

# $\eta$ -symbols in exceptional field theory

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We present the universal form of  $\eta$ -symbols that can be applied to an arbitrary  $E_{d(d)}$  exceptional field theory (EFT) up to  $d = 7$ . We then express the  $Y$ -tensor, which governs the gauge algebra of EFT, as a quadratic form of the  $\eta$ -symbols. The usual definition of the  $Y$ -tensor strongly depends on the dimension of the compactification torus while it is not the case for our  $Y$ -tensor. Furthermore, using the  $\eta$ -symbols, we propose a universal form of the linear section equation. In particular, in the  $SL(5)$  EFT, we explicitly show the equivalence to the known linear section equation.  
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## 1. Introduction

In double field theory (DFT) (Refs. [1–7]), for the purpose of the manifest  $T$ -duality covariance, we consider a  $2d$ -dimensional doubled space with the generalized coordinates  $x^I$  ( $I = 1, \dots, 2d$ ). In order to make contact with the conventional supergravity in  $d$ -dimensions, it is useful to decompose the generalized coordinates into the physical coordinates  $x^i$  ( $i = 1, \dots, d$ ) and the dual coordinates  $\tilde{x}_i$ ;  $(x^I) = (x^i, \tilde{x}_i)$ . By introducing the  $O(d, d)$   $T$ -duality-invariant metric,

$$(\eta_{IJ}) = \begin{pmatrix} 0 & \delta_j^i \\ \delta_i^j & 0 \end{pmatrix}, \quad (\eta^{IJ}) = \begin{pmatrix} 0 & \delta_j^i \\ \delta_i^j & 0 \end{pmatrix}, \quad (1.1)$$

the consistency condition of DFT, the so-called the section condition, is expressed as

$$\eta^{IJ} \partial_I \otimes \partial_J = 0. \quad (1.2)$$

Here,  $\otimes$  represents that

$$\eta^{IJ} \partial_I \partial_J A = 0, \quad \eta^{IJ} \partial_I A \partial_J B = 0 \quad (1.3)$$

are satisfied for arbitrary fields or gauge parameters  $A$  and  $B$ . Under the section condition, the gauge algebra generated by the following generalized Lie derivative is closed:

$$\hat{\mathcal{L}}_V W^I \equiv V^J \partial_J W^I - W^J \partial_J V^I + \eta^{IJ} \eta_{KL} \partial_J V^K W^L. \quad (1.4)$$

As a natural generalization of DFT, the  $E_{d(d)}$  exceptional field theories (EFTs) (Refs. [8–16]) have been formulated in a manifestly  $E_{d(d)}$   $U$ -duality covariant manner (see Refs. [17–20] for the

initial attempts). In EFT, the generalized coordinates  $x^I$  ( $I = 1, \dots, D$ ) are defined to transform in a fundamental representation, called the  $R_1$ -representation (see Appendix A.2). The generalized Lie derivative is defined by

$$\hat{\mathfrak{L}}_V W^I \equiv V^J \partial_J W^I - W^J \partial_J V^I + Y_{KL}^{IJ} \partial_J V^K W^L, \quad (1.5)$$

where the  $Y$ -tensor  $Y_{KL}^{IJ}$  for each  $d$  ( $4 \leq d \leq 7$ ) is given as follows (Ref. [11]; see Refs. [21–23] for the generalized Lie derivative in the context of exceptional generalized geometry):

| $E_{d(d)}$     | SL(5)                                             | SO(5, 5)                                                                       | $E_{6(6)}$           | $E_{7(7)}$                                             |       |
|----------------|---------------------------------------------------|--------------------------------------------------------------------------------|----------------------|--------------------------------------------------------|-------|
| $D = \dim R_1$ | 10                                                | 16                                                                             | 27                   | 56                                                     | (1.6) |
| $Y_{KL}^{IJ}$  | $\epsilon^{\mathbf{e}IJ} \epsilon_{\mathbf{e}KL}$ | $\frac{1}{2} \gamma_{\mathbf{A}}^{IJ} \bar{\gamma}_{\mathbf{KL}}^{\mathbf{A}}$ | $10 d^{IJM} d_{KLM}$ | $12 c^{IJ}_{KL} + \frac{1}{2} \Omega^{IJ} \Omega_{KL}$ |       |

Here, e.g.,  $\gamma_{\mathbf{A}}^{IJ}$  is the gamma matrix for the SO(5, 5) group and  $d^{IJK}$  is the totally symmetric tensor intrinsic to the  $E_{6(6)}$  group (see Appendix B for the details of these  $d$ -dependent tensors). The gauge algebra of the generalized diffeomorphism is closed if the following section conditions are satisfied (Refs. [11,22]):

$$\begin{aligned} d \leq 6 : \quad & Y_{KL}^{IJ} \partial_I \otimes \partial_J = 0, \\ d = 7 : \quad & Y_{KL}^{IJ} \partial_I \otimes \partial_J = 0, \quad \Omega^{IJ} \partial_I \otimes \partial_J = 0, \end{aligned} \quad (1.7)$$

where  $\Omega^{IJ}$  is the antisymmetric tensor intrinsic to the  $E_{7(7)}$  group. Under the section condition, all fields can depend on at most  $d$  coordinates (see Ref. [24] for a proof in the  $E_{7(7)}$  EFT).

In the above conventional formulation, the  $Y$ -tensor and the section condition strongly depend on the dimension  $d$ , and when we consider applications of the  $E_{d(d)}$  EFT, we need to specify the dimension  $d$  explicitly. A hypothetical “underlying EFT” (or 11D EFT), which reproduces all  $E_{d(d)}$  EFTs ( $d \leq 8$ ) from simple truncations, has been proposed in Ref. [25], but the program has not been completed yet. In this paper, we investigate such uniform formulations from a different approach. In our approach, the  $Y$ -tensor is expressed in terms of SL( $d$ ) [or SL( $d - 1$ )] tensors and  $E_{d(d)}$  tensors are not used. Accordingly, the truncation to lower  $d$  can be easily performed.

The present paper is organized as follows. In Sect. 2, we introduce  $\eta$ -symbols as a natural generalization of the O( $d, d$ )-invariant metric  $\eta_{IJ}$  in DFT and explain how the  $\eta$ -symbols are related to branes in M-theory/type IIB theory. The  $Y$ -tensor is expressed by using the  $\eta$ -symbols and the  $\Omega$ -tensor. In Sect. 3, we find the explicit form of the  $\eta$ -symbols and the  $\Omega$ -tensor. In Sect. 4, we show the explicit form of the section condition and the generalized Lie derivative. In Sect. 5, we propose a new linear section equation, and show that it reproduces the known linear section equation (Ref. [11]) in the case of the SL(5) EFT. Section 6 is devoted to conclusions and discussion.

## 2. A sketch of the basic idea

The section condition in DFT has been proposed on the basis of the level-matching constraint in string sigma model (Refs. [1–3]),

$$S = -\frac{1}{4\pi\alpha'} \int_{\Sigma} \sqrt{-\gamma} d^2\sigma (G_{ij} \gamma^{\bar{A}\bar{B}} + B_{ij} \epsilon^{\bar{A}\bar{B}}) \partial_{\bar{A}} X^i \partial_{\bar{B}} X^j \quad (\bar{A}, \bar{B} = \tau, \sigma). \quad (2.1)$$

In the canonical formulation, the level-matching constraint, or the momentum constraint  $\mathcal{H}_\sigma = 0$ , can be expressed as

$$\mathcal{H}_\sigma = P_i \partial_\sigma X^i = \frac{1}{4\pi\alpha'} \eta^{IJ} Z_I Z_J = 0, \quad (2.2)$$

where  $P_i(\sigma)$  are the conjugate momenta to  $X^i(\sigma)$  and  $Z_I(\sigma)$  are the generalized momenta,

$$Z_I(\sigma) = \left( \frac{2\pi\alpha' P_i(\sigma)}{\partial_\sigma X^i(\sigma)} \right). \quad (2.3)$$

By supposing that the operator

$$\mathbb{L}_V \equiv \int d\sigma V^I(X^J(\sigma)) Z_I(\sigma) \quad (2.4)$$

acts as the generator of the diffeomorphism along  $V^I \partial_I$ , we can roughly identify  $Z_I$  with  $\partial_I$ , and the momentum constraint corresponds to the section condition in DFT,  $\eta^{IJ} \partial_I \otimes \partial_J = 0$ .

A similar consideration has been given for M-theory branes, in Refs. [26,27]. In the case of an M2-brane wrapped on a 4-torus, the momentum constraint  $\mathcal{H}_A = 0$  ( $A = 1, 2$ : index for spatial coordinates on the M2-brane) is rewritten as

$$\eta^{IJ;k} Z_I Z_J = 0, \quad (2.5)$$

where

$$\eta^k \equiv (\eta^{IJ;k}) \equiv \begin{pmatrix} 0 & \frac{2! \delta_{j_1 j_2}^{k i}}{\sqrt{2!}} \\ \frac{2! \delta_{i_1 i_2}^{k j}}{\sqrt{2!}} & 0 \end{pmatrix}, \quad (Z_I) \equiv \begin{pmatrix} P_i \\ \frac{\frac{1}{2} \epsilon^{AB} \partial_A X^{[i_1} \partial_B X^{i_2]}}{\sqrt{2!}} \end{pmatrix}. \quad (2.6)$$

Again by supposing the generalized momenta  $Z_I$  to act as  $\partial_I \equiv \partial/\partial x^I$  with  $(x^I) = (x^i, \frac{y_{i_1 i_2}}{\sqrt{2!}})$ , the momentum constraint is expressed as the section condition,

$$\eta^{IJ;k} \partial_I \otimes \partial_J = 0. \quad (2.7)$$

Similarly, in the case of an M5-brane wrapped on a 5-torus, the momentum constraint,  $\mathcal{H}_A = 0$  ( $A = 1, \dots, 5$ ), has been expressed in a bilinear form (see Ref. [27] for the details),

$$a_k \eta^{IJ;k} Z_I Z_J + b_{k_1 \dots k_4} \eta^{IJ; k_1 \dots k_4} Z_I Z_J = 0, \quad (2.8)$$

where the matrices  $\eta^k \equiv (\eta^{IJ;k})$  and  $\eta^{k_1 \dots k_4} \equiv (\eta^{IJ; k_1 \dots k_4})$  have the form

$$\eta^k \equiv \begin{pmatrix} 0 & \frac{2! \delta_{j_1 j_2}^{k i}}{\sqrt{2!}} & 0 \\ \frac{2! \delta_{i_1 i_2}^{k j}}{\sqrt{2!}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \eta^{k_1 \dots k_4} \equiv \begin{pmatrix} 0 & 0 & \frac{5! \delta_{j_1 \dots j_5}^{i k_1 \dots k_4}}{\sqrt{5!}} \\ 0 & \frac{4! \delta_{i_1 i_2 j_1 j_2}^{k_1 \dots k_4}}{\sqrt{2!} 2!} & 0 \\ \frac{5! \delta_{i_1 \dots i_5}^{j k_1 \dots k_4}}{\sqrt{5!}} & 0 & 0 \end{pmatrix}. \quad (2.9)$$

Here,  $a_k$  and  $b_{k_1 \dots k_4}$  ( $= b_{[k_1 \dots k_4]}$ ) are arbitrary constants and the section conditions can be decomposed into two parts,

$$\eta^{IJ;k} \partial_I \otimes \partial_J = 0, \quad \eta^{IJ; k_1 \dots k_4} \partial_I \otimes \partial_J = 0. \quad (2.10)$$

The former condition is the same as the section condition coming from the M2-brane and the latter is intrinsic to the M5-brane.

A similar consideration for a D $p$ -brane in type II string theory was made in Ref. [28] (though the  $U$ -duality covariance is not manifest there), and the general rule we observe is that each  $p$ -brane provides the corresponding  $\eta$ -symbol  $\eta^{k_1 \cdots k_{p-1}}$  and the associated section condition  $\eta^{IJ; k_1 \cdots k_{p-1}} \partial_I \otimes \partial_J = 0$ . In fact, a set of multiple indices with one dimension fewer in the spatial dimension of branes is known to form the string multiplet of  $E_{d(d)}$  group. The dimension of the string multiplet for each  $U$ -duality group is given as follows (see Ref. [29] for a concise review):

| Duality group             | SL(5) | SO(5, 5) | $E_{6(6)}$ | $E_{7(7)}$ |
|---------------------------|-------|----------|------------|------------|
| Dimension of string mult. | 5     | 10       | 27         | 133        |

(2.11)

As is clear from the dimension, the string multiplet is the same as the  $R_2$ -representation that determines the section condition (Ref. [29]; see also Appendix A.2). Now, it is natural to expect that each brane in the string multiplet provides a particular  $\eta$ -symbol and the corresponding section condition, and the sum of all these section conditions is equivalent to the section condition in the  $E_{d(d)}$  EFT. We thus introduce the following set of  $\eta$ -symbols associated with branes in the string multiplet in M-theory and type IIB theory:

$$\begin{aligned}
 (\eta^{\text{I}}) &= \left( \underbrace{\eta^k}_{\text{M2}}, \underbrace{\frac{\eta^{k_1 \cdots k_4}}{\sqrt{4!}}}_{\text{M5}}, \underbrace{\frac{\eta^{k_1 \cdots k_6, l}}{\sqrt{6!}}}_{\text{KKM/8}}, \underbrace{\frac{\eta^{k_1 \cdots k_7, l_1 l_2 l_3}}{\sqrt{7! 3!}}}_{5^3}, \underbrace{\frac{\eta^{k_1 \cdots k_7, l_1 \cdots l_6}}{\sqrt{7! 6!}}}_{2^6}, \dots \right), \\
 (\eta^{\text{M}}) &= \left( \underbrace{\eta_\alpha}_{\text{F1/D1}}, \underbrace{\frac{\eta^{m_1 m_2}}{\sqrt{2!}}}_{\text{D3}}, \underbrace{\frac{\eta^{m_1 \cdots m_4}}{\sqrt{4!}}}_{\text{NS5/D5}}, \underbrace{\frac{\eta^{m_1 \cdots m_5, n}}{\sqrt{5!}}}_{\text{KKM/7}_2}, \underbrace{\frac{\eta^{m_1 \cdots m_6}}{\sqrt{6!}}}_{\text{Q7}}, \underbrace{\frac{\eta^{m_1 \cdots m_6, n_1 n_2}}{\sqrt{6! 2!}}}_{5_2^2/5_3^2}, \underbrace{\frac{\eta^{m_1 \cdots m_6, n_1 \cdots n_4}}{\sqrt{6! 4!}}}_{3_3^4}, \underbrace{\frac{\eta^{m_1 \cdots m_6, n_1 \cdots n_6}}{\sqrt{6! 6!}}}_{1_4^6/1_3^6}, \dots \right),
 \end{aligned}
 \tag{2.12}$$

where the multiple indices are totally antisymmetrized and the ranges of the indices are  $k, l = 1, \dots, d$ ,  $m, n = 1, \dots, d-1$ , and  $\alpha, \beta = 1, 2$ . Each  $\eta$ -symbol corresponds to a brane specified below the underbrace (see Ref. [29] and also Ref. [30] for the notation of exotic branes  $b_n^c$ ). The ellipses are relevant only for the  $E_{d(d)}$  EFT with  $d \geq 8$ , which is not considered here.

The above set of  $\eta$ -symbols would be essentially the same as the set of  $\eta$ -symbols introduced in an “F-theory” (Refs. [31–35]).<sup>1</sup> There, the  $\eta$ -symbols were introduced as the Clebsch–Gordan–Wigner coefficients connecting  $R_1 \otimes R_1$  and the  $R_2$ -representation, and the Virasoro-like constraint was expressed as

$$\eta^{IJ; \text{I}} \mathcal{P}_I \mathcal{P}_J = 0. \tag{2.13}$$

<sup>1</sup> The set of  $\eta$ -symbols was introduced in Ref. [22] as the projection  $\times_N : E \times E \rightarrow N$ , and in Ref. [36] as the wedge product  $\wedge : R_1 \otimes R_1 \rightarrow R_2$ . Note that the wedge product is defined for more general representations.

The generalized Lie derivative was obtained from the Virasoro-like constraint, and by comparing with the generalized Lie derivative, the  $Y$ -tensor in EFT was expressed as

$$Y_{KL}^{IJ} = \begin{cases} \eta^{IJ;\mathbb{I}} \eta_{KL;\mathbb{I}} & (d \leq 6), \\ \eta^{IJ;\mathbb{I}} \eta_{KL;\mathbb{I}} - \frac{1}{2} \Omega^{IJ} \Omega_{KL} & (d = 7), \end{cases} \quad (2.14)$$

where the singlet constraint  $\Omega^{IJ} \partial_I \otimes \partial_J = 0$  was introduced for  $d = 7$  from consistency with the EFT. The explicit form of the  $\eta$ -symbol was found in Ref. [34] using a different convention from ours.

In this paper, instead of attempting to translate the  $\eta$ -symbols found in Ref. [34] into our convention, we utilize the linear map considered in Ref. [30]. As has been well known (Refs. [12,37]), EFT can reproduce both M-theory and type IIB theory. Depending on which theory one has in mind, there are two natural parameterizations of the generalized coordinates:  $x^I$  for M-theory and  $x^M$  for type IIB theory. The linear map in Ref. [30] provides a relation between the two parameterizations:

$$x^I = S^I_N x^N, \quad x^M = (S^{-1})^M_J x^J. \quad (2.15)$$

When we consider the linear map, we decompose the physical coordinates  $x^i$  ( $i = 1, \dots, d$ ) for M-theory and  $x^m$  ( $m = 1, \dots, d-1$ ) for type IIB theory as

$$(x^i) = (x^a, x^\alpha), \quad (x^m) = (x^a, x^y) \quad (a = 1, \dots, d-2, \alpha = y, z). \quad (2.16)$$

Here,  $x^z$  in the M-theory side corresponds to the coordinate on the M-theory circle. If we adopt the type IIA picture (by compactifying the M-theory circle), the linear map corresponds to a single  $T$ -duality along the  $x^y$  or  $x^y$  directions in type IIA/IIB theory. Indeed, in Ref. [30], it was shown that the linear map between two generalized metrics,  $\mathcal{M}_{IJ}$  (M-theory) and  $M_{MN}$  (type IIB theory),

$$M_{MN} = S^I_M S^J_N \mathcal{M}_{IJ}, \quad (2.17)$$

precisely reproduces the well-known  $T$ -duality transformation rules for supergravity fields. In this paper, we apply this linear map to  $\eta$ -symbols in M-theory/type IIB theory.

To be more specific, following the convention used in Ref. [30], we parameterize the generalized coordinates as

$$\begin{aligned} \text{M-theory: } (x^I) &= \left( \underbrace{x^i}_P, \underbrace{\frac{y_{i_1 i_2}}{\sqrt{2!}}}_{M2}, \underbrace{\frac{y_{i_1 \dots i_5}}{\sqrt{5!}}}_{M5}, \underbrace{\frac{y_{i_1 \dots i_7, j}}{\sqrt{7!}}}_{KKM/8}, \dots \right), \\ \text{type IIB: } (x^M) &= \left( \underbrace{x^m}_P, \underbrace{y_m^\alpha}_{F1/D1}, \underbrace{\frac{y_{m_1 m_2 m_3}}{\sqrt{3!}}}_{D3}, \underbrace{\frac{y_{m_1 \dots m_5}^\alpha}{\sqrt{5!}}}_{NS5/D5}, \underbrace{\frac{y_{m_1 \dots m_6, n}}{\sqrt{6!}}}_{KKM/7_2}, \dots \right), \end{aligned} \quad (2.18)$$

where the coordinates other than the physical coordinates are winding coordinates associated with some branes specified below the underbrace and ellipses again are relevant only for the  $E_{d(d)}$  EFT with  $d \geq 8$ . In the above parameterized generalized coordinates  $x^I$  for M-theory, we begin by

considering two  $\eta$ -symbols,

$$\eta^k \equiv \begin{pmatrix} 0 & \frac{2! \delta_{j_1 j_2}^{k i}}{\sqrt{2!}} & 0 & 0 \\ \frac{2! \delta_{i_1 i_2}^{k j}}{\sqrt{2!}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \eta^{k_1 \dots k_4} \equiv \begin{pmatrix} 0 & 0 & \frac{5! \delta_{j_1 \dots j_5}^{i k_1 \dots k_4}}{\sqrt{5!}} & 0 \\ 0 & \frac{4! \delta_{i_1 i_2 j_1 j_2}^{k_1 \dots k_4}}{\sqrt{2!} 2!} & 0 & 0 \\ \frac{5! \delta_{i_1 \dots i_5}^{j k_1 \dots k_4}}{\sqrt{5!}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.19)$$

which are trivial extensions of the  $\eta$ -symbols associated with M2-/M5-branes shown in Eq. (2.9). Under a compactification on the M-theory circle, an M2-brane becomes a D2-brane or an F-string in type IIA theory, and under a  $T$ -duality, it can become a D1/D3-brane or an F-string. Correspondingly, under the linear map, the  $\eta$ -symbol ( $\eta^k = (\eta^a, \eta^\alpha)$ ) can be mapped to an  $\eta$ -symbol,  $\eta^{a y}$  or  $\eta_\alpha$ , associated with a D3-brane or an F/D-string in type IIB theory. Similarly, the  $\eta$ -symbol ( $\eta^{k_1 \dots k_4} = (\eta^{a_1 \dots a_4}, \eta^{a_1 a_2 a_3 \alpha}, \eta^{a_1 a_2 y z})$ ) can be mapped to an  $\eta$ -symbol,  $\eta^{a_1 \dots a_4 y, y}$ ,  $\eta_\alpha^{a_1 \dots a_3 y}$ , or  $\eta^{a_1 a_2}$ , associated with a Kaluza–Klein monopole (KKM), an NS/D5-brane, or a D3-brane in type IIB theory. Repeating the linear map, we can find almost all of the  $\eta$ -symbols described in Eq. (2.18). The only  $\eta$ -symbols that cannot straightforwardly be obtained from the linear map are  $\eta^{[k_1 \dots k_6, l]}$  and  $\eta^{[m_1 \dots m_5, n]}$ , which correspond to 8-branes in M-theory and  $7_2$ -branes in type IIB theory, respectively. These branes are not related to other branes described in Eq. (2.18) via  $T$ -duality transformations, but they are related to each other. In fact, by requiring the  $SL(d)$  or  $SL(d-1)$  covariance in the M-theory or type IIB theory sides, they also can be determined completely. Then, we find the explicit form of all  $\eta$ -symbols (and also the  $\Omega$ -tensor) and can construct the  $Y$ -tensor through Eq. (2.14).

We expect that, in the same manner as Refs. [26–28], all of the  $\eta$ -symbols obtained in this paper will also be read off from the momentum constraint (i.e., Virasoro-like constraint) in worldvolume theories of branes appearing in Eq. (2.12), but we leave the task for future work, and here we will concentrate on the determination of the  $\eta$ -symbols for the  $E_{d(d)}$  EFT ( $d \leq 7$ ).

### 3. Explicit form of $\eta$ -symbols

In this section, we begin by showing the explicit matrix form of  $\eta$ -symbols in two generalized coordinates,  $x^I$  and  $x^M$ , associated with M-theory and type IIB theory, respectively. Their derivations are explained in Sect. 3.3. In Appendix B, we explain how to reproduce the known  $Y$ -tensors from our  $\eta$ -symbols.

#### 3.1. M-theory parameterization

When we adopt the M-theory description, the decomposition of  $\eta^\mp$  becomes

$$(\eta^\mp) = \left( \eta^k, \frac{\eta^{k_1 \dots k_4}}{\sqrt{4!}}, \frac{\eta^{k_1 \dots k_6, l}}{\sqrt{6!}}, \frac{\eta^{k_1 \dots k_7, l_1 l_2 l_3}}{\sqrt{7!} 3!}, \frac{\eta^{k_1 \dots k_7, l_1 \dots l_6}}{\sqrt{7!} 6!} \right). \quad (3.1)$$

The explicit forms of each matrix,  $\eta^{\mathbb{I}} = (\eta^{IJ; \mathbb{I}})$ , are

$$\eta^k \equiv \begin{pmatrix} 0 & \frac{2! \delta_{j_1/2}^{ki}}{\sqrt{2!}} & 0 & 0 \\ \frac{2! \delta_{i_1/2}^{kj}}{\sqrt{2!}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.2)$$

$$\eta^{k_1 \dots k_4} \equiv \begin{pmatrix} 0 & 0 & \frac{5! \delta_{j_1 \dots j_5}^{ik_1 \dots k_4}}{\sqrt{5!}} & 0 \\ 0 & \frac{4! \delta_{i_1 i_2 j_1 j_2}^{k_1 \dots k_4}}{\sqrt{2!} 2!} & 0 & 0 \\ \frac{5! \delta_{i_1 \dots i_5}^{jk_1 \dots k_4}}{\sqrt{5!}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.3)$$

$$\eta^{k_1 \dots k_6, l} \equiv \eta_{\text{KKM}}^{k_1 \dots k_6, l} + \eta^{k_1 \dots k_6 l}, \quad (3.4)$$

$$\eta_{\text{KKM}}^{k_1 \dots k_6, l} \equiv \begin{pmatrix} 0 & 0 & 0 & \frac{7!}{\sqrt{7!}} \left( \delta_{j_1 \dots j_7}^{k_1 \dots k_6 l} \delta_j^l - \frac{\delta_{j_1 \dots j_7}^{k_1 \dots k_6 l} \delta_j^l}{7} \right) \\ 0 & 0 & \frac{-6! 2!}{\sqrt{2!} 5!} \left( \delta_{j_1 \dots j_5 k}^{k_1 \dots k_6} \delta_{i_1 i_2}^{kl} - \delta_{j_1 \dots j_5 i_1 i_2}^{k_1 \dots k_6 l} \right) & 0 \\ 0 & \frac{-6! 2!}{\sqrt{2!} 5!} \left( \delta_{i_1 \dots i_5 k}^{k_1 \dots k_6} \delta_{j_1 j_2}^{kl} - \delta_{i_1 \dots i_5 j_1 j_2}^{k_1 \dots k_6 l} \right) & 0 & 0 \\ \frac{7!}{\sqrt{7!}} \left( \delta_{i_1 \dots i_7}^{k_1 \dots k_6 l} \delta_i^l - \frac{\delta_{i_1 \dots i_7}^{k_1 \dots k_6 l} \delta_i^l}{7} \right) & 0 & 0 & 0 \end{pmatrix}, \quad (3.5)$$

$$\eta^{k_1 \dots k_7} \equiv \frac{1}{7\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 3 \frac{7! \delta_{j_1 \dots j_7}^{k_1 \dots k_7} \delta_j^l}{\sqrt{7!}} \\ 0 & 0 & \frac{7! \delta_{j_1 \dots j_5 i_1 i_2}^{k_1 \dots k_7}}{\sqrt{2!} 5!} & 0 \\ 0 & \frac{7! \delta_{i_1 \dots i_5 j_1 j_2}^{k_1 \dots k_7}}{\sqrt{2!} 5!} & 0 & 0 \\ 3 \frac{7! \delta_{i_1 \dots i_7}^{k_1 \dots k_7} \delta_i^l}{\sqrt{7!}} & 0 & 0 & 0 \end{pmatrix}, \quad (3.6)$$

$$\eta^{k_1 \dots k_7, l_1 l_2 l_3} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-7! 7!}{4! \sqrt{2!} 7!} \delta_{j_1 \dots j_7}^{l_1 l_2 l_3 m_1 \dots m_4} \times \delta_{j l_1 i_2 m_1 \dots m_4}^{k_1 \dots k_7} \\ 0 & 0 & \frac{7! 5!}{2! \sqrt{5!} 5!} \delta_{i_1 \dots i_5 m_1 m_2}^{k_1 \dots k_5 k_6 k_7} \times \delta_{j_1 \dots j_5}^{m_1 m_2 l_1 l_2 l_3} & 0 \\ 0 & \frac{-7! 7!}{4! \sqrt{2!} 7!} \delta_{i_1 \dots i_7}^{l_1 l_2 l_3 m_1 \dots m_4} \times \delta_{j l_1 j_2 m_1 \dots m_4}^{k_1 \dots k_7} & 0 & 0 \end{pmatrix}, \quad (3.7)$$

$$\eta^{k_1 \dots k_7, l_1 \dots l_6} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{7! 6! \delta_{j_1 \dots j_7}^{k_1 \dots k_7} \delta_{j l_1 \dots l_5}^{l_1 \dots l_6}}{\sqrt{5!} 7!} \\ 0 & 0 & \frac{7! 6! \delta_{i_1 \dots i_7}^{k_1 \dots k_7} \delta_{j l_1 \dots l_5}^{l_1 \dots l_6}}{\sqrt{5!} 7!} & 0 \end{pmatrix}. \quad (3.8)$$

Here,  $\eta_{\text{KKM}}^{k_1 \cdots k_6, l}$  is defined to satisfy  $\eta_{\text{KKM}}^{[k_1 \cdots k_6, l]} = 0$ . We also define the  $\eta$ -symbols  $\eta_{\text{I}} = (\eta_{IJ; \text{I}})$  as

$$\eta_{IJ; \text{I}} = \eta^{IJ; \text{I}}. \quad (3.9)$$

For example,  $\eta_k$  is defined as

$$\eta_k \equiv (\eta_{IJ; k}) \equiv \begin{pmatrix} 0 & \frac{2! \delta_{ji}^{1/2}}{\sqrt{2!}} & 0 & 0 \\ \frac{2! \delta_{ij}^{1/2}}{\sqrt{2!}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.10)$$

The position of the indices is converted but  $\eta_{\text{I}}$  and  $\eta^{\text{I}}$  have the same components as a matrix.

In the above expressions, we are supposing the case of the  $E_{7(7)}$  EFT but the expressions for the  $E_{d(d)}$  EFT with  $d \leq 6$  can be obtained via a simple truncation of the above matrices. For example, in the  $\text{SL}(5)$  EFT (where  $d = 4$ ), the generalized coordinates are given by  $x^I = (x^i, \frac{y_{i_1 i_2}}{\sqrt{2!}})$  and the nonvanishing  $\eta$ -symbols become

$$\eta^k = \begin{pmatrix} 0 & \frac{2! \delta_{j_1 j_2}^{ki}}{\sqrt{2!}} \\ \frac{2! \delta_{i_1 i_2}^{kj}}{\sqrt{2!}} & 0 \end{pmatrix}, \quad \eta^{k_1 \cdots k_4} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{4! \delta_{i_1 i_2 j_1 j_2}^{k_1 \cdots k_4}}{\sqrt{2! 2!}} \end{pmatrix}. \quad (3.11)$$

The number of the generalized coordinates, the  $\eta$ -symbols, and the corresponding brane charges for  $d \leq 7$  can be summarized as follows:

$$d = 2 : \quad \begin{matrix} (x^I) = (\underbrace{x^i}_{\text{P}[2]}, \underbrace{y_{i_1 i_2}}_{\text{M2}[1]}), \\ [3] \end{matrix} \quad \begin{matrix} (\eta^{\text{I}}) = (\underbrace{\eta^k}_{\text{M2}[2]}), \\ [2] \end{matrix} \quad (3.12)$$

$$d = 3 : \quad \begin{matrix} (x^I) = (\underbrace{x^i}_{\text{P}[3]}, \underbrace{y_{i_1 i_2}}_{\text{M2}[3]}), \\ [6] \end{matrix} \quad \begin{matrix} (\eta^{\text{I}}) = (\underbrace{\eta^k}_{\text{M2}[3]}), \\ [3] \end{matrix} \quad (3.13)$$

$$d = 4 : \quad \begin{matrix} (x^I) = (\underbrace{x^i}_{\text{P}[4]}, \underbrace{y_{i_1 i_2}}_{\text{M2}[6]}), \\ [10] \end{matrix} \quad \begin{matrix} (\eta^{\text{I}}) = (\underbrace{\eta^k}_{\text{M2}[4]}, \underbrace{\eta^{k_1 \cdots k_4}}_{\text{M5}[1]}), \\ [5] \end{matrix} \quad (3.14)$$

$$d = 5 : \quad \begin{matrix} (x^I) = (\underbrace{x^i}_{\text{P}[5]}, \underbrace{y_{i_1 i_2}}_{\text{M2}[10]}, \underbrace{y_{i_1 \cdots i_5}}_{\text{M5}[1]}), \\ [16] \end{matrix} \quad \begin{matrix} (\eta^{\text{I}}) = (\underbrace{\eta^k}_{\text{M2}[5]}, \underbrace{\eta^{k_1 \cdots k_4}}_{\text{M5}[5]}), \\ [10] \end{matrix} \quad (3.15)$$

$$d = 6 : \quad \begin{matrix} (x^I) = (\underbrace{x^i}_{\text{P}[6]}, \underbrace{y_{i_1 i_2}}_{\text{M2}[15]}, \underbrace{y_{i_1 \cdots i_5}}_{\text{M5}[6]}), \\ [27] \end{matrix} \quad \begin{matrix} (\eta^{\text{I}}) = (\underbrace{\eta^k}_{\text{M2}[6]}, \underbrace{\eta^{k_1 \cdots k_4}}_{\text{M5}[15]}, \underbrace{\eta^{k_1 \cdots k_6, l}}_{\text{KKM}[6]}), \\ [27] \end{matrix} \quad (3.16)$$

$$d = 7 : \quad \begin{matrix} (x^I) = (\underbrace{x^i}_{\text{P}[7]}, \underbrace{y_{i_1 i_2}}_{\text{M2}[21]}, \underbrace{y_{i_1 \cdots i_5}}_{\text{M5}[21]}), \\ [56] \end{matrix} \quad \begin{matrix} (\eta^{\text{I}}) = (\underbrace{\eta^k}_{\text{M2}[7]}, \underbrace{\eta^{k_1 \cdots k_4}}_{\text{M5}[35]}, \underbrace{\eta^{k_1 \cdots k_6, l}}_{\text{KKM}[8][49]}), \\ [133] \end{matrix} \quad (3.17)$$

$$\begin{matrix} \underbrace{y_{i_1 \cdots i_7, j}}_{\text{KKM}[7]}, & \underbrace{\eta^{k_1 \cdots k_7, l_1 l_2 l_3}}_{5^3[35]}, \underbrace{\eta^{k_1 \cdots k_7, l_1 \cdots l_6}}_{2^6[7]}, \end{matrix}$$

where the normalization coefficients like  $(1/\sqrt{p!})$  are not displayed for simplicity.



In the case of  $d = 7$ , in addition to the  $\eta$ -symbols, we also define antisymmetric matrices  $\Omega^{IJ}$  and  $\Omega_{IJ}$  appearing in Eq. (2.14). As we explain in Sect. 3.3, their matrix forms, in our convention, are

$$(\Omega_{IJ}) \equiv \begin{pmatrix} 0 & 0 & 0 & \frac{\epsilon^{i_1 \dots i_7} \delta_i^j}{\sqrt{7!}} \\ 0 & 0 & \frac{\epsilon^{i_1 i_2 j_1 \dots j_5}}{\sqrt{2! 5!}} & 0 \\ 0 & -\frac{\epsilon^{i_1 \dots i_5 j_1 j_2}}{\sqrt{2! 5!}} & 0 & 0 \\ -\frac{\epsilon^{i_1 \dots i_7} \delta_j^i}{\sqrt{7!}} & 0 & 0 & 0 \end{pmatrix}, \quad (3.18)$$

$$(\Omega^{IJ}) \equiv \begin{pmatrix} 0 & 0 & 0 & \frac{\epsilon_{j_1 \dots j_7} \delta_j^i}{\sqrt{7!}} \\ 0 & 0 & \frac{\epsilon_{i_1 i_2 j_1 \dots j_5}}{\sqrt{2! 5!}} & 0 \\ 0 & -\frac{\epsilon_{i_1 \dots i_5 j_1 j_2}}{\sqrt{2! 5!}} & 0 & 0 \\ -\frac{\epsilon_{i_1 \dots i_7} \delta_i^j}{\sqrt{7!}} & 0 & 0 & 0 \end{pmatrix}, \quad (3.19)$$

where the totally antisymmetric symbols  $\epsilon_{i_1 \dots i_7}$  and  $\epsilon^{i_1 \dots i_7}$  are defined as  $\epsilon^{1 \dots 7} = \epsilon_{1 \dots 7} = 1$ .

From the above  $\eta$ -symbols and the  $\Omega$ -tensor, we can obtain the  $Y$ -tensor as

$$\begin{aligned} Y_{KL}^{IJ} &= \eta^{IJ; \mathbb{I}} \eta_{KL; \mathbb{I}} - \frac{1}{2} \Omega^{IJ} \Omega_{KL} \\ &= \eta^{IJ; k} \eta_{KL; k} + \frac{\eta^{IJ; k_1 \dots k_4} \eta_{KL; k_1 \dots k_4}}{4!} + \frac{\eta^{IJ; k_1 \dots k_6, l} \eta_{KL; k_1 \dots k_6, l}}{6!} \\ &\quad + \frac{\eta^{IJ; k_1 \dots k_7, l_1 l_2 l_3} \eta_{KL; k_1 \dots k_7, l_1 l_2 l_3}}{7! 3!} + \frac{\eta^{IJ; k_1 \dots k_7, l_1 \dots l_6} \eta_{KL; k_1 \dots k_7, l_1 \dots l_6}}{7! 6!} \\ &\quad - \frac{1}{2} \Omega^{IJ} \Omega_{KL}. \end{aligned} \quad (3.20)$$

### 3.2. Type IIB parameterization

When we adopt the type IIB description, we consider the following decomposition of  $\eta$ -symbols:

$$(\eta^M) = \left( \eta_\gamma, \frac{\eta^{p_1 p_2}}{\sqrt{2!}}, \frac{\eta^{p_1 \dots p_4}}{\sqrt{4!}}, \frac{\eta^{p_1 \dots p_5, q}}{\sqrt{5!}}, \frac{\eta^{p_1 \dots p_6}_{(\gamma_1 \gamma_2)}}{\sqrt{6!}}, \frac{\eta^{p_1 \dots p_6, q_1 q_2}}{\sqrt{6! 2!}}, \frac{\eta^{p_1 \dots p_6, q_1 \dots q_4}}{\sqrt{6! 4!}}, \frac{\eta^{p_1 \dots p_6, q_1 \dots q_6}}{\sqrt{6! 6!}} \right), \quad (3.21)$$

where the matrices take the form

$$\eta_\gamma \equiv \begin{pmatrix} 0 & \delta_\gamma^\beta \delta_n^m & 0 & 0 & 0 \\ \delta_\gamma^\alpha \delta_m^n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.22)$$

$$\eta^{p_1 p_2} \equiv \begin{pmatrix} 0 & 0 & \frac{3! \delta_{n_1 n_2 n_3}^{mp_1 p_2}}{\sqrt{3!}} & 0 & 0 \\ 0 & -2! \epsilon^{\alpha\beta} \delta_{mn}^{p_1 p_2} & 0 & 0 & 0 \\ \frac{3! \delta_{m_1 m_2 m_3}^{np_1 p_2}}{\sqrt{3!}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.23)$$

$$\eta_{\gamma}^{p_1 \cdots p_4} \equiv \begin{pmatrix} 0 & 0 & 0 & \frac{-5! \delta_{\gamma}^{\beta} \delta_{n_1 \cdots n_5}^{p_1 \cdots p_4 m}}{\sqrt{5!}} & 0 \\ 0 & 0 & \frac{4! \delta_{\gamma}^{\alpha} \delta_{n_1 n_2 n_3 m}^{p_1 \cdots p_4}}{\sqrt{3!}} & 0 & 0 \\ 0 & \frac{4! \delta_{\gamma}^{\beta} \delta_{m_1 m_2 m_3 n}^{p_1 \cdots p_4}}{\sqrt{3!}} & 0 & 0 & 0 \\ \frac{-5! \delta_{\gamma}^{\alpha} \delta_{m_1 \cdots m_5}^{p_1 \cdots p_4 n}}{\sqrt{5!}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.24)$$

$$\eta^{p_1 \cdots p_5, q} \equiv \eta_{KKM}^{p_1 \cdots p_5, q} + \eta^{p_1 \cdots p_5 q}, \quad (3.25)$$

$$\eta_{KKM}^{p_1 \cdots p_5, q}$$

$$\equiv \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{6!}{\sqrt{6!}} \left( \delta_{n_1 \cdots n_6}^{mp_1 \cdots p_5} \delta_n^q + \frac{\delta_{n_1 \cdots p_5 q}^{p_1 \cdots p_5 q} \delta_n^m}{6} \right) \\ 0 & 0 & 0 & \frac{5! \epsilon^{\beta\alpha}}{\sqrt{5!}} \left( \delta_{n_1 \cdots n_5}^{p_1 \cdots p_5} \delta_m^q - \delta_{n_1 \cdots n_5 m}^{p_1 \cdots p_5 q} \right) & 0 \\ 0 & 0 & \frac{5! 3!}{2 \cdot 2! \sqrt{3!} 3!} \times \left( \delta_{m_1 m_2 m_3 r_1 r_2}^{p_1 p_2 p_3 p_4 p_5} \delta_{n_1 n_2 n_3}^{r_1 r_2 q} + \delta_{n_1 n_2 n_3 r_1 r_2}^{p_1 p_2 p_3 p_4 p_5} \delta_{m_1 m_2 m_3}^{r_1 r_2 q} \right) & 0 & 0 \\ 0 & \frac{5! \epsilon^{\alpha\beta}}{\sqrt{5!}} \left( \delta_{m_1 \cdots m_5}^{p_1 \cdots p_5} \delta_n^q - \delta_{m_1 \cdots m_5 n}^{p_1 \cdots p_5 q} \right) & 0 & 0 & 0 \\ \frac{6!}{\sqrt{6!}} \left( \delta_{m_1 \cdots m_6}^{np_1 \cdots p_5} \delta_m^q + \frac{\delta_{m_1 \cdots p_5 q}^{p_1 \cdots p_5 q} \delta_m^n}{6} \right) & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.26)$$

$$\eta^{p_1 \cdots p_6} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{6! \delta_{n_1 \cdots n_6}^{p_1 \cdots p_6} \delta_n^m}{3\sqrt{6!}} \\ 0 & 0 & 0 & \frac{6! \epsilon^{\beta\alpha} \delta_{n_1 \cdots n_5 m}^{p_1 \cdots p_6}}{6\sqrt{5!}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6! \epsilon^{\alpha\beta} \delta_{m_1 \cdots m_5 n}^{p_1 \cdots p_6}}{6\sqrt{5!}} & 0 & 0 & 0 \\ \frac{6! \delta_{m_1 \cdots m_6}^{p_1 \cdots p_6} \delta_m^n}{3\sqrt{6!}} & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.27)$$

$$\eta_{(\gamma_1 \gamma_2)}^{p_1 \dots p_6} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-6! \delta_{(\gamma_1}^\alpha \delta_{\gamma_2)}^\beta \delta_{n_1 \dots n_5 m}^{p_1 \dots p_6}}{\sqrt{5!}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-6! \delta_{(\gamma_1}^\alpha \delta_{\gamma_2)}^\beta \delta_{m_1 \dots m_5 n}^{p_1 \dots p_6}}{\sqrt{5!}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.28)$$

$$\eta_\gamma^{p_1 \dots p_6, q_1 q_2} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-6! 2!}{\sqrt{6!}} \delta_\gamma^\alpha \delta_{n_1 \dots n_6}^{p_1 \dots p_6} \delta_{mn}^{q_1 q_2} \\ 0 & 0 & 0 & \frac{6! 3!}{\sqrt{3! 5!}} \delta_\gamma^\beta \delta_{n_1 \dots n_5 r}^{p_1 \dots p_6} & 0 \\ & & & \times \delta_{m_1 m_2 m_3}^{r q_1 q_2} & \\ 0 & 0 & \frac{6! 3!}{\sqrt{3! 5!}} \delta_\gamma^\alpha \delta_{m_1 \dots m_5 r}^{p_1 \dots p_6} & 0 & 0 \\ & & \times \delta_{n_1 n_2 n_3}^{r q_1 q_2} & & \\ 0 & \frac{-6! 2!}{\sqrt{6!}} \delta_\gamma^\beta \delta_{m_1 \dots m_6}^{p_1 \dots p_6} \delta_{nm}^{q_1 q_2} & 0 & 0 & 0 \end{pmatrix}, \quad (3.29)$$

$$\eta^{p_1 \dots p_6, q_1 \dots q_4} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{6! 4! \delta_{n_1 \dots n_6}^{p_1 \dots p_6} \delta_{nm_1 m_2 m_3}^{q_1 \dots q_4}}{\sqrt{6! 3!}} \\ 0 & 0 & 0 & \frac{6! 5! \epsilon^{\alpha\beta} \delta_{m_1 \dots m_5 r}^{p_1 \dots p_6} \delta_{n_1 \dots n_5}^{q_1 \dots q_4 r}}{\sqrt{5! 5!}} & 0 \\ 0 & 0 & \frac{6! 4! \delta_{m_1 \dots m_6}^{p_1 \dots p_6} \delta_{mn_1 n_2 n_3}^{q_1 \dots q_4}}{\sqrt{3! 6!}} & 0 & 0 \end{pmatrix}, \quad (3.30)$$

$$\eta_\gamma^{p_1 \dots p_6, q_1 \dots q_6} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{6! 6! \delta_\gamma^\alpha \delta_{n_1 \dots n_6}^{p_1 \dots p_6} \delta_{m_1 \dots m_5 n}^{q_1 \dots q_6}}{\sqrt{5! 6!}} & 0 \\ 0 & 0 & 0 & \frac{6! 6! \delta_\gamma^\beta \delta_{m_1 \dots m_6}^{p_1 \dots p_6} \delta_{n_1 \dots n_5 m}^{q_1 \dots q_6}}{\sqrt{5! 6!}} & 0 & 0 \end{pmatrix}. \quad (3.31)$$

We also define the  $\eta$ -symbols  $\eta_M = (\eta_{MN;M})$  as

$$\eta_{MN;M} = \eta^{MN;M}. \quad (3.32)$$

A list of nonvanishing coordinates and  $\eta$ -symbols for each  $d$  is

$$d = 2 : (x^M) = (\underbrace{\mathbf{x}^m}_{[3] \text{ P}[1] \text{ F1/D1}[2]}, \underbrace{\mathbf{y}_m^\alpha}_{[2] \text{ F1/D1}[2]}), \quad (\eta^M) = (\underbrace{\eta_\gamma}_{[2] \text{ F1/D1}[2]}), \quad (3.33)$$

$$d = 3 : (x^M) = (\underbrace{\mathbf{x}^m}_{[6] \text{ P}[2] \text{ F1/D1}[4]}, \underbrace{\mathbf{y}_m^\alpha}_{[3] \text{ F1/D1}[2] \text{ D3}[1]}), \quad (\eta^M) = (\underbrace{\eta_\gamma}_{[3] \text{ F1/D1}[2]}, \underbrace{\eta^{p_1 p_2}}_{\text{D3}[1]}), \quad (3.34)$$

$$d = 4 : (x^M) = \left( \underbrace{x^m}_{[10]}, \underbrace{y_m^\alpha}_{P[3]}, \underbrace{y_{m_1 m_2 m_3}}_{F1/D1[6]}, \underbrace{y_{m_1 m_2 m_3}}_{D3[1]} \right), \quad (\eta^M) = \left( \underbrace{\eta_\gamma}_{[5]}, \underbrace{\eta^{p_1 p_2}}_{F1/D1[2]}, \underbrace{\eta^{p_1 p_2}}_{D3[3]} \right), \quad (3.35)$$

$$d = 5 : (x^M) = \left( \underbrace{x^m}_{[16]}, \underbrace{y_m^\alpha}_{P[4]}, \underbrace{y_{m_1 m_2 m_3}}_{F1/D1[8]}, \underbrace{y_{m_1 m_2 m_3}}_{D3[4]} \right), \quad (\eta^M) = \left( \underbrace{\eta_\gamma}_{[10]}, \underbrace{\eta^{p_1 p_2}}_{F1/D1[2]}, \underbrace{\eta^{p_1 \dots p_4}}_{D3[6]}, \underbrace{\eta^{p_1 \dots p_4}}_{NS5/D5[2]} \right), \quad (3.36)$$

$$d = 6 : (x^M) = \left( \underbrace{x^m}_{[27]}, \underbrace{y_m^\alpha}_{P[5]}, \underbrace{y_{m_1 m_2 m_3}}_{F1/D1[10]}, \underbrace{y_{m_1 \dots m_5}}_{D3[10]}, \underbrace{y_{m_1 \dots m_5}}_{NS5/D5[2]} \right), \quad (\eta^M) = \left( \underbrace{\eta_\gamma}_{[27]}, \underbrace{\eta^{p_1 p_2}}_{F1/D1[2]}, \underbrace{\eta^{p_1 \dots p_4}}_{D3[10]}, \underbrace{\eta^{p_1 \dots p_4}}_{NS5/D5[10]}, \underbrace{\eta^{p_1 \dots p_5, q}}_{KKM[5]} \right), \quad (3.37)$$

$$d = 7 : (x^M) = \left( \underbrace{x^m}_{[56]}, \underbrace{y_m^\alpha}_{P[6]}, \underbrace{y_{m_1 m_2 m_3}}_{F1/D1[12]}, \underbrace{y_{m_1 \dots m_6, n}}_{D3[20]}, \underbrace{y_{m_1 \dots m_5}}_{NS5/D5[12]}, \underbrace{y_{m_1 \dots m_6, n}}_{KKM[6]} \right), \quad (\eta^M) = \left( \underbrace{\eta_\gamma}_{[133]}, \underbrace{\eta^{p_1 p_2}}_{F1/D1[2]}, \underbrace{\eta^{p_1 \dots p_6}}_{D3[15]}, \underbrace{\eta^{p_1 \dots p_6, q_1 q_2}}_{NS5/D5[30]}, \underbrace{\eta^{p_1 \dots p_6, q_1 \dots q_4}}_{KKM/7_2[36]}, \underbrace{\eta^{p_1 \dots p_6, q_1 \dots q_6}}_{Q7[3]}, \underbrace{\eta^{p_1 \dots p_6, q_1 \dots q_6}}_{5_2^2/5_3^2[30]}, \underbrace{\eta^{p_1 \dots p_6, q_1 \dots q_6}}_{3_3^4[15]}, \underbrace{\eta^{p_1 \dots p_6, q_1 \dots q_6}}_{1_4^6/1_3^6[2]} \right), \quad (3.38)$$

Here again, the normalization coefficients like  $(1/\sqrt{p!})$  are not displayed. On the other hand, the matrices  $\Omega^{MN}$  and  $\Omega_{MN}$  in the type IIB side have the form

$$(\Omega^{MN}) \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{\epsilon_{n_1 \dots n_6} \delta_n^m}{\sqrt{6!}} \\ 0 & 0 & 0 & \frac{\epsilon^{\alpha\beta} \epsilon_{mn_1 \dots n_5}}{\sqrt{5!}} & 0 \\ 0 & 0 & -\frac{\epsilon_{m_1 m_2 m_3 n_1 n_2 n_3}}{\sqrt{3!} 3!} & 0 & 0 \\ 0 & -\frac{\epsilon^{\alpha\beta} \epsilon_{m_1 \dots m_5 n}}{\sqrt{5!}} & 0 & 0 & 0 \\ -\frac{\epsilon_{m_1 \dots m_6} \delta_m^n}{\sqrt{6!}} & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.39)$$

$$(\Omega_{MN}) \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{\epsilon^{n_1 \dots n_6} \delta_m^n}{\sqrt{6!}} \\ 0 & 0 & 0 & \frac{\epsilon_{\alpha\beta} \epsilon^{mn_1 \dots n_5}}{\sqrt{5!}} & 0 \\ 0 & 0 & -\frac{\epsilon_{m_1 m_2 m_3 n_1 n_2 n_3}}{\sqrt{3!} 3!} & 0 & 0 \\ 0 & -\frac{\epsilon_{\alpha\beta} \epsilon^{m_1 \dots m_5 n}}{\sqrt{5!}} & 0 & 0 & 0 \\ -\frac{\epsilon_{m_1 \dots m_6} \delta_m^n}{\sqrt{6!}} & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.40)$$

where the totally antisymmetric symbols  $\epsilon_{n_1 \dots n_6}$  and  $\epsilon^{n_1 \dots n_6}$  are defined as  $\epsilon^{1 \dots 6} = \epsilon_{1 \dots 6} = 1$ .

### 3.3. The linear map

We utilize the following linear map between generalized coordinates  $x^I$  (M-theory) and  $x^M$  (type IIB) (Ref. [30]):

$$(x^M) = \begin{pmatrix} x^a \\ x^y \\ y_a^\alpha \\ y_y^\alpha \\ \frac{y_{a_1 a_2 a_3}}{\sqrt{3!}} \\ \frac{y_{a_1 a_2 y}}{\sqrt{2!}} \\ \frac{y_{a_1 \dots a_5}^\alpha}{\sqrt{5!}} \\ \frac{y_{a_1 \dots a_4 y}^\alpha}{\sqrt{4!}} \\ \frac{y_{a_1 \dots a_5 y, a}}{\sqrt{5!}} \\ \frac{y_{a_1 \dots a_5 y, y}}{\sqrt{5!}} \end{pmatrix} = (S^{-1})^M_J \begin{pmatrix} x^b \\ x^\beta \\ \frac{y_{b_1 b_2}}{\sqrt{2!}} \\ y_{b\beta} \\ y_{yz} \\ \frac{y_{b_1 \dots b_5}}{\sqrt{5!}} \\ \frac{y_{b_1 \dots b_4 \beta}}{\sqrt{4!}} \\ \frac{y_{b_1 b_2 b_3 yz}}{\sqrt{3!}} \\ \frac{y_{b_1 \dots b_5 yz, b}}{\sqrt{5!}} \\ \frac{y_{b_1 \dots b_5 yz, \beta}}{\sqrt{5!}} \end{pmatrix} = (S^{-1})^M_J x^J, \quad (3.41)$$

$$(S^{-1})^M_J \equiv \begin{pmatrix} \delta_b^a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon^{\alpha\beta} \delta_a^b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta_\beta^\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_{a_1 a_2}^{b_1 b_2 b_3} & 0 & 0 \\ 0 & 0 & \delta_{a_1 a_2}^{b_1 b_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon^{\alpha\beta} \delta_{a_1 \dots a_5}^{b_1 \dots b_5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \epsilon^{\alpha\beta} \delta_{a_1 \dots a_4}^{b_1 \dots b_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_{a_1 \dots a_5}^{b_1 \dots b_5} \delta_a^b & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_{a_1 \dots a_5}^{b_1 \dots b_5} & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.42)$$

$$S^I_N \equiv \begin{pmatrix} \delta_b^a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_\beta^\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_{a_1 a_2}^{b_1 b_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_{\alpha\beta}^T \delta_a^b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_{a_1 \dots a_5}^{b_1 \dots b_5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon_{\alpha\beta}^T \delta_{a_1 \dots a_4}^{b_1 \dots b_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_{a_1 a_2 a_3}^{b_1 b_2 b_3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_{a_1 \dots a_5}^{b_1 \dots b_5} \delta_a^b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \epsilon_{\alpha\beta}^T \delta_{a_1 \dots a_5}^{b_1 \dots b_5} & 0 & 0 & 0 \end{pmatrix}. \quad (3.43)$$

Under the linear map, e.g., the matrix  $\eta^a = (\eta^{IJ;a})$  associated with an M2-brane is mapped to a matrix  $\eta^{aY} = (\eta^{MN;aY})$  associated with the D3-brane in type IIB theory,

$$\eta^{MN;aY} = -(S^{-1})^M{}_I \eta^{IJ;a} (S^{-T})_J{}^N, \quad (3.44)$$

where the minus sign is introduced by convention. Similarly, we can relate all of the  $\eta$ -symbols for M-theory and type IIB theory via the linear map  $S$ . By introducing a transformation matrix for the  $R_2$ -representation  $T^M{}_J$ , we can express the linear map for the  $\eta$ -symbols as

$$\eta^{MN;M} = T^M{}_J (S^{-1})^M{}_I \eta^{IJ;J} (S^{-T})_J{}^N, \quad \eta^{IJ;I} = (T^{-1})^I{}_N S^I{}_M \eta^{MN;N} (S^T)_N{}^J. \quad (3.45)$$

Here, the matrix  $T$  that maintains the  $SL(d-2)$  covariance can be summarized as follows:

$$\begin{pmatrix} \eta_\alpha \\ \frac{1}{\sqrt{2!}} \eta^{a_1 a_2} \\ \eta^{aY} \\ \frac{1}{\sqrt{4!}} \eta_\alpha^{a_1 \cdots a_4} \\ \frac{1}{\sqrt{3!}} \eta_\alpha^{a_1 a_2 a_3 Y} \\ \frac{1}{\sqrt{5!}} \eta^{a_1 \cdots a_5, c} \\ \frac{1}{\sqrt{5!}} \eta^{a_1 \cdots a_5, Y} \\ \frac{1}{\sqrt{4!}} \eta^{a_1 \cdots a_4 Y, c} \\ \frac{1}{\sqrt{4!}} \eta^{a_1 \cdots a_4 Y, Y} \\ \frac{1}{\sqrt{5!}} \eta_{(\alpha_1 a_2)}^{a_1 \cdots a_5 Y} \\ \frac{1}{\sqrt{5! 2!}} \eta_\alpha^{a_1 \cdots a_5 Y, c_1 c_2} \\ \frac{1}{\sqrt{5!}} \eta_\alpha^{a_1 \cdots a_5 Y, cY} \\ \frac{1}{\sqrt{5! 4!}} \eta^{a_1 \cdots a_5 Y, c_1 \cdots c_4} \\ \frac{1}{\sqrt{5! 3!}} \eta^{a_1 \cdots a_5 Y, c_1 c_2 c_3 Y} \\ \frac{1}{\sqrt{5! 5!}} \eta_\alpha^{a_1 \cdots a_5 Y, c_1 \cdots c_5 Y} \end{pmatrix} = T \begin{pmatrix} \eta^b \\ \eta^\beta \\ \frac{1}{\sqrt{4!}} \eta^{b_1 \cdots b_4} \\ \frac{1}{\sqrt{3!}} \eta^{b_1 b_2 b_3 \beta} \\ \frac{1}{\sqrt{2!}} \eta^{b_1 b_2 Y \beta} \\ \frac{1}{\sqrt{5!}} \eta^{b_1 \cdots b_5 \beta, d} \\ \frac{1}{\sqrt{5!}} \eta^{b_1 \cdots b_5 (\beta_1, \beta_2)} \\ \frac{1}{\sqrt{5!}} \eta^{b_1 \cdots b_5 [Y, Z]} \\ \frac{1}{\sqrt{4!}} \eta^{b_1 \cdots b_4 Y \beta, d} \\ \frac{1}{\sqrt{4!}} \eta^{b_1 \cdots b_4 Y \beta, \beta} \\ \frac{1}{\sqrt{5! 3!}} \eta^{b_1 \cdots b_5 Y \beta, d_1 d_2 d_3} \\ \frac{1}{\sqrt{5! 2!}} \eta^{b_1 \cdots b_5 Y \beta, d_1 d_2 \beta} \\ \frac{1}{\sqrt{5!}} \eta^{b_1 \cdots b_5 Y \beta, d Y \beta} \\ \frac{1}{\sqrt{5! 5!}} \eta^{b_1 \cdots b_5 Y \beta, d_1 \cdots d_5 \beta} \\ \frac{1}{\sqrt{5! 4!}} \eta^{b_1 \cdots b_5 Y \beta, d_1 \cdots d_4 Y \beta} \end{pmatrix}, \quad (3.46)$$

[illegible]

for  $d \leq 7$  is

$$\eta^{k_1 \cdots k_7} = \begin{pmatrix} 0 & 0 & 0 & \chi_1 \frac{7! \delta_{j_1 \cdots j_7}^{k_1 \cdots k_7} \delta_j^i}{\sqrt{7!}} \\ 0 & 0 & \chi_2 \frac{7! \delta_{j_1 \cdots j_5 i_1 i_2}^{k_1 \cdots k_7}}{\sqrt{2! 5!}} & 0 \\ 0 & \chi_2 \frac{7! \delta_{i_1 \cdots i_5 j_1 j_2}^{k_1 \cdots k_7}}{\sqrt{2! 5!}} & 0 & 0 \\ \chi_1 \frac{7! \delta_{i_1 \cdots i_7}^{k_1 \cdots k_7} \delta_i^j}{\sqrt{7!}} & 0 & 0 & 0 \end{pmatrix}, \quad (3.49)$$

$$\eta^{p_1 \cdots p_6} = \begin{pmatrix} 0 & 0 & 0 & 0 & \chi_3 \frac{6! \delta_{n_1 \cdots n_6}^{p_1 \cdots p_6} \delta_n^m}{\sqrt{6!}} \\ 0 & 0 & 0 & \chi_4 \frac{6! \epsilon^{\beta \alpha} \delta_{n_1 \cdots n_5 m}^{p_1 \cdots p_6}}{\sqrt{5!}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \chi_4 \frac{6! \epsilon^{\alpha \beta} \delta_{m_1 \cdots m_5 n}^{p_1 \cdots p_6}}{\sqrt{5!}} & 0 & 0 & 0 \\ \chi_3 \frac{6! \delta_{m_1 \cdots m_6}^{p_1 \cdots p_6} \delta_m^n}{\sqrt{6!}} & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.50)$$

From these ansatz, we define

$$\eta^{k_1 \cdots k_6, l} \equiv \eta_{\text{KKM}}^{k_1 \cdots k_6, l} + \eta^{k_1 \cdots k_6 l}, \quad \eta^{p_1 \cdots p_5, q} \equiv \eta_{\text{KKM}}^{p_1 \cdots p_5, q} + \eta^{p_1 \cdots p_5 q}. \quad (3.51)$$

Supposing that  $\eta^{k_1 \cdots k_6, l}$  and  $\eta^{p_1 \cdots p_5, q}$  are related with each other by the linear map, under the decomposition  $\{i\} \rightarrow \{a, \alpha\}$  and  $\{m\} \rightarrow \{a, y\}$ , the nontrivial components  $\eta^{a_1 \cdots a_5, y}$  and  $\eta^{a_1 \cdots a_4 y, c}$  (which include the contribution from  $\eta^{k_1 \cdots k_6 l}$ ) should be expanded with  $\eta^{k_1 \cdots k_6, l}$  in the general forms

$$\begin{aligned} \eta^{a_1 \cdots a_5, y} &= \lambda_1 \eta^{a_1 \cdots a_5 [y, z]} + \lambda_2 \eta^{[a_1 \cdots a_4 [y z, | a_5] ]}, \\ \eta^{a_1 \cdots a_4 y, c} &= \lambda_3 \eta^{a_1 \cdots a_4 y z, c} + \lambda_4 \eta^{[a_1 \cdots a_4 [y z, | c] ]} + \lambda_5 \eta^{a_1 \cdots a_4 c [y, z]}. \end{aligned} \quad (3.52)$$

This requires  $\chi_1 = 3\chi_2$  and  $\chi_3 = 2\chi_4$ . We can determine the overall constant (up to sign) of  $\eta^{k_1 \cdots k_7}$  and  $\eta^{p_1 \cdots p_6}$  (i.e.,  $\chi_2$  and  $\chi_4$ ) by further requiring the conditions

$$\eta^{IJ; \mathbb{I}} \eta_{IJ; \mathbb{J}} \propto \delta_{\mathbb{J}}^{\mathbb{I}}, \quad \eta^{MN; \mathbb{M}} \eta_{MN; \mathbb{N}} \propto \delta_{\mathbb{N}}^{\mathbb{M}}. \quad (3.53)$$

By choosing a sign convention, we obtain the  $\eta$ -symbols shown in the previous subsections. The coefficients  $\lambda_1, \dots, \lambda_5$  also can be determined and the result is shown in Eq. (3.47).

Similarly, we can also determine the matrix form of the  $\Omega$ -tensors that appear in  $d = 7$ . Supposing that they are also constructed from combinations of products of Kronecker deltas, the defining properties,

$$\Omega_{IJ} = \Omega^{IJ}, \quad \Omega^{KI} \Omega_{KJ} = \delta_J^I, \quad \Omega_{IJ} = \Omega_{[IJ]}, \quad (3.54)$$

require them to have the following form up to the overall sign convention:

$$(\Omega^{IJ}) = \begin{pmatrix} 0 & 0 & 0 & \frac{\epsilon_{j_1 \cdots j_7} \delta_j^i}{\sqrt{7!}} \\ 0 & 0 & \pm \frac{\epsilon_{i_1 i_2 j_1 \cdots j_5}}{\sqrt{2! 5!}} & 0 \\ 0 & \mp \frac{\epsilon_{i_1 \cdots i_5 j_1 j_2}}{\sqrt{2! 5!}} & 0 & 0 \\ -\frac{\epsilon_{i_1 \cdots i_7} \delta_i^j}{\sqrt{7!}} & 0 & 0 & 0 \end{pmatrix}. \quad (3.55)$$



In order for the  $\Omega$ -tensor in the type IIB side, namely  $\Omega^{MN} \equiv (S^{-1})^M_I \Omega^{IJ} (S^{-T})_J^N$ , to be expressed covariantly by means of the Kronecker deltas, we shall choose the upper sign, and then the  $\Omega$ -tensor in the type IIB side becomes Eq. (3.39). In this manner, we have determined all of the  $\eta$ -symbols and the  $\Omega$ -tensor.

### 3.4. Properties of $\eta$ -symbols

We can easily check that the identities

$$\eta_{IJ;\mathbb{I}} \eta^{IJ;\mathbb{J}} = 2(d-1) \delta_{\mathbb{I}}^{\mathbb{J}}, \quad \eta_{MN;\mathbb{M}} \eta^{MN;\mathbb{N}} = 2(d-1) \delta_{\mathbb{M}}^{\mathbb{N}} \quad (3.56)$$

are satisfied for  $d = 2, \dots, 7$ . We can also show the identities

$$\eta^{IK;\mathbb{I}} \eta_{KJ;\mathbb{I}} = D_{d-1} \delta_J^I, \quad \eta^{\text{MP};\mathbb{M}} \eta_{\text{PN};\mathbb{M}} = D_{d-1} \delta_{\mathbb{N}}^{\mathbb{M}}, \quad (3.57)$$

where  $D_{d-1}$  is given by  $D_2 = 2$ ,  $D_3 = 3$ ,  $D_4 = 5$ ,  $D_5 = 10$ , and  $D_6 = 57/2$ . From identity (3.56) or (3.57), the normalization of the  $Y$ -tensor in  $E_{d(d)}$  EFT becomes

$$Y_{IJ}^{IJ} = n_d \quad (n_3 = 12, n_4 = 30, n_5 = 80, n_6 = 270, n_7 = 1568). \quad (3.58)$$

In the case of  $E_{7(7)}$  EFT, we can check additional identities. If we define

$$\begin{aligned} (t^{\mathbb{I}})_{I'}^J &\equiv \Omega_{IK} \eta^{KJ;\mathbb{I}}, & K^{\mathbb{I}\mathbb{J}} &\equiv \frac{1}{12} (t^{\mathbb{I}})_{I'}^J (t^{\mathbb{J}})_{J'}^{I'}, \\ (t_{\mathbb{I}})_{I'}^J &\equiv \Omega^{JK} \eta_{IK;\mathbb{I}}, & K_{\mathbb{I}\mathbb{J}} &\equiv \frac{1}{12} (t_{\mathbb{I}})_{I'}^J (t_{\mathbb{J}})_{J'}^{I'}, \end{aligned} \quad (3.59)$$

we can show a relation that connects the two types of  $\eta$ -symbols,  $\eta_{IJ;\mathbb{I}}$  and  $\eta^{IJ;\mathbb{I}}$ ,

$$\eta_{IJ;\mathbb{I}} = -K_{\mathbb{I}\mathbb{J}} \Omega_{IK} \Omega_{JL} \eta^{KL;\mathbb{J}}, \quad t_{\mathbb{I}} = K_{\mathbb{I}\mathbb{J}} t^{\mathbb{J}}, \quad K^{\mathbb{I}\mathbb{K}} K_{\mathbb{K}\mathbb{J}} = \delta_{\mathbb{J}}^{\mathbb{I}}, \quad (3.60)$$

where  $t_{\mathbb{I}} \equiv ((t_{\mathbb{I}})_{I'}^J)$  and  $t^{\mathbb{I}} \equiv ((t^{\mathbb{I}})_{I'}^J)$ . The matrix  $K \equiv (K_{\mathbb{I}\mathbb{J}})$  becomes

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{\epsilon_{j_1 \dots j_7} \epsilon_{l_1 \dots l_6}}{\sqrt{7! 6!}} \\ 0 & 0 & 0 & -\frac{\epsilon_{j_1 \dots j_7} \epsilon_{i_1 \dots i_4 l_1 l_2 l_3}}{\sqrt{4! 7! 3!}} & 0 \\ 0 & 0 & \frac{\epsilon_{i_1 \dots i_6 l} \epsilon_{kj_1 \dots j_6}}{\sqrt{6! 6!}} & 0 & 0 \\ 0 & -\frac{\epsilon_{i_1 \dots i_7} \epsilon_{j_1 \dots j_4 k_1 k_2 k_3}}{\sqrt{4! 7! 3!}} & 0 & 0 & 0 \\ -\frac{\epsilon_{i_1 \dots i_7} \epsilon_{jk_1 \dots k_6}}{\sqrt{7! 6!}} & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.61)$$

which has the eigenvalues 70 “+1” and 63 “−1.” The same relations are also satisfied in the type IIB side, and there the matrix  $K = (K_{MN})$  becomes

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\epsilon^{n_6 \cdots n_6}}{\sqrt{6! 6!}} \times \epsilon^{q_6 \cdots q_6} \epsilon^{\alpha\beta} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\epsilon^{m_1 m_2 q_1 \cdots q_4}}{\sqrt{2! 6! 4!}} \times \epsilon^{n_1 \cdots n_6} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\epsilon^{\beta\alpha} \epsilon^{m_1 \cdots m_4 q_1 q_2}}{\sqrt{4! 6! 2!}} \times \epsilon^{n_1 \cdots n_6} & 0 & 0 \\ 0 & 0 & 0 & \frac{\epsilon^{m_1 \cdots m_5 q}}{\sqrt{5! 5!}} \times \epsilon^{p n_1 \cdots n_5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\epsilon_{\alpha_1 \beta_1} \epsilon_{\alpha_2 \beta_2} & 0 & 0 & 0 \\ 0 & 0 & \frac{\epsilon^{\alpha\beta} \epsilon^{m_1 \cdots m_6}}{\sqrt{6! 2! 4!}} \times \epsilon^{p_1 p_2 n_1 \cdots n_4} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\epsilon^{m_1 \cdots m_6}}{\sqrt{6! 4! 2!}} \times \epsilon^{p_1 \cdots p_4 n_1 n_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\epsilon^{m_6 \cdots m_6}}{\sqrt{6! 6!}} \times \epsilon^{p_6 \cdots p_6} \epsilon^{\beta\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.62)$$

In fact,  $t^\mathbb{I}$  corresponds to the generators of the  $E_{7(7)}$  group. By using the generators of the  $E_{7(7)}$  group shown in Appendix A.2,  $t^\mathbb{I}$  can be expressed as

$$\begin{aligned} t^k &= \frac{1}{6!} \epsilon^{kl_1 \cdots l_6} R_{l_1 \cdots l_6}, \\ t^{k_1 \cdots k_4} &= -\frac{1}{3!} \epsilon^{k_1 \cdots k_4 l_1 l_2 l_3} R_{l_1 l_2 l_3}, \\ t^{k_1 \cdots k_6, k} &= \left( \epsilon^{k_1 \cdots k_6 j} \delta_i^k - \frac{3 - \sqrt{2}}{21} \epsilon^{k_1 \cdots k_6 k} \delta_i^j \right) K^i_j, \\ t^{k_1 \cdots k_7, l_1 l_2 l_3} &= \epsilon^{k_1 \cdots k_7} R^{l_1 l_2 l_3}, \\ t^{k_1 \cdots k_7, l_1 \cdots l_6} &= -\epsilon^{k_1 \cdots k_7} R^{l_1 \cdots l_6}. \end{aligned} \quad (3.63)$$

Similarly, the  $t^M$  are related to the generators in the type IIB parameterization as

$$\begin{aligned} t_\gamma &= \frac{1}{6!} \epsilon_{\gamma\delta} \epsilon^{p_1 \cdots p_6} R_{p_1 \cdots p_6}^\delta, \\ t^{p_1 p_2} &= -\frac{1}{4!} \epsilon^{p_1 p_2 q_1 \cdots q_4} R_{q_1 \cdots q_4}, \\ t_\gamma^{p_1 \cdots p_4} &= \epsilon_{\gamma\delta} \epsilon^{p_1 \cdots p_4 q_1 q_2} R_{q_1 q_2}^\delta, \\ t^{p_1 \cdots p_5, p} &= -\left( \epsilon^{p_1 \cdots p_5 q} \delta_r^p - \frac{1}{4} \epsilon^{p_1 \cdots p_5 p} \delta_r^q \right) K^r_q, \end{aligned}$$

$$\begin{aligned}
t_{(\gamma\delta)}^{p_1 \cdots p_6} &= \epsilon^{p_1 \cdots p_6} R_{(\gamma\delta)}, \\
t_{\gamma}^{p_1 \cdots p_6, q_1 q_2} &= -\epsilon^{p_1 \cdots p_6} R_{\gamma}^{q_1 q_2}, \\
t^{p_1 \cdots p_6, q_1 \cdots q_4} &= \epsilon^{p_1 \cdots p_6} R^{q_1 \cdots q_4}, \\
t_{\gamma}^{p_1 \cdots p_6, q_1 \cdots q_6} &= \epsilon^{p_1 \cdots p_6} R_{\gamma}^{q_1 \cdots q_6}.
\end{aligned} \tag{3.64}$$

Then,  $t^{[k_1 \cdots k_6, k]}$  or  $t^{[p_1 \cdots p_5, p]}$  and  $t_{(12)}^{p_1 \cdots p_6}$  are Cartan generators, and the matrix  $K$  corresponds to the Cartan–Killing form.

We can also check the following identities (Ref. [14]):

$$\begin{aligned}
(t_{\mathbb{I}})_K^I (t^{\mathbb{I}})_L^J &= \frac{1}{2} \delta_K^I \delta_L^J + \delta_K^J \delta_L^I - \eta^{IJ; \mathbb{I}} \eta_{KL; \mathbb{I}} - \frac{1}{2} \Omega^{IJ} \Omega_{KL}, \\
(t_{\mathbb{M}})_P^M (t^{\mathbb{M}})_Q^N &= \frac{1}{2} \delta_P^M \delta_Q^N + \delta_P^N \delta_Q^M - \eta^{MN; \mathbb{M}} \eta_{PQ; \mathbb{I}} - \frac{1}{2} \Omega^{MN} \Omega_{PQ},
\end{aligned} \tag{3.65}$$

$$t^{\mathbb{I}} t_{\mathbb{J}} t_{\mathbb{I}} = \frac{21}{2} t_{\mathbb{J}}, \quad t^{\mathbb{M}} t_{\mathbb{N}} t_{\mathbb{M}} = \frac{21}{2} t_{\mathbb{N}}. \tag{3.66}$$

#### 4. Generalized Lie derivative

By using the obtained  $\eta$ -symbols, the section condition  $\eta^{IJ; \mathbb{I}} \partial_I \otimes \partial_J = 0$  can be expressed as follows (see Ref. [22] for a quite similar section condition and also Ref. [25] for a section condition in the “underlying EFT”):

$$\partial_i \otimes \partial^{ki} + \partial^{ki} \otimes \partial_i = 0, \tag{4.1}$$

$$\partial_i \otimes \partial^{ik_1 \cdots k_4} + 6 \partial^{[k_1 k_2} \otimes \partial^{k_3 k_4]} + \partial^{ik_1 \cdots k_4} \otimes \partial_i = 0, \tag{4.2}$$

$$\begin{aligned}
&\partial_i \otimes \partial^{k_1 \cdots k_6 i, l} + \mathbf{c}_1 \partial_i \otimes \partial^{k_1 \cdots k_6 l, i} - 6 \partial^{l[k_1} \otimes \partial^{k_2 \cdots k_6]} + 21 \mathbf{c}_2 \partial^{[k_1 k_2} \otimes \partial^{k_3 \cdots k_6 l]} \\
&- 6 \partial^{[k_1 \cdots k_5} \otimes \partial^{k_6 l]} + 21 \mathbf{c}_2 \partial^{[k_1 \cdots k_5} \otimes \partial^{k_6 l]} + \partial^{k_1 \cdots k_6 l, i} \otimes \partial_j + \mathbf{c}_1 \partial^{k_1 \cdots k_6 l, i} \otimes \partial_i = 0,
\end{aligned} \tag{4.3}$$

$$\epsilon_{j_1 \cdots j_7} (21 \partial_i \otimes \partial^{j_1 \cdots j_7, i} + \partial^{[i_1 i_2} \otimes \partial^{i_3 \cdots i_7]} - \partial^{[i_1 \cdots i_5} \otimes \partial^{i_6 i_7]} - 21 \partial^{i_1 \cdots i_7, i} \otimes \partial_i) = 0, \tag{4.4}$$

$$5 \partial^{[k_1 k_2]} \otimes \partial^{l_1 l_2 l_3 [k_3 \cdots k_6, k_7]} - \partial^{[k_1 \cdots k_5} \otimes \partial^{k_6 k_7] l_1 l_2 l_3} + 5 \partial^{l_1 l_2 l_3 [k_1 \cdots k_4, k_5} \otimes \partial^{k_6 k_7]} = 0, \tag{4.5}$$

$$\partial^{k_1 \cdots k_7, [l_1} \otimes \partial^{l_2 \cdots l_6]} + \partial^{[l_2 \cdots l_6]} \otimes \partial^{k_1 \cdots k_7, |l_1]} = 0, \tag{4.6}$$

where  $\mathbf{c}_1$  and  $\mathbf{c}_2$  are defined in Eq. (3.48). In particular, when we consider, e.g., the SL(5) EFT, the above section conditions are truncated easily to get

$$\partial_i \otimes \partial^{ki} + \partial^{ki} \otimes \partial_i = 0, \quad \partial^{[k_1 k_2} \otimes \partial^{k_3 k_4]} = 0. \tag{4.7}$$

The section condition in the type IIB parameterization,  $\eta^{MN; \mathbb{M}} \partial_{\mathbb{M}} \otimes \partial_{\mathbb{N}} = 0$ , can also be rewritten in a similar manner, though we will not show this explicitly.

There are two well-known solutions to the section condition. One is the solution, called the M-theory section, where

$$\partial^{i_1 i_2} = 0, \quad \partial^{i_1 \cdots i_5} = 0, \quad \partial^{i_1 \cdots i_7, j} = 0 \tag{4.8}$$

are satisfied; namely, on the M-theory section, all fields depend only on the  $d$  coordinates  $x^i$ . The other solution is called the type IIB section, where

$$\partial_{\alpha}^m = 0, \quad \partial^{m_1 m_2 m_3} = 0, \quad \partial_{\alpha}^{m_1 \cdots m_5} = 0, \quad \partial^{m_1 \cdots m_6, n} = 0 \tag{4.9}$$

are satisfied. In the type IIB section, fields depend only on the  $d - 1$  coordinates  $\mathbf{x}^m$  (see Ref. [37] for the type IIB section in the SL(5) EFT and also Refs. [12–14,16] for a higher  $E_{d(d)}$  EFT).

On the M-theory section, the generalized Lie derivative reduces to the exceptional Dorfman bracket (Refs. [21–23,38]). Indeed, by using the  $Y$ -tensor,

$$Y_{KL}^{IJ} = \eta^{IJ;\mathbb{T}} \eta_{KL;\mathbb{T}} - \frac{1}{2} \Omega^{IJ} \Omega_{KL}, \quad (4.10)$$

and the explicit form of the  $\eta$ -symbols and the  $\Omega$ -tensor, we obtain

$$\begin{pmatrix} \delta_V W^i \\ \frac{\delta_V w_{i_1 i_2}}{\sqrt{2!}} \\ \frac{\delta_V w_{i_1 \dots i_5}}{\sqrt{5!}} \\ \frac{\delta_V w_{i_1 \dots i_7, i}}{\sqrt{7!}} \end{pmatrix} = \begin{pmatrix} \mathfrak{L}_V W^i \\ \frac{\mathfrak{L}_V w_{i_1 i_2} - w^k (dv_2)_{ki_1 i_2}}{\sqrt{2!}} \\ \frac{\mathfrak{L}_V w_{i_1 \dots i_5} + 10 (dv_2)_{[i_1 i_2 i_3} w_{i_4 i_5]} - w^k (dv_5)_{ki_1 \dots i_5}}{\sqrt{5!}} \\ \frac{\mathfrak{L}_V w_{i_1 \dots i_7, i} + 21 (dv_2)_{i[i_1 i_2} w_{i_3 \dots i_7]} + 7 w_{i[i_1} (dv_5)_{i_2 \dots i_7]}}{\sqrt{7!}} \end{pmatrix}, \quad (4.11)$$

where  $(dv_2)_{i_1 i_2 i_3} \equiv 3 \partial_{[i_1} v_{i_2 i_3]}$  and  $(dv_5)_{i_1 \dots i_6} \equiv 6 \partial_{[i_1} v_{i_2 \dots i_6]}$ . In the last line, we have repeatedly used the Schouten-like identities such as

$$\partial_{[i_1} v^k w_{i_2 \dots i_8], k} = 0, \quad (4.12)$$

which are satisfied for  $d \leq 7$ . This result precisely matches with the known result (Refs. [21–23,38]).

For a gauge parameter of the form  $V^I = \eta^{IJ;\mathbb{T}} \partial_J f_{\mathbb{T}} = \partial_J f^{IJ}$ , where  $f^{IJ} \equiv \eta^{IJ;\mathbb{T}} f_{\mathbb{T}}$ , the generalized Lie derivative becomes

$$\hat{\mathfrak{L}}_V W^I = (Y_{KL}^{IJ} Y_{RS}^{KP} - Y_{RS}^{IJ} \delta_L^P) \partial_{(J} \partial_{P)} f^{RS} W^L. \quad (4.13)$$

In fact, a condition,

$$(Y_{KL}^{IJ} Y_{RS}^{KP} - Y_{RS}^{IJ} \delta_L^P) \partial_{(J} \otimes \partial_{P)} = 0, \quad (4.14)$$

is necessary for the closure of the gauge algebra (Ref. [11]), and for  $d \leq 7$ , it is indeed satisfied under the section condition (1.7) (Ref. [11]). Therefore, a gauge parameter of the form  $V^I = \eta^{IJ;\mathbb{T}} \partial_J f_{\mathbb{T}}$  is a generalized Killing vector for an arbitrary  $f_{\mathbb{T}}$ . Moreover,  $V^I = \Omega^{IJ} \chi_J$  with  $\chi_J$  satisfying

$$\eta^{IJ;\mathbb{T}} \chi_I \otimes \partial_J = 0, \quad \Omega^{IJ} \chi_I \otimes \partial_J = 0 \quad (4.15)$$

is also a trivial generalized Killing vector (Ref. [14]),

$$\begin{aligned} \hat{\mathfrak{L}}_V W^I &= (Y_{PL}^{IJ} \Omega^{PK} - \Omega^{IK} \delta_L^J) \partial_J \chi_K W^L \\ &= - \left[ \eta^{KJ;\mathbb{T}} (t_{\mathbb{T}})_L^I - \frac{1}{2} \delta_L^I \Omega^{JK} \right] \partial_J \chi_K W^L = 0, \end{aligned} \quad (4.16)$$

where the identity (3.65) is used in the second equality and Eq. (4.15) is used in the last equality.

On the other hand, if we choose the type IIB section, the generalized Lie derivative takes the form

$$\begin{pmatrix} \delta_V w^m \\ \delta_V w_m^\alpha \\ \frac{\delta_V w_{m_1 m_2 m_3}}{\sqrt{3!}} \\ \frac{\delta_V w_{m_1 \dots m_5}^\alpha}{\sqrt{5!}} \\ \frac{\delta_V w_{m_1 \dots m_6, m}}{\sqrt{6!}} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_V w^m \\ \mathcal{L}_V w_m^\alpha - w^n (dv_1^\alpha)_{nm} \\ \frac{\mathcal{L}_V w_{m_1 m_2 m_3} - 3 \epsilon_{\gamma\delta} (dv_1^\gamma)_{[m_1 m_2} w_{m_3]}^\delta - w^n (dv_3)_{nm_1 m_2 m_3}}{\sqrt{3!}} \\ \frac{\mathcal{L}_V w_{m_1 \dots m_5}^\alpha + 10 (dv_1^\alpha)_{[m_1 m_2} w_{m_3 m_4 m_5]}^\alpha - 5 (dv_3)_{[m_1 \dots m_4} w_{m_5]}^\alpha - w^n (dv_5^\alpha)_{nm_1 \dots m_5}}{\sqrt{5!}} \\ \frac{\mathcal{L}_V w_{m_1 \dots m_6, m} + 6 \epsilon_{\gamma\delta} (dv_1^\gamma)_{m[m_1} w_{m_2 \dots m_6]}^\delta - 20 (dv_3)_{m[m_1 m_2 m_3} w_{m_4 m_5 m_6]} + \epsilon_{\gamma\delta} (dv_5^\gamma)_{m_1 \dots m_6} w_m^\delta}{\sqrt{6!}} \end{pmatrix}. \quad (4.17)$$

Again,  $V^M = \eta^{MN;M} \partial_N f_M$  and  $V^M = \Omega^{MN} \chi_N$  are trivial gauge parameters.

## 5. Linear section equation

In the  $O(d, d)$  DFT, the section condition is expressed as  $\eta^{IJ} \partial_I \otimes \partial_J = 0$ . This condition states that  $\partial_I$  is restricted to a  $d$ -dimensional maximal null subspace in the generalized tangent bundle. We can specify the maximal null subspace by introducing a set of independent  $d$  generalized vectors  $\lambda^a = (\lambda_I^a) = (\lambda_i^a, \lambda^{i;a})$  ( $a = 1, \dots, d$ ) satisfying

$$\lambda_I^a \eta^{IJ} \lambda_J^b = 0, \quad \lambda_I^a \eta^{IJ} \partial_J = 0. \quad (5.1)$$

If we consider a particular solution to the first equation,  $\lambda^a = \hat{\lambda}^a$ , that takes the form

$$\hat{\lambda}^a = (\hat{\lambda}_I^a) = \begin{pmatrix} \delta_i^a \\ 0 \end{pmatrix}, \quad (5.2)$$

the second equation in Eq. (5.1) gives  $\tilde{\partial}^i = 0$ , which is the commonly used section to reproduce the usual supergravity from DFT. More generally, if the  $d \times d$  matrix  $\lambda_i^a$  is invertible, we can always realize  $\lambda_i^a = \delta_i^a$  by a redefinition of  $\lambda^a$ ;  $\lambda^a \rightarrow \Lambda^a_b \lambda^b$ . Then, by introducing an antisymmetric tensor  $\beta^{ij} = \beta^{[ij]}$ , the general solution to the first equation in (5.1) becomes

$$\lambda^a = \begin{pmatrix} \delta_i^a \\ \beta^{ia} \end{pmatrix} = \begin{pmatrix} \delta_i^j & 0 \\ \beta^{ij} & \delta_j^i \end{pmatrix} \begin{pmatrix} \delta_j^a \\ 0 \end{pmatrix}, \quad (5.3)$$

which is just an  $O(d, d)$  rotation of the generalized vector  $\hat{\lambda}^a$ . For this general  $\lambda^a$ , the second equation in Eq. (5.1) becomes

$$\tilde{\partial}^i = \beta^{ij} \partial_j. \quad (5.4)$$

We can easily show that this leads to the section condition

$$\eta^{IJ} \partial_I \otimes \partial_J = \partial_i \otimes \tilde{\partial}^i + \tilde{\partial}^i \otimes \partial_i = (\beta^{ij} + \beta^{ji}) \partial_i \otimes \partial_j = 0. \quad (5.5)$$

In fact, in the context of generalized geometry, essentially the same set of generalized vectors has been considered in Ref. [39] (see also Ref. [40]). There, the maximal null subspace has been called the Dirac manifold or the Dirac structure, and the set of generalized vectors  $\lambda^a$  has been called the basis representation of the Dirac structure. In addition, it has been shown that the Dirac structure

can be characterized by an antisymmetric tensor, which is denoted by  $\beta^{ij}$  here. Alternatively, we can also characterize the Dirac structure by using a pure spinor (Ref. [41]).

A linear differential equation similar to Eq. (5.1), which reproduces the section condition, is called the linear section equation in Ref. [11]. There, a linear section equation in  $E_{d(d)}$  EFT for  $d \leq 7$  was proposed, but the equation strongly depends on the dimension  $d$  and it becomes complicated for higher  $d$ . Here, using the  $\eta$ -symbols, we propose a simple linear section equation, and show that it is equivalent to the proposal of Ref. [11] for the SL(5) EFT.

Our linear section equations take the form

$$\lambda^a \eta \partial = 0, \quad \lambda^a \Omega \partial = 0, \quad (5.6)$$

where  $\lambda^a$  ( $a = 1, \dots, N$ ) is a set of generalized vectors satisfying the null conditions

$$\lambda^a \eta \lambda^b = 0, \quad \lambda^a \Omega \lambda^b = 0. \quad (5.7)$$

If we show all of the indices explicitly, the linear section equations in the M-theory/type IIB parameterization become

$$\begin{aligned} \text{M-theory :} \quad & \lambda_I^a \eta^{IJ; \perp} \partial_J = 0, \quad \lambda_I^a \Omega^{IJ} \partial_J = 0, \\ \text{Type IIB :} \quad & \lambda_M^a \eta^{MN; \text{M}} \partial_N = 0, \quad \lambda_M^a \Omega^{MN} \partial_N = 0. \end{aligned} \quad (5.8)$$

The number of independent null generalized vectors  $N$  depends on the choice of the section. As was shown in Ref. [24],  $N$  cannot be greater than  $d$ , but we can always choose  $N = d$  or  $N = d - 1$ , which correspond to the M-theory section and the type IIB section, respectively. In fact, the M-theory section and the type IIB section can be described by the following set of null vectors,  $\hat{\lambda}^a$  ( $N = d$ ) and  $\hat{\kappa}^a$  ( $N = d - 1$ ):

$$\hat{\lambda}^a = \begin{pmatrix} \hat{\lambda}_i^a \\ \frac{(\hat{\lambda}^a)_{i_1 i_2}}{\sqrt{2!}} \\ \frac{(\hat{\lambda}^a)_{i_1 \dots i_5}}{\sqrt{5!}} \\ \frac{(\hat{\lambda}^a)_{i_1 \dots i_7, i}}{\sqrt{7!}} \end{pmatrix} = \begin{pmatrix} \delta_i^a \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{\kappa}^a = \begin{pmatrix} \hat{\kappa}_m^a \\ \frac{(\hat{\kappa}^a)_\alpha^m}{\sqrt{3!}} \\ \frac{(\hat{\kappa}^a)_{m_1 m_2 m_3}}{\sqrt{3!}} \\ \frac{(\hat{\kappa}^a)_{m_1 \dots m_5}}{\sqrt{5!}} \\ \frac{(\hat{\kappa}^a)_{m_1 \dots m_6, m}}{\sqrt{6!}} \end{pmatrix} = \begin{pmatrix} \delta_m^a \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (5.9)$$

In the former case,  $\hat{\lambda}^a \eta^k \partial = 0$  and  $\hat{\lambda}^a \eta^{k_1 \dots k_4} \partial = 0$  require  $\partial^{i_1 i_2} = \partial^{i_1 \dots i_5} = 0$ . On the other hand,  $\hat{\lambda}^a \eta^{k_1 \dots k_7, l_1 l_2 l_3} \partial = 0$  and  $\hat{\lambda}^a \eta^{k_1 \dots k_7, l_1 \dots l_6} \partial = 0$  are trivially satisfied. The remaining conditions,  $\hat{\lambda}^a \eta^{k_1 \dots k_6, l} \partial = 0$  and  $\hat{\lambda}^a \Omega \partial = 0$ , require  $\partial^{i_1 \dots i_7, i} = 0$ . Therefore,  $\hat{\lambda}^a$  describes the M-theory section where all fields depend only on  $x^i$ . The quadratic section condition  $\eta^{IJ; \perp} \partial_I \otimes \partial_J = 0$  is trivially satisfied on this section. Similarly, in the latter case, we can easily show that all fields depend only on  $x^m$ , and  $\hat{\kappa}^a$  describes the type IIB section.

In order to describe a more general section, we can rotate the above canonical sections,  $\hat{\lambda}^a$  and  $\hat{\kappa}^a$ , by  $U$ -duality transformations:

$$\hat{\lambda}_I^a \rightarrow \lambda_I^a \equiv a_I^J \hat{\lambda}_J^a, \quad \hat{\kappa}_M^a \rightarrow \kappa_M^a \equiv b_M^N \hat{\kappa}_N^a. \quad (5.10)$$

Since  $\eta^{IJ;\mathbb{I}}$  and  $\eta^{MN;\mathbb{M}}$  behave as the Clebsch–Gordan–Wigner coefficients, they satisfy

$$a_K^I a_L^J \eta^{KL;\mathbb{I}} = \hat{a}^{\mathbb{I}}_{\mathbb{J}} \eta^{IJ;\mathbb{J}}, \quad b_P^M b_Q^N \eta^{PQ;\mathbb{M}} = \hat{b}^{\mathbb{M}}_{\mathbb{N}} \eta^{MN;\mathbb{N}}, \quad (5.11)$$

where  $\hat{a}^{\mathbb{I}}_{\mathbb{J}}$  and  $\hat{b}^{\mathbb{M}}_{\mathbb{N}}$  are certain  $U$ -duality-transformation matrices in the  $R_2$ -representation. Moreover, the  $\Omega$ -tensor is invariant under the  $U$ -duality transformations. Then, the transformed generalized vector,  $\lambda_I^a$  or  $\bar{\lambda}_M^a$ , also satisfies the null conditions

$$\begin{aligned} \lambda^a \eta^{\mathbb{I}} \lambda^b &= \hat{a}^{\mathbb{I}}_{\mathbb{J}} (\hat{\lambda}^a \eta^{\mathbb{J}} \hat{\lambda}^b) = 0, & \lambda^a \Omega \lambda^b &= \hat{\lambda}^a \Omega \hat{\lambda}^b = 0, \\ \bar{\lambda}^a \eta^{\mathbb{M}} \bar{\lambda}^b &= \hat{b}^{\mathbb{M}}_{\mathbb{N}} (\hat{\bar{\lambda}}^a \eta^{\mathbb{N}} \hat{\bar{\lambda}}^b) = 0, & \bar{\lambda}^a \Omega \bar{\lambda}^b &= \hat{\bar{\lambda}}^a \Omega \hat{\bar{\lambda}}^b = 0, \end{aligned} \quad (5.12)$$

which are the required properties for writing the linear section equations. In fact, the linear section equations specified by the transformed  $\lambda^a$  or  $\bar{\lambda}^a$  lead to the quadratic section condition  $\eta^{IJ;\mathbb{I}} \partial_I \otimes \partial_J = 0$  or  $\eta^{MN;\mathbb{M}} \partial_M \otimes \partial_N = 0$ . Indeed, the linear section equations require that the dual components of the transformed derivatives  $\partial'_I \equiv (a^{-1})_I{}^J \partial_J$  or  $\partial'_M \equiv (b^{-1})_M{}^N \partial_N$  vanish, and from this, we can easily show  $\eta^{IJ;\mathbb{I}} \partial'_I \otimes \partial'_J = 0$  and  $\Omega^{IJ} \partial'_I \otimes \partial'_J = 0$  or  $\eta^{MN;\mathbb{M}} \partial'_M \otimes \partial'_N = 0$  and  $\Omega^{MN} \partial'_M \otimes \partial'_N = 0$ , which are equivalent to the quadratic section condition.

In an example of the  $E_{6(6)}$  EFT, the null conditions for the generalized vectors,  $(\lambda_I^a) = (\lambda_i^a, \frac{\lambda^{i_1 i_2; a}}{\sqrt{2!}}, \frac{\lambda^{i_1 \dots i_5; a}}{\sqrt{5!}})$  are

$$\begin{aligned} \lambda_i^a \lambda^{ki; b} + \lambda^{ki; a} \lambda_i^b &= 0, & \lambda_i^a \lambda^{ik_1 \dots k_4; b} + 6 \lambda^{[k_1 k_2]; a} \lambda^{[k_3 k_4]; b} + \lambda^{ik_1 \dots k_4; a} \lambda_i^b &= 0, \\ \lambda^{[k_1]; a} \lambda^{[k_2 \dots k_6]; b} - \lambda^{[k_1 \dots k_5]; a} \lambda^{[k_6]; b} &= 0. \end{aligned} \quad (5.13)$$

If  $\lambda_i^a$  is invertible, we can choose  $\lambda_i^a = \delta_i^a$  and then the first equation requires  $\lambda^{ka; b} = -\lambda^{kb; a}$ . Then, we can express  $\lambda^{ij; k}$  by using a 3-vector (i.e., antisymmetric third-rank tensor)  $\omega^{ijk}$ ,  $\lambda^{ij; k} = -\omega^{ijk}$ , and the second equation becomes

$$\lambda^{k_1 \dots k_4 k_5; k_6} + 6 \omega^{k_5 [k_1 k_2} \omega^{k_3 k_4] k_6} + \lambda^{k_6 k_1 \dots k_4; k_5} = 0. \quad (5.14)$$

This leads to

$$\lambda^{k_1 \dots k_5; k_6} = \omega^{k_1 \dots k_6} - 5 \omega^{[k_1 k_2 k_3} \omega^{k_4 k_5] k_6}, \quad (5.15)$$

where  $\omega^{k_1 \dots k_6}$  is an arbitrary 6-vector. The last equation in Eq. (5.13) is trivially satisfied. Therefore, the most general parameterization is given by

$$\begin{aligned} (\lambda_I^a) &= \begin{pmatrix} \lambda_i^a \\ \frac{\lambda^{i_1 i_2; a}}{\sqrt{2!}} \\ \frac{\lambda^{i_1 \dots i_5; a}}{\sqrt{5!}} \end{pmatrix} = \begin{pmatrix} \delta_i^a \\ -\frac{\omega^{i_1 i_2 a}}{\sqrt{2!}} \\ \frac{\omega^{i_1 \dots i_5 a} - 5 \omega^{[i_1 i_2 i_3} \omega^{i_4 i_5] a}}{\sqrt{5!}} \end{pmatrix} \\ &= \exp\left(\frac{1}{6!} \omega^{i_1 \dots i_6} R_{i_1 \dots i_6}\right) \exp\left(\frac{1}{3!} \omega^{ijk} R_{ijk}\right) \hat{\lambda}^a. \end{aligned} \quad (5.16)$$

In this way, when  $\lambda_i^a$  is invertible, the most general parameterization of  $\lambda^a$  is obtained from  $\hat{\lambda}^a$  via a  $U$ -duality transformation generated only by negative-root generators  $R_{i_1 i_2 i_3}$  and  $R_{i_1 \dots i_6}$  (GL( $d$ ) generators  $K^i_j$  are not necessary).<sup>2</sup> It is also the case for lower exceptional groups  $d \leq 5$ . The same

<sup>2</sup> The positive-root generators  $R^{i_1 i_2 i_3}$  and  $R^{i_1 \dots i_6}$  do not rotate  $\hat{\lambda}^a$ .

will be the case for  $E_{7(7)}$  also, and in that case,  $\lambda^a$  will be specified by 42 ( $= 35 + 7$ ) parameters  $\omega^{i_1 i_2 i_3}$  and  $\omega^{i_1 \dots i_6}$ . Similarly, in the case of the type IIB section, if  $\bar{\lambda}_m^a$  is invertible, the most general parameterization of  $\bar{\lambda}^a$  will be given by

$$\bar{\lambda}^a = \exp\left(\frac{1}{2!} \omega_\alpha^{m_1 m_2} R_{m_1 m_2}^\alpha\right) \exp\left(\frac{1}{4!} \omega^{m_1 \dots m_4} R_{m_1 \dots m_4}\right) \exp\left(\frac{1}{6!} \omega_\alpha^{m_1 \dots m_6} R_{m_1 \dots m_6}^\alpha\right) \hat{\lambda}^a. \quad (5.17)$$

In the following, we show that our linear section reproduces the known linear section equation in the SL(5) EFT, both for the M-theory and the type IIB sections.

### 5.1. M-theory section in SL(5) EFT

In the SL(5) EFT, the generalized coordinates  $x^I$  ( $I = 1, \dots, 10$ ) are frequently parameterized as  $x^I = x^{ab}$  ( $= x^{[ab]}$ ) ( $a, b = 1, \dots, 5$ ). In this parameterization, the section condition takes the form (Ref. [8])

$$\epsilon^{abcde} \partial_{bc} \otimes \partial_{de} = 0. \quad (5.18)$$

On the other hand, the linear section equation is expressed as (Ref. [11])

$$\Lambda_{[a} \partial_{bc]} = 0 \quad (a, b, c = 1, \dots, 5), \quad (5.19)$$

where  $\Lambda_a$  are arbitrary parameters that specify the section (which is considered to be a generalized notion of the pure spinor that specifies a generalized notion of the Dirac structure (Ref. [11])). Before comparing this equation with our linear section equation, let us consider the number of independent equations. In order for the linear section equation to be meaningful,  $\Lambda_a$  should not be a zero-vector, and let us suppose  $\Lambda_5 \neq 0$ . Then, we can decompose the linear section equation as

$$\Lambda_{[i} \partial_{jk]} = 0, \quad \partial_{ij} = -\frac{2}{\Lambda_5} \Lambda_{[i} \partial_{j]5}. \quad (5.20)$$

Since the first equation in Eq. (5.20) is satisfied when the second equation is satisfied, the second equation is equivalent to the linear section equation (although the SL(5) covariance is lost). Moreover, the linear section (5.19) is sufficient for the section condition (5.18) since Eq. (5.18) is automatically satisfied from the second equation.

On the other hand, our linear section equations are given by

$$\lambda_I^a \eta^{IJ; \mathbb{I}} \partial_J = 0, \quad (5.21)$$

where  $\lambda_I^a = (\lambda_I^a, \frac{\lambda^{i_1 i_2; a}}{\sqrt{2}})$  satisfies

$$\lambda_I^a \eta^{IJ; \mathbb{I}} \lambda_J^b = 0, \quad (5.22)$$

namely

$$\begin{aligned} \lambda_I^a \eta^{IJ; k} \lambda_J^b &= \lambda_k^a \lambda^{ik; b} + \lambda^{ik; a} \lambda_k^b = 0, \\ \lambda_I^a \eta^{IJ; k_1 \dots k_4} \lambda_J^b &= \epsilon_{i_1 i_2 j_1 j_2} \lambda^{i_1 i_2; a} \lambda^{j_1 j_2; b} = 0. \end{aligned} \quad (5.23)$$

If we consider a case where  $\lambda_k^a$  is invertible, we can choose  $\lambda_k^a = \delta_k^a$  and the first equation shows  $\lambda^{ij; a} = -\omega^{ija}$  with  $\omega^{ijk} = \omega^{[ijk]}$ . The second equation is then automatically satisfied since the following identity is satisfied in  $d = 4$ :

$$\epsilon_{k_1 \dots k_4} \omega^{k_1 k_2 i} \omega^{k_3 k_4 j} = 0. \quad (5.24)$$



Therefore, the set of the null vectors becomes

$$\lambda^a = \begin{pmatrix} \lambda_i^a \\ \frac{\lambda^{i_1 i_2; a}}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \delta_i^a \\ -\frac{\omega^{i_1 i_2 a}}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \delta_i^j & 0 \\ -\frac{\omega^{i_1 i_2 j}}{\sqrt{2}} & \delta_{j_1 j_2}^{i_1 i_2} \end{pmatrix} \begin{pmatrix} \delta_j^a \\ 0 \end{pmatrix}. \quad (5.25)$$

The linear section equations  $\lambda_J^a \eta^{IJ; k} \partial_J = 0$  and  $\lambda_J^a \eta^{IJ; k_1 \dots k_4} \partial_J = 0$  then become

$$\begin{aligned} \lambda_k^a \partial^{ik} - \lambda^{ik; a} \partial_k &= \partial^{ia} + \omega^{iak} \partial_k = 0, \\ \epsilon_{i_1 i_2 j_1 j_2} \lambda^{i_1 i_2; a} \partial^{j_1 j_2} &= -\epsilon_{i_1 i_2 j_1 j_2} \omega^{i_1 i_2 a} \partial^{j_1 j_2} = 0. \end{aligned} \quad (5.26)$$

The first condition is precisely the second equation in Eq. (5.20) if we make the identifications

$$\frac{\Lambda_i}{\Lambda_5} \equiv \frac{1}{3!} \epsilon_{ij_1 j_2 j_3} \omega^{j_1 j_2 j_3} \quad (\Lambda_5 \neq 0), \quad \partial_{i5} \equiv -\partial_{5i} \equiv \partial_i, \quad \partial_{ij} \equiv \frac{1}{2!} \epsilon_{ijkl} \partial^{kl}. \quad (5.27)$$

The second condition follows from the first. In this sense, when  $\lambda_i^a$  is invertible, our linear section equations are equivalent to Eq. (5.19).

For completeness, let us see the number of independent parameters that specify a section. The linear section equation (5.19) includes 5 parameters  $\Lambda_a$ , but as we can see from Eq. (5.20), only the 4 ratios  $\Lambda_i/\Lambda_5$  specify the section. This matches with the number of independent parameters  $\omega^{ijk}$  entering in our section equations.

If we consider a case where  $\lambda_i^a$  is not invertible, we may find an inequivalent section. For example, when  $(\lambda_i^a) = \text{diag}(1, 1, 0, 0)$ ,  $(\lambda^{34; 3}) = 1$ , and other components vanish ( $\lambda^4$  is a zero-vector in this case), the null condition is trivially satisfied, and the linear section equation shows fields can depend only on  $x^1, x^2$ , and  $y_{34}$ . This is the well-known type IIB section considered in the next section with a different parameterization of the generalized coordinates. Note that if the number of non-vanishing components of  $\lambda^a$  is too small, the linear section equation is not sufficient to reproduce the section condition.

## 5.2. Type IIB section in SL(5) EFT

The known linear section equation for the IIB section is (Refs. [42,43])

$$\Lambda^{ab} \partial_{bc} = 0, \quad (5.28)$$

where  $\Lambda^{ab}$  is defined to satisfy  $\epsilon_{eabcd} \Lambda^{ab} \Lambda^{cd} = 0$ . If  $\Lambda^{34} \neq 0$ , the condition  $\epsilon_{eabcd} \Lambda^{ab} \Lambda^{cd} = 0$  determines components  $\Lambda^{st}$  ( $s, t = 1, 2, 5$ ) as

$$\Lambda^{st} = \frac{2 \Lambda^{[s|3} \Lambda^{|t]4}}{\Lambda^{34}}. \quad (5.29)$$

Then, the linear section equations become

$$\partial_{s3} = \frac{\Lambda^{t4}}{\Lambda^{34}} \partial_{ts}, \quad \partial_{s4} = -\frac{\Lambda^{t3}}{\Lambda^{34}} \partial_{ts}, \quad \partial_{34} = \frac{\Lambda^{s3} \Lambda^{t4}}{(\Lambda^{34})^2} \partial_{st}, \quad (5.30)$$

and from these, we can show the section condition,  $\epsilon^{abcd} \partial_{ab} \partial_{cd} = 0$ . In this approach, a section is specified by 6 parameters  $\Lambda^{s3}/\Lambda^{34}$  and  $\Lambda^{s4}/\Lambda^{34}$ . In particular,  $\Lambda^{s3}/\Lambda^{34} = \Lambda^{s4}/\Lambda^{34} = 0$  corresponds to the type IIB section where fields depend on 3 coordinates  $\{x^{15}, x^{25}, x^{12}\}$ . The generalized coordinates  $x^{ab}$  in the literature are related to our generalized coordinates  $(x^I) = (x^i, \frac{y_{i_1 i_2}}{\sqrt{2!}})$  or

$(\mathbf{x}^M) = (\mathbf{x}^m, y_m^\alpha, \frac{y_{m_1 m_2 m_3}}{\sqrt{3!}})$  as follows, and the coordinates  $\{x^{15}, x^{25}, x^{12}\}$  correspond to the physical coordinates  $\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3\}$  in the type IIB parameterization:

| $x^{ab}$        | $x^{15}$       | $x^{25}$       | $x^{12}$       | $x^{13}$  | $x^{23}$ | $x^{53}$ | $x^{14}$ | $x^{24}$  | $x^{34}$  | $x^{45}$ |
|-----------------|----------------|----------------|----------------|-----------|----------|----------|----------|-----------|-----------|----------|
| M-theory        | $x^1$          | $x^2$          | $y_{34}$       | $-y_{24}$ | $y_{14}$ | $-x^3$   | $y_{23}$ | $-y_{13}$ | $y_{12}$  | $x^4$    |
| Type IIB theory | $\mathbf{x}^1$ | $\mathbf{x}^2$ | $\mathbf{x}^3$ | $-y_2^1$  | $y_1^1$  | $-y_3^1$ | $-y_2^2$ | $y_1^2$   | $y_{123}$ | $y_3^2$  |

(5.31)

Our linear section equations are specified by

$$\begin{aligned} \bar{\lambda}^a = \begin{pmatrix} \bar{\lambda}_m^a \\ \bar{\lambda}_\alpha^{m;a} \\ \frac{\bar{\lambda}_\alpha^{m_1 m_2 m_3; a}}{\sqrt{3!}} \end{pmatrix} &= \begin{pmatrix} \delta_m^n & 0 & 0 \\ \omega_\alpha^{mn} & \delta_\alpha^\beta \delta_n^m & 0 \\ -\frac{\frac{3}{2} \epsilon^{\gamma\delta} \omega_\gamma^{[m_1 m_2} \omega_\delta^{m_3]n}}{\sqrt{3!}} & \frac{3 \epsilon^{\beta\gamma} \delta_n^{[m_1} \omega_\gamma^{m_2 m_3]}}{\sqrt{3!}} & \delta_{n_1 n_2 n_3}^{m_1 m_2 m_3} \end{pmatrix} \begin{pmatrix} \delta_n^a \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \delta_m^a \\ \omega_\alpha^{ma} \\ -\frac{\frac{3}{2} \epsilon^{\gamma\delta} \omega_\gamma^{[m_1 m_2} \omega_\delta^{m_3]a}}{\sqrt{3!}} \end{pmatrix}, \end{aligned} \quad (5.32)$$

which satisfies the null conditions

$$\bar{\lambda}_m^a \bar{\lambda}_\gamma^{m;b} + \bar{\lambda}_\gamma^{m;a} \bar{\lambda}_m^b = 0, \quad \bar{\lambda}_m^a \bar{\lambda}^{mp_1 p_2; b} - 2! \epsilon^{\alpha\beta} \bar{\lambda}_\alpha^{[p_1; a} \bar{\lambda}_\beta^{p_2]; b} + \bar{\lambda}^{np_1 p_2; a} \bar{\lambda}_n^b = 0. \quad (5.33)$$

By using the explicit form of  $\bar{\lambda}^a$ , the linear section equations become

$$\partial_\alpha^m = -\omega_\alpha^{mn} \partial_n, \quad \partial^{123} = \epsilon^{\gamma\delta} (\omega_\gamma^{12} \omega_\delta^{13} \partial_1 + \omega_\gamma^{12} \omega_\delta^{23} \partial_2 + \omega_\gamma^{13} \omega_\delta^{23} \partial_3). \quad (5.34)$$

These are precisely equations (5.30) if we make the following identifications:

$$\omega_1^{s3} = \frac{\Lambda^{s4}}{\Lambda^{34}}, \quad \omega_2^{s3} = -\frac{\Lambda^{s3}}{\Lambda^{34}}, \quad \omega_1^{12} = -\frac{\Lambda^{54}}{\Lambda^{34}}, \quad \omega_2^{12} = -\frac{\Lambda^{35}}{\Lambda^{34}} \quad (s = 1, 2). \quad (5.35)$$

In this sense, our linear section equations are equivalent to the linear section equation (5.28) for the type IIB section in the SL(5) EFT.

## 6. Conclusions and discussion

In this paper, we obtained a set of  $\eta$ -symbols associated with branes in the string multiplet, and reproduced the known  $Y$ -tensor in  $E_{d(d)}$  EFT with  $d \leq 7$ . Our expression does not depend on the  $E_{d(d)}$  tensors for a particular  $d$ , and a reduction to lower  $d$  can be easily performed. Using the  $\eta$ -symbols (and the  $\Omega$ -tensor), we proposed linear section equations that reproduce the usual quadratic section condition. Equivalence to the known linear section for the M-theory and the type IIB sections in the SL(5) EFT are shown.

Our considerations are limited to  $d \leq 7$ , but we can also consider the  $E_{8(8)}$  EFT, where the number of  $\eta$ -symbols will be the same as the dimension of the  $R_2$ -representation of  $E_{8(8)}$ , namely 3875. According to Ref. [29], the branes in the string multiplet can be summarized as in Table 1. There, each brane in the table is wrapping a certain cycle in the 8-torus  $T^8$  and behaves as a string with a

**Table 1.** M-theory branes in the string multiplet for the  $E_{8(8)}$  EFT. In each column, all of the indices  $\{i_1, \dots, i_p, j_1, \dots, j_q, k_1, \dots, k_r\}$  must be different. Branes in the left and right columns are dual to each other (Ref. [29]).

| Brane                | Tension $\mathcal{T}$                                                                  | Number of degeneracy | Brane       | Tension $\mathcal{T}$                                                                  | Number of degeneracy |
|----------------------|----------------------------------------------------------------------------------------|----------------------|-------------|----------------------------------------------------------------------------------------|----------------------|
| M2                   | $\frac{R_i}{\ell_{11}^3}$                                                              | 8                    | 1 (7,1,0)   | $\frac{R_{i_1}^4 \dots R_{i_7}^4 R_j^3}{\ell_{11}^{33}}$                               | 8                    |
| M5                   | $\frac{R_{i_1} \dots R_{i_4}}{\ell_{11}^6}$                                            | 70                   | 1 (3,4,0)   | $\frac{R_{i_1}^4 \dots R_{i_4}^4 R_{j_1}^3 \dots R_{j_4}^3}{\ell_{11}^{30}}$           | 70                   |
| KKM                  | $\frac{R_i^2 R_{j_1} \dots R_{j_5}}{\ell_{11}^9}$                                      | 168                  | 1 (2,5,1)   | $\frac{R_{i_1}^4 R_{i_2}^4 R_{j_1}^3 \dots R_{j_5}^3 R_k^2}{\ell_{11}^{27}}$           | 168                  |
| 7×M8                 | $\frac{R_{i_1} \dots R_{i_7}}{\ell_{11}^9}$                                            | 7×8                  | 7×1 (1,7,0) | $\frac{R_i^4 R_{j_1}^3 \dots R_{j_7}^3}{\ell_{11}^{27}}$                               | 7×8                  |
| 5 <sup>3</sup>       | $\frac{R_{i_1}^2 R_{i_2}^2 R_{i_3}^2 R_{j_1} \dots R_{j_4}}{\ell_{11}^{12}}$           | 280                  | 1 (1,4,3)   | $\frac{R_i^4 R_{j_1}^3 \dots R_{j_4}^3 R_{k_1}^2 R_{k_2}^2 R_{k_3}^2}{\ell_{11}^{24}}$ | 280                  |
| 8 <sup>(1,0)</sup>   | $\frac{R_i^3 R_{j_1} \dots R_{j_7}}{\ell_{11}^{12}}$                                   | 8                    | 2 (7,0)     | $\frac{R_{i_1}^3 \dots R_{i_7}^3 R_j}{\ell_{11}^{24}}$                                 | 8                    |
| 7×7 <sup>2</sup>     | $\frac{R_{i_1}^2 R_{i_2}^2 R_{j_1} \dots R_{j_6}}{\ell_{11}^{12}}$                     | 7×28                 | 7×1 (6,2)   | $\frac{R_{i_1}^3 \dots R_{i_6}^3 R_{j_1}^2 R_{j_2}^2}{\ell_{11}^{24}}$                 | 7×28                 |
| 2 <sup>6</sup>       | $\frac{R_{i_1}^2 \dots R_{i_6}^2 R_{j_1} R_{j_2}}{\ell_{11}^{15}}$                     | 56                   | 1 (2,6)     | $\frac{R_{i_1}^3 R_{i_2}^3 R_{j_1}^2 \dots R_{j_6}^2}{\ell_{11}^{21}}$                 | 56                   |
| 5 <sup>(1,3)</sup>   | $\frac{R_i^3 R_{j_1}^2 R_{j_2}^2 R_{j_3}^2 R_{k_1} \dots R_{k_4}}{\ell_{11}^{15}}$     | 280                  | 2 (4,3)     | $\frac{R_{i_1}^3 \dots R_{i_4}^3 R_{j_1}^2 R_{j_2}^2 R_{j_3}^2 R_k}{\ell_{11}^{21}}$   | 280                  |
| 7×4 <sup>5</sup>     | $\frac{R_{i_1}^2 \dots R_{i_5}^2 R_{j_1} R_{j_2} R_{j_3}}{\ell_{11}^{15}}$             | 7×56                 | 7×1 (3,5)   | $\frac{R_{i_1}^3 R_{i_2}^3 R_{i_3}^3 R_{j_1}^2 \dots R_{j_5}^2}{\ell_{11}^{21}}$       | 7×56                 |
| 3 <sup>(2,4)</sup>   | $\frac{R_{i_1}^3 R_{i_2}^3 R_{j_1}^2 \dots R_{j_4}^2 R_{k_1} R_{k_2}}{\ell_{11}^{18}}$ | 420                  |             |                                                                                        |                      |
| 7×2 <sup>(1,6)</sup> | $\frac{R_i^3 R_{j_1}^2 \dots R_{j_6}^2 R_k}{\ell_{11}^{18}}$                           | 7×56                 |             |                                                                                        |                      |
| 35×1 <sup>8</sup>    | $\frac{R_{i_1}^2 \dots R_{i_8}^2}{\ell_{11}^{18}}$                                     | 35×1                 |             |                                                                                        |                      |

tension  $\mathcal{T}$  in the uncompactified spacetime. We call the brane a “ $b^{(c,d,e)}$ -brane” if the tension of the string takes the form

$$\mathcal{T} = \frac{(R_{i_1} \dots R_{i_c})^4 (R_{j_1} \dots R_{j_d})^3 (R_{k_1} \dots R_{k_e})^2 R_{l_1} \dots R_{l_{b-1}}}{\ell_{11}^{b+4c+3d+2e+1}}, \quad (6.1)$$

where  $R_i$  denotes the radius along the  $x^i$ -direction ( $i = 1, \dots, 8$ ), and  $\ell_{11}$  is the 11-dimensional Planck length. We also define  $b^{(d,e)} \equiv b^{(0,d,e)}$  and  $b^e \equiv b^{(0,e)}$ . It will be interesting to determine all of the  $\eta$ -symbols associated with the 3875 branes.

In this paper, we have not discussed the role of the  $\eta$ -symbols in worldvolume theories in detail, but in fact, they play an important role. In the  $T$ -duality manifest formulation of the string, the equations of motion can be expressed as the self-duality relation (Ref. [44])

$$\mathcal{M}_{IJ} *_{\gamma} \mathcal{P}^J = \eta_{IJ} \mathcal{P}^J, \quad (6.2)$$

where  $*_{\gamma}$  is the Hodge star operator on the worldsheet associated with the metric  $\gamma$ , and

$$(\mathcal{M}_{IJ}) \equiv \begin{pmatrix} G_{ij} - B_{ik} G^{kl} B_{lj} & B_{ik} G^{kj} \\ -G^{ik} B_{kj} & G^{ij} \end{pmatrix}, \quad (\mathcal{P}^I) \equiv \begin{pmatrix} dX^i \\ d\tilde{X}_i \end{pmatrix}. \quad (6.3)$$

As a generalization of this relation, if we consider a membrane theory in the approach of Ref. [45], the equations of motion can be expressed as

$$\mathcal{M}_{IJ} *_\gamma \mathcal{P}^J = \eta_{IJ}^{(M2)} \wedge \mathcal{P}^J, \quad \eta_{IJ}^{(M2)} \equiv \frac{1}{2} \eta_{IJ;k} dX^k \quad (6.4)$$

by using the  $\eta$ -symbol  $\eta_k$  associated with an M2-brane. It will be interesting to see whether this kind of self-duality relation is satisfied for all of the branes in the string multiplet. It is also interesting to see how the  $\Omega$ -tensor appears in the brane worldvolume theories.

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## Appendix A. Conventions and formulas

### A.1. Combinatoric factors

We shall use the following convention for multiple indices. When we consider M-theory, the generalized vector is parameterized as

$$(V^I) = \left( v^i, \frac{v_{i_1 i_2}}{\sqrt{2!}}, \frac{v_{i_1 \dots i_5}}{\sqrt{5!}}, \frac{v_{i_1 \dots i_7, j}}{\sqrt{7!}} \right), \quad (W_I) = \left( w_i, \frac{w^{i_1 i_2}}{\sqrt{2!}}, \frac{w^{i_1 \dots i_5}}{\sqrt{5!}}, \frac{w^{i_1 \dots i_7, j}}{\sqrt{7!}} \right). \quad (A.1)$$

The combinatoric factors are introduced such that the indices are summed with weight 1 when we consider the ordered multiple indices  $\overline{i_1 \dots i_p}$ , which satisfy  $i_1 < \dots < i_p$ . For example, the inner product between  $V^I$  and  $W_I$  becomes

$$\begin{aligned} V^I W_I &= v^i w_i + \frac{1}{2!} v_{i_1 i_2} w^{i_1 i_2} + \frac{1}{5!} v_{i_1 \dots i_5} w^{i_1 \dots i_5} + \frac{1}{7!} v_{i_1 \dots i_7, j} w^{i_1 \dots i_7, j} \\ &= v^i w_i + v_{\overline{i_1 i_2}} w^{\overline{i_1 i_2}} + v_{\overline{i_1 \dots i_5}} w^{\overline{i_1 \dots i_5}} + v_{\overline{i_1 \dots i_7, j}} w^{\overline{i_1 \dots i_7, j}}, \end{aligned} \quad (A.2)$$

and in the second line, all components are summed with weight 1. Similarly, the generalized coordinates and derivatives are defined as

$$(x^I) = \left( x^i, \frac{y_{i_1 i_2}}{\sqrt{2!}}, \frac{y_{i_1 \dots i_5}}{\sqrt{5!}}, \frac{y_{i_1 \dots i_7, j}}{\sqrt{7!}} \right), \quad (\partial_I) = \left( \partial_i, \frac{\partial^{i_1 i_2}}{\sqrt{2!}}, \frac{\partial^{i_1 \dots i_5}}{\sqrt{5!}}, \frac{\partial^{i_1 \dots i_7, j}}{\sqrt{7!}} \right), \quad (A.3)$$

$$(x^{\bar{I}}) = \left( x^i, y_{\overline{i_1 i_2}}, y_{\overline{i_1 \dots i_5}}, y_{\overline{i_1 \dots i_7, j}} \right), \quad (\partial_{\bar{I}}) = \left( \partial_i, \partial^{\overline{i_1 i_2}}, \partial^{\overline{i_1 \dots i_5}}, \partial^{\overline{i_1 \dots i_7, j}} \right). \quad (A.4)$$

We define the derivative as

$$\partial^{i_1 \dots i_p} y_{j_1 \dots j_p} = \delta_{j_1 \dots j_p}^{i_1 \dots i_p} \equiv \delta_{j_1}^{i_1} \dots \delta_{j_p}^{i_p}, \quad \partial^{\overline{i_1 \dots i_p}} y_{\overline{j_1 \dots j_p}} = \delta_{\overline{j_1 \dots j_p}}^{\overline{i_1 \dots i_p}} \equiv p! \delta_{j_1 \dots j_p}^{i_1 \dots i_p}, \quad (A.5)$$

which gives, e.g.,  $\partial^{12}y_{12} = 1/2$  and  $\partial^{\overline{12}}y_{\overline{12}} = 1$ . If we define the Kronecker delta as

$$(\delta_I^J) = \begin{pmatrix} \delta_i^j & 0 & 0 & 0 \\ 0 & \delta_{j_1 j_2}^{i_1 i_2} & 0 & 0 \\ 0 & 0 & \delta_{j_1 \dots j_5}^{i_1 \dots i_5} & 0 \\ 0 & 0 & 0 & \delta_{j_1 \dots j_7}^{i_1 \dots i_7} \delta_j^i \end{pmatrix}, \quad (\delta_{\bar{I}}^{\bar{J}}) = \begin{pmatrix} \delta_i^j & 0 & 0 & 0 \\ 0 & \delta_{j_1 j_2}^{\overline{i_1 i_2}} & 0 & 0 \\ 0 & 0 & \delta_{j_1 \dots j_5}^{\overline{i_1 \dots i_5}} & 0 \\ 0 & 0 & 0 & \delta_{j_1 \dots j_7}^{\overline{i_1 \dots i_7}} \delta_j^i \end{pmatrix}, \quad (\text{A.6})$$

they satisfy

$$\partial_I x^J = \delta_I^J, \quad \partial_{\bar{I}} x^{\bar{J}} = \delta_{\bar{I}}^{\bar{J}}, \quad \delta_I^I = \delta_{\bar{I}}^{\bar{I}} = D. \quad (\text{A.7})$$

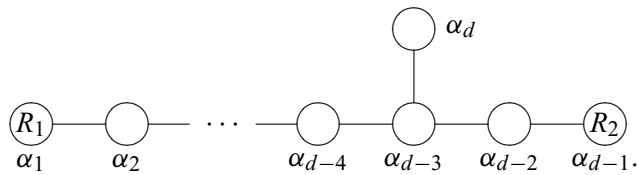
If we use the ordered indices, e.g., the matrix  $\eta^{k_1 \dots k_4}$  has a simpler form. Indeed, the complicated numerical factors disappear:

$$\eta^{\bar{I}\bar{J}; \overline{k_1 \dots k_4}} \equiv \begin{pmatrix} 0 & 0 & \delta_{j_1 \dots j_5}^{\overline{i k_1 \dots k_4}} & 0 \\ 0 & \delta_{i_1 i_2 j_1 j_2}^{\overline{k_1 \dots k_4}} & 0 & 0 \\ \delta_{i_1 \dots i_5}^{\overline{j k_1 \dots k_4}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.8})$$

In fact, all of the  $\eta$ -symbols except those associated with KKM and 8-branes (or KKM and 7<sub>2</sub>-branes in the type IIB side) have a simple form without complicated numerical factors. If we stick to the unordered multiple indices, as in the main text, the rule for the numerical factor is as follows: For a  $\{i_1 \dots i_p, k_1 \dots k_q\} - \{j_1 \dots j_r, l_1 \dots l_s\}$  component of  $\eta^\Gamma$ , we introduce  $\frac{1}{\sqrt{p! q! r! s!}}$  (where  $q$  or  $s$  may be 0). For each  $\delta_{j_1 \dots j_p}^{i_1 \dots i_p}$  inside  $\eta^\Gamma$ , we introduce  $p!$ . If there are contractions of multiple indices in the Kronecker deltas like  $\delta_{\dots i_1 \dots i_p}^{\dots i_1 \dots i_p}$ , we additionally introduce  $1/p!$ . This rule reproduces (almost) all of the numerical factors in  $\eta^\Gamma$  and  $\Omega$ .

## A.2. $E_{d(d)}$ group

The simple roots of the  $E_{d(d)}$  group are denoted by  $\alpha_n$  ( $n = 1, \dots, d$ ) and their relation is shown in the following Dynkin diagram:



In this convention, the  $R_1$ -/ $R_2$ -representations are defined by the following Dynkin labels:

$$\begin{aligned} R_1\text{-representation (particle multiplet):} & \quad (1, 0, \dots, 0), \\ R_2\text{-representation (string multiplet):} & \quad (0, \dots, 0, 1, 0). \end{aligned} \quad (\text{A.9})$$

The generators of the  $E_{d(d)}$  group ( $d \leq 7$ ) can be parameterized in two different ways, depending on whether we are considering M-theory or type IIB theory (Refs. [10,18,46,47]):

$$\begin{aligned} \text{M-theory: } & \{K^i_j, R^{i_1 i_2 i_3}, R_{i_1 i_2 i_3}, R^{i_1 \dots i_6}, R_{i_1 \dots i_6}\}, \\ \text{type IIB: } & \{K^m_n, R_{\alpha\beta}, R^\alpha_{m_1 m_2}, R^\alpha_{m_1 m_2}, R^{m_1 \dots m_4}, R_{m_1 \dots m_4}, R^\alpha_{m_1 \dots m_6}, R^\alpha_{m_1 \dots m_6}\}, \end{aligned} \quad (\text{A.10})$$

where  $i, j = 1, \dots, d$ ,  $m, n = 1, \dots, d-1$ , and  $\alpha, \beta = 1, 2$ . By considering  $R_{\alpha\beta} = R_{(\alpha\beta)}$ , the number of the above generators is the same as the dimension of the  $E_{d(d)}$  group ( $d \leq 7$ ).

In the M-theory parameterization, the explicit forms of the generators are given by

$$(K^{k_1}_{k_2})_I^J \equiv \begin{pmatrix} \delta_i^{k_1} \delta_{k_2}^j & 0 & 0 & 0 \\ 0 & -\frac{2! 2! \delta_{k_2}^{i_1 i_2} \delta_{j_1 j_2}^{k_1 l}}{\sqrt{2! 2!}} & 0 & 0 \\ 0 & 0 & -\frac{5! 5! \delta_{k_2 l_1 \dots l_4}^{i_1 \dots i_5} \delta_{j_1 \dots j_5}^{k_1 l_1 \dots l_4}}{4! \sqrt{5! 5!}} & 0 \\ 0 & 0 & 0 & -\frac{2 \cdot 7! \delta_{j_1 \dots j_7}^{i_1 \dots i_7} \delta_i^{(i} \delta_{k_2}^{k_1)}}{\sqrt{7! 7!}} \end{pmatrix} + \frac{\delta_{k_2}^{k_1}}{9-d} \delta_I^J, \quad (\text{A.11})$$

$$(R^{k_1 k_2 k_3})_I^J \equiv \begin{pmatrix} 0 & -\frac{3! \delta_{ij_1 j_2}^{k_1 k_2 k_3}}{\sqrt{2!}} & 0 & 0 \\ 0 & 0 & \frac{5! \delta_{j_1 \dots j_5}^{i_1 i_2 k_1 k_2 k_3}}{\sqrt{2! 5!}} & 0 \\ 0 & 0 & 0 & \frac{7! 3! \delta_{j_1 \dots j_7}^{i_1 \dots i_5 l_1 l_2} \delta_{l_1 l_2 j}^{k_1 k_2 k_3}}{2! \sqrt{5! 7!}} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{A.12})$$

$$(R_{k_1 k_2 k_3})_I^J \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{3! \delta_{k_1 k_2 k_3}^{i_1 i_2 j}}{\sqrt{2!}} & 0 & 0 & 0 \\ 0 & \frac{5! \delta_{j_1 j_2 k_1 k_2 k_3}^{i_1 \dots i_5}}{\sqrt{2! 5!}} & 0 & 0 \\ 0 & 0 & \frac{7! 3! \delta_{j_1 \dots j_5 l_1 l_2}^{i_1 \dots i_7} \delta_{k_1 k_2 k_3}^{l_1 l_2 i}}{2! \sqrt{5! 7!}} & 0 \end{pmatrix}, \quad (\text{A.13})$$

$$(R^{k_1 \dots k_6})_I^J \equiv \begin{pmatrix} 0 & 0 & -\frac{6! \delta_{ij_1 \dots j_5}^{k_1 \dots k_6}}{\sqrt{5!}} & 0 \\ 0 & 0 & 0 & \frac{7! 6! \delta_{j_1 \dots j_7}^{i_1 i_2 l_1 \dots l_5} \delta_{l_1 \dots l_5 j}^{k_1 \dots k_6}}{5! \sqrt{2! 7!}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{A.14})$$

$$(R_{k_1 \dots k_6})_I^J \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{6! \delta_{k_1 \dots k_6}^{i_1 \dots i_5 j}}{\sqrt{5!}} & 0 & 0 & 0 \\ 0 & \frac{7! 6! \delta_{j_1 j_2 l_1 \dots l_5}^{i_1 \dots i_7} \delta_{k_1 \dots k_6}^{l_1 \dots l_5 i}}{5! \sqrt{2! 7!}} & 0 & 0 \end{pmatrix}. \quad (\text{A.15})$$

On the other hand, in the type IIB parameterization, the explicit forms of the generators are given by

$$(K^{p_1}_{p_2})_M^N \equiv \begin{pmatrix} \delta_m^{p_1} \delta_n^{p_2} & 0 & 0 & 0 & 0 \\ 0 & -\delta_\alpha^\beta \delta_{p_2}^m \delta_n^{p_1} & 0 & 0 & 0 \\ 0 & 0 & -\frac{3!3! \delta_{p_2 q_1 q_2}^{m_1 m_2 m_3} \delta_{n_1 n_2 n_3}^{p_1 q_1 q_2}}{2! \sqrt{3!3!}} & 0 & 0 \\ 0 & 0 & 0 & -\frac{5!5! \delta_\alpha^\beta \delta_{p_2 q_1 \dots q_4}^{m_1 \dots m_5} \delta_{n_1 \dots n_5}^{p_1 q_1 \dots q_4}}{4! \sqrt{5!5!}} & 0 \\ 0 & 0 & 0 & 0 & -\frac{2 \cdot 6! \delta_{n_1 \dots n_6}^{m_1 \dots m_6} \delta_{p_2}^{(m} \delta_n^{p_1)}}{\sqrt{6!6!}} \end{pmatrix} + \frac{\delta_{p_2}^{p_1}}{9-d} \delta_M^N, \quad (\text{A.16})$$

$$(R_{\gamma\delta})_M^N \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon_{\alpha(\gamma} \delta_{\delta)}^\beta \delta_n^m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_{\alpha(\gamma} \delta_{\delta)}^\beta \delta_{n_1 \dots n_5}^{m_1 \dots m_5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{A.17})$$

$$(R_\gamma^{p_1 p_2})_M^N \equiv \begin{pmatrix} 0 & -2! \delta_\gamma^\alpha \delta_{mn}^{p_1 p_2} & 0 & 0 & 0 \\ 0 & 0 & \frac{3! \epsilon_{\alpha\gamma} \delta_{n_1 n_2 n_3}^{mp_1 p_2}}{\sqrt{3!}} & 0 & 0 \\ 0 & 0 & 0 & \frac{5! \delta_\gamma^\beta \delta_{n_1 \dots n_5}^{m_1 m_2 m_3 p_1 p_2}}{\sqrt{3!5!}} & 0 \\ 0 & 0 & 0 & 0 & -\frac{6!2! \epsilon_{\alpha\gamma} \delta_{n_1 \dots n_6}^{m_1 \dots m_5 q} \delta_{qn}^{p_1 p_2}}{\sqrt{5!6!}} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{A.18})$$

$$(R_{p_1 p_2}^\gamma)_M^N \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2! \delta_\alpha^\gamma \delta_{p_1 p_2}^{mn} & 0 & 0 & 0 & 0 \\ 0 & \frac{3! \epsilon^{\beta\gamma} \delta_{n p_1 p_2}^{m_1 m_2 m_3}}{\sqrt{3!}} & 0 & 0 & 0 \\ 0 & 0 & \frac{5! \delta_\alpha^\gamma \delta_{n_1 n_2 n_3 p_1 p_2}^{m_1 \dots m_5}}{\sqrt{3!5!}} & 0 & 0 \\ 0 & 0 & 0 & -\frac{6!2! \epsilon^{\beta\gamma} \delta_{n_1 \dots n_5 q}^{m_1 \dots m_6} \delta_{p_1 p_2}^{qm}}{\sqrt{5!6!}} & 0 \end{pmatrix}, \quad (\text{A.19})$$

$$(R^{p_1 \dots p_4})_M^N \equiv \begin{pmatrix} 0 & 0 & -\frac{4! \delta_{mn_1 n_2 n_3}^{p_1 \dots p_4}}{\sqrt{3!}} & 0 & 0 \\ 0 & 0 & 0 & -\frac{5! \delta_\alpha^\beta \delta_{n_1 \dots n_5}^{mp_1 \dots p_4}}{\sqrt{5!}} & 0 \\ 0 & 0 & 0 & 0 & -\frac{6!4! \delta_{n_1 \dots n_6}^{m_1 m_2 m_3 q_1 q_2 q_3} \delta_{q_1 q_2 q_3 n}^{p_1 \dots p_4}}{3! \sqrt{3!6!}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{A.20})$$

$$(R_{p_1 \dots p_4})_M^N \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{4! \delta_{p_1 \dots p_4}^{m_1 m_2 m_3 n}}{\sqrt{3!}} & 0 & 0 & 0 & 0 \\ 0 & -\frac{5! \delta_\alpha^\beta \delta_{np_1 \dots p_4}^{m_1 \dots m_5}}{\sqrt{5!}} & 0 & 0 & 0 \\ 0 & 0 & -\frac{6! 4! \delta_{n_1 n_2 n_3 q_1 q_2 q_3}^{m_1 \dots m_6} \delta_{p_1 \dots p_4}^{q_1 q_2 q_3 m}}{3! \sqrt{3!} 6!} & 0 & 0 \end{pmatrix}, \quad (\text{A.21})$$

$$(R_\gamma^{p_1 \dots p_6})_M^N \equiv \begin{pmatrix} 0 & 0 & 0 & -\frac{6! \delta_\gamma^\beta \delta_{mn_1 \dots n_5}^{p_1 \dots p_6}}{\sqrt{5!}} & 0 \\ 0 & 0 & 0 & 0 & \frac{6! 6! \epsilon_{\alpha\gamma} \delta_{n_1 \dots n_6}^{mq_1 \dots q_5} \delta_{q_1 \dots q_5 n}^{p_1 \dots p_6}}{5! \sqrt{6!}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{A.22})$$

$$(R_{p_1 \dots p_6}^\gamma)_M^N \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{6! \delta_\alpha^\gamma \delta_{p_1 \dots p_6}^{m_1 \dots m_5 n}}{\sqrt{5!}} & 0 & 0 & 0 & 0 \\ 0 & \frac{6! 6! \epsilon^{\beta\gamma} \delta_{nq_1 \dots q_5}^{m_1 \dots m_6} \delta_{p_1 \dots p_6}^{q_1 \dots q_5 m}}{5! \sqrt{6!}} & 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.23})$$

## Appendix B. Comparison with known $Y$ -tensors

In this appendix, we reproduce known  $Y$ -tensors from our result.

### B.1. $Y$ -tensor in $SL(5)$ EFT

In the  $SL(5)$  EFT, we have 5 nonvanishing  $\eta$ -symbols, which can be redefined as

$$\epsilon^k \equiv \eta^k = \begin{pmatrix} 0 & \frac{2! \delta_{1/2}^{ki}}{\sqrt{2!}} \\ \frac{2! \delta_{1/2}^{kj}}{\sqrt{2!}} & 0 \end{pmatrix}, \quad \epsilon^5 \equiv \frac{1}{4!} \epsilon_{k_1 \dots k_4} \eta^{k_1 \dots k_4} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\epsilon_{i_1 i_2 j_1 j_2}}{\sqrt{2!} 2!} \end{pmatrix}. \quad (\text{B.1})$$

If we redefine the coordinates as  $(x^I) = (x^i, \frac{x^{i_1 i_2}}{\sqrt{2!}})$  with  $x^{i_1 i_2} \equiv \frac{1}{2!} \epsilon^{i_1 i_2 j_1 j_2} y_{j_1 j_2}$ , they become

$$\epsilon^k = \begin{pmatrix} 0 & \frac{\epsilon^{kij_1 j_2}}{\sqrt{2!}} \\ \frac{\epsilon^{kji_1 i_2}}{\sqrt{2!}} & 0 \end{pmatrix}, \quad \epsilon^5 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\epsilon^{i_1 i_2 j_1 j_2}}{\sqrt{2!} 2!} \end{pmatrix}. \quad (\text{B.2})$$

These can be neatly summarized as follows by introducing indices  $\mathbf{a}, \mathbf{b}, \mathbf{c} = 1, \dots, 5$  and a totally antisymmetric tensor  $\epsilon^{\mathbf{a}_1 \dots \mathbf{a}_5}$  satisfying  $\epsilon^{i_1 \dots i_4 5} = \epsilon^{i_1 \dots i_4}$ :

$$(\epsilon^{\mathbf{c}}) = (\epsilon^k, \epsilon^5), \quad \epsilon^{\mathbf{c}} = \begin{pmatrix} 0 & \frac{\epsilon^{ci_5 j_1 j_2}}{\sqrt{2!}} \\ \frac{\epsilon^{cj_5 i_1 i_2}}{\sqrt{2!}} & \frac{\epsilon^{ci_1 i_2 j_1 j_2}}{\sqrt{2!} 2!} \end{pmatrix}. \quad (\text{B.3})$$



By further using the conventional parameterization  $x^I \equiv x^{a_1 a_2}$  ( $x^{i5} \equiv x^i$ ), they become

$$\epsilon^c = (\epsilon^{cIJ}) = \left( \frac{\epsilon^{ca_1 a_2 b_1 b_2}}{\sqrt{2! 2!}} \right). \quad (B.4)$$

We also define  $\epsilon_c = (\epsilon_{cIJ}) = \left( \frac{\epsilon_{ca_1 a_2 b_1 b_2}}{\sqrt{2! 2!}} \right)$ , and then  $Y_{KL}^{IJ}$  becomes

$$Y_{KL}^{IJ} = \frac{\epsilon^{ea_1 a_2 b_1 b_2}}{\sqrt{2! 2!}} \frac{\epsilon_{ec_1 c_2 d_1 d_2}}{\sqrt{2! 2!}}, \quad (B.5)$$

which is summarized as  $Y_{KL}^{IJ} = \epsilon^{eIJ} \epsilon_{eKL}$  in Eq. (1.6).

### B.2. $Y$ -tensor in $SO(5, 5)$ EFT

In the  $SO(5, 5)$  EFT, we have 10  $\eta$ -symbols, which can be redefined as

$$\gamma^k \equiv \sqrt{2} \eta^k = \sqrt{2} \begin{pmatrix} 0 & \frac{2! \delta_{j_1 j_2}^{ki}}{\sqrt{2!}} & 0 \\ \frac{2! \delta_{i_1 i_2}^{kj}}{\sqrt{2!}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (B.6)$$

$$\gamma_k \equiv \sqrt{2} \frac{\epsilon_{kk_1 \dots k_4} \eta^{k_1 \dots k_4}}{4!} = \sqrt{2} \begin{pmatrix} 0 & 0 & \frac{5 \delta_{[j_1}^i \epsilon_{j_2 \dots j_5] k}}{\sqrt{5!}} \\ 0 & \frac{\epsilon_{ki_1 i_2 j_1 j_2}}{\sqrt{2! 2!}} & 0 \\ \frac{5 \delta_{[i_1}^j \epsilon_{i_2 \dots i_5] k}}{\sqrt{5!}} & 0 & 0 \end{pmatrix}. \quad (B.7)$$

In the coordinates  $(x^I) = (x^i, \frac{y_{i_1 i_2}}{\sqrt{2}}, z)$  with  $z \equiv \frac{1}{5!} \epsilon^{i_1 \dots i_5} y_{j_1 \dots j_5}$ , they become

$$\gamma^k = \sqrt{2} \begin{pmatrix} 0 & \frac{2! \delta_{j_1 j_2}^{ki}}{\sqrt{2!}} & 0 \\ \frac{2! \delta_{i_1 i_2}^{kj}}{\sqrt{2!}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \gamma_k = \sqrt{2} \begin{pmatrix} 0 & 0 & \delta_k^i \\ 0 & \frac{\epsilon_{ki_1 i_2 j_1 j_2}}{\sqrt{2! 2!}} & 0 \\ \delta_k^j & 0 & 0 \end{pmatrix}. \quad (B.8)$$

We also define matrices

$$\bar{\gamma}^k \equiv (\bar{\gamma}_{IJ}^k) \equiv \sqrt{2} \frac{\epsilon^{kk_1 \dots k_4} \eta_{k_1 \dots k_4}}{4!}, \quad \bar{\gamma}_k \equiv (\bar{\gamma}_{kIJ}) \equiv \sqrt{2} \eta_k, \quad (B.9)$$

or more explicitly,

$$\bar{\gamma}^k = \sqrt{2} \begin{pmatrix} 0 & 0 & \delta_i^k \\ 0 & \frac{\epsilon^{ki_1 i_2 j_1 j_2}}{\sqrt{2! 2!}} & 0 \\ \delta_j^k & 0 & 0 \end{pmatrix}, \quad \bar{\gamma}_k = \sqrt{2} \begin{pmatrix} 0 & \frac{2! \delta_{ki}^{j_1 j_2}}{\sqrt{2!}} & 0 \\ \frac{2! \delta_{kj}^{i_1 i_2}}{\sqrt{2!}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (B.10)$$

Then, the  $Y$ -tensor can be expressed as

$$Y_{KL}^{IJ} = \eta^{kJ} \eta_{kKL} + \frac{1}{4!} \eta^{k_1 \dots k_4 IJ} \eta_{k_1 \dots k_4 KL} = \frac{1}{2} (\gamma^{kJ} \bar{\gamma}_{kKL} + \gamma_k^{IJ} \bar{\gamma}_{IJ}^k). \quad (B.11)$$

If we further define

$$(\gamma^A) \equiv (\gamma^i, \gamma_i), \quad (\bar{\gamma}^A) \equiv (\bar{\gamma}^i, \bar{\gamma}_i), \quad (\eta^{AB}) \equiv \begin{pmatrix} 0 & \delta_j^i \\ \delta_i^j & 0 \end{pmatrix}, \quad (\text{B.12})$$

which satisfy the relation

$$(\gamma^A \bar{\gamma}^B + \gamma^B \bar{\gamma}^A)^I{}_J = 2 \eta^{AB} \delta_J^I, \quad (\text{B.13})$$

the  $Y$ -tensor can be expressed in the conventional form (1.6),

$$Y_{KL}^{IJ} = \frac{1}{2} \gamma_A^{IJ} \bar{\gamma}_{KL}^A \quad (\gamma_A \equiv \eta_{AB} \gamma^B). \quad (\text{B.14})$$

### B.3. $Y$ -tensor in $E_{6(6)}$ EFT

In the  $E_{6(6)}$  EFT, we have 27  $\eta$ -symbols, which can be redefined as

$$d^k \equiv -\frac{\epsilon_{k_1 \dots k_6} \eta^{k_1 \dots k_6, k}}{6! \sqrt{10}} = \frac{1}{\sqrt{10}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{2! \epsilon_{j_1 \dots j_5 k} \delta_{i_1 i_2}^{kl}}{\sqrt{2! 5!}} \\ 0 & \frac{2! \epsilon_{i_1 \dots i_5 k} \delta_{j_1 j_2}^{kl}}{\sqrt{2! 5!}} & 0 \end{pmatrix}, \quad (\text{B.15})$$

$$d_{k_1 k_2} \equiv \frac{\epsilon_{k_1 k_2 l_1 \dots l_4} \eta^{l_1 \dots l_4}}{4! \sqrt{10}} = \frac{1}{\sqrt{10}} \begin{pmatrix} 0 & 0 & \frac{5 \delta_{[l_1}^i \epsilon_{j_2 \dots j_5] k_1 k_2}}{\sqrt{5!}} \\ 0 & \frac{\epsilon_{k_1 k_2 i_1 i_2 j_1 j_2}}{\sqrt{2! 2!}} & 0 \\ \frac{5 \delta_{[i_1}^j \epsilon_{i_2 \dots i_5] k_1 k_2}}{\sqrt{5!}} & 0 & 0 \end{pmatrix}, \quad (\text{B.16})$$

$$d^{\bar{k}} \equiv \frac{1}{\sqrt{10}} \eta^k = \frac{1}{\sqrt{10}} \begin{pmatrix} 0 & \frac{2! \delta_{j_1 j_2}^{ki}}{\sqrt{2!}} & 0 \\ \frac{2! \delta_{i_1 i_2}^{kj}}{\sqrt{2!}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (\text{B.17})$$

In the coordinates  $(x^i, \frac{y_{i_1 i_2}}{\sqrt{2}}, z^i)$  with  $z^i \equiv -\frac{1}{5!} \epsilon^{ij_1 \dots j_5} y_{j_1 \dots j_5}$ , the matrices become

$$d^k = \frac{1}{\sqrt{10}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{2! \delta_{i_1 i_2}^{kj}}{\sqrt{2!}} \\ 0 & \frac{2! \delta_{j_1 j_2}^{ki}}{\sqrt{2!}} & 0 \end{pmatrix}, \quad d^{\bar{k}} = \frac{1}{\sqrt{10}} \begin{pmatrix} 0 & \frac{2! \delta_{j_1 j_2}^{ki}}{\sqrt{2!}} & 0 \\ \frac{2! \delta_{i_1 i_2}^{kj}}{\sqrt{2!}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\text{B.18})$$

$$d_{k_1 k_2} = \frac{1}{\sqrt{10}} \begin{pmatrix} 0 & 0 & 2! \delta_{k_1 k_2}^{ij} \\ 0 & \frac{\epsilon_{k_1 k_2 i_1 i_2 j_1 j_2}}{\sqrt{2! 2!}} & 0 \\ 2! \delta_{k_1 k_2}^{ji} & 0 & 0 \end{pmatrix}. \quad (\text{B.19})$$

They are components of the conventional totally symmetric tensor  $d^{IJK} \equiv (d^{IJ; k}, \frac{d_{k_1 k_2}^{IJ}}{\sqrt{2!}}, d^{IJ; \bar{k}})$  (see, e.g., Eq. (4.42) in Ref. [13]). By defining  $d_{IJK}$  in a similar manner, they satisfy

$$d^{IKL} d_{JKL} = \delta_J^I, \quad d^{IJK} d_{IJK} = 27. \quad (\text{B.20})$$

They also satisfy the relation

$$10 d^{M(I|P} d_{QPL} d^{Q|JK)} - d^{M(IJ} \delta_L^K) = \frac{1}{3} d^{IJK} \delta_L^M, \quad (\text{B.21})$$

which ensures the following relation for the  $Y$ -tensor:

$$Y_{RS}^{(I|P} Y_{PL}^{J|K)} - Y_{RS}^{(IJ} \delta_L^K) = \frac{10}{3} d_{LRS} d^{IJK}. \quad (\text{B.22})$$

#### B.4. $Y$ -tensor in $E_{7(7)}$ EFT

In the  $E_{7(7)}$  case, we have 133  $\eta$ -symbols, which can be redefined as

$$t_{k_1 \dots k_4} \equiv \frac{\epsilon_{i_1 \dots i_7} \epsilon_{k_1 \dots k_4 j_1 j_2 j_3} \eta^{i_1 \dots i_7, j_1 j_2 j_3}}{3! 7!}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{7! \epsilon_{i_1 i_2 j_1 \dots j_4} \epsilon_{j_5 j_6 j_7} k_1 \dots k_4}{3! 4! \sqrt{2!} 7!} \\ 0 & 0 & \frac{5! \epsilon_{i_1 \dots i_5 j_1 j_2} \epsilon_{j_3 j_4 j_5} k_1 \dots k_4}{2! 3! \sqrt{5!} 5!} & 0 \\ 0 & -\frac{7! \epsilon_{j_1 j_2 i_1 \dots i_4} \epsilon_{i_5 i_6 i_7} k_1 \dots k_4}{3! 4! \sqrt{2!} 7!} & 0 & 0 \end{pmatrix}, \quad (\text{B.23})$$

$$t_{k_1 k_2 k_3 8} \equiv -\frac{1}{4!} \epsilon_{k_1 k_2 k_3 l_1 \dots l_4} \eta^{l_1 \dots l_4}$$

$$= \begin{pmatrix} 0 & 0 & -\frac{5}{\sqrt{5!}} \delta_{[j_1}^i \epsilon_{j_2 \dots j_5] k_1 k_2 k_3} & 0 \\ 0 & -\frac{\epsilon_{k_1 k_2 k_3 i_1 j_1 j_2}}{\sqrt{2!} 2!} & 0 & 0 \\ -\frac{5}{\sqrt{5!}} \delta_{[i_1}^j \epsilon_{i_2 \dots i_5] k_1 k_2 k_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{B.24})$$

$$t_8^k \equiv -\eta^k = \begin{pmatrix} 0 & -\frac{2!}{\sqrt{2!}} \delta_{j_1 j_2}^{ki} & 0 & 0 \\ -\frac{2!}{\sqrt{2!}} \delta_{i_1 i_2}^{kj} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{B.25})$$

$$t_k^l \equiv \frac{1}{6!} \epsilon_{k k_1 \dots k_6} \left( \eta^{k_1 \dots k_6, l} - \frac{\sqrt{2}}{4} \eta^{[k_1 \dots k_6, l]} \right)$$

$$\equiv \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{7!}}(\delta_k^i \epsilon_{j_1 \dots j_7} \delta_j^l) \\ & & & -\frac{1}{4} \delta_k^l \epsilon_{j_1 \dots j_7} \delta_j^i \\ 0 & 0 & \frac{-1}{\sqrt{2!5!}}(2! \delta_{i_1 i_2}^{lm} \epsilon_{j_1 \dots j_5 km} \\ & & -\frac{1}{4} \delta_k^l \epsilon_{i_1 i_2 j_1 \dots j_5}) & 0 \\ 0 & \frac{-1}{\sqrt{2!5!}}(2! \delta_{j_1 j_2}^{lm} \epsilon_{i_1 \dots i_5 km} \\ & -\frac{1}{4} \delta_k^l \epsilon_{j_1 j_2 i_1 \dots i_5}) & 0 & 0 \\ \frac{1}{\sqrt{7!}}(\delta_k^j \epsilon_{i_1 \dots i_7} \delta_i^l) & 0 & 0 & 0 \\ -\frac{1}{4} \delta_k^l \epsilon_{i_1 \dots i_7} \delta_i^j & & & \end{pmatrix}, \quad (\text{B.26})$$

$$t_k^8 \equiv \frac{1}{6! 7!} \epsilon_{k_1 \dots k_7} \epsilon_{kl_1 \dots l_6} \eta^{k_1 \dots k_7, l_1 \dots l_6} \quad (\text{B.27})$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\epsilon_{j_1 \dots j_7} \epsilon_{k j i_1 \dots i_5}}{2\sqrt{5!}} \\ 0 & 0 & \frac{\epsilon_{i_1 \dots i_7} \epsilon_{k i j_1 \dots j_5}}{2\sqrt{5!}} & 0 \end{pmatrix}. \quad (\text{B.28})$$

If we use the conventional parameterization of the generalized coordinates,

$$(x^J) = \left( \frac{x_{i_1 i_2}^{\hat{1}}}{\sqrt{2!}}, \frac{x_{i_1 i_2}^{\hat{2}}}{\sqrt{2!}} \right) \quad (\hat{i} = 1, \dots, 8), \quad (\text{B.29})$$

where

$$x^{i8} \equiv x^i, \quad x^{i_1 i_2} \equiv -\frac{1}{5!} \epsilon^{i_1 i_2 j_1 \dots j_5} y_{j_1 \dots j_5}, \quad x_{i8} \equiv \frac{1}{7!} \epsilon^{j_1 \dots j_7} y_{j_1 \dots j_7, i}, \quad x_{i_1 i_2} \equiv y_{i_1 i_2}, \quad (\text{B.30})$$

the above matrices take the following forms:

$$t_{k_1 \dots k_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{4!}{\sqrt{2! 2!}} \delta_{k_1 \dots k_4}^{i_1 i_2 j_1 j_2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{\epsilon_{i j_1 j_2 k_1 \dots k_4}}{\sqrt{2!}} \\ 0 & 0 & -\frac{\epsilon_{i_1 i_2 j k_1 \dots k_4}}{\sqrt{2!}} & 0 \end{pmatrix}, \quad (\text{B.31})$$

$$t_{k_1 k_2 k_3 8} = \begin{pmatrix} 0 & \frac{3!}{\sqrt{2!}} \delta_{k_1 k_2 k_3}^{i j_1 j_2} & 0 & 0 \\ \frac{3!}{\sqrt{2!}} \delta_{k_1 k_2 k_3}^{i_1 i_2 j} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\epsilon_{i_1 i_2 j_1 j_2 k_1 k_2 k_3}}{\sqrt{2! 2!}} \end{pmatrix}, \quad (\text{B.32})$$

$$t_8^k = \begin{pmatrix} 0 & 0 & 0 & -\frac{2!}{\sqrt{2!}} \delta_{j1j2}^{ki} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{2!}{\sqrt{2!}} \delta_{i1i2}^{kj} & 0 & 0 & 0 \end{pmatrix}, \quad (\text{B.33})$$

$$t_k^l = \begin{pmatrix} 0 & 0 & \delta_k^i \delta_j^l - \frac{1}{4} \delta_k^l \delta_j^i & 0 \\ 0 & 0 & 0 & 2 \delta_{km}^{i1i2} \delta_{j1j2}^{lm} - \frac{1}{4} \delta_k^l \delta_{j1j2}^{i1i2} \\ \delta_k^j \delta_i^l - \frac{1}{4} \delta_k^l \delta_i^j & 0 & 0 & 0 \\ 0 & 2 \delta_{km}^{j1j2} \delta_{i1i2}^{lm} - \frac{1}{4} \delta_k^l \delta_{i1i2}^{j1j2} & 0 & 0 \end{pmatrix}, \quad (\text{B.34})$$

$$t_k^8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2!}{\sqrt{2!}} \delta_{kj}^{i1i2} & 0 \\ 0 & -\frac{2!}{\sqrt{2!}} \delta_{ki}^{j1j2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{B.35})$$

These can be summarized as the following familiar matrices (see, e.g., Appendix A.2 in Ref. [48]):

$$t_{\hat{k}_1 \dots \hat{k}_4} = \begin{pmatrix} \frac{4!}{\sqrt{2!} 2!} \delta_{\hat{k}_1 \dots \hat{k}_4}^{\hat{i}_1 \hat{i}_2 \hat{j}_1 \hat{j}_2} & 0 \\ 0 & -\frac{\epsilon_{\hat{i}_1 \hat{i}_2 \hat{j}_1 \hat{j}_2 \hat{k}_1 \dots \hat{k}_4}}{\sqrt{2!} 2!} \end{pmatrix}, \quad (\text{B.36})$$

$$t_{\hat{k}}^{\hat{l}} = \begin{pmatrix} 0 & 2 \delta_{\hat{k}\hat{m}}^{\hat{i}_1 \hat{i}_2} \delta_{\hat{j}_1 \hat{j}_2}^{\hat{l}\hat{m}} - \frac{1}{4} \delta_{\hat{k}}^{\hat{l}} \delta_{\hat{j}_1 \hat{j}_2}^{\hat{i}_1 \hat{i}_2} \\ 2 \delta_{\hat{k}\hat{m}}^{\hat{j}_1 \hat{j}_2} \delta_{\hat{i}_1 \hat{i}_2}^{\hat{l}\hat{m}} - \frac{1}{4} \delta_{\hat{k}}^{\hat{l}} \delta_{\hat{i}_1 \hat{i}_2}^{\hat{j}_1 \hat{j}_2} & 0 \end{pmatrix}, \quad (\text{B.37})$$

where

$$t_{\hat{k}}^{\hat{k}} = 0, \quad \epsilon_{\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}\hat{6}\hat{7}\hat{8}} = \epsilon_{1234567}. \quad (\text{B.38})$$

From these matrices, the  $Y$ -tensor and the generalized Lie derivative have been explicitly computed in Ref. [38].

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