In Appendix A, it is shown that the VEV $\langle (A^3_\mu)^2 \rangle$ vanishes.

4. Inclusion of monopoles

The Lagrangian $\mathcal{L}_\varphi$ is invariant under the BRS transformation δ_B, if the field φ transforms as $\delta_B \varphi = g\varphi \times c$. That is, φ behaves like a Higgs field in the adjoint representation. As φ acquires the VEV $\langle \varphi^A \rangle = (v/g)\delta^{A3}$, we can introduce Abelian monopoles in the $A = 3$ direction [14]. Let us consider the classical field $b^A_\mu = \tilde{C}_\mu \delta^{A3}$, which satisfies the equation of motion

$$\partial^2_\nu b^A_\mu - \partial_\mu \partial_\nu b^A_\nu = 0.$$

Except for a string singularity, it is known that the configuration

$$\tilde{C}_\mu = \frac{n}{g}(\cos\theta - a)\partial_\mu \phi$$

satisfies this equation, where n is an integer, and (r, θ, ϕ) are spherical coordinates. If $a = 1$ ($a = -1$), the Dirac string appears on the negative (positive) z-axis. In Sect. 6, we choose $a = 1$ as a concrete example, i.e.,

$$\tilde{C}_\mu = \frac{n}{g}(\cos\theta - 1)\partial_\mu \phi = \frac{n}{g}\frac{z - r}{r(x^2 + y^2)}(0, -y, x, 0). \tag{4.1}$$

This monopole satisfies the equation of motion

$$\partial_\nu(\partial_\nu \tilde{C}_\mu - \partial_\mu \tilde{C}_\nu) = \epsilon_{\nu\mu\alpha\beta}\partial_\nu \frac{n_\alpha}{n_\rho \partial_\rho} k_\beta,$$

where

$$n_\alpha = \delta_{\alpha 3}, \quad k_\beta = \delta_{\beta 0}\frac{4\pi n}{g}\delta(x)\delta(y)\delta(z).$$

We call $\tilde{C}_\mu$ a magnetic potential.

In Sect. 3, we have shown that the VEV $\Phi_4 = \langle A^+_\mu A^-_\mu \rangle$ removes the tachyonic mass for the gluon. What happens for the classical solution b^A_μ? If we substitute $A^A_\mu = a^A_\mu + b^A_\mu$ into Eq. (3.5), Eq. (3.12) changes to

$$V(\Phi_4) + (g^2\Phi_4 - m^2_4)[a^+_\mu a^-_\mu + (a^3_\mu)^2] + M^2_4 a^+_\mu a^-_\mu + g^2\Phi_4[2b^3_\mu a^3_\mu + (b^3_\mu)^2]. \tag{4.2}$$

As Φ_4 is given by Eq. (3.5), the mass term for the magnetic potential appears as

$$g^2\Phi_4(b^3_\mu)^2 = m^2_4 \tilde{C}_\mu \tilde{C}_\mu. \tag{4.3}$$

In the next section, we study the mass for $\tilde{C}_\mu$ by using the background covariant gauge.

5. Massive magnetic potential

First we derive the Lagrangian with the field φ^A and the magnetic potential $\tilde{C}_\mu$ in the background covariant gauge [15]. We divide A^A_μ into the classical part b^A_μ and the quantum fluctuation a^A_μ as[3]

$$A^A_\mu = b^A_\mu + a^A_\mu, \quad b^A_\mu = \tilde{C}_\mu \delta^{A3}.$$

Then the gauge transformation

$$\delta A_\mu = D_\mu(A)\varepsilon \tag{5.1}$$

is satisfied by

$$\delta a_\mu = D_\mu(b+a)\varepsilon,\ \delta b_\mu = 0, \tag{5.2}$$

where the notation $D_\mu(A) = (\partial_\mu + gA_\mu \times)$ has been used. Choosing the gauge-fixing function

$$\tilde{G}(a) = D_\mu(b)a_\mu + \varphi - w = (\partial_\mu + gb_\mu \times)a_\mu + \varphi - w,$$

and applying the transformation (5.2), we obtain the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{inv}}(b+a) + \tilde{\mathcal{L}}_\varphi(a,b),$$

$$\tilde{\mathcal{L}}_\varphi(a,b) = -\frac{\alpha_1}{2}B^2 + B\cdot[D_\mu(b)a_\mu + \varphi - w] + i\bar{c}\cdot[D_\mu(b)D_\mu(b+a) + g\varphi\times]c + \frac{\varphi^2}{2\alpha_2}, \tag{5.3}$$

where w is a constant. We require that φ^A belongs to the adjoint representation, and it transforms as $\delta\varphi = g\varphi \times \varepsilon$. The last term $\varphi^2/(2\alpha_2)$ is added because it is gauge-invariant and necessary to yield the quartic ghost interaction term $-\alpha_2\bar{B}^2/2$ in Eq. (2.1).

Since the ghost condensation happens by the quartic ghost interaction [16], the condensate $v = g\varphi_0$ appears irrespective of b_μ. Thus, after the ghost condensation, $\tilde{\mathcal{L}}_\varphi(a,b)$ becomes

$$\tilde{\mathcal{L}}_\varphi(a,b) = -\frac{\alpha_1}{2}B^2 + B\cdot[D_\mu(b)a_\mu + \tilde{\varphi}] + i\bar{c}\cdot[D_\mu(b)D_\mu(b+a) + g(\varphi_0+\tilde{\varphi})\times]c, \tag{5.4}$$

where $\varphi(x) = \varphi_0 + \tilde{\varphi}(x)$ and $w = \varphi_0$. We note that Eq. (5.4) is obtained from Eq. (2.3) by replacing ∂_μ to $D_\mu(b)$. As we explained in Sect. 4, the VEV $g\langle\varphi^A\rangle = v\delta^{A3}$ selects the unbroken U(1) direction, and the Abelian monopole $b^A_\mu = \tilde{C}_\mu\delta^{A3}$ is included in the Lagrangian consistently.

Next we study mass terms. Without b_μ, a ghost loop brings about the tachyonic gluon mass term (3.1). We show that Eq. (3.1) is obtained even if b_μ exists. To show this, we introduce the background gauge transformation

$$\delta b_\mu = D_\mu(b)\varepsilon, \quad \delta a_\mu = ga_\mu \times \varepsilon, \quad \delta\tilde{\varphi} = g\tilde{\varphi}\times\varepsilon, \tag{5.5}$$

which satisfies the transformation (5.1). When $v = 0$, the Lagrangian $\mathcal{L}_{\text{inv}}(b+a)$ and $\tilde{\mathcal{L}}_\varphi(a,b)$ in Eq. (5.4) are invariant under the transformation (5.5), if $\bar{c}^A$, c^A, and B^A transform in the adjoint representation. When $v\delta^{A3} \neq 0$, the ghost determinant

$$\int DcD\bar{c}\exp[-\int dx i\bar{c}\cdot(D_\mu(b)D_\mu(b+a) + v\times)c] = \det[D_\mu(b)D_\mu(b+a) + v\times] \tag{5.6}$$

[3] In this paper, as we are interested in the massive magnetic potential, we choose $b^A_\mu = \tilde{C}_\mu\delta^{A3}$. However, instead of $\tilde{C}_\mu$, we can introduce other classical solutions B_μ in the form $b^A_\mu = B_\mu\delta^{A3}$.

breaks the symmetry (5.5). However, if we restrict Eq. (5.5) to the U(1) transformation

$$\delta b_\mu = -\frac{1}{g}\partial_\mu \varepsilon, \quad \delta a_\mu^A = -\varepsilon f_{A3B} a_\mu^B, \quad |\varepsilon| \ll 1,$$

the determinant (5.6) is invariant. As tachyonic mass terms come from Eq. (5.6), they must respect this U(1) symmetry. So the terms b_μ^2 and $b_\mu a_\mu^3$ are forbidden, and the terms $-L_1^2(a_\mu^a)^2$ and $-L_2^2(a_\mu^3)^2$ are allowed. We calculate the determinant (5.6) in Appendix B, and obtain the tachyonic mass term

$$-m_4^2[a_\mu^+ a_\mu^- + (a_\mu^3)^2] \tag{5.7}$$

as expected.

Finally, to remove the tachyonic mass, the procedure in Sect. 3.2 is applied. We replace $a_\mu^+ a_\mu^-$ with $\Phi_4 + a_\mu^+ a_\mu^-$, where $\Phi_4 = \langle a_\mu^+ a_\mu^- \rangle$, and add the source term $M_4^2 a_\mu^+ a_\mu^-$. As $\mathcal{L}_{\text{inv}}(b+a)$ contains the term

$$\begin{aligned}\frac{g^2}{4}\{f_{ABC}(b+a)_\mu^B(b+a)_\nu^C\}^2 &= \frac{g^2}{2}(a_\mu^+ a_\mu^-)^2 + g^2(a_\mu^+ a_\mu^-)\{(b_\nu^3 + a_\nu^3)^2\} \\ &\quad - \frac{g^2}{2}(a_\mu^+)^2(a_\nu^-)^2 - g^2\{a_\mu^+(b_\mu^3 + a_\mu^3)\}\{a_\nu^-(b_\nu^3 + a_\nu^3)\},\end{aligned} \tag{5.8}$$

Eqs. (5.7) and (5.8) lead to

$$V(\Phi_4) + (g^2\Phi_4 - m_4^2)\{a_\mu^+ a_\mu^- + (a_\mu^3)^2\} + M_4^2 a_\mu^+ a_\mu^- + g^2\Phi_4\{2b_\mu^3 a_\mu^3 + (b_\mu^3)^2\}. \tag{5.9}$$

As $V(\Phi_4)$ takes the minimum value at $g^2\Phi_4 = m_4^2$, Eq. (5.9) becomes

$$M_4^2 a_\mu^+ a_\mu^- + m_4^2\{2b_\mu^3 a_\mu^3 + (b_\mu^3)^2\}.$$

Namely, the magnetic potential $\tilde{C}_\mu$ has the mass term (4.3).

Except for string singularity, the magnetic potential $\tilde{C}_\mu$ must be modified to satisfy the massive equation of motion[4]

$$\left(\frac{\delta \mathcal{L}}{\delta \tilde{C}_\mu} - \partial_\nu \frac{\delta \mathcal{L}}{\delta \partial_\nu \tilde{C}_\mu}\right)_{a_\rho^A=0} = \partial_\nu^2 \tilde{C}_\mu - \partial_\mu \partial_\nu \tilde{C}_\nu - 2m_4^2 \tilde{C}_\nu = 0. \tag{5.10}$$

Then the term linear with respect to a_μ^A

$$a_\mu^3 \left(\frac{\delta \mathcal{L}}{\delta \tilde{C}_\mu} - \partial_\nu \frac{\delta \mathcal{L}}{\delta \partial_\nu \tilde{C}_\mu}\right)_{a_\rho^A=0}$$

vanishes. Thus the final Lagrangian with massive magnetic potential becomes

$$\frac{1}{4}\left(\partial_\mu \tilde{C}_\nu - \partial_\nu \tilde{C}_\mu\right)^2 + m_4^2 \tilde{C}_\mu \tilde{C}_\mu + \mathcal{L}_{\text{inv}}(a) + M_4^2 a_\mu^+ a_\mu^- + \cdots, \tag{5.11}$$

where interaction terms with a_μ^A and $\tilde{C}_\mu$ are neglected.

[4] Equation (4.1) no longer satisfies Eq. (5.10). If we use a dual potential, a modified solution is obtained. A concrete solution will be given in the next paper.

6. Massive non-Abelian magnetic potential

The magnetic potential $\tilde{C}_\mu$ has string singularity. We remove it by a singular gauge transformation [14]. In this section, we use the matrix notation $D_\mu(b) = \partial_\mu - igb_\mu^A T_A$ with $(T_A)_{BC} = if_{BAC}$. We perform a singular gauge transformation for $A_\mu^A = b_\mu^A + a_\mu^A$ and $\varphi^A = \varphi_0\delta^{A3} + \tilde{\varphi}^A$ as

$$A'^A_\mu T_A = U^\dagger A_\mu^A T_A U + \frac{i}{g} U^\dagger \partial_\mu U\,, \quad \varphi'^A T_A = U^\dagger(\varphi_0\delta^{A3} + \tilde{\varphi}^A)T_A U. \tag{6.1}$$

The matrix U, which corresponds to the monopole $\tilde{C}_\mu$ in Eq. (4.1), is $U = e^{-in\phi T_3}e^{i\theta T_2}e^{in\phi T_3}$. As we explain in Appendix C, using this matrix U and the monopole solution $b_\mu^A T_A = \tilde{C}_\mu T_3$, Eq. (6.1) becomes

$$A'^A_\mu T_A = \left[a_\mu^1\hat{n}_1^A + a_\mu^2\hat{n}_2^A + a_\mu^3\hat{n}^A - \frac{1}{g}(\hat{n}\times\partial_\mu\hat{n})^A\right]T_A \tag{6.2}$$

and

$$\varphi'^A T_A = \left(\varphi_0\hat{n}^A + \tilde{\varphi}^B\hat{n}_B^A\right)T_A, \tag{6.3}$$

where $\hat{n}_1, \hat{n}_2$, and $\hat{n}_3 = \hat{n}$ are defined in Eq. (C.2) with $\alpha = \theta$ and $\beta = -\gamma = n\phi$. We note that the right-hand side of Eq. (6.2) is the gauge field decomposition by Cho [6], and Eq. (6.3) shows that the color vector $\hat{n}^A$ comes from the VEV $\varphi_0\delta^{A3}$.

Now we regard the transformation (6.1) as the background gauge transformation. The classical field has the inhomogeneous part as

$$b'_\mu = U^\dagger b_\mu U + \frac{i}{g}U^\dagger\partial_\mu U, \quad a'_\mu = U^\dagger a_\mu U. \tag{6.4}$$

Then, as we show in Appendix C, Eqs. (6.2) and (6.4) indicate that the background gauge transformation is realized simply by the replacement

$$a_\mu^A \to a_\mu^B\hat{n}_B^A, \quad \tilde{C}_\mu\delta^{3A} \to C_\mu^A = -\frac{1}{g}(\hat{n}\times\partial_\mu\hat{n})^A. \tag{6.5}$$

We call C_μ^A a non-Abelian magnetic potential. Let us give an example. If we choose $n = 1$, $\hat{n}$ and C_μ^A become

$$\hat{n} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \quad C_\mu^A = \epsilon_{A\mu k}\frac{x^k}{gr^2}.$$

Although this configuration found by Wu and Yang [17] is singular at the origin, it has no string.

Applying Eq. (6.5) to the Lagrangian $\mathcal{L}_{\rm inv}(b+a)$ in Eq. (5.3), we obtain

$$\begin{aligned}\mathcal{L}_{\rm inv} =& \frac{1}{4}(F_{\mu\nu}+H_{\mu\nu})^2 + \frac{1}{4}(\hat{D}_\mu X_\nu - \hat{D}_\nu X_\mu)^2 \\ &+ \frac{g}{2}(F_{\mu\nu}+H_{\mu\nu})\hat{n}\cdot(X_\mu\times X_\nu) + \frac{g^2}{4}(X_\mu\times X_\nu)^2,\end{aligned} \tag{6.6}$$

where the following notation in Ref. [6] has been used:

$$\hat{A}_\mu^A = a_\mu^3 \hat{n}^A + C_\mu^A, \quad X_\mu^A = a_\mu^1 \hat{n}_1^A + a_\mu^2 \hat{n}_2^A,$$
$$\hat{D}_\mu = \partial_\mu + g\hat{A}_\mu \times, \quad F_{\mu\nu} = \partial_\mu a_\nu^3 - \partial_\nu a_\mu^3,$$
$$H_{\mu\nu}\hat{n}^A = \partial \mu C_\nu^A - \partial_\nu C_\mu^A + g(C_\mu \times C_\nu)^A = -\frac{1}{g}(\partial_\mu \hat{n} \times \partial_\nu \hat{n})^A. \tag{6.7}$$

Equation (6.6) is the Lagrangian of the extended QCD [6]. Next we apply the transformation (6.5) to $\tilde{\mathcal{L}}_\varphi(a, b)$ in Eq. (5.4) as well. However, as $\hat{n}_B \cdot \hat{n}_C = \delta_{BC}$, the tachyonic mass term (5.7) is obtained again. Replacing $a_\mu^+ a_\mu^-$ with $a_\mu^+ a_\mu^- + \Phi_4$, and adding the source term $M_4^2 a_\mu^+ a_\mu^-$, the part that depends on Φ_4, m_4^2, or M_4^2 becomes

$$V(\Phi_4) + (g^2\Phi_4 - m_4^2)[a_\mu^+ a_\mu^- + (a_\mu^3)^2] + M_4^2 a_\mu^+ a_\mu^- + g^2\Phi_4[2C_\mu^A a_\mu^3 \hat{n}^A + (C_\mu^A)^2]. \tag{6.8}$$

Equation (6.8) is derived directly by applying the transformation (6.5) to Eq. (5.9). However, there is a difference between Eqs. (5.9) and (6.8). Since $C_\mu^A \hat{n}^A = 0$, the cross-term $g^2\Phi_4[2C_\mu^A a_\mu^3 \hat{n}^A]$ vanishes without using the equation of motion for C_μ^A. Now we set $g^2\Phi_4 = m_4^2$. Then, from Eqs. (6.6) and (6.8), we obtain the Lagrangian

$$\frac{1}{4}(F_{\mu\nu} + H_{\mu\nu})^2 + m_4^2(C_\mu^A)^2 + \frac{1}{4}(\hat{D}_\mu X_\nu - \hat{D}_\nu X_\mu)^2 + \frac{M_4^2}{2}(X_\mu^A)^2 + \cdots, \tag{6.9}$$

where $2a_\mu^+ a_\mu^- = (X_\mu^A)^2$ has been used. Equation (6.9) is the extended QCD with the massive non-Abelian magnetic potential and the massive off-diagonal gluons.

We make a comment. Neglecting the quantum fields, the Lagrangian (6.9) becomes

$$\mathcal{L}_{\mathrm{SF}} = \frac{1}{4}H_{\mu\nu}^2 + m_4^2(C_\mu^A)^2 = \frac{1}{4g^2}(\partial_\mu \hat{n} \times \partial_\mu \hat{n})^2 + \frac{m_4^2}{g^2}(\partial_\mu \hat{n})^2. \tag{6.10}$$

This is the Skyrme–Faddeev Lagrangian [18]. The equation of motion for $\hat{n}$ comes from

$$Q^A(\hat{n})\delta\hat{n}^A = 0, \quad Q^A(\hat{n}) = \left(\frac{\delta\mathcal{L}_{\mathrm{SF}}}{\delta\hat{n}^A} - \partial_\mu \frac{\delta\mathcal{L}_{\mathrm{SF}}}{\delta\partial_\mu \hat{n}^A}\right).$$

As $\delta\hat{n}^A$ satisfies $\hat{n} \cdot \delta\hat{n} = 0$, this equation means that the component of $Q^A(\hat{n})$ that is perpendicular to $\hat{n}^A$ vanishes. So, from Eq. (6.10), we obtain

$$Q^A(\hat{n}) = \frac{1}{g^2}\partial_\mu[\partial_\nu \hat{n} \times (\partial_\mu \hat{n} \times \partial_\nu \hat{n})]^A + \frac{m_4^2}{g^2}\partial_\mu^2 \hat{n}^A, \tag{6.11}$$

and the perpendicular component gives [6]

$$-\frac{1}{g}\partial_\mu(H_{\mu\nu})\partial_\nu \hat{n} + \frac{m_4^2}{g^2}\hat{n} \times \partial_\mu^2 \hat{n} = 0. \tag{6.12}$$

Now we give an example. The color vector $\hat{n}$ in Eq. (C.7) satisfies $\partial_\mu(H_{\mu\nu}) = 0$. When $m_4^2 \neq 0$, it must be modified to satisfy Eq. (6.12). However, we find that it satisfies

$$\partial_\mu^2 \hat{n} = -\frac{2}{r^2}\hat{n} + \frac{1 - n^2}{r^2 \sin\theta}\begin{pmatrix} \cos n\phi \\ \sin n\phi \\ 0 \end{pmatrix}.$$

So, if we choose $n = \pm 1$, the equations

$$\partial_\mu (H_{\mu\nu}) = 0, \quad \hat{n} \times \partial_\mu^2 \hat{n} = 0$$

hold, and Eq. (6.12) is fulfilled. As $\int F_{\mu\nu} H_{\mu\nu} dx = -2 \int a_\nu^3 \partial_\mu (H_{\mu\nu}) dx = 0$, Eq. (6.9) becomes

$$\frac{1}{4} H_{\mu\nu}^2 + m_4^2 (C_\mu^A)^2 + \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{4} (\hat{D}_\mu X_\nu - \hat{D}_\nu X_\mu)^2 + \frac{M_4^2}{2} (X_\mu^A)^2 + \cdots .$$

7. Comment on the $D = 3$ case

In $D = 3$, ghost condensation happens as well. From Eq. (2.5), the tachyonic gluon mass term is

$$-m_3^2 \left\{ a_\mu^+ a_\mu^- + \frac{\kappa_3}{2} (a_\mu^3)^2 \right\}, \quad m_3^2 = \frac{\alpha_2}{24\pi^2} g_3^4 = \frac{\sqrt{2v_3} g_3^2}{12\pi}, \quad \kappa_3 = \frac{3}{2}.$$

As in Sect. 5, we divide the gluon as $A_\mu^A = b_\mu^A + a_\mu^A$, add the source term $M_3^2 a_\mu^+ a_\mu^-$, and introduce Φ_3 as

$$a_\mu^+ a_\mu^- \to \Phi_3 + a_\mu^+ a_\mu^-, \quad \Phi_3 = \langle a_\mu^+ a_\mu^- \rangle.$$

The terms that contain Φ_3, m_3^2, or M_3^2 are

$$V(\Phi_3) + (g_3^2 \Phi_3 - m_3^2 + M_3^2) a_\mu^+ a_\mu^- + \left(g_3^2 \Phi_3 - \frac{\kappa_3}{2} m_3^2 \right) (a_\mu^3)^2 + g_3^2 \Phi_3 \{ 2 b_\mu^3 a_\mu^3 + (b_\mu^3)^2 \}, \tag{7.1}$$

where

$$V(\Phi_3) = \frac{g_3^2}{2} \Phi_3^2 - m_3^2 \Phi_3, \quad \Phi_3 = \langle x | (-\Delta + M_3^2)^{-1}_{\mu\mu} | x \rangle.$$

The minimum of $V(\Phi_3)$ determines the VEV Φ_3 and the mass M_3 as

$$\Phi_3 = \langle a_\mu^+ a_\mu^- \rangle = \frac{m_3^2}{g_3^2} = \frac{\sqrt{2v_3}}{12\pi}, \quad \frac{m_3^2}{g_3^2} = \langle x | (-\Delta + M_3^2)^{-1}_{\mu\mu} | x \rangle. \tag{7.2}$$

Substituting Eq. (7.2) into Eq. (7.1), we find the mass terms

$$M_3^2 a_\mu^+ a_\mu^- + \frac{m_3^2}{4} (a_\mu^3)^2 + m_3^2 \{ 2 b_\mu^3 a_\mu^3 + (b_\mu^3)^2 \}. \tag{7.3}$$

Namely, the components $a_\mu^\pm$ have the mass M_3, and the classical part b_μ^A acquires the mass $\sqrt{2} m_3$. However, unlike the $D = 4$ case, the component a_μ^3 acquires the mass $m_3/\sqrt{2} = O(g_3^2)$. The value $\kappa_3 = \frac{3}{2} \neq 2$ is crucial.

As the gauge field in $D = 3$ (or $D = 2 + 1$) is expected to have masses of this order (see, e.g., Refs. [19,20]). It is known that the $D = 3 + 1$ gauge theory at high temperature becomes the effective $D = 3$ gauge theory with the coupling constant $g_3 = g\sqrt{T}$ [21]. The mass of $O(g^2 T)$, which is called the magnetic mass, is expected [22]. The mass terms $M_3^2 a_\mu^+ a_\mu^-$ and $m_3^2 (a_\mu^3)^2/4$ may imply the existence of the magnetic mass at high T.

8. Summary and comments

We have studied the SU(2) gauge theory in the nonlinear gauge. In 4D Euclidean space, the ghost condensation $\langle\varphi^A\rangle = \varphi_0\delta^{A3} \neq 0$ happens under the scale μ_0, and the ghost loop yields the tachyonic gluon mass term $-m_4^2\{A_\mu^+A_\mu^- + (A_\mu^3)^2\}$. We considered the effective potential for the LCO $A_\mu^+A_\mu^-$ up to $O(\hbar^2)$, and showed that $g^2\langle A_\mu^+A_\mu^-\rangle = m_4^2$ gives the minimum of the potential. The VEV $\Phi_4 = \langle A_\mu^+A_\mu^-\rangle$ makes the diagonal gluon A_μ^3 massless. In contrast, the off-diagonal gluons $A_\mu^\pm$ acquire the mass M_4 determined by Eq. (3.10).

The field φ^A belongs to the adjoint representation of SU(2). Since the VEV $\varphi_0\delta^{A3}$ selects the $A = 3$ direction, the Abelian monopole solution $\tilde{C}_\mu$ can be introduced in this direction. Combining the VEV $\langle A_\mu^+A_\mu^-\rangle$ and $\tilde{C}_\mu$, we find that the magnetic potential $\tilde{C}_\mu$ becomes massive.

To remove the string singularity of the monopole solution $\tilde{C}_\mu$, we performed the singular gauge transformation in the background covariant gauge. As $\varphi_0\delta^{A3}$ transforms into $\varphi_0\hat{n}^A$, the field $\hat{n}^A$ that specifies the color direction appears, and the extended QCD [6] is derived. In addition, the non-Abelian magnetic potential C_μ^A transformed from $\tilde{C}_\mu\delta^{A3}$ becomes massive. If quantum fields are neglected, the Skyrme–Faddeev Lagrangian is obtained.

In 3D Euclidean space, phenomena similar to the $D = 4$ case happen. However, unlike the $D = 4$ case, the diagonal component a_μ^3 becomes massive.

We make some comments.

(1) In the maximally Abelian gauge, the lattice simulation shows that the off-diagonal gluons are massive and the diagonal gluons are nearly massless [23,24]. In the present model, the VEV φ_0 breaks the global SU(2) symmetry to U(1), so the result that $A_\mu^\pm$ and A_μ^3 have different masses may be reasonable. In addition, the masslessness of the component A_μ^3 may be related to the remaining U(1) symmetry.

(2) Since the magnetic potential $\tilde{C}_\mu$ couples with the off-diagonal components $A_\mu^\pm$, $\tilde{C}_\mu$ can become massive by the VEV $\langle A_\mu^+A_\mu^-\rangle$. Namely we can obtain a massive magnetic potential without monopole condensation.

(3) To show the quark confinement, Abelian dominance is often assumed [25,26]. In this assumption, the off-diagonal components $A_\mu^\pm$ are neglected. Our result insists that these components play an important role in realizing the Abelian dominance.[5]

(4) In Ref. [27], based on the LCO formalism in Refs. [28,29], the LCO $(\bar{c}\times c)^A$ is studied in the Landau gauge, together with the LCO $(A_\mu^A)^2 = (A_\mu^a)^2 + (A_\mu^3)^2$. This formalism introduces sources ω^A and J for $\bar{c}\times c$ and $(A_\mu^A)^2$, respectively, in a BRS-exact form. From the renormalizability, the terms $(\omega^A)^2$ and J^2 are necessary. To delete these terms, the Hubbard–Stratonovich transformation is applied, and the fields ϕ^A and σ, which correspond to $\bar{c}\times c$ and $(A_\mu^A)^2$, respectively, are introduced. The final Lagrangian they used is $\mathcal{L} = \mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{ext}}$, where

$$\mathcal{L}_{\mathrm{YM}} = \frac{1}{4}(F_{\mu\nu}^A)^2, \quad \mathcal{L}_{\mathrm{GF}} = B^A(\partial_\mu A_\mu^A),$$
$$\mathcal{L}_{\mathrm{ext}} = \mathcal{L}_{\mathrm{ext}}(\phi) + \mathcal{L}_{\mathrm{ext}}(\sigma),$$

[5] The components $\rho d\hat{n}$ and $\sigma d\hat{n}\times\hat{n}$ in Ref. [18] correspond to $A_\mu^\pm$ in this paper. Condensations related to ρ and σ are discussed in Ref. [18].

$$\mathcal{L}_{\rm ext}(\phi) = \frac{(\phi^A)^2}{2g^2\rho} + \frac{1}{\rho}\phi^A(\bar{c}\times c)^A + \frac{g^2}{2\rho}(\bar{c}\times c)^2 - \omega^A\frac{\phi^A}{g},$$
$$\mathcal{L}_{\rm ext}(\sigma) = \frac{\sigma}{2g^2\zeta} + \frac{\sigma}{2g\zeta}(A^A_\mu)^2 + \frac{1}{8\zeta}(A^A_\mu A^A_\mu)^2 - J\frac{\sigma}{g}.$$

The one-loop effective potential $V(\phi)+V(\sigma)$ was calculated by using the free propagators derived from $\mathcal{L}_{\rm YM}+\mathcal{L}_{\rm GF}$ and the interactions in $\mathcal{L}_{\rm ext}$. Then the conditions $\frac{dV(\phi)}{d\phi}=0$ and $\frac{dV(\sigma)}{d\sigma}=0$ give the VEVs $\langle\phi^3\rangle = -g^2\langle(\bar{c}\times c)^3\rangle$ and $\langle\sigma\rangle = -\frac{g}{2}\langle(A^A_\mu)^2\rangle$, respectively.

When the VEV $\langle(\bar{c}\times c)^3\rangle$ exists, the ghost loop yields the tachyonic gluon masses. As in Sect. 2, the magnitude for A^3_μ is twice that for $A^\pm_\mu$. In the presence of the VEV $\langle(A^A_\mu)^2\rangle$, gluons acquire a mass. The latter mass compensates the former tachyonic masses. Thus gluons have the usual masses, and the mass for A^3_μ is different from that for $A^\pm_\mu$.

In their approach, the interactions that yield these VEVs are introduced irrelevant to the original Lagrangian $\mathcal{L}_{\rm YM}+\mathcal{L}_{\rm GF}$. As a result, the VEV $\langle(A^A_\mu)^2\rangle$ appears irrespective of the VEV $\langle(\bar{c}\times c)^3\rangle$.

(5) In our procedure, we need the action in the nonlinear gauge. The quartic ghost interaction yields the VEV $\langle(\bar{c}\times c)^3\rangle$, and the ghost loop brings about the tachyonic gluon masses. The VEV $\langle A^+_\mu A^-_\mu\rangle = \frac{1}{2}\langle A^a_\mu A^a_\mu\rangle$ appears to eliminate the tachyonic mass for $A^\pm_\mu$.

In the Landau gauge, there is no quartic ghost interaction. However, in the low energy region, gauge field configurations on the Gribov horizon contribute to the partition function Z. These configurations give rise to zero-modes of the ghost operator $\partial_\mu D_\mu$, and make Z vanish.

However, these zero-modes can produce the quartic ghost interaction [10]. The Landau gauge changes automatically to the nonlinear gauge. Thus, even if we start from the Landau gauge, effective quartic ghost interaction is produced, and we can apply the scenario proposed in this article.

We note that the scale μ_0 in Eq. (2.4) depends on the parameter α_2 at the cut-off scale Λ. One-loop calculation shows that α_2 has the ultraviolet fixed point $\alpha_2 = \beta_0/2$, where β_0 is the coefficient of the beta function $\beta = -\frac{\beta_0}{16\pi^2}g^3$. By substituting $\alpha_2 = \beta_0/2$ into μ_0, we find that μ_0 coincides with the QCD scale parameter $\Lambda_{\rm QCD}$ [8].

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Appendix A. $\langle(A^3_\mu)^2\rangle = 0$

To eliminate the tachyonic mass terms

$$-m_D^2 A^+_\mu A^-_\mu - m_D^2\frac{\kappa_D}{2}(A^3_\mu)^2, \tag{A.1}$$

the VEV $\langle A^+_\mu A^-_\mu\rangle$ is necessary. In this appendix, we show, in contrast with $\langle A^+_\mu A^-_\mu\rangle$, the VEV $\langle(A^3_\mu)^2\rangle$ vanishes.

To apply the procedure in Sect. 3.2, we substitute

$$A^+_\mu A^-_\mu \to \Phi_D + A^+_\mu A^-_\mu,\ \Phi_D = \langle A^+_\mu A^-_\mu\rangle, \quad (A^3_\mu)^2 \to \Psi_D + (A^3_\mu)^2,\ \Psi_D = \langle(A^3_\mu)^2\rangle \tag{A.2}$$

into Eqs. (3.5) and (A.1). Then the part that contains Φ_D or Ψ_D is given by

$$V(\Phi_D,\Psi_D) = V_1 + V_2, \quad V_1 = \frac{g^2}{2}\Phi_D^2 - m_D^2\Phi_D, \quad V_2 = \left(g^2\Phi_D - m_D^2\frac{\kappa_D}{2}\right)\Psi_D.$$

If we set $g^2\Phi_D = \lambda m_D^2$, we obtain

$$V(\Phi_D, \Psi_D) = V_1(\lambda) + V_2(\lambda), \quad V_1(\lambda) = \frac{m_D^4}{2g^2}\lambda(\lambda - 2), \quad V_2(\lambda) = m_D^2\left(\lambda - \frac{\kappa_D}{2}\right)\Psi_D. \tag{A.3}$$

The potential V_1 has the minimum value $-\frac{m_D^4}{2g^2}$ at $\lambda = 1$. As $\Psi_D \geq 0$, and V_2 is linear with respect to Ψ_D, the coefficient must satisfy $(g^2\Phi_D - \kappa_D m_D^2/2) \geq 0$. Thus we find that

$$V_2(\lambda) = m_D^2\left(\lambda - \frac{\kappa_D}{2}\right)\Psi_D \geq 0. \tag{A.4}$$

Since the local minimum condition $\frac{\delta V}{\delta\Phi_D} = 0$ leads to

$$g^2\Phi_D + g^2\Psi_D - m_D^2 = (\lambda - 1)m_D^2 + g^2\Psi_D = 0, \tag{A.5}$$

from Eqs. (A.4) and (A.5), we obtain

$$\frac{\kappa_D}{2} \leq \lambda \leq 1.$$

Now we show that $\Psi_D = 0$. When $D = 4$, as $\kappa_4 = 2$, V_1 and V_2 have their minimum values at $\lambda = 1$, and Eq. (A.5) leads to $\Psi_4 = 0$. When $D = 3$, there are two possibilities to minimize V_2. As $\kappa_3 = 3/2$, one way is to choose $\lambda = 3/4$. However, since $V_1(\lambda = 3/4) > V_1(\lambda = 1)$, V_1 does not have a minimum value. Another way is to set $\Psi_3 = 0$. Then Eq. (A.5) gives $\lambda = 1$, and V_1 is minimized.

We make a comment. Historically, the VEV $\langle (A_\mu^A)^2\rangle$ was discussed in the framework of the operator product expansion (OPE) [30,31], and the relation with monopoles was considered [32]. When $k_\mu \to \infty$, the OPE gives [30]

$$\langle T(A_\nu^B(-k)A_\rho^C(k))\rangle = (c_0)_{\nu\rho}^{BC}(k) + (c_2)_{\nu\rho}^{BC}(k)\langle : (A_\mu^A)^2 : \rangle + \cdots .$$

However, the VEV $\langle : (A_\mu^A)^2 : \rangle$ is different from $\Phi_D = \langle A_\mu^+ A_\mu^- \rangle$, because Φ_D appears as $k_\mu \to 0$, and the component A_μ^3 does not condense. To make $A_\mu^\pm$ massive, the VEVs $\langle A_\mu^+ A_\mu^- \rangle$ and $\langle \bar{c}^a c^a \rangle$ were considered in Ref. [33]. In our approach, although $A_\mu^\pm$ becomes massive, the significant role of Φ_D is to remove the tachyonic masses.

Appendix B. Derivation of Eq. (5.7)

We calculate the ghost determinant (5.6) given by

$$\det[D_\mu(b)D_\mu(A) + v\times] = \exp[-\int dx V_{\mathrm{gh}}],$$

$$V_{\mathrm{gh}} = -\mathrm{tr}\int \frac{d^Dk}{(2\pi)^D}\ln[D_\mu(b)D_\mu(A) + v\times],$$

where the classical part is $b_\mu^A = b_\mu^3\delta^{A3}$, and $A_\mu^A = b_\mu^A + a_\mu^A$. Up to the second order of the fields, we find

$$\begin{aligned} V_{\mathrm{gh}} = -\int \frac{d^Dk}{(2\pi)^D}\ln[&k^4 + v^2 + 2vigk_\mu(A_\mu^3 + b_\mu^3) \\ &- g^2k_\mu k_\nu\{A_\mu^a A_\nu^a + (A_\mu^3 + b_\mu^3)(A_\nu^3 + b_\nu^3)\} + 2g^2k^2A_\mu^3 b_\mu^3 + \cdots]. \end{aligned} \tag{B.1}$$

The term $-g^2 k_\mu k_\nu \{A^a_\mu A^a_\nu + (A^3_\mu + b^3_\mu)(A^3_\nu + b^3_\nu)\}$ in Eq. (B.1) gives

$$-\int \frac{d^D k}{(2\pi)^D} \frac{-g^2 k_\mu k_\nu}{k^4 + v^2} \{A^a_\mu A^a_\nu + (A^3_\mu + b^3_\mu)(A^3_\nu + b^3_\nu)\}.$$

Using

$$\int \frac{d^4 k}{(2\pi)^4} \frac{k_\mu k_\nu}{k^4 + v^2} = -\frac{\delta_{\mu\nu}}{2} \frac{v}{64\pi}, \quad \int \frac{d^4 k}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^4 + v^2)^2} = \frac{\delta_{\mu\nu}}{4} \frac{v}{64\pi},$$

we obtain [7]

$$V_{\mathrm{gh}(1)} = \frac{1}{2} \left(-\frac{g^2 v}{64\pi} \right) \{(A^a_\mu)^2 + (A^3_\mu + b^3_\mu)^2\}.$$

In the same way, the term $2vigk_\mu (A^3_\mu + b^3_\mu)$ in Eq. (B.1) gives

$$V_{\mathrm{gh}(2)} = -\int \frac{d^D k}{(2\pi)^D} \frac{2g^2 v^2 k_\mu k_\nu}{(k^4 + v^2)^2} (A^3_\mu + b^3_\mu)(A^3_\nu + b^3_\nu) = \frac{1}{2} \left(-\frac{g^2 v}{64\pi} \right) (A^3_\mu + b^3_\mu)^2,$$

and the term $2k^2 g^2 A^3_\mu b^3_\mu$ in Eq. (B.1) gives

$$V_{\mathrm{gh}(3)} = -\int \frac{d^D k}{(2\pi)^D} \frac{2g^2 k^2}{k^4 + v^2} A^3_\mu b^3_\mu = \frac{4g^2 v}{64\pi} A^3_\mu b^3_\mu.$$

Thus, substituting $A^a_\mu = a^a_\mu$ and $A^3_\mu = b^3_\mu + a^3_\mu$, we obtain

$$V_{\mathrm{gh}} = \sum_{n=1}^{3} V_{\mathrm{gh}(n)} = \frac{1}{2} \left(-\frac{g^2 v}{64\pi} \right) \{(a^a_\mu)^2 + 2(a^3_\mu)^2\}.$$

Appendix C. Singular gauge transformation

Following Ref. [4], we consider the gauge transformation with the matrix

$$U = e^{i\gamma T_3} e^{i\alpha T_2} e^{i\beta T_3},$$

where $(T_B)_{AC} = if_{ABC}$. This matrix U satisfies

$$\frac{\langle \varphi \rangle}{\varphi_0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = U\hat{n}, \quad \hat{n} = \begin{pmatrix} \sin\alpha \cos\beta \\ \sin\alpha \sin\beta \\ \cos\alpha \end{pmatrix} \tag{C.1}$$

and $U^\dagger T_A U = T_B \hat{n}^B_A$, where

$$\hat{n}_1 = \begin{pmatrix} \cos\alpha \cos\beta \cos\gamma - \sin\beta \sin\gamma \\ \cos\alpha \sin\beta \cos\gamma + \cos\beta \sin\gamma \\ -\sin\alpha \cos\gamma \end{pmatrix}, \hat{n}_2 = \begin{pmatrix} -\cos\alpha \cos\beta \sin\gamma - \sin\beta \cos\gamma \\ -\cos\alpha \sin\beta \sin\gamma + \cos\beta \cos\gamma \\ \sin\alpha \sin\gamma \end{pmatrix}, \hat{n}_3 = \hat{n}. \tag{C.2}$$

These color vectors satisfy the orthonormality

$$\hat{n}^C_A \hat{n}^C_B = \delta_{AB}.$$

Under this transformation, the gauge field transforms as

$$\begin{aligned} A'^{A}_{\mu} T_A &= U^{\dagger} A^A_{\mu} T_A U + \frac{i}{g} U^{\dagger} \partial_{\mu} U \\ &= \left[A^1_{\mu} \hat{n}^A_1 + A^2_{\mu} \hat{n}^A_2 + A^3_{\mu} \hat{n}^A - \frac{1}{g} \left\{ (\hat{n} \times \partial_{\mu} \hat{n})^A + (\cos\alpha\, \partial_{\mu}\beta + \partial_{\mu}\gamma) \hat{n}^A \right\} \right] T_A. \end{aligned} \tag{C.3}$$

If we write

$$A^A_{\mu} = b^A_{\mu} + a^A_{\mu}, \quad b^A_{\mu} = \frac{1}{g} (\cos\alpha\, \partial_{\mu}\beta + \partial_{\mu}\gamma) \delta^{A3}, \tag{C.4}$$

Eq. (C.3) becomes

$$A'^{A}_{\mu} T_A = \left[a^1_{\mu} \hat{n}^A_1 + a^2_{\mu} \hat{n}^A_2 + a^3_{\mu} \hat{n}^A - \frac{1}{g} \left\{ (\hat{n} \times \partial_{\mu} \hat{n})^A \right\} \right] T_A. \tag{C.5}$$

If we regard the transformation (C.5) as the background gauge transformation

$$U^{\dagger} b_{\mu} U + \frac{i}{g} U^{\dagger} \partial_{\mu} U, \quad U^{\dagger} a_{\mu} U,$$

Eq. (C.5) implies that a^A_{μ} and b^A_{μ} transform as

$$a^A_{\mu} \to a^B_{\mu} \hat{n}^A_B, \quad b^A_{\mu} \to -\frac{1}{g} (\hat{n} \times \partial_{\mu} \hat{n})^A. \tag{C.6}$$

Now, using the spherical coordinates (r, θ, ϕ) and integer n, we set the angles as $\alpha = \theta, \beta = -\gamma = n\phi$. Then b^A_{μ} and $\hat{n}^A$ become

$$b^A_{\mu} = \tilde{C}_{\mu} \delta^{3A}, \quad \hat{n} = \begin{pmatrix} \sin\theta \cos n\phi \\ \sin\theta \sin n\phi \\ \cos\theta \end{pmatrix}, \tag{C.7}$$

where $\tilde{C}_{\mu}$ is the Abelian monopole in Eq. (4.1). The corresponding non-Abelian monopole is

$$C^A_{\mu} = -\frac{1}{g} (\hat{n} \times \partial_{\mu} \hat{n})^A = \frac{1}{g} \begin{pmatrix} \sin n\phi\, \partial_{\mu}\theta + \sin\theta \cos\theta \cos n\phi\, \partial_{\mu}(n\phi) \\ -\cos n\phi\, \partial_{\mu}\theta + \sin\theta \cos\theta \sin n\phi\, \partial_{\mu}(n\phi) \\ -\sin^2\theta\, \partial_{\mu}(n\phi) \end{pmatrix}. \tag{C.8}$$

The field strength $H_{\mu\nu}$ is defined in Eq. (6.7). If we use Eq. (C.8), it becomes

$$H_{\mu\nu} = -\frac{1}{g} \sin\theta \{ \partial_{\mu}\theta\, \partial_{\nu}(n\phi) - \partial_{\nu}\theta\, \partial_{\mu}(n\phi) \},$$

and it satisfies $\partial_{\mu} H_{\mu\nu} = 0$. In the same way, $\hat{n}$ in Eq. (C.7) satisfies

$$\partial^2_{\mu} \hat{n} = -\frac{2}{r^2} \hat{n} + \frac{1 - n^2}{r^2 \sin\theta} \begin{pmatrix} \cos n\phi \\ \sin n\phi \\ 0 \end{pmatrix}.$$

Appendix D. BRS symmetry and global gauge symmetry

In this appendix, for the sake of explanation, we use the operator formalism [34]. The BRS transformation is δ_B and the BRS charge is Q_B. A state $|\mathrm{phys}\rangle$ in the physical subspace satisfies the condition $Q_B|\mathrm{phys}\rangle = 0$.

D.1. *BRS symmetry*

First we show that the constant w in Eq. (2.2) must be chosen as $w^A = \varphi_0\delta^{A3}$ to preserve the BRS symmetry. The Lagrangian (2.2) is invariant under the BRS transformation

$$\delta_B A_\mu = D_\mu c,\ \delta_B c = -\frac{g}{2} c \times c,\ \delta_B \bar{c} = iB,\ \delta_B \varphi = g\varphi \times c,\ \delta_B w = 0.$$

Using the equation of motion for B^A, we obtain

$$\langle 0|\alpha_1 B^A|0\rangle = \varphi_0\delta^{A3} - w^A, \tag{D.1}$$

where $\langle 0|A_\mu|0\rangle = 0$ and $\langle 0|\varphi^A|0\rangle = \varphi_0\delta^{A3}$ have been used. As $B = -i\delta_B\bar{c}$, we must set $w^A = \varphi_0\delta^{A3}$ to obtain $\langle 0|\delta_B\bar{c}|0\rangle = \langle 0|\{iQ_B, \bar{c}\}|0\rangle = 0$.

Next we consider the VEV $\langle 0|A^+_\mu A^-_\mu|0\rangle$. Since the operator $A^+_\mu A^-_\mu$ satisfies

$$\delta_B(A^+_\mu A^-_\mu) = A^a_\mu (D_\mu c)^a \neq 0, \tag{D.2}$$

it is not BRS-invariant. If there is an operator Ω that satisfies $\delta_B\Omega = A^+_\mu A^-_\mu$, the BRS symmetry is broken spontaneously by the VEV $\langle 0|A^+_\mu A^-_\mu|0\rangle$. However, as $\delta_B^2 = 0$, Eq. (D.2) implies that such an operator Ω does not exist.[6] So the VEV $\langle 0|A^+_\mu A^-_\mu|0\rangle \neq 0$ does not contradict the BRS invariance of the vacuum.

We make two comments. First, since there is no ghost-number-violating interaction nor such a condensate, the ghost number should be conserved. Therefore, although $\delta_B A^+_\mu A^-_\mu \neq 0$, its VEV satisfies

$$\delta_B\langle 0|A^+_\mu A^-_\mu|0\rangle = \langle 0|A^a_\mu (D_\mu c)^a|0\rangle = 0.$$

Second, the anti-BRS symmetry is broken in this model [10]. However, the unitarity of the model is guaranteed by the BRS symmetry.

D.2. *Global SU(2) symmetry*

Using the constant small parameter θ, the global color transformation is defined by $\delta_\theta\Sigma = \theta \times \Sigma$, where Σ represents all the fields in $\mathcal{L}_\varphi$. Since $\langle 0|\delta_\theta\varphi^A|0\rangle = f^{AB3}\theta^B\varphi_0$, this symmetry breaks down spontaneously to U(1), and the fields $\varphi^\pm$ are Goldstone bosons. In addition, as $\delta_\theta\mathcal{L}_\varphi = -w\cdot(\theta \times B)$, the nonzero constant w breaks this symmetry at the Lagrangian level. However, as $B = -i\delta_B\bar{c}$, this breaking term $\delta_\theta\mathcal{L}_\varphi = -i(w\times\theta)\cdot\delta_B\bar{c}$ is BRS-exact. Since the physical states satisfy $Q_B|\mathrm{phys}\rangle = 0$, we find that

$$\langle \mathrm{phys}_2|\delta_\theta\mathcal{L}_\varphi|\mathrm{phys}_1\rangle = (w \times \theta)\cdot\langle \mathrm{phys}_2|\{Q_B, \bar{c}\}|\mathrm{phys}_1\rangle = 0.$$

Thus the breaking term does not contribute to amplitudes between physical states [10].

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[6] In the case of φ^A, the equation of motion for B^A gives Eq. (D.1). However, in the case of $\langle 0|A^+_\mu A^-_\mu|0\rangle$, any colored fields do not give the equation of motion with $(A^a_\mu)^2$.

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