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Electroweak symmetry breaking and mass spectra in six-dimensional gauge–Higgs grand unification

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The mass spectra of the standard model particles are reproduced in the $SO(11)$ gauge–Higgs grand unification in six-dimensional warped space without introducing exotic light fermions. Light neutrino masses are explained by the gauge–Higgs seesaw mechanism. We evaluate the effective potential of the four-dimensional Higgs boson appearing as a fluctuation mode of the Aharonov–Bohm phase θ_H in the extra-dimensional space, and show that the dynamical electroweak symmetry breaking takes place with the Higgs boson mass $m_H \sim 125\,\text{GeV}$ and $\theta_H \sim 0.1$. The Kaluza–Klein mass scale in the fifth dimension is approximately given by $m_{\text{KK}} \sim 1.230\,\text{TeV}/\sin\theta_H$.

Subject Index B40, B41, B42, B43

1. Introduction

The discovery of the last of the standard model (SM) particles, the Higgs boson, seems to imply that the non-vanishing vacuum expectation value (VEV) of the Higgs field spontaneously breaks the electroweak (EW) gauge symmetry $SU(2)_L \times U(1)_Y$ to the electromagnetic gauge symmetry $U(1)_{\text{EM}}$. Almost all observational data from low-energy experiments are consistent with the SM. The SM is a good low-energy effective theory. However, it is not clear whether or not the Higgs boson is a genuine fundamental scalar field, which usually suffer from the so-called gauge hierarchy problem.

There are several proposals to overcome the problem by making use of symmetries. One of them is the gauge–Higgs unification. In this theory, the Higgs boson is identified with part of the extra-dimensional component of gauge fields in higher-dimensional spacetime [1–5]. It is described as a four-dimensional (4D) fluctuation mode of the Aharonov–Bohm (AB) phase θ_H along the extra-dimensional space.

The $SU(2)_L \times U(1)$ EW unification of the SM is formulated as the $SO(5) \times U(1)$ gauge–Higgs unification in the five-dimensional (5D) Randall–Sundrum (RS) warped space [6–16]. According to Refs. [11–14], its phenomenology at low energies under the mass scale of the first Kaluza–Klein (KK) modes is almost the same as in the SM for the AB phase $\theta_H \lesssim 0.1$. Z' bosons, which are the first KK modes of γ, Z, and Z_R, are predicted around the $7 \sim 9\,\text{TeV}$ range for $\theta_H = 0.1 \sim 0.07$. Z' bosons can be produced at the 14 TeV LHC [15]. At electron–positron linear colliders at energies of $250\,\text{GeV} \sim 1\,\text{TeV}$, the interference effects among γ, Z, and Z' bosons give distinct signals of the gauge–Higgs unification [16].

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To incorporate the $SU(3)_C$ gauge symmetry in higher-dimensional gauge theories and gauge–Higgs unification scenario, grand unified theories (GUTs) based on a GUT gauge group $G_{\rm GUT}$ ($\supset G_{\rm SM} := SU(3)_C \times SU(2)_L \times U(1)$) have been discussed in Refs. [17–37]. The $SO(11)$ gauge–Higgs grand unified theory (GHGUT) is proposed in the 5D RS warped space in Ref. [32]. The EW Higgs boson is identified with part of the fifth-dimensional component of the $SO(11)$ gauge bosons. The $SO(11)$ gauge symmetry is reduced first by two different orbifold boundary conditions (BCs) on the ultraviolet (UV) and infrared (IR) branes in the RS space. It is reduced to $SO(10)$ by the BC on the UV brane, and to $SO(4) \times SO(7)$ by the BC on the IR brane, the resultant symmetry being $SO(4) \times SO(6)$. Secondly, the brane scalar on the UV brane, which is an $SO(10)$ spinor **16**, spontaneously breaks $SO(10)$ to $SU(5)$ by the Higgs mechanism on the UV brane. As a net result the $SO(11)$ gauge symmetry is reduced to the SM gauge symmetry $G_{\rm SM}$, which is dynamically broken to $SU(3)_C \times U(1)_{\rm EM}$ by the Hosotani mechanism.

Zero modes of 5D $SO(11)$ **32** fermions in the bulk are identified with each generation of quarks and leptons. In Ref. [35] it is found that the observed mass spectra of the quarks and leptons can be reproduced, while additional exotic particles appear having unacceptably small masses.

Recently, to avoid these light exotic particles, $SO(11)$ GHGUT in six-dimensional (6D) hybrid warped space has been proposed in Ref. [38]. For neutrinos a new seesaw mechanism in 6D hybrid warped space has been formulated by using a 5D symplectic Majorana fermion [39], which generalizes the well-known 4D seesaw mechanism [40].

This paper is organized as follows. In Sect. 2, 6D $SO(11)$ GHGUT is introduced. The matter content is specified and the action is given that contains both 6D bulk and 5D brane terms. In Sect. 3, a summary is given for the mass spectrum of the 4D gauge and scalar bosons originating from the 6D $SO(11)$ gauge bosons. In Sect. 4, we derive the mass spectrum of 4D SM fermions. By using these mass spectra, we evaluate the effective potential $V_{\rm eff}(\theta_H)$ in Sect. 5 to show that the dynamical EW symmetry breaking takes place and the 4D Higgs boson mass $m_H = 125.1$ GeV is obtained. Section 6 is devoted to a summary and discussions. In the Appendix, the basics of the KK expansion in 6D warped space are explained.

2. Six-dimensional gauge–Higgs grand unification theory

We construct an $SO(11)$ gauge-Higgs grand unified model on the six-dimensional hybrid warped space introduced in Ref. [38]. The metric of generalized RS space [38,41] is given by

$$ds^2 = e^{-2\sigma(y)} \left(\eta_{\mu\nu} dx^\mu dx^\nu + dv^2\right) + dy^2, \tag{2.1}$$

where $e^{-2\sigma(y)}$ is a warped factor and $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$. $\sigma(y)$ satisfies $\sigma(y) = \sigma(-y) = \sigma(y + 2L_5)$ and $\sigma(y) = k|y|$ for $|y| \le L_5$. The fifth dimension with the coordinate y behaves as the EW dimension, whereas the sixth dimension with the coordinate v is S^1 ($v \sim v + 2\pi R_6$) and behaves as the GUT dimension. Two spacetime points (x^μ, y, v) and $(x^\mu, -y, -v)$ are identified by the $\mathbb{Z}_2$ transformation. As a result, the spacetime has the same topology as the orbifold $M^4 \times T^2/\mathbb{Z}_2$. The spacetime in Eq. (2.1) solves the Einstein equations with brane tensions at $y = 0$ and $y = L_5$. The bulk region is the anti-de-Sitter space with a negative cosmological constant $\Lambda = -10k^2$. The five-dimensional branes at $y = 0$ and $y = L_5$ have the same topology as $M^4 \times S^1$.

The extra-dimensional space has four fixed points under $\mathbb{Z}_2$: $(y_0, v_0) = (0, 0)$, $(y_1, v_1) = (L_5, 0)$, $(y_2, v_2) = (0, \pi R_6)$, and $(y_3, v_3) = (L_5, \pi R_6)$. In terms of the conformal coordinate $z = e^{ky}$ ($1 \le$

$z \le z_L = e^{kL_5}$) in the region $0 \le y \le L_5$, the metric becomes

$$ds^2 = \frac{1}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dv^2 + \frac{dz^2}{k^2}\right). \tag{2.2}$$

The fifth- and sixth-dimensional KK mass scales are given by $m_{\mathrm{KK}_5} = \pi k/(e^{kL_5} - 1)$ and $m_{\mathrm{KK}_6} = R_6^{-1}$. The warp factor is supposed to be large; $z_L = e^{kL_5} \gg 1$. m_{KK_6} is expected to be a GUT scale while m_{KK_5} is $O(10)$ TeV so that $m_{\mathrm{KK}_6} \gg m_{\mathrm{KK}_5}$.

Parity transformations P_j ($j = 0, 1, 2, 3$) around the four fixed points are defined as $(x^\mu, y_j + y, v_j + v) \to (x^\mu, y_j - y, v_j - v)$. Only three of the four parity transformations P_j are independent. They satisfy the relation $P_3 = P_2P_0P_1 = P_1P_0P_2$.

We adopt orbifold BCs such that $P_0 = P_1$ and $P_2 = P_3$, which enables us to avoid the problem of unwanted light exotic fermions mentioned in the Introduction. More specifically, we choose BCs such that $P_0 = P_1$ and $P_2 = P_3$ break $SO(11)$ to $SO(4) \times SO(7)$ and $SO(10)$, respectively. The orbifold BCs reduce $SO(11)$ symmetry to the Pati–Salam symmetry $SO(4) \times SO(6) \simeq SU(2)_L \times SU(2)_R \times SU(4)_C =: G_{\mathrm{PS}}$. We note that the fifth- and sixth-dimensional loop translations $U_5 : (x^\mu, y, v) \to (x^\mu, y + 2L_5, v)$ and $U_6 : (x^\mu, y, v) \to (x^\mu, y, v + 2\pi R_6)$ are related to the P_js by $U_5 = P_1P_0 = P_3P_2$ and $U_6 = P_2P_0 = P_3P_1$.

2.1. *6D bulk and 5D brane fields and orbifold boundary conditions*

The matter content in the $SO(11)$ GHGUT consists of 6D $SO(11)$ gauge bosons A_M, 6D $SO(11)$ **32** Weyl fermions $\Psi^\alpha_{\mathbf{32}}(x, y, v)$ ($\alpha = 1, 2, 3, 4$), 6D $SO(11)$ **11** Dirac fermions $\Psi^\beta_{\mathbf{11}}(x, y, v)$ and $\Psi'^\beta_{\mathbf{11}}(x, y, v)$ ($\beta = 1, 2, 3$) in the six-dimensional bulk space, and a 5D $SO(11)$ **32** brane scalar boson $\Phi_{\mathbf{32}}(x, v)$ and 5D $SO(11)$ **1** brane symplectic Majorana fermions $\chi^\beta_{\mathbf{1}}(x, v)$ ($\beta = 1, 2, 3$) on the UV brane $y = 0$ [38]. Their orbifold BCs are listed below:

- For the 6D $SO(11)$ gauge boson A_M, the orbifold BCs are given by

$$\begin{pmatrix} A_\mu \\ A_y \\ A_v \end{pmatrix}(x, y_j - y, v_j - v) = P_j \begin{pmatrix} A_\mu \\ -A_y \\ -A_v \end{pmatrix}(x, y_j + y, v_j + v)P_j^{-1}, \tag{2.3}$$

where in the $SO(11)$ vector representation we take the orbifold BCs $P_0 = P_1$ and $P_2 = P_3$ as

$$P_0^{\mathrm{vec}} = P_1^{\mathrm{vec}} = \mathrm{diag}(I_4, -I_7), \quad P_2^{\mathrm{vec}} = P_3^{\mathrm{vec}} = \mathrm{diag}(I_{10}, -I_1). \tag{2.4}$$

By using ($P_0 = P_1$, $P_2 = P_3$), the parity assignment of A_μ, A_y, and A_v are summarized in Table 1.

Table 1. Parity assignment $(P_0, P_2) = (P_1, P_3)$ of A_μ, A_y, and A_v in $G_{\mathrm{PS}} = SU(2)_L \times SU(2)_R \times SU(4)_C$.

G_{PS}	A_μ	A_y	A_v
(3,1,1)	$(+,+)$	$(-,-)$	$(-,-)$
(1,3,1)	$(+,+)$	$(-,-)$	$(-,-)$
(1,1,15)	$(+,+)$	$(-,-)$	$(-,-)$
(2,2,6)	$(-,+)$	$(+,-)$	$(+,-)$
(2,2,1)	$(-,-)$	$(+,+)$	$(+,+)$
(1,1,6)	$(+,-)$	$(-,+)$	$(-,+)$

Table 2. Parity assignment $\begin{pmatrix} P_2 & P_3 \\ P_0 & P_1 \end{pmatrix}$ in Eq. (2.7) of $\Psi_{\mathbf{32}}^{\alpha}$ ($\alpha = 1,2,3,4$) in $G_{\rm PS}$.

$\Psi_{\mathbf{32}}^{\alpha=1,2,3}$

$G_{\rm PS}$	Left	Right	name
$(\mathbf{2},\mathbf{1},\mathbf{4})$	$\begin{pmatrix} + & + \\ + & + \end{pmatrix}$	$\begin{pmatrix} - & - \\ - & - \end{pmatrix}$	$\begin{matrix} \nu & u_j \\ e & d_j \end{matrix}$
$(\mathbf{1},\mathbf{2},\overline{\mathbf{4}})$	$\begin{pmatrix} + & + \\ - & - \end{pmatrix}$	$\begin{pmatrix} - & - \\ + & + \end{pmatrix}$	$\begin{matrix} \hat{e} & \hat{d}_j \\ \hat{\nu} & \hat{u}_j \end{matrix}$
$(\mathbf{2},\mathbf{1},\overline{\mathbf{4}})$	$\begin{pmatrix} - & - \\ + & + \end{pmatrix}$	$\begin{pmatrix} + & + \\ - & - \end{pmatrix}$	$\begin{matrix} \hat{e}' & \hat{d}_j' \\ \hat{\nu}' & \hat{u}_j' \end{matrix}$
$(\mathbf{1},\mathbf{2},\mathbf{4})$	$\begin{pmatrix} - & - \\ - & - \end{pmatrix}$	$\begin{pmatrix} + & + \\ + & + \end{pmatrix}$	$\begin{matrix} \nu' & u_j' \\ e' & d_j' \end{matrix}$

$\Psi_{\mathbf{32}}^{\alpha=4}$

$G_{\rm PS}$	Left	Right
$(\mathbf{2},\mathbf{1},\mathbf{4})$	$\begin{pmatrix} + & - \\ + & - \end{pmatrix}$	$\begin{pmatrix} - & + \\ - & + \end{pmatrix}$
$(\mathbf{1},\mathbf{2},\overline{\mathbf{4}})$	$\begin{pmatrix} + & - \\ - & + \end{pmatrix}$	$\begin{pmatrix} - & + \\ + & - \end{pmatrix}$
$(\mathbf{2},\mathbf{1},\overline{\mathbf{4}})$	$\begin{pmatrix} - & + \\ + & - \end{pmatrix}$	$\begin{pmatrix} + & - \\ - & + \end{pmatrix}$
$(\mathbf{1},\mathbf{2},\mathbf{4})$	$\begin{pmatrix} - & + \\ - & + \end{pmatrix}$	$\begin{pmatrix} + & - \\ + & - \end{pmatrix}$

- For the four 6D $SO(11)$ spinor **32** bulk Weyl fermions $\Psi_{\mathbf{32}}^{\alpha}(x,y,v)$ ($\alpha = 1,2,3,4$), the orbifold BCs are given by

$$\Psi_{\mathbf{32}}^{\alpha}(x,y_j-y,v_j-v) = \eta_j^{\alpha}\overline{\gamma}P_j^{\rm sp}\Psi_{\mathbf{32}}^{\alpha}(x,y_j+y,v_j+v), \tag{2.5}$$

where $\eta_j^{\alpha} = \pm 1$. Six-dimensional Dirac matrices γ^a ($a = 1,2,\ldots,6$) satisfy $\{\gamma^a,\gamma^b\} = 2\eta^{ab}$ ($\eta^{ab} = \mathrm{diag}(-I_1, I_5)$), and $\overline{\gamma} := -i\gamma^5\gamma^6 = \gamma_{6D}^7\gamma_{4D}^5 = \gamma_{4D}^5\gamma_{6D}^7$, $\gamma_{4D}^5 = I_2\otimes\sigma^3\otimes I_2$, $\gamma_{6D}^7 = I_4\otimes\sigma^3$. The $P_j^{\rm sp}$s are

$$P_0^{\rm sp} = P_1^{\rm sp} = I_2\otimes\sigma^3\otimes I_8, \quad P_2^{\rm sp} = P_3^{\rm sp} = I_{16}\otimes\sigma^3. \tag{2.6}$$

To ensure the 6D $SO(11)$ chiral anomaly cancellation, we assign $\gamma_{6D}^7 = +1$ for $\alpha = 1,2$; $\gamma_{6D}^7 = -1$ for $\alpha = 3,4$. We take $\eta_j^{1,2} = -1$, $\eta_j^3 = 1$, and $\eta_{0,2}^4 = -\eta_{1,3}^4 = 1$. The parity assignment

$$\begin{pmatrix} \eta_2^{\alpha}\overline{\gamma}P_2^{\rm sp} & \eta_3^{\alpha}\overline{\gamma}P_3^{\rm sp} \\ \eta_0^{\alpha}\overline{\gamma}P_0^{\rm sp} & \eta_1^{\alpha}\overline{\gamma}P_1^{\rm sp} \end{pmatrix} \tag{2.7}$$

of the 4D left- and right-handed components of $\Psi_{\mathbf{32}}^{\alpha}$ are summarized in Table 2. We find that $\Psi_{\mathbf{32}}^{\alpha}$ ($\alpha = 1,2,3$) has zero modes, corresponding to one generation of quarks and leptons for each α. The corresponding names adopted in Ref. [35] are also listed in Table 2 for $\alpha = 1,2,3$.
- For the three 6D $SO(11)$ vector **11** bulk Dirac fermions $\Psi_{\mathbf{11}}^{\beta}(x,y,v)$ and $\Psi_{\mathbf{11}}'^{\beta}(x,y,v)$ ($\beta = 1,2,3$), the orbifold BCs are given by

$$\Psi_{\mathbf{11}}^{(\prime)\beta}(x,y_j-y,v_j-v) = \eta_j^{(\prime)\beta}\overline{\gamma}P_j^{\rm vec}\Psi_{\mathbf{11}}^{(\prime)\beta}(x,y_j+y,v_j+v), \tag{2.8}$$

where $\eta_{0,1}^{\beta} = -\eta_{2,3}^{\beta} = -1$ for $\Psi_{\mathbf{11}}^{\beta}$; $\eta_{0,1}'^{\beta} = \eta_{2,3}'^{\beta} = -1$ for $\Psi_{\mathbf{11}}'^{\beta}$. The parity assignments ($\eta_0'^{\beta}\overline{\gamma}P_0^{\rm vec} = \eta_1'^{\beta}\overline{\gamma}P_1^{\rm vec}$, $\eta_2'^{\beta}\overline{\gamma}P_2^{\rm vec} = \eta_3'^{\beta}\overline{\gamma}P_3^{\rm vec}$) of the 4D left- and right-handed components of $\Psi_{\mathbf{11}}^{(\prime)\beta}$ are summarized in Table 3.
- For the 5D $SO(11)$ spinor **32** brane scalar field on the UV brane at $y = 0$, $\Phi_{\mathbf{32}}(x,v)$, the orbifold BCs are given by

$$\Phi_{\mathbf{32}}(x,v_j-v) = \eta_j P_j^{\rm sp}\Phi_{\mathbf{32}}(x,v_j+v), \tag{2.9}$$

Table 3. Parity assignment $(P_0, P_2) = (P_1, P_3)$ for $\Psi_{\mathbf{11}}^{(\prime)\beta}$ $(\beta = 1, 2, 3)$ in $G_{\rm PS}$.

$\Psi_{\mathbf{11}}^{\beta}$

$G_{\rm PS}$	Left$_+$	Right$_+$	Left$_-$	Right$_-$	name
$(\mathbf{2},\mathbf{2},\mathbf{1})$	$(+,-)$	$(-,+)$	$(-,+)$	$(+,-)$	N $\hat{E}$ E $\hat{N}$
$(\mathbf{1},\mathbf{1},\mathbf{6})$	$(-,-)$	$(+,+)$	$(+,+)$	$(-,-)$	$D_j, \hat{D}_j$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(-,+)$	$(+,-)$	$(+,-)$	$(-,+)$	S

$\Psi_{\mathbf{11}}^{\prime\beta}$

$G_{\rm PS}$	Left$_+$	Right$_+$	Left$_-$	Right$_-$	name
$(\mathbf{2},\mathbf{2},\mathbf{1})$	$(+,+)$	$(-,-)$	$(-,-)$	$(+,+)$	N' $\hat{E}'$ E' $\hat{N}'$
$(\mathbf{1},\mathbf{1},\mathbf{6})$	$(-,+)$	$(+,-)$	$(+,-)$	$(-,+)$	$D_j', \hat{D}_j'$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(-,-)$	$(+,+)$	$(+,+)$	$(-,-)$	S'

Table 4. Parity assignment (P_0, P_2) for $\Phi_{\mathbf{32}}$ in $G_{\rm PS}$.

$\Phi_{\mathbf{32}}$

$G_{\rm PS}$	BCs
$(\mathbf{2},\mathbf{1},\mathbf{4})$	$(-,+)$
$(\mathbf{1},\mathbf{2},\overline{\mathbf{4}})$	$(+,+)$
$(\mathbf{2},\mathbf{1},\overline{\mathbf{4}})$	$(-,-)$
$(\mathbf{1},\mathbf{2},\mathbf{4})$	$(+,-)$

where $j = 0, 2$, $\eta_0 = -\eta_2 = -1$. The components of $G_{\rm PS}$ $(\mathbf{1},\mathbf{2},\overline{\mathbf{4}})$ have zero modes, one of which corresponds to the $SU(5)$ $\mathbf{1}$ of $SO(10)$ $\mathbf{16}$. It is responsible for reducing $SO(11)$ to $SU(5)$ at the UV brane $y = 0$. The BCs of $\Phi_{\mathbf{32}}$ are summarized in Table 4.

- For the 5D $SO(11)$ singlet fermions $\chi^\beta(x, v) = \chi_{\mathbf{1}}^\beta(x, v)$ $(\beta = 1, 2, 3)$ at $y = 0$, their orbifold BCs are given by

$$\chi^\beta(x, v_j - v) = \overline{\gamma}\chi^\beta(x, v_j + v) \ \ (j = 0, 2). \tag{2.10}$$

These brane fields also satisfy the 5D symplectic Majorana condition $\chi^C = \widetilde{\chi}$, $\widetilde{\chi} := i\overline{\Gamma}\chi$, where $\overline{\Gamma} := \gamma_{4D}^5\gamma^6$:

$$\chi = \begin{pmatrix} \xi_+ \\ \eta_+ \\ \xi_- \\ \eta_- \end{pmatrix}, \quad \chi^C = \begin{pmatrix} +\eta_+^C \\ -\xi_+^C \\ -\eta_-^C \\ +\xi_-^C \end{pmatrix} = e^{i\delta_C}\begin{pmatrix} -\sigma^2\eta_+^* \\ -\sigma^2\xi_+^* \\ +\sigma^2\eta_-^* \\ +\sigma^2\xi_-^* \end{pmatrix} = \begin{pmatrix} +\xi_- \\ -\eta_- \\ -\xi_+ \\ +\eta_+ \end{pmatrix} = \widetilde{\chi}. \tag{2.11}$$

Here the generation index β has been suppressed.

2.2. Action

The action consists of the 6D bulk and 5D brane terms.

2.2.1. Bulk terms

The bulk part of the action is given by

$$S_{\rm bulk} = S_{\rm bulk}^{\rm gauge} + S_{\rm bulk}^{\rm fermion}, \tag{2.12}$$

where $S_{\rm bulk}^{\rm gauge}$ and $S_{\rm bulk}^{\rm fermion}$ are bulk actions of gauge and fermion fields, respectively. The action of $SO(11)$ gauge field $\mathbf{55}$ $A_M(x,y,v)$ is

$$S_{\rm bulk}^{\rm gauge} = \int d^6x\sqrt{-\det G}\Big[-{\rm tr}\left(\frac{1}{4}F^{MN}F_{MN} + \frac{1}{2\xi}(f_{\rm gf})^2 + \mathcal{L}_{\rm gh}\right)\Big], \tag{2.13}$$

where $\sqrt{-\det G} = 1/kz^6$, $z = e^{ky}$, $M, N = 0, 1, 2, 3, 5, 6$, and the field strength F_{MN} is defined by

$$F_{MN} := \partial_M A_N - \partial_N A_M - ig[A_M, A_N]. \tag{2.14}$$

We take the following gauge-fixing and ghost terms:

$$\begin{aligned} f_{\rm gf} &= z^2\left\{\eta^{\mu\nu}\mathcal{D}_\mu^c A_\nu^q + \mathcal{D}_6^c A_6^q + \xi k^2 z^2 \mathcal{D}_z^c\left(\frac{A_z^q}{z^2}\right)\right\}, \\ \mathcal{L}_{\rm gh} &= \bar{c}\left\{\eta^{\mu\nu}\mathcal{D}_\mu^c \mathcal{D}_\nu^{c+q} + \mathcal{D}_6^c\mathcal{D}_6^{c+q} + \xi k^2 z^2 \mathcal{D}_z^c \frac{1}{z^2}\mathcal{D}_z^{c+q}\right\} c, \end{aligned} \tag{2.15}$$

where $\mu, \nu = 0, 1, 2, 3$, $\eta^{\mu\nu} = \eta_{\mu\nu} = {\rm diag}(-1, 1, 1, 1)$, and $A_M = A_M^c + A_M^q$. $\mathcal{D}_M^c B = \partial_M B - ig[A_M^c, B]$ and $\mathcal{D}_M^{c+q} B = \partial_M B - ig[A_M, B]$, where $B = A_\mu^q, A_z^q/z^2, A_6^q$ and c.

The action of fermions in $SO(11)$ spinor and vector representations $\mathbf{32}$ and $\mathbf{11}$, $\Psi_{\mathbf{32}}^\alpha(x,y,v)$, $\Psi_{\mathbf{11}}^\beta(x,y,v)$, and $\Psi_{\mathbf{11}}'^\beta(x,y,v)$, is given by

$$\begin{aligned} S_{\rm bulk}^{\rm fermion} = \int d^6x\sqrt{-\det G}\ &\left\{\sum_{\alpha=1}^{4}\overline{\Psi_{\mathbf{32}}^\alpha}\mathcal{D}(c_{\Psi_{\mathbf{32}}^\alpha})\Psi_{\mathbf{32}}^\alpha\right. \\ &\left. + \sum_{\beta=1}^{3}\overline{\Psi_{\mathbf{11}}^\beta}\mathcal{D}(c_{\Psi_{\mathbf{11}}^\beta})\Psi_{\mathbf{11}}^\beta + \sum_{\beta=1}^{3}\overline{\Psi_{\mathbf{11}}'^\beta}\mathcal{D}(c_{\Psi_{\mathbf{11}}'^\beta})\Psi_{\mathbf{11}}'^\beta\right\}, \end{aligned} \tag{2.16}$$

where $\overline{\Psi_{\mathbf{32}}^\alpha} := i\Psi_{\mathbf{32}}^{\alpha\dagger}\gamma^0$. $c_{\Psi_{\mathbf{32}}^\alpha}$, $c_{\Psi_{\mathbf{11}}^\beta}$, and $c_{\Psi_{\mathbf{11}}'^\beta}$ are bulk mass parameters, and $\mathcal{D}(c)$ is a covariant derivative given by

$$\mathcal{D}(c) = z\left\{\gamma^\mu D_\mu + \gamma^6 D_v - \frac{5}{2}\frac{\sigma'}{z}\gamma^5 + \sigma'\gamma^5 D_z + ic\frac{\sigma'}{z}\gamma^6\right\}, \tag{2.17}$$

where $\sigma'(y) := d\sigma(y)/dy$ and $\sigma'(y) = k$ for $0 < y < L_5$.

Here, we introduce $\check{\Psi}$ defined by

$$\check{\Psi} := \frac{1}{z^{5/2}}\Psi,\ \left(D_z - \frac{5}{2}\frac{1}{z}\right)\Psi = z^{5/2}D_z\check{\Psi}, \tag{2.18}$$

which turns out to be convenient for discussing mass spectra for fermions in Sect. 4. The bulk part of the bulk fermion action becomes

$$S = \int d^4x \int_0^{2\pi R_6} dv \int_1^{z_L} \frac{dz}{k} \Big[\overline{\check{\Psi}^{\alpha}_{\mathbf{32}}} \left(\gamma^\mu D_\mu + \gamma^6 D_v + \sigma' \gamma^5 D_z + i c_{\Psi_{32}} \frac{\sigma'}{z} \gamma^6 \right) \check{\Psi}^{\alpha}_{\mathbf{32}}$$
$$+ \overline{\check{\Psi}^{\beta}_{\mathbf{11}}} \left(\gamma^\mu D_\mu + \gamma^6 D_v + \sigma' \gamma^5 D_z + i c_{\Psi_{11}} \frac{\sigma'}{z} \gamma^6 \right) \check{\Psi}^{\beta}_{\mathbf{11}}$$
$$+ \overline{\check{\Psi}'^{\beta}_{\mathbf{11}}} \left(\gamma^\mu D_\mu + \gamma^6 D_v + \sigma' \gamma^5 D_z + i c_{\Psi'_{11}} \frac{\sigma'}{z} \gamma^6 \right) \check{\Psi}'^{\beta}_{\mathbf{11}} \Big]. \tag{2.19}$$

2.2.2. *Action for the brane scalar $\Phi_{\mathbf{32}}$ and the Higgs mechanism*

We consider the 5D brane terms for a brane scalar $\Phi_{\mathbf{32}}(x, v)$:

$$S^{\text{brane}}_{\text{5D brane}} = \int d^6x \sqrt{-\det G}\, \delta(y) \mathcal{L}_{\Phi_{32}},$$
$$\mathcal{L}_{\Phi_{32}} := -(D_\mu \Phi_{\mathbf{32}})^\dagger D^\mu \Phi_{\mathbf{32}} - (D_v \Phi_{\mathbf{32}})^\dagger D^v \Phi_{\mathbf{32}} - \lambda_{\Phi_{32}} \left(\Phi^\dagger_{\mathbf{32}} \Phi_{\mathbf{32}} - |w|^2 \right)^2, \tag{2.20}$$

where $\lambda_{\Phi_{32}}$ and w are constants, and

$$D_\mu \Phi_{\mathbf{32}} = \left(\partial_\mu - i g A^{SO(11)}_\mu \right) \Phi_{\mathbf{32}} = \left\{ \partial_\mu - \frac{ig}{\sqrt{2}} \sum_{j<k}^{11} A^{(jk)}_\mu T^{\text{sp}}_{jk} \right\} \Phi_{\mathbf{32}},$$
$$D_v \Phi_{\mathbf{32}} = \left(\partial_v - i g A^{SO(11)}_v \right) \Phi_{\mathbf{32}} = \left\{ \partial_v - \frac{ig}{\sqrt{2}} \sum_{j<k}^{11} A^{(jk)}_v T^{\text{sp}}_{jk} \right\} \Phi_{\mathbf{32}}. \tag{2.21}$$

We denote the 5D brane scalar field $\Phi_{\mathbf{32}}$ as

$$\Phi_{\mathbf{32}} := \begin{pmatrix} \Phi_{\mathbf{16}} \\ \Phi_{\overline{\mathbf{16}}} \end{pmatrix} \tag{2.22}$$

and assume that the component $(\mathbf{1}, \mathbf{2}, \overline{\mathbf{4}})$ of G_{PS} $(= SU(2)_L \times SU(2)_R \times SU(4)_C)$ for $\Phi_{\mathbf{32}}$ develops the non-vanishing VEV:

$$\langle \Phi_{\mathbf{16}} \rangle = \begin{pmatrix} 0_4 \\ 0_4 \\ v_4 \\ 0_4 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ w \end{pmatrix}, \quad \langle \Phi_{\overline{\mathbf{16}}} \rangle = 0, \tag{2.23}$$

where one component of $(\mathbf{1}, \mathbf{2}, \overline{\mathbf{4}})$ of G_{PS} $(\subset SO(10))$ corresponds to $\mathbf{1}_{+5}$ of $SU(5) \times U(1)_Z$ $(\subset SO(10))$. The VEV breaks $SO(11)$ to $SU(5)$. Note that branching rules of $SO(11)$ $\mathbf{32}$ and $\mathbf{55}$ representations for $SO(11) \supset SO(10) \supset SU(5)$ are given by $\mathbf{32} = (\mathbf{16}) \oplus (\overline{\mathbf{16}}) = (\mathbf{10} \oplus \overline{\mathbf{5}} \oplus \mathbf{1}) \oplus (\overline{\mathbf{10}} \oplus \mathbf{5} \oplus \mathbf{1})$ and $\mathbf{55} = (\mathbf{45}) \oplus (\mathbf{10}) = (\mathbf{24} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{1}) \oplus (\mathbf{5} \oplus \overline{\mathbf{5}})$, respectively. (For further information see, e.g., Refs. [42–44].)

We also use the 5D brane scalar field $\widetilde{\Phi}_{\mathbf{32}}$ related to $\Phi_{\mathbf{32}}$ by

$$\widetilde{\Phi}_{\mathbf{32}} = \begin{pmatrix} -\widehat{R} \Phi^*_{\overline{\mathbf{16}}} \\ \widehat{R} \Phi^*_{\mathbf{16}} \end{pmatrix}, \quad \widehat{R} = \sigma^2 \otimes \sigma^3 \otimes \sigma^2 \otimes \sigma^3. \tag{2.24}$$

For $\langle\Phi_{\mathbf{32}}\rangle \neq 0$,

$$\widehat{R}\langle\Phi^*_{\mathbf{16}}\rangle = \begin{pmatrix} 0_4 \\ 0_4 \\ 0_4 \\ \tilde{v}_4 \end{pmatrix}, \quad \tilde{v}_4 = \begin{pmatrix} 0 \\ 0 \\ w^* \\ 0 \end{pmatrix}. \tag{2.25}$$

The combination of the non-vanishing VEV $\langle\Phi_{\mathbf{32}}\rangle$ on the UV brane (at $y = 0$) and the orbifold BCs P_j ($j = 0, 1, 2, 3$) reduces $SO(11)$ to the SM gauge group $G_{\rm SM}$ ($= SU(3)_C \times SU(2)_L \times U(1)_Y$).

2.2.3. *Action for the singlet brane fermion* $\chi_{\mathbf{1}}$

The action for $\chi_{\mathbf{1}}^{\beta}(x, v)$, which satisfies the symplectic Majorana condition (2.11), is

$$\begin{aligned}
S^{\rm brane}_{\rm 5D\ brane} &= \int d^6x \sqrt{-\det G}\, \delta(y) \mathcal{L}_{\chi_{\mathbf{1}}}, \\
\mathcal{L}_{\chi_{\mathbf{1}}} &:= \frac{1}{2} \overline{\chi}_{\mathbf{1}}^{\beta} \left(\gamma^{\mu}\partial_{\mu} + \gamma^6 \partial_v\right) \chi_{\mathbf{1}}^{\beta} - \frac{1}{2} M^{\beta\beta'} \overline{\chi}_{\mathbf{1}}^{\beta} \chi_{\mathbf{1}}^{\beta'},
\end{aligned} \tag{2.26}$$

where $M^{\beta\beta'}$ is a constant matrix.

2.2.4. *Brane masses and interactions*

On the UV brane there can be $SO(11)$-invariant brane interactions among the bulk fermions, the singlet brane fermion, and the brane scalar. We consider

$$\begin{aligned}
S^{\rm fermion}_{\rm 5D\ brane} = \int d^6x &\sqrt{-\det G} \delta(y) \Big(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 + \mathcal{L}_6 + \mathcal{L}_7 + \mathcal{L}_8\Big), \\
\mathcal{L}_1 &:= -\kappa^{\alpha\beta} \overline{\Psi^{\alpha}_{\mathbf{32}}} \Gamma^a \Phi_{\mathbf{32}} \cdot (\Psi^{\beta}_{\mathbf{11}})_a - \kappa^{\alpha\beta *} (\overline{\Psi^{\beta}_{\mathbf{11}}})_a \Phi^{\dagger}_{\mathbf{32}} \Gamma^a (\Psi^{\alpha}_{\mathbf{32}})_a, \\
\mathcal{L}_2 &:= -\widetilde{\kappa}^{\alpha\beta} \overline{\Psi^{\alpha}_{\mathbf{32}}} \Gamma^a \widetilde{\Phi}_{\mathbf{32}} \cdot (\Psi^{\beta}_{\mathbf{11}})_a - \widetilde{\kappa}^{\alpha\beta *} (\overline{\Psi^{\beta}_{\mathbf{11}}})_a \widetilde{\Phi}^{\dagger}_{\mathbf{32}} \Gamma^a (\Psi^{\alpha}_{\mathbf{32}})_a, \\
\mathcal{L}_3 &:= -2\mu^{\beta\beta'}_{\mathbf{11}} \overline{\Psi^{\beta}_{\mathbf{11}}} \Psi^{\beta'}_{\mathbf{11}}, \\
\mathcal{L}_4 &:= -\kappa'^{\alpha\beta} \overline{\Psi^{\alpha}_{\mathbf{32}}} \Gamma^a \Phi_{\mathbf{32}} \cdot (\Psi'^{\beta}_{\mathbf{11}})_a - \kappa'^{\alpha\beta *} (\overline{\Psi'^{\beta}_{\mathbf{11}}})_a \Phi^{\dagger}_{\mathbf{32}} \Gamma^a (\Psi^{\alpha}_{\mathbf{32}})_a, \\
\mathcal{L}_5 &:= -\widetilde{\kappa}'^{\alpha\beta} \overline{\Psi^{\alpha}_{\mathbf{32}}} \Gamma^a \widetilde{\Phi}_{\mathbf{32}} \cdot (\Psi'^{\beta}_{\mathbf{11}})_a - \widetilde{\kappa}'^{\alpha\beta *} (\overline{\Psi'^{\beta}_{\mathbf{11}}})_a \widetilde{\Phi}^{\dagger}_{\mathbf{32}} \Gamma^a (\Psi^{\alpha}_{\mathbf{32}})_a, \\
\mathcal{L}_6 &:= -2\mu'^{\beta\beta'}_{\mathbf{11}} \overline{\Psi'^{\beta}_{\mathbf{11}}} \Psi'^{\beta'}_{\mathbf{11}}, \\
\mathcal{L}_7 &:= -\widetilde{\kappa}^{\alpha\beta}_{\mathbf{1}} \overline{\chi^{\beta}_{\mathbf{1}}} \widetilde{\Phi}^{\dagger}_{\mathbf{32}} \Psi^{\alpha}_{\mathbf{32}} - \widetilde{\kappa}^{\alpha\beta *}_{\mathbf{1}} \overline{\Psi^{\alpha}_{\mathbf{32}}} \widetilde{\Phi}_{\mathbf{32}} \chi^{\beta}_{\mathbf{1}}, \\
\mathcal{L}_8 &:= -\kappa^{\alpha\beta}_{\mathbf{1}} \overline{\chi^{\beta}_{\mathbf{1}}} \Phi^{\dagger}_{\mathbf{32}} \Psi^{\alpha}_{\mathbf{32}} - \kappa^{\alpha\beta *}_{\mathbf{1}} \overline{\Psi^{\alpha}_{\mathbf{32}}} \Phi_{\mathbf{32}} \chi^{\beta}_{\mathbf{1}},
\end{aligned} \tag{2.27}$$

where the κs and μs are coupling constants. Note that $\overline{\Psi^{\alpha}_{\mathbf{11}}} \Psi'^{\beta}_{\mathbf{11}}$ and $\overline{\Psi'^{\beta}_{\mathbf{11}}} \Psi^{\alpha}_{\mathbf{11}}$ are forbidden by the parity assignment of $\Psi^{\alpha}_{\mathbf{11}}$ and $\Psi'^{\beta}_{\mathbf{11}}$ shown in Table 3.

2.2.5. Mass terms for fermions on the UV brane

From the action in Eqs. (2.26) and (2.27), the quadratic mass terms on the UV brane for the bulk and brane fermions with $\langle\Phi_{32}\rangle \neq 0$ are given by

$$S_{\text{brane mass}}^{\text{fermion}} = \int d^4x \int_0^{2\pi R_6} dv \left(\sum_{j=1}^{8} \mathcal{L}_j^m + \mathcal{L}_{\chi_1}^m \right), \tag{2.28}$$

where

$$\begin{aligned}
\mathcal{L}_1^m =& - \mu_1^{\alpha\beta} \left\{ -i2(\overline{\check{\hat{e}}'^{\alpha}} \check{\hat{E}}^{\beta} + \overline{\check{\hat{\nu}}'^{\alpha}} \check{\hat{N}}^{\beta}) - 2\overline{\check{d}'^{\alpha}} \check{D}^{\beta} + \sqrt{2}\overline{\check{\hat{\nu}}^{\alpha}} \check{S}^{\beta} \right\} + \text{h.c.}, \\
\mathcal{L}_2^m =& - \widetilde{\mu}_1^{\alpha\beta} \left\{ i2(\overline{\check{e}^{\alpha}} \check{E}^{\beta} + \overline{\check{\nu}^{\alpha}} \check{N}^{\beta}) - 2\overline{\check{\hat{d}}^{\alpha}} \check{\hat{D}}^{\beta} + \sqrt{2}\overline{\check{\nu}'^{\alpha}} \check{S}^{\beta} \right\} + \text{h.c}, \\
\mathcal{L}_3^m =& - 2\mu_{\mathbf{11}}^{\beta\beta'} \Big\{ \overline{\check{D}^{\beta}} \check{D}^{\beta'} + \overline{\check{\hat{D}}^{\beta}} \check{\hat{D}}^{\beta'} + \overline{\check{E}^{\beta}} \check{E}^{\beta'} + \overline{\check{\hat{E}}^{\beta}} \check{\hat{E}}^{\beta'} \\
& \qquad + \overline{\check{N}^{\beta}} \check{N}^{\beta'} + \overline{\check{\hat{N}}^{\beta}} \check{\hat{N}}^{\beta'} + \overline{\check{S}^{\beta}} \check{S}^{\beta'} \Big\}, \\
\mathcal{L}_4^m =& - \mu_2^{\alpha\beta} \left\{ -i2(\overline{\check{\hat{e}}'^{\alpha}} \check{\hat{E}}'^{\beta} + \overline{\check{\hat{\nu}}'^{\alpha}} \check{\hat{N}}'^{\beta}) - 2\overline{\check{d}'^{\alpha}} \check{D}'^{\beta} + \sqrt{2}\overline{\check{\hat{\nu}}^{\alpha}} \check{S}'^{\beta} \right\} + \text{h.c.}, \\
\mathcal{L}_5^m =& - \widetilde{\mu}_2^{\alpha\beta} \left\{ i2(\overline{\check{e}^{\alpha}} \check{E}'^{\beta} + \overline{\check{\nu}^{\alpha}} \check{N}'^{\beta}) - 2\overline{\check{\hat{d}}^{\alpha}} \check{\hat{D}}'^{\beta} + \sqrt{2}\overline{\check{\nu}'^{\alpha}} \check{S}'^{\beta} \right\} + \text{h.c}, \\
\mathcal{L}_6^m =& - 2\mu_{\mathbf{11}}'^{\beta\beta'} \Big\{ \overline{\check{D}'^{\beta}} \check{D}'^{\beta'} + \overline{\check{\hat{D}}'^{\beta}} \check{\hat{D}}'^{\beta'} + \overline{\check{E}'^{\beta}} \check{E}'^{\beta'} + \overline{\check{\hat{E}}'^{\beta}} \check{\hat{E}}'^{\beta'} \\
& \qquad + \overline{\check{N}'^{\beta}} \check{N}'^{\beta'} + \overline{\check{\hat{N}}'^{\beta}} \check{\hat{N}}'^{\beta'} + \overline{\check{S}'^{\beta}} \check{S}'^{\beta'} \Big\}, \\
\mathcal{L}_7^m =& - \frac{m_B^{\alpha\beta}}{\sqrt{k}} (\overline{\chi^{\beta}}\, \check{\nu}'^{\alpha} + \overline{\check{\nu}'^{\alpha}}\, \chi^{\beta}), \\
\mathcal{L}_8^m =& - \frac{\hat{m}_B^{\alpha\beta}}{\sqrt{k}} (\overline{\chi^{\beta}}\, \check{\hat{\nu}}^{\alpha} + \overline{\check{\hat{\nu}}^{\alpha}}\, \chi^{\beta}), \\
\mathcal{L}_{\chi_1}^m =& - \frac{1}{2} M^{\beta\beta'}\, \overline{\chi^{\beta}}\, \chi^{\beta'}.
\end{aligned} \tag{2.29}$$

Here, $2\mu_1^{\alpha\beta} := \sqrt{2}\kappa^{\alpha\beta} w$, $2\widetilde{\mu}_1^{\alpha\beta} := \sqrt{2}\widetilde{\kappa}^{\alpha\beta} w$, $2\mu_2^{\alpha\beta} := \sqrt{2}\kappa'^{\alpha\beta} w$, $2\widetilde{\mu}_2^{\alpha\beta} := \sqrt{2}\widetilde{\kappa}'^{\alpha\beta} w$, $m_B^{\alpha\beta}/\sqrt{k} := \widetilde{\kappa}_{\mathbf{1}}^{\alpha\beta} w$, and $\hat{m}_B^{\alpha\beta}/\sqrt{k} := \kappa_{\mathbf{1}}^{\alpha\beta} w$. For the sake of simplicity $\chi_{\mathbf{1}}^{\beta}$ has been denoted as χ^{β}. All the μ parameters are dimensionless quantities, whereas $m_B^{\alpha\beta}$, $\hat{m}_B^{\alpha\beta}$, and $M^{\beta\beta'}$ have the dimension of mass.

2.2.6. Mass terms for gauge bosons on the UV brane

By replacing $\Phi_{\mathbf{32}}$ with its VEV $\langle\Phi_{\mathbf{32}}\rangle$ in Eq. (2.20), the mass terms for the 4D components of the $SO(11)$ gauge fields A_μ can be read off as

$$\begin{aligned}
\mathcal{L}_{\text{brane mass}}^{\text{4D gauge}} =& - \frac{g^2 w^2}{8} \Big\{ \left(A_\mu^{15} - A_\mu^{26}\right)^2 + \left(A_\mu^{16} + A_\mu^{25}\right)^2 \\
& + \left(A_\mu^{17} - A_\mu^{28}\right)^2 + \left(A_\mu^{18} + A_\mu^{27}\right)^2 + \left(A_\mu^{19} - A_\mu^{2,10}\right)^2 + \left(A_\mu^{1,10} + A_\mu^{29}\right)^2
\end{aligned}$$

$$
\begin{aligned}
&+\left(A_\mu^{35}+A_\mu^{46}\right)^2+\left(A_\mu^{36}-A_\mu^{45}\right)^2+\left(A_\mu^{37}+A_\mu^{48}\right)^2+\left(A_\mu^{38}-A_\mu^{47}\right)^2\\
&+\left(A_\mu^{39}+A_\mu^{4,10}\right)^2+\left(A_\mu^{3,10}-A_\mu^{49}\right)^2+\left(A_\mu^{57}-A_\mu^{68}\right)^2+\left(A_\mu^{58}+A_\mu^{67}\right)^2\\
&+\left(A_\mu^{59}-A_\mu^{6,10}\right)^2+\left(A_\mu^{5,10}+A_\mu^{69}\right)^2+\left(A_\mu^{79}-A_\mu^{8,10}\right)^2+\left(A_\mu^{7,10}+A_\mu^{89}\right)^2\\
&+\left(A_\mu^{23}-A_\mu^{14}\right)^2+\left(A_\mu^{31}-A_\mu^{24}\right)^2+\left(A_\mu^{12}-A_\mu^{34}+A_\mu^{56}+A_\mu^{78}+A_\mu^{9,10}\right)^2\\
&+\left(A_\mu^{1,11}\right)^2+\left(A_\mu^{2,11}\right)^2+\left(A_\mu^{3,11}\right)^2+\left(A_\mu^{4,11}\right)^2+\left(A_\mu^{5,11}\right)^2\\
&+\left(A_\mu^{6,11}\right)^2+\left(A_\mu^{7,11}\right)^2+\left(A_\mu^{8,11}\right)^2+\left(A_\mu^{9,11}\right)^2+\left(A_\mu^{10,11}\right)^2\Bigg\},
\end{aligned}
\tag{2.30}
$$

where the notation is the same as in Ref. [35]. We omit them here. All the 31 components of $SO(11)/SU(5)$ obtain large brane masses via $\langle\Phi_{\mathbf{32}}\rangle \neq 0$. Each mass term changes the fifth-dimensional BCs on the UV brane for the corresponding fields. Similarly, from the action in Eq. (2.20), we find the mass terms for the sixth-dimensional component of the A_v:

$$
\begin{aligned}
\mathcal{L}_{\text{brane mass}}^{\text{6th dim gauge}} = -\frac{g^2w^2}{8}\Bigg\{&\left(A_v^{15}-A_v^{26}\right)^2+\left(A_v^{16}+A_v^{25}\right)^2\\
&+\left(A_v^{17}-A_v^{28}\right)^2+\left(A_v^{18}+A_v^{27}\right)^2+\left(A_v^{19}-A_v^{2,10}\right)^2+\left(A_v^{1,10}+A_v^{29}\right)^2\\
&+\left(A_v^{35}+A_v^{46}\right)^2+\left(A_v^{36}-A_v^{45}\right)^2+\left(A_v^{37}+A_v^{48}\right)^2+\left(A_v^{38}-A_v^{47}\right)^2\\
&+\left(A_v^{39}+A_v^{4,10}\right)^2+\left(A_v^{3,10}-A_v^{49}\right)^2+\left(A_v^{57}-A_v^{68}\right)^2+\left(A_v^{58}+A_v^{67}\right)^2\\
&+\left(A_v^{59}-A_v^{6,10}\right)^2+\left(A_v^{5,10}+A_v^{69}\right)^2+\left(A_v^{79}-A_v^{8,10}\right)^2+\left(A_v^{7,10}+A_v^{89}\right)^2\\
&+\left(A_v^{23}-A_v^{14}\right)^2+\left(A_v^{31}-A_v^{24}\right)^2+\left(A_v^{12}-A_v^{34}+A_v^{56}+A_v^{78}+A_v^{9,10}\right)^2\\
&+\left(A_v^{1,11}\right)^2+\left(A_v^{2,11}\right)^2+\left(A_v^{3,11}\right)^2+\left(A_v^{4,11}\right)^2+\left(A_v^{5,11}\right)^2\\
&+\left(A_v^{6,11}\right)^2+\left(A_v^{7,11}\right)^2+\left(A_v^{8,11}\right)^2+\left(A_v^{9,11}\right)^2+\left(A_v^{10,11}\right)^2\Bigg\}.
\end{aligned}
\tag{2.31}
$$

These mass terms give masses for the corresponding fields by changing the BCs. We shall see more details in Section 3.

2.3. EW Higgs boson and twisted gauge

The orbifold BCs break $SO(11)$ to G_{PS}, and, further, the non-vanishing VEV of the 5D brane scalar Φ_{32} reduces G_{PS} to G_{SM}. The bilinear terms of the action of gauge fields in Eq. (2.13) are written down as

$$
\begin{aligned}
&\left(S_{\text{bulk}}^{\text{gauge}}\right)_{\text{quadratic}}\\
&= \int d^4x dv \frac{dz}{kz^2} \sum_{j<k} \left[\frac{1}{2} A_\lambda^{(jk)} \left\{\hat{\eta}^{\lambda\rho}\left(\Box + k^2\mathcal{P}_5 + \partial_v^2\right) - \left(1-\frac{1}{\xi}\right)\partial^\lambda\partial^\rho\right\} A_\rho^{(jk)}\right.\\
&\qquad \left. + \frac{1}{2}k^2 A_z^{(jk)}\left(\Box + \xi k^2\mathcal{P}_z + \partial_v^2\right) A_z^{(jk)} + \bar{c}^{(jk)}\left(\Box + \xi k^2\mathcal{P}_5 + \partial_v^2\right) c^{(jk)}\right],
\end{aligned}
$$

$$\hat{\eta}_{\lambda\rho} = \hat{\eta}^{\lambda\rho} = \mathrm{diag}(-1, +1, +1, +1, +1),$$

$$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu,\ \partial^\lambda = \hat{\eta}^{\lambda\rho}\partial_\rho,\ \mathcal{P}_5 = z^2\frac{\partial}{\partial z}\frac{1}{z^2}\frac{\partial}{\partial z},\ \mathcal{P}_z = \frac{\partial}{\partial z}z^2\frac{\partial}{\partial z}\frac{1}{z^2}. \tag{2.32}$$

Here, λ, ρ run over $0, 1, 2, 3, 6$ so that $A_\lambda\hat{\eta}^{\lambda\rho}B_\rho = A_\mu\eta^{\mu\nu}B_\nu + A_v B_v$. The fourth- and sixth-dimensional components A_μ and A_v have additional bilinear terms that come from the 5D brane scalar term $\mathcal{L}_{\Phi_{\mathbf{32}}}$ in Eq. (2.20) with $\langle\Phi_{\mathbf{32}}\rangle \neq 0$. The parity assignments of A_μ, A_z, and A_v are summarized in Table 1. The components $(\mathbf{2}, \mathbf{2}, \mathbf{1})$ of A_z and A_v have parity $(+, +)$. The components $(\mathbf{2}, \mathbf{2}, \mathbf{1})$ of A_v obtain masses through the brane mass terms in Eq. (2.31), so only the components $(\mathbf{2}, \mathbf{2}, \mathbf{1})$ of A_z, which is identified with the SM Higgs scalar fields, have zero modes. Three of them are absorbed by W and Z bosons. The remaining neutral component, the observed Higgs boson, also becomes massive by radiative corrections through the Hosotani mechanism as shown below. We note that the components $(\mathbf{2}, \mathbf{2}, \mathbf{1})$ of A_v get additional masses of order gR_6^{-1} by radiative corrections at one loop. The components $(\mathbf{3}, \mathbf{1}, \mathbf{1})$, $(\mathbf{1}, \mathbf{3}, \mathbf{1})$, and $(\mathbf{1}, \mathbf{1}, \mathbf{15})$ of A_μ have parity $(+, +)$. G_{PS} is spontaneously broken to G_{SM} by the non-vanishing VEV of the component $(\mathbf{1}, \mathbf{2}, \bar{\mathbf{4}})$ of $\Phi_{\mathbf{32}}$.

The zero modes of A_z are physical degrees of freedom which cannot be gauged away. By using the KK mode expansion discussed in Appendix A.1, the sixth-dimensional $n = 0$ modes (v-independent modes) of $A_z^{a\,11}$ are expanded, in terms of mode functions $\{h_n^{(++)}(z)\}$ for the parity $(+, +)$ boundary condition, as

$$A_z^{a\,11} = \frac{1}{\sqrt{2\pi R_6}}\left\{\phi_H^{a(0)}(x)u_H(z) + \sum_{n=1}^{\infty}\phi_H^{a(n)}(x)h_n^{(++)}(z)\right\}\ \ (a = 1, 2, 3, 4), \tag{2.33}$$

where the zero mode function in the fifth dimension is given by $u_H(z) = [3/k(z_L^3 - 1)]^{1/2} z^2$. The four-component real field $\phi_H^{a(0)}(x)$ is identified with the EW Higgs doublet field in the SM. We will show later that the $\phi_H := \phi_H^{4(0)}$ dynamically obtain the non-vanishing VEV $\langle\phi_H\rangle \neq 0$. The VEV breaks G_{SM} to $SU(3)_C \times U(1)_{\mathrm{EM}}$. $\phi_H^{1,2,3(0)}$ are absorbed by W and Z bosons.

Under a general gauge transformation $A'_M = \Omega A_M\Omega^{-1} + (i/g)\Omega\partial_M\Omega^{-1}$, new gauge potentials satisfy the following new BCs:

$$\begin{aligned}
\begin{pmatrix} A'_\mu \\ A'_y \\ A'_v \end{pmatrix}(x, y_j - y, v_j - v) &= P'_j\begin{pmatrix} A'_\mu \\ -A'_y \\ -A'_v \end{pmatrix}(x, y_j + y, v_j + v)P_j'^{-1} \\
&\quad + \frac{i}{g}P'_j\begin{pmatrix} \partial_\mu \\ -\partial_y \\ -\partial_v \end{pmatrix}P_j'^{-1}, \\
P'_j &= \Omega(x, y_j - y, v_j - v)P_j\Omega(x, y_j + y, v_j + v)^{-1}.
\end{aligned} \tag{2.34}$$

It is convenient to work in the twisted gauge discussed, e.g., in Ref. [35]. With the following large gauge transformation $\Omega(y; \alpha)$, $\hat{\theta}_H$ is transformed to $\hat{\theta}'_H = 0$:

$$\begin{aligned}
\Omega(y) &= \exp\left\{i\frac{g\theta_H f_H}{\sqrt{4\pi R_6}}\int_y^L dy\tilde{u}_H(y)T_{4,11}\right\} = \exp\left\{i\theta(z)T_{4,11}\right\}, \\
\theta(z) &= \theta_H\frac{z_L^3 - z^3}{z_L^3 - 1}\ \ \text{for } 1 \leq z \leq z_L.
\end{aligned} \tag{2.35}$$

The new BC matrices P'_j are given by

$$\tilde{P}_j = \Omega(0)^2 P_j = e^{2i\theta_H T_{4,11}} P_j \ \ (j = 0, 2),$$
$$\tilde{P}_k = P_k \ \ (k = 1, 3). \tag{2.36}$$

For fields in the $SO(11)$ vector and spinor representations, the BC matrices are given by

$$\tilde{P}_0^{\rm vec} = \begin{cases} \begin{pmatrix} \cos\theta_H & -\sin\theta_H \\ -\sin\theta_H & -\cos\theta_H \end{pmatrix} & \text{in the 4–11 subspace,} \\ I_3 & \text{in the } 1,2,3 \text{ subspace,} \\ -I_6 & \text{in the 5–10 subspace,} \end{cases}$$

$$\tilde{P}_2^{\rm vec} = \begin{cases} \begin{pmatrix} \cos\theta_H & -\sin\theta_H \\ -\sin\theta_H & -\cos\theta_H \end{pmatrix} & \text{in the 4–11 subspace,} \\ I_9 & \text{otherwise,} \end{cases}$$

$$\tilde{P}_0^{\rm sp} = \begin{pmatrix} \pm\cos\frac{\theta_H}{2} & -i\sin\frac{\theta_H}{2} \\ i\sin\frac{\theta_H}{2} & \mp\cos\frac{\theta_H}{2} \end{pmatrix} \quad \text{for} \begin{cases} \psi \\ \hat{\psi}, \end{cases}$$

$$\tilde{P}_2^{\rm sp} = \begin{pmatrix} \cos\frac{\theta_H}{2} & \mp i\sin\frac{\theta_H}{2} \\ \pm i\sin\frac{\theta_H}{2} & -\cos\frac{\theta_H}{2} \end{pmatrix} \quad \text{for} \begin{cases} \psi \\ \hat{\psi}, \end{cases} \tag{2.37}$$

where

$$\psi = \begin{pmatrix} \nu \\ \nu' \end{pmatrix}, \begin{pmatrix} e \\ e' \end{pmatrix}, \begin{pmatrix} u_j \\ u'_j \end{pmatrix}, \begin{pmatrix} d_j \\ d'_j \end{pmatrix},$$
$$\hat{\psi} = \begin{pmatrix} \hat{\nu} \\ \hat{\nu}' \end{pmatrix}, \begin{pmatrix} \hat{e} \\ \hat{e}' \end{pmatrix}, \begin{pmatrix} \hat{u}_j \\ \hat{u}'_j \end{pmatrix}, \begin{pmatrix} \hat{d}_j \\ \hat{d}'_j \end{pmatrix}. \tag{2.38}$$

3. Mass spectrum of bosons

In this section we examine the spectrum of gauge fields, particularly for the sixth-dimensional $n = 0$ KK modes (v-independent modes), because we are interested in the mass spectrum of the SM particles and the effective potential $V_{\rm eff}(\theta_H)$. In the following we omit the modes which are odd under the sixth-dimensional loop translation $U_6 = P_2 P_0 = P_3 P_1 = -1$. Those modes have masses $\geq \frac{1}{2} m_{{\rm KK}_6}$.

The BCs for gauge fields in the absence of the brane terms are given by

$$\begin{cases} N: & \frac{\partial}{\partial z} A_\mu = 0 & \text{for parity} = +, \\ D: & A_\mu = 0 & \text{for parity} = -, \end{cases}$$

$$\begin{cases} N: & \frac{\partial}{\partial z}\left(\frac{1}{z^2} A_z\right) = 0 & \text{for parity} = +, \\ D: & A_z = 0 & \text{for parity} = -, \end{cases}$$

$$\begin{cases} N: & \frac{\partial}{\partial z} A_v = 0 \quad \text{for parity} = +, \\ D: & A_v = 0 \quad \text{for parity} = - \end{cases} \tag{3.1}$$

at $z = 1$ ($y = 0$) and $z = z_L$ ($y = L$). In the presence of the brane mass terms $|gA_\mu \langle \Phi_{32} \rangle|^2$ on the UV brane, the Neumann BC N is modified to an effective Dirichlet BC D_{eff} for the fifth-dimensional zero mode of the $SO(11)/SU(5)$ components of A_μ and A_v [35]. The equation of motion for A_μ^a in the y-coordinate is given in the form

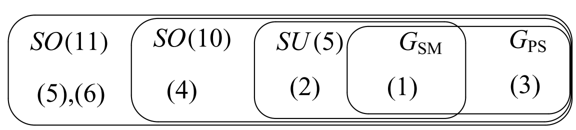
$$\left(\Box + e^{\sigma(y)} \frac{\partial}{\partial y} e^{-3\sigma(y)} \frac{\partial}{\partial y} + \frac{\partial^2}{\partial v^2} - \frac{g^2 w^2}{2} \delta(y)\right) A_\mu^a - \left(1 - \frac{1}{\xi}\right) \partial_\mu (\partial^\nu A_\nu^a + \partial_v A_v^a) = \cdots , \tag{3.2}$$

where the right-hand side involves interaction terms. Suppose that A_μ^a is parity even, namely $A_\mu^a(x, -y, -v) = +A_\mu^a(x, y, v)$. Note that A_v^a is parity odd so that $\partial_v A_v^a$ is parity even. By integrating the equation $\int_{-\epsilon}^{\epsilon} dy \cdots$ and taking the limit $\epsilon \to 0$, one finds $\partial A_\mu^a / \partial y|_{y=\epsilon} = (g^2 w^2/4) A_\mu^a|_{y=0}$. In the z-coordinate the Neumann condition is changed to the effective Dirichlet condition

$$D_{\text{eff}}(\omega): \ \left(\frac{\partial}{\partial z} - \omega\right) A_\mu^a = 0, \ \ \omega = \frac{g^2 w^2}{4k} \quad \text{at } z = 1^+. \tag{3.3}$$

Note that in the current six-dimensional model the mass dimensions of g and w are $[g] = M^{-1}$, $[w] = M^{3/2}$, so that ω is dimensionless. When $\omega \neq 0$, A_μ^a develops a cusp at $y = 0$. The lowest mode of A_μ^a becomes massive, with mass $O(m_{\text{KK}_5})$. The value of ω depends on the brane mass terms. The BCs for the gauge fields are summarized in Table 5.

Table 5. Venn diagram for the symmetry-breaking pattern and summary of the BCs $\begin{pmatrix} P_2 & P_3 \\ P_0 & P_1 \end{pmatrix}$ of the $SO(11)$ gauge field. N and D stand for Neumann and Dirichlet conditions, respectively. D_{eff} represents the effective Dirichlet condition given by Eq. (3.3).



		No. of generators	A_μ	A_z	A_v
(1)	G_{SM}	12	$\begin{pmatrix} N & N \\ N & N \end{pmatrix}$	$\begin{pmatrix} D & D \\ D & D \end{pmatrix}$	$\begin{pmatrix} D & D \\ D & D \end{pmatrix}$
(2)	$SU(5)/G_{\text{SM}}$	12	$\begin{pmatrix} N & N \\ D & D \end{pmatrix}$	$\begin{pmatrix} D & D \\ N & N \end{pmatrix}$	$\begin{pmatrix} D & D \\ N & N \end{pmatrix}$
(3)	$G_{\text{PS}}/G_{\text{SM}}$	9	$\begin{pmatrix} D_{\text{eff}} & N \\ D_{\text{eff}} & N \end{pmatrix}$	$\begin{pmatrix} D & D \\ D & D \end{pmatrix}$	$\begin{pmatrix} D & D \\ D & D \end{pmatrix}$
(4)	$SO(10)/(SU(5) \cup G_{\text{PS}})$	12	$\begin{pmatrix} D_{\text{eff}} & N \\ D & D \end{pmatrix}$	$\begin{pmatrix} D & D \\ N & N \end{pmatrix}$	$\begin{pmatrix} D & D \\ D_{\text{eff}} & N \end{pmatrix}$
(5)	$SO(5)/SO(4)$	4	$\begin{pmatrix} D & D \\ D & D \end{pmatrix}$	$\begin{pmatrix} N & N \\ N & N \end{pmatrix}$	$\begin{pmatrix} D_{\text{eff}} & N \\ D_{\text{eff}} & N \end{pmatrix}$
(6)	$SO(7)/SO(6)$	6	$\begin{pmatrix} D & D \\ D_{\text{eff}} & N \end{pmatrix}$	$\begin{pmatrix} N & N \\ D & D \end{pmatrix}$	$\begin{pmatrix} D_{\text{eff}} & N \\ D & D \end{pmatrix}$

In the twisted gauge $\tilde{A}_M = \Omega(z) A_M \Omega(z)^{-1} + \frac{i}{g}\Omega(z)\partial_M \Omega(z)^{-1}$, where $\Omega(z) = e^{i\theta(z)T_{4,11}}$. $\theta(z)$ is defined in Eq. (2.35). The $(k,4)$, $(k,11)$, and $(4,11)$ components of the $SO(11)$ gauge fields are changed as

$$A_M^{k4} = \cos\theta(z)\tilde{A}_M^{k4} - \sin\theta(z)\tilde{A}_M^{k,11} \ (k \neq 4, 11),$$
$$A_M^{k,11} = \sin\theta(z)\tilde{A}_M^{k4} + \sin\theta(z)\tilde{A}_M^{k,11},$$
$$A_z^{4,11} = \tilde{A}_z^{4,11} - \frac{\sqrt{2}}{g}\theta'(z) = \tilde{A}_z^{4,11} + \frac{3\sqrt{2}}{g}\frac{z^2}{z_L^3 - 1}\theta_H, \tag{3.4}$$

while the other components remain unchanged. As in Ref. [35], the BCs at $z = z_L$ determine wave functions for $\tilde{A}_\mu$, $\tilde{A}_z$, and $\tilde{A}_v$:

$$\tilde{A}_\mu \ \ N: \ C(z;\lambda), \ D: \ S(z;\lambda),$$
$$\tilde{A}_z \ \ N: \ S'(z;\lambda), \ D: \ C'(z;\lambda),$$
$$\tilde{A}_v \ \ N: \ S'(z;\lambda), \ D: \ C'(z;\lambda), \tag{3.5}$$

where $C(z;\lambda)$ and $S(z;\lambda)$ are defined by Eqs. (A.2) and (A.3) in Appendix A.1.

From the BCs for the gauge fields summarized in Table 5, the components (2) $SU(5)/G_{\rm SM}$, (4) $SO(10)/(SU(5) \cup G_{\rm PS})$, and (6) $SO(7)/SO(6)$ are parity odd under the sixth-dimensional loop translation U_6, while the components (1) $G_{\rm SM}$, (3) $G_{\rm PS}/G_{\rm SM}$, and (5) $SO(5)/SO(4)$ are parity even under U_6. Thus, we find that the low-lying modes of the components (2), (4), and (6) of A_M have masses of $O(m_{{\rm KK}_6})$, while the low-lying modes of the components (1), (3), and (5) of A_M have masses less than or around $O(m_{{\rm KK}_5})$.

We summarize the formulas determining the mass spectrum of the sixth-dimensional $n = 0$ modes of A_M, for which the arguments in Ref. [35] remain intact. We use the same notation as Ref. [35]. $A_M^{a_L}$ and $A_M^{a_R}$ $(a_L, a_R = 1, 2, 3)$ stand for $SU(2)_L$ and $SU(2)_R$ components of A_M, respectively.

- A_μ components.
  (i) $(\tilde{A}_\mu^{a_L}, \tilde{A}_\mu^{a_R}, \tilde{A}_\mu^{a,11})$ $(a = 1, 2)$: For W and W_R towers, there are sixth-dimensional $n = 0$ modes, and there are also fifth-dimensional zero modes in the absence of brane terms. In the presence of the brane terms in Eq. (2.30), their mass spectra of the fifth-dimensional modes are given by

$$2C'(SC' + \lambda\sin^2\theta_H) - \omega C(2SC' + \lambda\sin^2\theta_H) = 0, \tag{3.6}$$

where $\omega := g^2 w^2/4k$. Let us denote the average magnitude of $C(1;\lambda)$ and $C'(1;\lambda)$ by $\overline{C}$ and $\overline{C'}$, respectively. When $\omega\overline{C} \gg \overline{C'}$, the spectrum is determined by

$$W \text{ tower: } 2S(1;\lambda)C'(1;\lambda) + \lambda\sin^2\theta_H = 0,$$
$$W_R \text{ tower: } C(1;\lambda) = 0. \tag{3.7}$$

Indeed, in the typical cases examined below, $\overline{C}/\overline{C'} \sim 10^6$ so that the condition $\omega\overline{C} \gg \overline{C'}$ is well satisfied for $w \gg (m_{{\rm KK}_5})^{3/2}$. The mass of the W boson $m_W = m_{W^{(0)}}$ is given by

$$m_W \simeq \sqrt{\frac{3}{2}} k z_L^{-3/2} \sin\theta_H = \frac{\sqrt{3}\sin\theta_H}{\sqrt{2}\pi\sqrt{z_L}} m_{{\rm KK}_5}. \tag{3.8}$$

(ii) $(\tilde{A}_\mu^{3_L}, \tilde{C}_\mu, \tilde{B}_\mu^Y, \tilde{A}_\mu^{3,11})$: For γ, Z, and Z_R towers, there are sixth-dimensional $n = 0$ modes, and there are also fifth-dimensional zero modes in the absence of brane terms. Here, $C_\mu = \sqrt{2/5}A_\mu^{3_R} + \sqrt{3/5}A_\mu^{0_C}$ and $B_\mu^Y = \sqrt{3/5}A_\mu^{3_R} - \sqrt{2/5}A_\mu^{0_C}$, where $A_\mu^{0_C} = (A_\mu^{56} + A_\mu^{78} + A_\mu^{9\,10})/\sqrt{3}$. In the presence of the brane terms in Eq. (2.30), the mass spectra of the fifth-dimensional modes are given by

$$C' \left\{2C'(SC' + \lambda \sin^2\theta_H) - \omega C(5SC' + 4\lambda \sin^2\theta_H)\right\} = 0. \tag{3.9}$$

For $\omega\overline{C} \gg \overline{C'}$,

$$\begin{aligned} \gamma \text{ tower: } & C'(1;\lambda) = 0, \\ Z \text{ tower: } & 5S(1;\lambda)C'(1;\lambda) + 4\lambda \sin^2\theta_H = 0, \\ Z_R \text{ tower: } & C(1;\lambda) = 0. \end{aligned} \tag{3.10}$$

The mass of the Z boson $m_Z = m_{Z^{(0)}}$ is given by

$$m_Z \simeq \sqrt{\frac{12}{5}} k z_L^{-3/2} \sin\theta_H = \frac{m_W}{\cos\theta_W}, \quad \sin^2\theta_W = \frac{3}{8}. \tag{3.11}$$

The relation for the Z tower in Eq. (3.10) can be written, by using $\cos^2\theta_W = \frac{5}{8}$, as

$$Z \text{ tower: } 2S(1;\lambda)C'(1;\lambda) + \frac{\lambda}{\cos^2\theta_W} \sin^2\theta_H = 0. \tag{3.12}$$

The value $\sin^2\theta_W = \frac{3}{8}$ is valid at the GUT scale. The gauge coupling constants evolve as the energy scale, following the renormalization group equation (RGE). It is not clear whether $\sin^2\theta_W$ evolves to the observed value at low energies as there are KK modes which do not respect $SU(5)$ below the GUT scale. We leave solving the RGE for future investigation. When we evaluate the effective potential $V_{\text{eff}}(\theta_H)$ at the EW scale in Sect. 5, we adopt the formula in Eq. (3.12), where the observed value, $\sin^2\theta_W \simeq 0.2312$, at the EW scale is inserted.

(iii) $\tilde{A}_\mu^{4,11}$: For the $\hat{A}^4$ tower, there are sixth-dimensional $n = 0$ modes, but there are no fifth-dimensional zero modes. The mass spectra of the fifth-dimensional modes are given by

$$\hat{A}^4 \text{ tower: } S(1;\lambda) = 0. \tag{3.13}$$

(iv) For $SU(3)_C$ gluons, there are sixth-dimensional $n = 0$ modes, and there are also fifth-dimensional zero modes which correspond to 4D gluons. The mass spectrum of the fifth-dimensional massive modes is given by

$$\text{gluon tower: } C'(1;\lambda) = 0. \tag{3.14}$$

(v) For X-gluons, there are sixth-dimensional $n = 0$ modes, and there are also fifth-dimensional zero modes in the absence of brane terms. In the presence of the brane terms in Eq. (2.30), the mass spectrum of the fifth-dimensional modes is given by

$$X\text{-gluon tower: } C'(1;\lambda) - \omega C(1;\lambda) = 0. \tag{3.15}$$

For $\omega\overline{C} \gg \overline{C'}$,

$$X\text{-gluon tower: } C(1;\lambda) \simeq 0. \tag{3.16}$$

(vi) For X-bosons, there are no sixth-dimensional $n = 0$ modes.
(vii) For X'-bosons, there are no sixth-dimensional $n = 0$ modes.
(viii) For Y, Y'-bosons, there are no sixth-dimensional $n = 0$ modes.

- A_z components.
  (i) A_z^{ab} ($1 \le a < b \le 3$) and A_z^{jk} ($5 \le j < k \le 10$): There are sixth-dimensional $n = 0$ modes, but there are no fifth-dimensional zero modes. The mass spectrum of the fifth-dimensional modes is given by

$$C'(1;\lambda) = 0. \tag{3.17}$$

  (ii) A_z^{a4}, $A_z^{a\,11}$ ($a = 1, 2, 3$): There are sixth-dimensional $n = 0$ modes for A_z^{a4}, $A_z^{a\,11}$ ($a = 1, 2, 3$). There are no fifth-dimensional zero modes for A_z^{a4}, while there are fifth-dimensional zero modes for $A_z^{a\,11}$. The mass spectra of the fifth-dimensional modes are given by

$$S(1;\lambda)C'(1;\lambda) + \lambda \sin^2\theta_H = 0. \tag{3.18}$$

  (iii) $A_z^{4,11}$: Higgs tower. There are sixth-dimensional $n = 0$ modes, and there is also a fifth-dimensional zero mode. The mass spectrum of the fifth-dimensional modes is given by

$$\text{Higgs tower: } S(1;\lambda) = 0. \tag{3.19}$$

  (iv) A_z^{ak} ($a = 1, 2, 3,\ k = 5, \ldots, 10$): There are no sixth-dimensional $n = 0$ modes.
  (v) A_z^{k4}, A_z^{k11} ($k = 5, \ldots, 10$): There are no sixth-dimensional $n = 0$ modes.

- A_v components.
  (i) A_v^{ab} ($1 \le a < b \le 3$) and A_v^{jk} ($5 \le j < k \le 10$): There are sixth-dimensional $n = 0$ modes, but there are no fifth-dimensional zero modes. The mass spectra of the fifth-dimensional modes are given by

$$C'(1;\lambda) = 0. \tag{3.20}$$

  (ii) A_v^{a4}, $A_v^{a\,11}$ ($a = 1, 2, 3$): There are sixth-dimensional $n = 0$ modes for A_v^{a4} and $A_v^{a\,11}$ ($a = 1, 2, 3$). There are no fifth-dimensional zero modes for A_v^{a4}, while there are fifth-dimensional zero modes for $A_v^{a\,11}$ ($a = 1, 2, 3$) in the absence of brane terms. In the presence of the brane terms in Eq. (2.31), one finds that at $z = 1$,

$$\begin{pmatrix} \cos\theta_H C' & -\sin\theta_H S' \\ \sin\theta_H (C - \omega C') & \cos\theta_H (S - \omega S') \end{pmatrix} \begin{pmatrix} \beta^v_{a4} \\ \beta^v_{a11} \end{pmatrix} = 0. \tag{3.21}$$

Thus, the mass spectra of the fifth-dimensional modes are given by

$$C'(1;\lambda)\left\{S(1;\lambda) - \omega S'(1;\lambda)\right\} + \lambda \sin^2\theta_H = 0. \tag{3.22}$$

The average magnitude of $S(1;\lambda)$ $(= \overline{S})$ is much larger than that of $S'(1;\lambda)$ $(= \overline{S'})$. In typical cases $\overline{S}/\overline{S'} \sim 10^4$. Hence the spectrum is determined by

$$C'(1;\lambda)S(1;\lambda) + \lambda \sin^2\theta_H = 0. \tag{3.23}$$

(iii) $A_v^{4,11}$: A_v-Higgs tower. There are sixth-dimensional $n=0$ modes, and there are also fifth-dimensional zero modes in the absence of brane terms. In the presence of the brane terms in Eq. (2.31), the mass spectra of the fifth-dimensional modes are given by

$$A_v\text{-Higgs tower: } S(1;\lambda) - \omega S'(1;\lambda) = 0, \tag{3.24}$$

which is well approximated by

$$A_v\text{-Higgs tower: } S(1;\lambda) = 0. \tag{3.25}$$

The components $A_v^{a\,11}$ ($a = 1, \ldots, 4$) acquire large one-loop corrections of order $g_w M_{\mathrm{KK}_6}$ to their masses so that their contributions to the θ_H-dependent part of $V_{\mathrm{eff}}(\theta_H)$ become negligible and may be dropped.

(iv) A_v^{ak} ($a = 1, 2, 3,\ k = 5, \ldots, 10$): There are no sixth-dimensional $n = 0$ modes.

(v) A_v^{k4}, A_v^{k11} ($k = 5, \ldots, 10$): There are no sixth-dimensional $n = 0$ modes.

4. Mass spectrum of fermions

In this section we determine the mass spectrum of quarks and leptons. For up-type quarks, there are no brane interactions on the UV brane ($y = 0$) as in the 5D $SO(11)$ GHGUT. For down-type quarks, charged leptons, and neutrinos, both 6D bulk mass terms and brane mass terms on the UV brane are important to reproduce the observed mass spectra.

In the following, we will calculate mass spectra for one set of a 6D $SO(11)$ **32** Weyl fermion $\Psi_{\mathbf{32}}^{\alpha}$ ($\alpha = 1, 2,$ or 3), 6D $SO(11)$ **11** Dirac fermions $\Psi_{\mathbf{11}}^{\beta}$ and $\Psi_{\mathbf{11}}^{\prime\beta}$ ($\beta = \alpha$), and 5D $SO(11)$ singlet $\chi_{\mathbf{1}}^{\alpha}$, which is identified with one generation of the SM quarks and leptons. We will also discuss mass spectra for the 6D $SO(11)$ **32** Weyl fermion $\Psi_{\mathbf{32}}^{\alpha=4}$ denoted by $\Psi'_{\mathbf{32}}$. We will omit the superscripts α and β. We are interested in EW-scale physics and we keep only the sixth-dimensional $n = 0$ KK modes because the masses of sixth-dimensional $n \neq 0$ modes are much larger than $O(m_{\mathrm{KK}_5})$ as $m_{\mathrm{KK}_6} \gg m_{\mathrm{KK}_5}$. That is, we will discuss mass spectra for the G_{PS} $(\mathbf{2}, \mathbf{1}, \mathbf{4})$ and $(\mathbf{1}, \mathbf{2}, \mathbf{4})$ components of $\Psi_{\mathbf{32}}$, the G_{PS} $(\mathbf{1}, \mathbf{1}, \mathbf{6})$ components of $\Psi_{\mathbf{11}}$, the G_{PS} $(\mathbf{2}, \mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{1}, \mathbf{1})$ components of $\Psi'_{\mathbf{11}}$, and the G_{PS} $(\mathbf{2}, \mathbf{1}, \mathbf{4})$ and $(\mathbf{1}, \mathbf{2}, \mathbf{4})$ components of $\Psi'_{\mathbf{32}}$. We will denote $c_{\Psi_{\mathbf{32}}}^{\alpha\neq 4}$, $c_{\Psi_{\mathbf{11}}}^{\beta}$, $c_{\Psi_{\mathbf{11}}}^{\prime\beta}$, and $c_{\Psi_{\mathbf{32}}}^{\alpha=4}$ as c_0, c_1, c_2, and c'_0, respectively. In this paper we assume that $c_0, c_1, c'_0 \geq 0$ while c_2 can be negative. The calculation method employed in the following is the same as employed in the 5D $SO(11)$ GHGUT discussed in Ref. [35].

4.1. *Up-type quark*

(i) $Q_{\mathrm{EM}} = +2/3$: u_j, u'_j ($\Psi_{\mathbf{32}}$)

From the action in Eq. (2.19), we find the equations of motion for up-type quarks with $\gamma_{6D}^{7} = +1$:

$$\begin{aligned}
-i\delta \begin{pmatrix} u_{+L}^{\dagger} \\ u_{+L}^{\prime\dagger} \end{pmatrix} &: \left(-k\hat{D}_{-}(c_0) + i\partial_v\right) \begin{pmatrix} \check{u}_{+R} \\ \check{u}'_{+R} \end{pmatrix} + \sigma^{\mu}\partial_{\mu} \begin{pmatrix} \check{u}_{+L} \\ \check{u}'_{+L} \end{pmatrix} = 0, \\
i\delta \begin{pmatrix} u_{+R}^{\dagger} \\ u_{+R}^{\prime\dagger} \end{pmatrix} &: \overline{\sigma}^{\mu}\partial_{\mu} \begin{pmatrix} \check{u}_{+R} \\ \check{u}'_{+R} \end{pmatrix} + \left(-k\hat{D}_{+}(c_0) + i\partial_v\right) \begin{pmatrix} \check{u}_{+L} \\ \check{u}'_{+L} \end{pmatrix} = 0.
\end{aligned} \tag{4.1}$$

Here, $\sigma^\mu = (I_2, \vec{\sigma})$, $\overline{\sigma}^\mu = (-I_2, \vec{\sigma})$, and

$$\hat{D}_\pm(c) = \pm\left(\frac{\partial}{\partial z} + i\theta'(z)T_{4,11}\right) + \frac{c}{z},$$

$$T_{4,11} = \begin{cases} \frac{1}{2}\tau_1 & \text{for } \psi, \\ -\frac{1}{2}\tau_1 & \text{for } \hat{\psi}, \end{cases} \tag{4.2}$$

where ψ, $\hat{\psi}$ are the pairs defined in Eq. (2.38). For an up-type quark with $\gamma^7_{6D} = -1$, namely for the top quark, the equations become

$$\left(k\hat{D}_+(c_0) - i\partial_v\right)\begin{pmatrix} \check{u}_{-R} \\ \check{u}'_{-R} \end{pmatrix} + \sigma^\mu\partial_\mu \begin{pmatrix} \check{u}_{-L} \\ \check{u}'_{-L} \end{pmatrix} = 0,$$

$$\overline{\sigma}^\mu\partial_\mu \begin{pmatrix} \check{u}_{-R} \\ \check{u}'_{-R} \end{pmatrix} + \left(k\hat{D}_-(c_0) - i\partial_v\right)\begin{pmatrix} \check{u}_{-L} \\ \check{u}'_{-L} \end{pmatrix} = 0. \tag{4.3}$$

In the twisted gauge, the equations of motion become

$$\begin{cases} \overline{\sigma}^\mu\partial_\mu\tilde{u}_{+R} = (+kD_+(c_0) - i\partial_v)\,\tilde{u}_{+L}, \\ \sigma^\mu\partial_\mu\tilde{u}_{+L} = (+kD_-(c_0) - i\partial_v)\,\tilde{u}_{+R}, \end{cases} \quad \begin{cases} \overline{\sigma}^\mu\partial_\mu\tilde{u}'_{+R} = (+kD_+(c_0) - i\partial_v)\,\tilde{u}'_{+L}, \\ \sigma^\mu\partial_\mu\tilde{u}'_{+L} = (+kD_-(c_0) - i\partial_v)\,\tilde{u}'_{+R}, \end{cases}$$

$$\begin{cases} \overline{\sigma}^\mu\partial_\mu\tilde{u}_{-R} = (-kD_-(c_0) + i\partial_v)\,\tilde{u}_{-L}, \\ \sigma^\mu\partial_\mu\tilde{u}_{-L} = (-kD_+(c_0) + i\partial_v)\,\tilde{u}_{-R}, \end{cases} \quad \begin{cases} \overline{\sigma}^\mu\partial_\mu\tilde{u}'_{-R} = (-kD_-(c_0) + i\partial_v)\,\tilde{u}'_{-L}, \\ \sigma^\mu\partial_\mu\tilde{u}'_{-L} = (-kD_+(c_0) + i\partial_v)\,\tilde{u}'_{-R}, \end{cases} \tag{4.4}$$

where

$$D_\pm(c) = \pm\frac{\partial}{\partial z} + \frac{c}{z}, \tag{4.5}$$

and the relation between the twisted gauge and the original one is given by

$$\begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} \cos\frac{1}{2}\theta(z) & -i\sin\frac{1}{2}\theta(z) \\ -i\sin\frac{1}{2}\theta(z) & \cos\frac{1}{2}\theta(z) \end{pmatrix}\begin{pmatrix} \tilde{u} \\ \tilde{u}' \end{pmatrix} =: \Omega(z)^{-1}\begin{pmatrix} \tilde{u} \\ \tilde{u}' \end{pmatrix}. \tag{4.6}$$

Let us check KK mode expansions for u_L and u_R. Since u_L has the parity assignment $\begin{pmatrix} P_2 & P_3 \\ P_0 & P_1 \end{pmatrix} = \begin{pmatrix} + & + \\ + & + \end{pmatrix}$, we can expand it by using parity even combinations:

$$u_L(x,y,v) = \sum_{n=0}^{\infty} u^C_{nL}(x,y)f^C_n(v) + \sum_{n=1}^{\infty} u^S_{nL}(x,y)f^S_n(v)$$

$$= \frac{1}{\sqrt{2\pi R_6}}\sum_{n=-\infty}^{\infty} u_{nL}(x,y)e^{inv/R_6}, \tag{4.7}$$

where

$$u_{nL}(x,y) = \begin{cases} \dfrac{1}{\sqrt{2}}\left\{u^C_{nL}(x,y) - iu^S_{nL}(x,y)\right\} & (n > 0) \\ u^C_{0L}(x,y) & (n = 0) \\ \dfrac{1}{\sqrt{2}}\left\{u^C_{-nL}(x,y) + iu^S_{-nL}(x,y)\right\} & (n < 0), \end{cases}$$

$$u_{nL}(x,-y) = u_{-nL}(x,y), \;\; u_{nL}(x,L-y) = u_{-nL}(x,L+y). \tag{4.8}$$

Note that $u^C_{nL}(x,y)$ and $u^S_{nL}(x,y)$ satisfy the same parity property as $\widetilde{\Phi}^C_n(x,y)$ and $\widetilde{\Phi}^S_n(x,y)$ in Eq. (A.13), and $f^C_n(v), f^S_n(v)$ are defined in Eq. (A.14). On the other hand, u_R has the parity assignment $\begin{pmatrix} P_2 & P_3 \\ P_0 & P_1 \end{pmatrix} = \begin{pmatrix} - & - \\ - & - \end{pmatrix}$ so that one can expand it by using parity odd combinations:

$$\begin{aligned} u_R(x,y,v) &= \sum_{n=0}^{\infty} u^S_{nR}(x,y) f^C_n(v) + \sum_{n=1}^{\infty} u^C_{nR}(x,y) f^S_n(v) \\ &= \frac{1}{\sqrt{2\pi R_6}} \sum_{n=-\infty}^{\infty} u_{nR}(x,y) e^{inv/R_6}, \end{aligned} \tag{4.9}$$

where

$$u_{nR}(x,y) = \begin{cases} \dfrac{1}{\sqrt{2}}\left\{u^S_{nR}(x,y) - iu^C_{nR}(x,y)\right\} & (n > 0) \\ u^S_{0R}(x,y) & (n = 0) \\ \dfrac{1}{\sqrt{2}}\left\{u^S_{-nR}(x,y) + iu^C_{-nR}(x,y)\right\} & (n < 0), \end{cases}$$

$$u_{nR}(x,-y) = -u_{-nR}(x,y), \;\; u_{nR}(x,L-y) = -u_{-nR}(x,L+y). \tag{4.10}$$

To investigate the physics at energies $\ll m_{\mathrm{KK}_6}$, we keep only sixth-dimensional $n = 0$ KK modes, and replace $u^{(\prime)}_{L/R}(x,y,v)$ by $u^{(\prime)}_{0L/R}(x,y)$:

$$\begin{pmatrix} u_L(x,y,v) \\ u_R(x,y,v) \\ u'_L(x,y,v) \\ u'_R(x,y,v) \end{pmatrix} \Rightarrow \frac{1}{\sqrt{2\pi R_6}} \begin{pmatrix} u_{0L}(x,y) \\ u_{0R}(x,y) \\ u'_{0L}(x,y) \\ u'_{0R}(x,y) \end{pmatrix}. \tag{4.11}$$

They have the following BCs at $(y_0, y_1) = (0, L_5)$:

$$\begin{cases} u_{0L}(x, y_j - y) = +u_{0L}(x, y_j + y), \\ u_{0R}(x, y_j - y) = -u_{0R}(x, y_j + y), \\ u'_{0L}(x, y_j - y) = -u'_{0L}(x, y_j + y), \\ u'_{0R}(x, y_j - y) = +u'_{0R}(x, y_j + y), \end{cases} \tag{4.12}$$

where $j = 0, 1$. The equations of motion in the twisted gauge of Eq. (4.6) for the $\gamma_{6D}^7 = +1$ fields become

$$\begin{cases} \overline{\sigma}^\mu \partial_\mu \tilde{\tilde{u}}_{+0R} = +kD_+(c_0)\tilde{\tilde{u}}_{+0L}, \\ \sigma^\mu \partial_\mu \tilde{\tilde{u}}_{+0L} = +kD_-(c_0)\tilde{\tilde{u}}_{+0R}, \end{cases}$$

$$\begin{cases} \overline{\sigma}^\mu \partial_\mu \tilde{\tilde{u}}'_{+0R} = +kD_+(c_0)\tilde{\tilde{u}}'_{+0L}, \\ \sigma^\mu \partial_\mu \tilde{\tilde{u}}'_{+0L} = +kD_-(c_0)\tilde{\tilde{u}}'_{+0R}. \end{cases} \tag{4.13}$$

It follows that $k^2 D_+ D_- \tilde{\tilde{u}}_{+0R} = \Box \tilde{\tilde{u}}_{+0R} = m^2 \tilde{\tilde{u}}_{+0R}$, etc. The BCs at $z = z_L$ in the twisted gauge are the same as those in the original gauge:

$$\tilde{\tilde{u}}_{+0R} = 0,\ D_+\tilde{\tilde{u}}_{+0L} = 0,\ \tilde{\tilde{u}}'_{+0L} = 0,\ D_-\tilde{\tilde{u}}'_{+0R} = 0. \tag{4.14}$$

In terms of the basis functions $C_{R/L}(z; \lambda, c)$, $S_{R/L}(z; \lambda, c)$ given in Appendix B of Ref. [35], which satisfy

$$D_+(c)\begin{pmatrix} C_L \\ S_L \end{pmatrix} = \lambda \begin{pmatrix} S_R \\ C_R \end{pmatrix},\ D_-(c)\begin{pmatrix} C_R \\ S_R \end{pmatrix} = \lambda \begin{pmatrix} S_L \\ C_L \end{pmatrix},$$

$$C_R = C_L = 1,\ \ S_R = S_L = 0 \quad \text{at } z = z_L, \tag{4.15}$$

one can write the mode functions for $\gamma_{6D}^7 = +1$ as

$$\begin{pmatrix} \tilde{\tilde{u}}_{+0R} \\ \tilde{\tilde{u}}'_{+0R} \end{pmatrix} = \begin{pmatrix} \alpha_R^u S_R(z; \lambda, c_0) \\ \alpha_R'^u C_R(z; \lambda, c_0) \end{pmatrix} f_R(x),\ \ \bar{\sigma}^\mu \partial_\mu f_R(x) = m f_L(x),$$

$$\begin{pmatrix} \tilde{\tilde{u}}_{+0L} \\ \tilde{\tilde{u}}'_{+0L} \end{pmatrix} = \begin{pmatrix} \alpha_L^u C_L(z; \lambda, c_0) \\ \alpha_L'^u S_L(z; \lambda, c_0) \end{pmatrix} f_L(x),\ \ \sigma^\mu \partial_\mu f_L(x) = m f_R(x). \tag{4.16}$$

The equations of motion in Eq. (4.13) in the bulk $0 < y < L$ $(1 < z < z_L)$ yield

$$\begin{pmatrix} -k\lambda & m \\ m & -k\lambda \end{pmatrix} \begin{pmatrix} \alpha_R^u \\ \alpha_L^u \end{pmatrix} = 0,\ \text{etc.}, \tag{4.17}$$

so that one finds

$$m = k\lambda. \tag{4.18}$$

The BCs at $z = 1$ in the twisted gauge follow from those in the original gauge:

$$u_{+0R} = 0\ \Rightarrow \tilde{u}_{+0R} \cos\frac{\theta_H}{2} - i\tilde{u}'_{+0R} \sin\frac{\theta_H}{2} = 0, \tag{4.19}$$

$$D_- u'_{+0R} = 0\ \Rightarrow -iD_-\tilde{u}_{+0R} \sin\frac{\theta_H}{2} + D_-\tilde{u}'_{+0R} \cos\frac{\theta_H}{2} = 0. \tag{4.20}$$

We write the above equations in matrix form:

$$M_u \begin{pmatrix} \alpha_R^u \\ \alpha_R'^u \end{pmatrix} := \begin{pmatrix} \cos\frac{\theta_H}{2} S_R^0 & -i\sin\frac{\theta_H}{2} C_R^0 \\ -i\sin\frac{\theta_H}{2} \lambda C_L^0 & \cos\frac{\theta_H}{2} \lambda S_L^0 \end{pmatrix} \begin{pmatrix} \alpha_R^u \\ \alpha_R'^u \end{pmatrix} = 0, \tag{4.21}$$

where we have used shorthand notation such as S_L^0 standing for $S_L(1;\lambda,c_0)$. In the following we will use the same notation. From $\det M_u = 0$, we find the mass formula of the up-type quarks:

$$S_L^0 S_R^0 + \sin^2\frac{\theta_H}{2} = 0. \tag{4.22}$$

The same formula holds for $\Psi_{\mathbf{32}}$ with $\gamma_{6D}^7 = -1$. The formula in Eq. (4.22) is the same as in the 5D $SO(11)$ GHGUT in Ref. [35]. For $\lambda z_L \ll 1$, the up-type quark mass spectrum is given by

$$m_u \simeq \begin{cases} kz_L^{-1}\sqrt{1-4c_0^2}\,\sin\frac{1}{2}\theta_H & \text{for } c_0 < \frac{1}{2}, \\ kz_L^{-1/2-c_0}\sqrt{4c_0^2-1}\,\sin\frac{1}{2}\theta_H & \text{for } c_0 > \frac{1}{2}. \end{cases} \tag{4.23}$$

4.2. Down-type quark

(ii) $Q_{\rm EM} = -\frac{1}{3}$: $d_j, d'_j, D_{j+}, D_{j-}$ $(\Psi_{\mathbf{32}}, \Psi_{\mathbf{11}})$

Consider $\Psi_{\mathbf{32}}$ with $\gamma_{6D}^7 = +1$. Parity even modes at $y=0$ with $(P_0,P_2)=(+,+)$ are d_{+L}, d'_{+R}, D_{+R}, and D_{-L}. From the action in Eq. (2.19) and the $\mathcal{L}_1^m$ and $\mathcal{L}_3^m$ terms in Eq. (2.28), one finds the equations of motion for down-type quarks:

$$\begin{aligned}
-i\delta\begin{pmatrix} d_{+L}^\dagger \\ d_{+L}^{\prime\dagger}\end{pmatrix} &: \big(-k\hat{D}_-(c_0)+i\partial_v\big)\begin{pmatrix}\check{d}_{+R}\\ \check{d}'_{+R}\end{pmatrix} + \sigma^\mu\partial_\mu\begin{pmatrix}\check{d}_{+L}\\ \check{d}'_{+L}\end{pmatrix} = 0,\\
i\delta\begin{pmatrix} d_{+R}^\dagger \\ d_{+R}^{\prime\dagger}\end{pmatrix} &: \overline{\sigma}^\mu\partial_\mu\begin{pmatrix}\check{d}_{+R}\\ \check{d}'_{+R}\end{pmatrix} + \big(-k\hat{D}_+(c_0)+i\partial_v\big)\begin{pmatrix}\check{d}_{+L}\\ \check{d}'_{+L}\end{pmatrix} = 2\mu_1\,\delta(y)\begin{pmatrix}0\\ \check{D}_{-L}\end{pmatrix},\\
-i\delta D_{+L}^\dagger &: \big(-k\hat{D}_-(c_1)+i\partial_v\big)\check{D}_{+R} + \sigma^\mu\partial_\mu\check{D}_{+L} = 0,\\
i\delta D_{+R}^\dagger &: \overline{\sigma}^\mu\partial_\mu\check{D}_{+R} + \big(-k\hat{D}_+(c_1)+i\partial_v\big)\check{D}_{+L} = 2\mu_{\mathbf{11}}\,\delta(y)\,\check{D}_{-L},\\
-i\delta D_{-L}^\dagger &: \big(k\hat{D}_+(c_1)-i\partial_v\big)\check{D}_{-R} + \sigma^\mu\partial_\mu\check{D}_{-L} = \delta(y)\left\{2\mu_{\mathbf{11}}\check{D}_{+R} + 2\mu_1\check{d}'_{+R}\right\},\\
i\delta D_{-R}^\dagger &: \overline{\sigma}^\mu\partial_\mu\check{D}_{-R} + \big(k\hat{D}_-(c_1)-i\partial_v\big)\check{D}_{-L} = 0.
\end{aligned} \tag{4.24}$$

For sixth-dimensional $n=0$ KK modes, the equations of motion reduce to

$$\begin{aligned}
\begin{matrix}\text{(a)}\\ \text{(b)}\end{matrix} &: -k\hat{D}_-(c_0)\begin{pmatrix}\check{d}_{+R}\\ \check{d}'_{+R}\end{pmatrix} + \sigma^\mu\partial_\mu\begin{pmatrix}\check{d}_{+L}\\ \check{d}'_{+L}\end{pmatrix} = 0,\\
\begin{matrix}\text{(c)}\\ \text{(d)}\end{matrix} &: \overline{\sigma}^\mu\partial_\mu\begin{pmatrix}\check{d}_{+R}\\ \check{d}'_{+R}\end{pmatrix} - k\hat{D}_+(c_0)\begin{pmatrix}\check{d}_{+L}\\ \check{d}'_{+L}\end{pmatrix} = 2\mu_1\delta(y)\begin{pmatrix}0\\ \check{D}_{-L}\end{pmatrix},\\
\text{(e)} &: -k\hat{D}_-(c_1)\check{D}_{+R} + \sigma^\mu\partial_\mu\check{D}_{+L} = 0,\\
\text{(f)} &: \overline{\sigma}^\mu\partial_\mu\check{D}_{+R} - k\hat{D}_+(c_1)\check{D}_{+L} = 2\mu_{\mathbf{11}}\delta(y)\check{D}_{-L},\\
\text{(g)} &: k\hat{D}_+(c_1)\check{D}_{-R} + \sigma^\mu\partial_\mu\check{D}_{-L} = 2\mu_{\mathbf{11}}\delta(y)\check{D}_{+R} + 2\mu_1\delta(y)\check{d}'_{+R},\\
\text{(h)} &: \overline{\sigma}^\mu\partial_\mu\check{D}_{-R} + k\hat{D}_-(c_1)\check{D}_{-L} = 0.
\end{aligned} \tag{4.25}$$

To obtain BCs at $y = 0$, we integrate the above (a), (d), (f), and (g) in the vicinity of $y = 0$ for parity-odd fields:

$$
\begin{aligned}
&\text{(a)} \ \Rightarrow 2\check{d}_{+R}(x,\epsilon) = 0, \\
&\text{(d)} \ \Rightarrow -2\check{d}'_{+L}(x,\epsilon) = 2\mu_1 \check{D}_{-L}(x,0), \\
&\text{(f)} \ \Rightarrow -2\check{D}_{+L}(x,\epsilon) = 2\mu_{\mathbf{11}} \check{D}_{-L}(x,0), \\
&\text{(g)} \ \Rightarrow 2\check{D}_{-R}(x,\epsilon) = 2\mu_{\mathbf{11}} \check{D}_{+R}(x,0) + 2\mu_1 \check{d}'_{+R}(x,0).
\end{aligned}
\tag{4.26}
$$

For parity-even fields, we evaluate the equations of motion at $y = +\epsilon$ by using the above conditions:

$$
\begin{aligned}
&\text{(c)} \ \Rightarrow \hat{D}_+ \check{d}_{+L}(x,\epsilon) = 0, \\
&\text{(b)} \ \Rightarrow -k\hat{D}_- \check{d}'_{+R} + \mu_1 k D_+ \check{D}_{-R} = 0, \\
&\text{(e)} \ \Rightarrow -kD_- \check{D}_{+R}(x,\epsilon) + \mu_{\mathbf{11}} k D_+ \check{D}_{-R} = 0, \\
&\text{(h)} \ \Rightarrow kD_- \check{D}_{-L} + \mu_{\mathbf{11}} k D_+ \check{D}_{+L} + \mu_1 k \hat{D}_+ \check{d}'_{+L} = 0,
\end{aligned}
\tag{4.27}
$$

where we used the equations of motion (d) and (g) at $y = +\epsilon$.

We recall that in the twisted gauge all fields obey free-field equations in the bulk. Their eigenmodes are determined by the BCs on the IR brane. The mass spectra can be fixed by the BCs at the UV brane. In the twisted gauge the mode functions are given by

$$
\begin{pmatrix} \tilde{\check{d}}_{+R} \\ \tilde{\check{d}}'_{+R} \\ \tilde{\check{D}}_{+R} \\ \tilde{\check{D}}_{-R} \end{pmatrix}
= \begin{pmatrix} \alpha_R^d S_R(z;\lambda,c_0) \\ \alpha_R^{d'} C_R(z;\lambda,c_0) \\ \alpha_R^{D_+} C_R(z;\lambda,c_1) \\ \alpha_R^{D_-} S_L(z;\lambda,c_1) \end{pmatrix} f_R(x),
$$

$$
\begin{pmatrix} \tilde{\check{d}}_{+L} \\ \tilde{\check{d}}'_{+L} \\ \tilde{\check{D}}_{+L} \\ \tilde{\check{D}}_{-L} \end{pmatrix}
= \begin{pmatrix} \alpha_L^d C_L(z;\lambda,c_0) \\ \alpha_L^{d'} S_L(z;\lambda,c_0) \\ \alpha_L^{D_+} S_L(z;\lambda,c_1) \\ \alpha_L^{D_-} C_R(z;\lambda,c_1) \end{pmatrix} f_L(x),
\tag{4.28}
$$

where $f_R(x), f_L(x)$ satisfy the relations in Eq. (4.16) and $m = k\lambda$. The BCs at $z = 1^+$ in the twisted gauge are converted to

$$
K \begin{pmatrix} \alpha_R^d \\ \alpha_R^{d'} \\ \alpha_R^{D_+} \\ \alpha_R^{D_-} \end{pmatrix} = 0,
$$

$$
K = \begin{pmatrix}
\cos\frac{\theta_H}{2} S_R^0 & -i\sin\frac{\theta_H}{2} C_R^0 & 0 & 0 \\
-i\sin\frac{\theta_H}{2} \lambda C_L^0 & \cos\frac{\theta_H}{2} \lambda S_L^0 & 0 & -\mu_1 \lambda C_R^1 \\
0 & 0 & \lambda S_L^1 & -\mu_{\mathbf{11}} \lambda C_R^1 \\
i\mu_1 \sin\frac{\theta_H}{2} S_R^0 & -\mu_1 \cos\frac{\theta_H}{2} C_R^0 & -\mu_{\mathbf{11}} C_R^1 & S_L^1
\end{pmatrix}.
\tag{4.29}
$$

From $\det K = 0$, we find the mass spectrum formula for the down-type quarks:

$$S_L^0 S_R^0 + \sin^2 \frac{\theta_H}{2} = -\frac{\mu_1^2 S_R^0 C_R^0 S_L^1 C_R^1}{\mu_{\mathbf{11}}^2 (C_R^1)^2 - (S_L^1)^2}. \tag{4.30}$$

The same formula is obtained for $\Psi_{\mathbf{32}}$ with $\gamma_{6D}^7 = -1$. For $\lambda z_L \ll 1$, the down-type quark mass spectrum is given by

$$m_d \simeq \begin{cases} k z_L^{c_0 - c_1 - 1} \dfrac{\mu_{\mathbf{11}}}{\mu_1} \sqrt{(1-2c_0)(1+2c_1)}\, \sin\dfrac{\theta_H}{2} & \text{for } c_0 < \frac{1}{2}, \\ k z_L^{-c_1 - 1/2} \dfrac{\mu_{\mathbf{11}}}{\mu_1} \sqrt{(2c_0-1)(2c_1+1)}\, \sin\dfrac{\theta_H}{2} & \text{for } c_0 > \frac{1}{2}. \end{cases} \tag{4.31}$$

Combining Eqs. (4.23) and (4.31), one finds

$$\frac{m_d}{m_u} = z_L^{c_0 - c_1} \frac{\mu_{\mathbf{11}}}{\mu_1} \sqrt{\frac{1+2c_1}{1+2c_0}}. \tag{4.32}$$

4.3. *Charged lepton*

(iii) $Q_{\rm EM} = -1$: e, e', E'_+, E'_- $(\Psi_{\mathbf{32}}, \Psi'_{\mathbf{11}})$

Parity even modes at $y = 0$ with $(P_0, P_2) = (+,+)$ are e_{+L}, e'_{+R}, E'_{+L}, and E'_{-R}. From the action in Eq. (2.19) and the $\mathcal{L}_5^m$ and $\mathcal{L}_6^m$ terms in Eq. (2.28), one finds the equations of motion for charged leptons:

$$\begin{aligned}
-i\delta \begin{pmatrix} e_{+L}^\dagger \\ e_{+L}'^\dagger \end{pmatrix} &: \big(-k\hat{D}_-(c_0) + i\partial_v\big) \begin{pmatrix} \check{e}_{+R} \\ \check{e}'_{+R} \end{pmatrix} + \sigma^\mu \partial_\mu \begin{pmatrix} \check{e}_{+L} \\ \check{e}'_{+L} \end{pmatrix} = 2i\tilde{\mu}_2\, \delta(y) \begin{pmatrix} \check{E}'_{-R} \\ 0 \end{pmatrix}, \\
i\delta \begin{pmatrix} e_{+R}^\dagger \\ e_{+R}'^\dagger \end{pmatrix} &: \overline{\sigma}^\mu \partial_\mu \begin{pmatrix} \check{e}_{+R} \\ \check{e}'_{+R} \end{pmatrix} + \big(-k\hat{D}_+(c_0) + i\partial_v\big) \begin{pmatrix} \check{e}_{+L} \\ \check{e}'_{+L} \end{pmatrix} = 0, \\
-i\delta E_{+L}'^\dagger &: \big(-k\hat{D}_-(c_2) + i\partial_v\big)\check{E}'_{+R} + \sigma^\mu \partial_\mu \check{E}'_{+L} = 2\mu'_{\mathbf{11}} \delta(y) \check{E}'_{-R}, \\
i\delta E_{+R}'^\dagger &: \overline{\sigma}^\mu \partial_\mu \check{E}'_{+R} + \big(-k\hat{D}_+(c_2) + i\partial_v\big)\check{E}'_{+L} = 0, \\
-i\delta E_{-L}'^\dagger &: \big(k\hat{D}_+(c_2) - i\partial_v\big)\check{E}'_{-R} + \sigma^\mu \partial_\mu \check{E}'_{-L} = 0, \\
i\delta E_{-R}'^\dagger &: \overline{\sigma}^\mu \partial_\mu \check{E}'_{-R} + \big(k\hat{D}_-(c_2) - i\partial_v\big)\check{E}'_{-L} = \delta(y) \left\{-2i\tilde{\mu}_2 \check{e}_{+L} + 2\mu'_{\mathbf{11}} \check{E}'_{+L}\right\}.
\end{aligned} \tag{4.33}$$

For sixth-dimensional $n = 0$ KK modes, the equations of motion for charged leptons reduce to

$$\begin{aligned}
\begin{matrix}\text{(a)} \\ \text{(b)}\end{matrix} &: -k\hat{D}_-(c_0) \begin{pmatrix} \check{e}_{+R} \\ \check{e}'_{+R} \end{pmatrix} + \sigma^\mu \partial_\mu \begin{pmatrix} \check{e}_{+L} \\ \check{e}'_{+L} \end{pmatrix} = 2i\tilde{\mu}_2\, \delta(y) \begin{pmatrix} \check{E}'_{-R} \\ 0 \end{pmatrix}, \\
\begin{matrix}\text{(c)} \\ \text{(d)}\end{matrix} &: \overline{\sigma}^\mu \partial_\mu \begin{pmatrix} \check{e}_{+R} \\ \check{e}'_{+R} \end{pmatrix} - k\hat{D}_+(c_0) \begin{pmatrix} \check{e}_{+L} \\ \check{e}'_{+L} \end{pmatrix} = 0, \\
\text{(e)} &: -k\hat{D}_-(c_2)\check{E}'_{+R} + \sigma^\mu \partial_\mu \check{E}'_{+L} = 2\mu'_{\mathbf{11}} \delta(y) \check{E}'_{-R}, \\
\text{(f)} &: \overline{\sigma}^\mu \partial_\mu \check{E}'_{+R} - k\hat{D}_+(c_2)\check{E}'_{+L} = 0, \\
\text{(g)} &: k\hat{D}_+(c_2)\check{E}'_{-R} + \sigma^\mu \partial_\mu \check{E}'_{-L} = 0, \\
\text{(h)} &: \overline{\sigma}^\mu \partial_\mu \check{E}'_{-R} + k\hat{D}_-(c_2)\check{E}'_{-L} = \delta(y) \left\{-2i\tilde{\mu}_2 \check{e}_{+L} + 2\mu'_{\mathbf{11}} \check{E}'_{+L}\right\}.
\end{aligned} \tag{4.34}$$

To obtain boundary conditions at $y=0$, we integrate the above (a), (d), (e), (h) in the vicinity of $y=0$ for parity-odd fields:

$$
\begin{aligned}
&\text{(a)} \ \Rightarrow 2\check{e}_{+R}(x,\epsilon) = 2i\widetilde{\mu}_2 \check{E}'_{-R}(x,0),\\
&\text{(d)} \ \Rightarrow -2\check{e}'_{+L}(x,\epsilon) = 0,\\
&\text{(e)} \ \Rightarrow 2\check{E}'_{+R}(x,\epsilon) = 2\mu'_{\mathbf{11}} \check{E}'_{-R}(x,0),\\
&\text{(h)} \ \Rightarrow -2\check{E}'_{-L}(x,\epsilon) = -i2\widetilde{\mu}_2 \check{e}_{+L}(x,0) + 2\mu'_{\mathbf{11}} \check{E}'_{+L}(x,0).
\end{aligned} \tag{4.35}
$$

For parity-even fields, we calculate the equations of motion at $y=+\epsilon$ by using the above conditions:

$$
\begin{aligned}
&\text{(b)} \ \Rightarrow \hat{D}_- \check{e}'_{+R}(x,\epsilon) = 0,\\
&\text{(c)} \ \Rightarrow \hat{D}_+ \check{e}_{+L} + i\widetilde{\mu}_2 D_- \check{E}'_{-L} = 0,\\
&\text{(f)} \ \Rightarrow D_+ \check{E}'_{+L}(x,\epsilon) + \mu'_{\mathbf{11}} D_- \check{E}'_{-L} = 0,\\
&\text{(g)} \ \Rightarrow D_+ \check{E}'_{-R} + i\widetilde{\mu}_2 \hat{D}_- \check{e}_{+R} - \mu'_{\mathbf{11}} D_- \check{E}'_{+R} = 0,
\end{aligned} \tag{4.36}
$$

where we used the equations of motion (d) and (h) at $y=+\epsilon$.

By using the BCs on the IR brane, the mode functions of charged leptons in the twisted gauge are given by

$$
\begin{pmatrix} \tilde{\check{e}}_{+R} \\ \tilde{\check{e}}'_{+R} \\ \tilde{\check{E}}'_{+R} \\ \tilde{\check{E}}'_{-R} \end{pmatrix}
= \begin{pmatrix} \alpha_R^e S_R(z;\lambda,c_0) \\ \alpha_R^{e'} C_R(z;\lambda,c_0) \\ \alpha_R^{E'_+} S_R(z;\lambda,c_2) \\ \alpha_R^{E'_-} C_L(z;\lambda,c_2) \end{pmatrix} f_R(x),
$$

$$
\begin{pmatrix} \tilde{\check{e}}_{+L} \\ \tilde{\check{e}}'_{+L} \\ \tilde{\check{E}}'_{+L} \\ \tilde{\check{E}}'_{-L} \end{pmatrix}
= \begin{pmatrix} \alpha_L^e C_L(z;\lambda,c_0) \\ \alpha_L^{e'} S_L(z;\lambda,c_0) \\ \alpha_L^{E'_+} C_L(z;\lambda,c_2) \\ \alpha_L^{E'_-} S_R(z;\lambda,c_2) \end{pmatrix} f_L(x). \tag{4.37}
$$

As in the case of down-type quarks, the BCs at $z=1^+$ in the twisted gauge are converted to

$$
K \begin{pmatrix} \alpha_R^e \\ \alpha_R^{e'} \\ \alpha_R^{E'_+} \\ \alpha_R^{E'_-} \end{pmatrix} = 0,
$$

$$
K = \begin{pmatrix}
\cos\frac{\theta_H}{2} S_R^0 & -i\sin\frac{\theta_H}{2} C_R^0 & 0 & -i\widetilde{\mu}_2 C_L^2 \\
-i\sin\frac{\theta_H}{2}\lambda C_L^0 & \cos\frac{\theta_H}{2}\lambda S_L^0 & 0 & 0 \\
0 & 0 & S_R^2 & -\mu'_{\mathbf{11}} C_L^2 \\
i\widetilde{\mu}_2 \cos\frac{\theta_H}{2}\lambda C_L^0 & \widetilde{\mu}_2 \sin\frac{\theta_H}{2}\lambda S_L^0 & -\mu'_{\mathbf{11}}\lambda C_L^2 & \lambda S_R^2
\end{pmatrix}. \tag{4.38}
$$

From $\det K = 0$, we find the mass spectrum formula for the charged leptons:

$$S_L^0 S_R^0 + \sin^2 \frac{\theta_H}{2} = -\frac{\tilde{\mu}_2^2 S_L^0 C_L^0 S_R^2 C_L^2}{\mu_{\mathbf{11}}'^2 (C_L^2)^2 - (S_R^2)^2}. \tag{4.39}$$

The mass spectrum of charged leptons, for which $\lambda z_L \ll 1$, is given by

$$m_e \simeq \begin{cases} k z_L^{-1+c_2-c_0} \dfrac{\mu'_{\mathbf{11}}}{\tilde{\mu}_2} \sqrt{(2c_0+1)(1-2c_2)} \sin \dfrac{\theta_H}{2} & \text{for } c_2 < \frac{1}{2}, \\ k z_L^{-\frac{1}{2}-c_0} \dfrac{\mu'_{\mathbf{11}}}{\tilde{\mu}_2} \sqrt{(2c_0+1)(2c_2-1)} \sin \dfrac{\theta_H}{2} & \text{for } c_2 > \frac{1}{2}. \end{cases} \tag{4.40}$$

Here we have made use of $(m_t/m_\tau)^2, (m_c/m_\mu)^2, (m_u/m_e)^2 \gg 1$, which assures $|S_L^0 S_R^0| \ll \sin^2 \frac{1}{2}\theta_H$, and have assumed that $\lambda^2 \ll \mu_{\mathbf{11}}'^2$ for $c_2 > \frac{1}{2}$ and $\lambda^2 \ll (\mu'_{\mathbf{11}} z_L^{2c_2-1})^2$ for $c_2 < \frac{1}{2}$ so that $\mu_{\mathbf{11}}'^2 (C_L^2)^2 \gg (S_R^2)^2$. Combining Eqs. (4.23) and (4.40), one finds

$$\frac{m_e}{m_u} = \begin{cases} z_L^{c_2-c_0} \dfrac{\mu'_{\mathbf{11}}}{\tilde{\mu}_2} \sqrt{\dfrac{1-2c_2}{1-2c_0}} & \text{for } c_0 < \frac{1}{2},\ c_2 < \frac{1}{2}, \\ z_L^{\frac{1}{2}-c_0} \dfrac{\mu'_{\mathbf{11}}}{\tilde{\mu}_2} \sqrt{\dfrac{2c_2-1}{1-2c_0}} & \text{for } c_0 < \frac{1}{2},\ c_2 > \frac{1}{2}, \\ z_L^{c_2-\frac{1}{2}} \dfrac{\mu'_{\mathbf{11}}}{\tilde{\mu}_2} \sqrt{\dfrac{1-2c_2}{2c_0-1}} & \text{for } c_0 > \frac{1}{2},\ c_2 < \frac{1}{2}, \\ \dfrac{\mu'_{\mathbf{11}}}{\tilde{\mu}_2} \sqrt{\dfrac{2c_2-1}{2c_0-1}} & \text{for } c_0 > \frac{1}{2},\ c_2 > \frac{1}{2}. \end{cases} \tag{4.41}$$

It will be seen below that in a typical example for the third generation $c_2 = -0.7$ and $\mu'_{\mathbf{11}} C_L^2/S_R^2 \sim 2.6$.

One comment is in order about the masses of exotic charged leptons $\hat{e}, \hat{e}', \hat{E}'_\pm$ with charge $Q_{\rm EM} = +1$. No zero modes exist for $\hat{e}$ and $\hat{e}'$, and all modes have masses larger than $\frac{1}{2} m_{{\rm KK}_6}$. On the other hand $\hat{E}'_\pm$ have zero modes. They acquire masses through the Hosotani mechanism and brane interactions. We suppose that $c_2 < \frac{1}{2}$ and/or $\mu'_{\mathbf{11}}$ is sufficiently large that their lightest masses are $O(m_{{\rm KK}_5})$.

4.4. Neutrino

(iv) $Q_{\rm EM} = 0$: $\nu, \nu', N'_\pm, \hat{N}'_\pm, S'_\pm, \eta_-$ $(\Psi_{\mathbf{32}}, \Psi'_{\mathbf{11}}, \chi_{\mathbf{1}})$

The 6D $SO(11)$ spinor and vector bulk fermion $\Psi_{\mathbf{32}}$ and $\Psi'_{\mathbf{11}}$ and a 5D $SO(11)$ singlet symplectic Majorana brane fermion χ appear in this sector. From the action in Eqs. (2.19) and $\mathcal{L}_5^m, \mathcal{L}_6^m, \mathcal{L}_7^m$ in Eq. (2.28), the equations of motion for the bulk fermions become

$$-i\delta \begin{pmatrix} \nu_{+L}^\dagger \\ \nu_{+L}'^\dagger \end{pmatrix} : \big(-k\hat{D}_-(c_0) + i\partial_v\big) \begin{pmatrix} \check{\nu}_{+R} \\ \check{\nu}'_{+R} \end{pmatrix} + \sigma^\mu \partial_\mu \begin{pmatrix} \check{\nu}_{+L} \\ \check{\nu}'_{+L} \end{pmatrix} = \delta(y) \begin{pmatrix} 2i\tilde{\mu}_2 \check{N}'_{-R} \\ (m_B/\sqrt{k})\,\xi_- \end{pmatrix},$$

$$i\delta \begin{pmatrix} \nu_{+R}^\dagger \\ \nu_{+R}'^\dagger \end{pmatrix} : \overline{\sigma}^\mu \partial_\mu \begin{pmatrix} \check{\nu}_{+R} \\ \check{\nu}'_{+R} \end{pmatrix} + \big(-k\hat{D}_+(c_0) + i\partial_v\big) \begin{pmatrix} \check{\nu}_{+L} \\ \check{\nu}'_{+L} \end{pmatrix}$$
$$= \delta(y) \begin{pmatrix} 0 \\ -\sqrt{2}\tilde{\mu}_2 \check{S}'_{-L} + (m_B/\sqrt{k})\,\eta_- \end{pmatrix},$$

$$-i\delta \begin{pmatrix} \hat{N}'^{\dagger}_{+L} \\ N'^{\dagger}_{+L} \\ S'^{\dagger}_{+L} \end{pmatrix} : \big(-k\hat{D}_-(c_2) + i\partial_v \big) \begin{pmatrix} \check{\hat{N}}'_{+R} \\ \check{N}'_{+R} \\ \check{S}'_{+R} \end{pmatrix} + \sigma^\mu \partial_\mu \begin{pmatrix} \check{\hat{N}}'_{+L} \\ \check{N}'_{+L} \\ \check{S}'_{+L} \end{pmatrix} = \delta(y) \begin{pmatrix} 2\mu'_{\mathbf{11}} \check{\hat{N}}'_{-R} \\ 2\mu'_{\mathbf{11}} \check{N}'_{-R} \\ 0 \end{pmatrix},$$

$$i\delta \begin{pmatrix} \hat{N}'^{\dagger}_{+R} \\ N'^{\dagger}_{+R} \\ S'^{\dagger}_{+R} \end{pmatrix} : \overline{\sigma}^\mu \partial_\mu \begin{pmatrix} \check{\hat{N}}'_{+R} \\ \check{N}'_{+R} \\ \check{S}'_{+R} \end{pmatrix} + \big(-k\hat{D}_+(c_2) + i\partial_v \big) \begin{pmatrix} \check{\hat{N}}'_{+L} \\ \check{N}'_{+L} \\ \check{S}'_{+L} \end{pmatrix} = \delta(y) \begin{pmatrix} 0 \\ 0 \\ 2\mu'_{\mathbf{11}} \check{S}'_{-L} \end{pmatrix},$$

$$-i\delta \begin{pmatrix} \hat{N}'^{\dagger}_{-L} \\ N'^{\dagger}_{-L} \\ S'^{\dagger}_{-L} \end{pmatrix} : \big(k\hat{D}_+(c_2) - i\partial_v \big) \begin{pmatrix} \check{\hat{N}}'_{-R} \\ \check{N}'_{-R} \\ \check{S}'_{-R} \end{pmatrix} + \sigma^\mu \partial_\mu \begin{pmatrix} \check{\hat{N}}'_{-L} \\ \check{N}'_{-L} \\ \check{S}'_{-L} \end{pmatrix}$$
$$= \delta(y) \begin{pmatrix} 0 \\ 0 \\ -\sqrt{2}\widetilde{\mu}_2 \check{\nu}'_{+R} + 2\mu'_{\mathbf{11}} \check{S}'_{+R} \end{pmatrix},$$

$$i\delta \begin{pmatrix} \hat{N}'^{\dagger}_{-R} \\ N'^{\dagger}_{-R} \\ S'^{\dagger}_{-R} \end{pmatrix} : \overline{\sigma}^\mu \partial_\mu \begin{pmatrix} \check{\hat{N}}'_{-R} \\ \check{N}'_{-R} \\ \check{S}'_{-R} \end{pmatrix} + \big(k\hat{D}_-(c_2) - i\partial_v \big) \begin{pmatrix} \check{\hat{N}}'_{-L} \\ \check{N}'_{-L} \\ \check{S}'_{-L} \end{pmatrix}$$
$$= \delta(y) \begin{pmatrix} 2\mu'_{\mathbf{11}} \check{\hat{N}}'_{+L} \\ -2i\widetilde{\mu}_2 \check{\nu}_{+L} + 2\mu'_{\mathbf{11}} \check{N}'_{+L} \\ 0 \end{pmatrix}. \tag{4.42}$$

The equations for the 5D brane symplectic Majorana fermion $\chi_{\mathbf{1}}$ are

$$-i\delta\eta_-^\dagger : \sigma^\mu \partial_\mu \eta_- - i\partial_v \xi_- - \frac{m_B}{\sqrt{k}}\, \nu'_{+R} - M\eta_-^C = 0,$$
$$i\delta\xi_-^\dagger : \overline{\sigma}^\mu \partial_\mu \xi_- - i\partial_v \eta_- - \frac{m_B}{\sqrt{k}}\, \nu'_{+L} - M\xi_-^C = 0. \tag{4.43}$$

Note that η_- has a zero mode but ξ_- has no zero mode in the sixth-dimensional direction.

For sixth-dimensional $n = 0$ KK modes, the equations of motion become

$$\begin{matrix}\text{(a)}\\ \text{(b)}\end{matrix} : -k\hat{D}_-(c_0) \begin{pmatrix} \check{\nu}_{+R} \\ \check{\nu}'_{+R} \end{pmatrix} + \sigma^\mu \partial_\mu \begin{pmatrix} \check{\nu}_{+L} \\ \check{\nu}'_{+L} \end{pmatrix} = \delta(y) \begin{pmatrix} 2i\widetilde{\mu}_2 \check{N}'_{-R} \\ 0 \end{pmatrix},$$

$$\begin{matrix}\text{(c)}\\ \text{(d)}\end{matrix} : \overline{\sigma}^\mu \partial_\mu \begin{pmatrix} \check{\nu}_{+R} \\ \check{\nu}'_{+R} \end{pmatrix} - k\hat{D}_+(c_0) \begin{pmatrix} \check{\nu}_{+L} \\ \check{\nu}'_{+L} \end{pmatrix} = \delta(y) \begin{pmatrix} 0 \\ -\sqrt{2}\widetilde{\mu}_2 \check{S}'_{-L} + (m_B/\sqrt{k})\, \eta_- \end{pmatrix},$$

$$\begin{matrix}\text{(e)}\\ \text{(f)}\\ \text{(g)}\end{matrix} : -k\hat{D}_-(c_2) \begin{pmatrix} \check{\hat{N}}'_{+R} \\ \check{N}'_{+R} \\ \check{S}'_{+R} \end{pmatrix} + \sigma^\mu \partial_\mu \begin{pmatrix} \check{\hat{N}}'_{+L} \\ \check{N}'_{+L} \\ \check{S}'_{+L} \end{pmatrix} = \delta(y) \begin{pmatrix} 2\mu'_{\mathbf{11}} \check{\hat{N}}'_{-R} \\ 2\mu'_{\mathbf{11}} \check{N}'_{-R} \\ 0 \end{pmatrix},$$

$$\begin{matrix}\text{(h)}\\ \text{(i)}\\ \text{(j)}\end{matrix} : \overline{\sigma}^\mu \partial_\mu \begin{pmatrix} \check{\hat{N}}'_{+R} \\ \check{N}'_{+R} \\ \check{S}'_{+R} \end{pmatrix} - k\hat{D}_+(c_2) \begin{pmatrix} \check{\hat{N}}'_{+L} \\ \check{N}'_{+L} \\ \check{S}'_{+L} \end{pmatrix} = \delta(y) \begin{pmatrix} 0 \\ 0 \\ 2\mu'_{\mathbf{11}} \check{S}'_{-L} \end{pmatrix},$$

$$
\begin{array}{l}
\text{(k)} \\
\text{(l)} \\
\text{(m)}
\end{array}
: k\hat{D}_+(c_2)\begin{pmatrix}\check{\hat{N}}'_{-R}\\ \check{N}'_{-R}\\ \check{S}'_{-R}\end{pmatrix} + \sigma^\mu\partial_\mu\begin{pmatrix}\check{\hat{N}}'_{-L}\\ \check{N}'_{-L}\\ \check{S}'_{-L}\end{pmatrix} = \delta(y)\begin{pmatrix}0\\ 0\\ -\sqrt{2}\tilde{\mu}_2\check{\nu}'_{+R} + 2\mu'_{\mathbf{11}}\check{S}'_{+R}\end{pmatrix},
$$

$$
\begin{array}{l}
\text{(n)} \\
\text{(o)} \\
\text{(p)}
\end{array}
: \overline{\sigma}^\mu\partial_\mu\begin{pmatrix}\check{\hat{N}}'_{-R}\\ \check{N}'_{-R}\\ \check{S}'_{-R}\end{pmatrix} + k\hat{D}_-(c_2)\begin{pmatrix}\check{\hat{N}}'_{-L}\\ \check{N}'_{-L}\\ \check{S}'_{-L}\end{pmatrix} = \delta(y)\begin{pmatrix}2\mu'_{\mathbf{11}}\check{\hat{N}}'_{+L}\\ -2i\tilde{\mu}_2\check{\nu}_{+L} + 2\mu'_{\mathbf{11}}\check{N}'_{+L}\\ 0\end{pmatrix},
$$

$$
\text{(q)}: \left\{\sigma^\mu\partial_\mu\eta_- - \frac{m_B}{\sqrt{k}}\,\nu'_{+R} - M\eta_-^C\right\}\delta(y) = 0. \tag{4.44}
$$

Here, $\hat{D}_\pm(c)$ is given by Eq. (4.2). To find the form of $\hat{D}_\pm(c_2)$ for $(\hat{N}', N', S')$ in $\Psi'_{\mathbf{11}}$, let us recall that the original and twisted gauges are related by $\Psi'_{\mathbf{11}} = \Omega(z)\widetilde{\Psi}'_{\mathbf{11}}$ where $\Omega(z) = e^{i\theta(z)T_{4,11}}$. Hence,

$$
\psi'_3 = \widetilde{\psi}'_3, \quad \begin{pmatrix}\psi'_4\\ \psi'_{11}\end{pmatrix} = \begin{pmatrix}\cos\theta(z) & \sin\theta(z)\\ -\sin\theta(z) & \cos\theta(z)\end{pmatrix}\begin{pmatrix}\widetilde{\psi}'_4\\ \widetilde{\psi}'_{11}\end{pmatrix}. \tag{4.45}
$$

As

$$
\psi'_4 = \frac{1}{\sqrt{2}}(\hat{N}' - N'), \quad \psi'_3 = \frac{i}{\sqrt{2}}(\hat{N}' + N'), \quad \psi'_{11} = S', \tag{4.46}
$$

one finds

$$
\begin{pmatrix}\tilde{\check{\hat{N}}}'\\ \tilde{\check{N}}'\\ \tilde{\check{S}}'\end{pmatrix} = \bar{\Omega}(z)\begin{pmatrix}\check{\hat{N}}'\\ \check{N}'\\ \check{S}'\end{pmatrix},
$$

$$
\bar{\Omega}^{-1}(z) = \begin{pmatrix}
\dfrac{1+\cos\theta(z)}{2} & \dfrac{1-\cos\theta(z)}{2} & \dfrac{\sin\theta(z)}{\sqrt{2}}\\
\dfrac{1-\cos\theta(z)}{2} & \dfrac{1+\cos\theta(z)}{2} & -\dfrac{\sin\theta(z)}{\sqrt{2}}\\
-\dfrac{\sin\theta(z)}{\sqrt{2}} & \dfrac{\sin\theta(z)}{\sqrt{2}} & \cos\theta(z)
\end{pmatrix}. \tag{4.47}
$$

It follows that

$$
\hat{D}_-(c_2)\begin{pmatrix}\check{\hat{N}}'_{+R}\\ \check{N}'_{+R}\\ \check{S}'_{+R}\end{pmatrix} = \bar{\Omega}^{-1}(z)\,D_-(c_2)\begin{pmatrix}\tilde{\check{\hat{N}}}'_{+R}\\ \tilde{\check{N}}'_{+R}\\ \tilde{\check{S}}'_{+R}\end{pmatrix}, \tag{4.48}
$$

and so on.

To obtain boundary conditions at $y=0$, we integrate the above (a), (d), (e), (f), (j), (m), (n), and (o) ($\frac{1}{2}\int_{-\epsilon}^{+\epsilon} dy \cdots$) in the vicinity of $y=0$ for parity-odd (in $x^5=y$) fields $\nu_{+R}, \nu'_{+L}, \hat{N}'_{+R}, N'_{+R}, S'_{+L}, S'_{-R}, \hat{N}'_{-L}, N'_{-L}$:

$$
\begin{aligned}
&\text{(a)} \;\Rightarrow\; +\check{\nu}_{+R}(x,\epsilon) = i\widetilde{\mu}_2 \check{N}'_{-R}(x,0),\\
&\text{(d)} \;\Rightarrow\; -\check{\nu}'_{+L}(x,\epsilon) = -\frac{\widetilde{\mu}_2}{\sqrt{2}}\check{S}'_{-L}(x,0) + \frac{m_B}{2\sqrt{k}}\,\eta_-(x),\\
&\text{(e)} \;\Rightarrow\; +\check{\hat{N}}'_{+R}(x,\epsilon) = \mu'_{\mathbf{11}}\check{\hat{N}}'_{-R}(x,0),\\
&\text{(f)} \;\Rightarrow\; +\check{N}'_{+R}(x,\epsilon) = \mu'_{\mathbf{11}}\check{N}'_{-R}(x,0),\\
&\text{(j)} \;\Rightarrow\; -\check{S}'_{+L}(x,\epsilon) = \mu'_{\mathbf{11}}\check{S}'_{-L}(x,0),\\
&\text{(m)} \;\Rightarrow\; +\check{S}'_{-R}(x,\epsilon) = -\frac{\widetilde{\mu}_2}{\sqrt{2}}\check{\nu}'_{+R}(x,0) + \mu'_{\mathbf{11}}\check{S}'_{+R}(x,0),\\
&\text{(n)} \;\Rightarrow\; -\check{\hat{N}}'_{-L}(x,\epsilon) = \mu'_{\mathbf{11}}\check{\hat{N}}'_{+L}(x,0),\\
&\text{(o)} \;\Rightarrow\; -\check{N}'_{-L}(x,\epsilon) = -i\widetilde{\mu}_2\check{\nu}_{+L}(x,0) + \mu'_{\mathbf{11}}\check{N}'_{+L}(x,0).
\end{aligned}
\tag{4.49}
$$

For parity-even (in $x^5=y$) fields $\nu_{+L}, \nu'_{+R}, \hat{N}'_{+L}, N'_{+L}, S'_{+R}, S'_{-L}, \hat{N}'_{-R}, N'_{-R}$, we evaluate the equations of motion at $y=+\epsilon$ by making use of the above conditions to find:

$$
\begin{aligned}
&\text{(b)} \;\Rightarrow\; -\hat{D}_-(c_0)\check{\nu}'_{+R} - \frac{\widetilde{\mu}_2}{\sqrt{2}}\hat{D}_+\check{S}'_{-R} - \frac{m_B^2}{2k^2}\,\check{\nu}'_{+R} - \frac{m_B M}{2k^{3/2}}\,\eta_-^C = 0,\\
&\text{(c)} \;\Rightarrow \hat{D}_+(c_0)\check{\nu}_{+L} + i\widetilde{\mu}_2\hat{D}_-\check{N}'_{-L} = 0,\\
&\text{(g)} \;\Rightarrow\; -\hat{D}_-(c_0)\check{S}'_{+R} + \mu'_{\mathbf{11}}\hat{D}_+\check{S}'_{-R} = 0,\\
&\text{(h)} \;\Rightarrow \hat{D}_+(c_0)\check{\hat{N}}'_{+L} + \mu'_{\mathbf{11}}\hat{D}_-\check{\hat{N}}'_{-R} = 0,\\
&\text{(i)} \;\Rightarrow \hat{D}_+(c_0)\check{N}'_{+L} + \mu'_{\mathbf{11}}\hat{D}_-\check{N}'_{-R} = 0,\\
&\text{(k)} \;\Rightarrow \hat{D}_+(c_2)\check{\hat{N}}'_{-R} - \mu'_{\mathbf{11}}\hat{D}_-\check{\hat{N}}'_{+R} = 0,\\
&\text{(l)} \;\Rightarrow \hat{D}_+(c_2)\check{N}'_{-R} + i\widetilde{\mu}_2\hat{D}_-\check{\nu}_{+R} - \mu'_{\mathbf{11}}\hat{D}_-\check{N}'_{+R} = 0,\\
&\text{(p)} \;\Rightarrow \hat{D}_-(c_2)\check{S}'_{-L} - \frac{\widetilde{\mu}_2}{\sqrt{2}}\hat{D}_+\check{\nu}'_{+L} + \mu'_{\mathbf{11}}\hat{D}_+\check{S}'_{+L} = 0
\end{aligned}
\tag{4.50}
$$

at $y=+\epsilon$.

There are nine fields in the neutral fermion sector which intertwine with each other. To simplify the discussions, we consider the case in which $m_B^2/k^2, m_B M/k^2, \mu'_{\mathbf{11}} \gg \widetilde{\mu}_2$. In the $\widetilde{\mu}_2 = 0$ limit, the equations of motion and boundary conditions in the neutral sector, Eqs. (4.44), (4.49), and (4.50), split into two parts. The first sector, the neutrino sector-1, contains $\check{\nu}'_{+R}$, $\check{\nu}'_{+L}$ in $\Psi_{\mathbf{32}}$ and η_- in $\chi_{\mathbf{1}}$, while the second sector, the neutrino sector-2, contains other components in $\Psi_{\mathbf{32}}$ and $\Psi'_{\mathbf{11}}$. $\widetilde{\mu}_2 \neq 0$ mixes the neutrino sector-1 and sector-2.

In the twisted gauge, the mode functions are determined by the BCs on the IR brane. The mass spectra can be fixed by the BCs on the UV brane. For the neutrino sector-1, by using the boundary conditions at $z = z_L$, their mode functions can be written as

$$\begin{pmatrix} \widetilde{\check{\nu}}_R \\ \widetilde{\check{\nu}}'_R \\ \eta_-^C \end{pmatrix} = \begin{pmatrix} \alpha_\nu S_R(z;\lambda,c_0) \\ i\alpha_{\nu'} C_R(z;\lambda,c_0) \\ -i\alpha_\eta^*/\sqrt{k} \end{pmatrix} f_R(x), \quad \overline{\sigma}^\mu \partial_\mu f_R(x) = k\lambda f_L(x),$$

$$\begin{pmatrix} \widetilde{\check{\nu}}_L \\ \widetilde{\check{\nu}}'_L \\ \eta_- \end{pmatrix} = \begin{pmatrix} \alpha_\nu C_L(z;\lambda,c_0) \\ i\alpha_{\nu'} S_L(z;\lambda,c_0) \\ i\alpha_\eta/\sqrt{k} \end{pmatrix} f_L(x), \quad \sigma^\mu \partial_\mu f_L(x) = k\lambda f_R(x). \tag{4.51}$$

Here, $f_{R/L}(x)$ are chosen in the neutrino sector-1 such that $f_L^C(x) = -e^{i\delta_C}\sigma^2 f_L(x)^* = f_R(x)$ is satisfied where δ_C is defined in Eq. (2.11). One can take $\alpha_\nu, \alpha_{\nu'}, \alpha_\eta$ to be real. In this case, $\sigma^\mu\partial_\mu \eta_- = -k\lambda\eta_-^C$ is satisfied so that equation (q) in Eq. (4.44) implies that

$$\frac{m_B}{\sqrt{k}}\, \check{\nu}'_{+R}\big|_{y=0} + (k\lambda + M)\eta_-^C = 0. \tag{4.52}$$

With this identity the first relation in Eq. (4.50) can be rewritten as

$$\hat{D}_-(c_0)\check{\nu}'_{+R} + \frac{\widetilde{\mu}_2}{\sqrt{2}}\hat{D}_+\check{S}'_{-R} - \frac{m_B\lambda}{2\sqrt{k}}\,\eta_-^C = 0. \tag{4.53}$$

Setting $\widetilde{\mu}_2 = 0$, one finds that the BCs at $z = 1^+$ in the twisted gauge can be written as

$$K_{\nu_1}\begin{pmatrix} \alpha_\nu \\ \alpha_{\nu'} \\ \alpha_\eta \end{pmatrix} = \begin{pmatrix} \cos\dfrac{\theta_H}{2} S_R^0 & \sin\dfrac{\theta_H}{2} C_R^0 & 0 \\ -\sin\dfrac{\theta_H}{2} C_L^0 & \cos\dfrac{\theta_H}{2} S_L^0 & \dfrac{m_B}{2k} \\ -\sin\dfrac{\theta_H}{2} S_R^0 & \cos\dfrac{\theta_H}{2} C_R^0 & -\dfrac{k\lambda + M}{m_B} \end{pmatrix}\begin{pmatrix} \alpha_\nu \\ \alpha_{\nu'} \\ \alpha_\eta \end{pmatrix} = 0. \tag{4.54}$$

From $\det K_{\nu_1} = 0$, we find the mass spectrum formula for the neutrino sector-1:

$$\det K_{\nu_1} = -\frac{k\lambda + M}{m_B}\left\{S_L^0 S_R^0 + \sin^2\frac{\theta_H}{2}\right\} - \frac{m_B}{2k}\, S_R^0 C_R^0 = 0. \tag{4.55}$$

This gives the gauge–Higgs seesaw mechanism for the small neutrino masses. For $\lambda z_L \ll 1$ and $k\lambda \ll |M|$, the neutrino mass is given by

$$m_\nu \simeq -\frac{2m_u^2 M z_L^{2c_0+1}}{(2c_0+1)m_B^2}, \tag{4.56}$$

where m_u is given by Eq. (4.23). As shown in Ref. [38], the gauge–Higgs seesaw mechanism is characterized by a 3×3 mass matrix

$$\begin{pmatrix} & m_D & \\ m_D & & \tilde{m}_B \\ & \tilde{m}_B & M \end{pmatrix} \tag{4.57}$$

in the 4D effective theory in which $m_D = m_u$ and $\tilde{m}_B \sim m_B z_L^{-c_0-1/2}$. The Majorana mass M may take a moderate value.

Next, we consider the neutrino sector-2. The BCs at the IR brane determine the mode functions in the twisted gauge to be

$$
\begin{pmatrix} \tilde{\tilde{\hat{N}}}{}'_{+R} \\ \tilde{\tilde{N}}{}'_{+R} \\ \tilde{\tilde{S}}{}'_{+R} \\ \tilde{\tilde{\hat{N}}}{}'_{-R} \\ \tilde{\tilde{N}}{}'_{-R} \\ \tilde{\tilde{S}}{}'_{-R} \end{pmatrix} = \begin{pmatrix} \alpha_{\hat{N}'_+} S_R(z;\lambda,c_2) \\ \alpha_{N'_+} S_R(z;\lambda,c_2) \\ \alpha_{S'_+} C_R(z;\lambda,c_2) \\ \alpha_{\hat{N}'_-} C_L(z;\lambda,c_2) \\ \alpha_{N'_-} C_L(z;\lambda,c_2) \\ \alpha_{S'_-} S_L(z;\lambda,c_2) \end{pmatrix} f_R(x),
$$

$$
\begin{pmatrix} \tilde{\tilde{\hat{N}}}{}'_{+L} \\ \tilde{\tilde{N}}{}'_{+L} \\ \tilde{\tilde{S}}{}'_{+L} \\ \tilde{\tilde{\hat{N}}}{}'_{-L} \\ \tilde{\tilde{N}}{}'_{-L} \\ \tilde{\tilde{S}}{}'_{-L} \end{pmatrix} = \begin{pmatrix} \alpha_{\hat{N}'_+} C_L(z;\lambda,c_2) \\ \alpha_{N'_+} C_L(z;\lambda,c_2) \\ \alpha_{S'_+} S_L(z;\lambda,c_2) \\ -\alpha_{\hat{N}'_-} S_R(z;\lambda,c_2) \\ -\alpha_{N'_-} S_R(z;\lambda,c_2) \\ -\alpha_{S'_-} C_R(z;\lambda,c_2) \end{pmatrix} f_L(x). \tag{4.58}
$$

By making use of Eqs. (4.47) and (4.48), the BCs at the UV brane are expressed as

$$
K_{\nu_2} \left(\alpha_{\hat{N}'_+} \ \alpha_{N'_+} \ \alpha_{S'_+} \ \alpha_{\hat{N}'_-} \ \alpha_{N'_-} \ \alpha_{S'_-} \right)^{\mathrm{T}} = 0, \tag{4.59}
$$

where

$$
K_{\nu_2} = \begin{pmatrix} A & -\mu'_{\mathbf{11}} B \\ -\mu'_{\mathbf{11}} B & A \end{pmatrix},
$$

$$
A = \begin{pmatrix} \frac{1+\cos\theta_H}{2} S_R^2 & \frac{1-\cos\theta_H}{2} S_R^2 & \frac{\sin\theta_H}{\sqrt{2}} C_R^2 \\ \frac{1-\cos\theta_H}{2} S_R^2 & \frac{1+\cos\theta_H}{2} S_R^2 & -\frac{\sin\theta_H}{\sqrt{2}} C_R^2 \\ -\frac{\sin\theta_H}{\sqrt{2}} C_L^2 & \frac{\sin\theta_H}{\sqrt{2}} C_L^2 & \cos\theta_H S_L^2 \end{pmatrix},
$$

$$
B = \begin{pmatrix} \frac{1+\cos\theta_H}{2} C_L^2 & \frac{1-\cos\theta_H}{2} C_L^2 & \frac{\sin\theta_H}{\sqrt{2}} S_L^2 \\ \frac{1-\cos\theta_H}{2} C_L^2 & \frac{1+\cos\theta_H}{2} C_L^2 & -\frac{\sin\theta_H}{\sqrt{2}} S_L^2 \\ -\frac{\sin\theta_H}{\sqrt{2}} S_R^2 & \frac{\sin\theta_H}{\sqrt{2}} S_R^2 & \cos\theta_H C_R^2 \end{pmatrix}. \tag{4.60}
$$

Note that

$$
K'_{\nu_2} = \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix} K_{\nu_2} \begin{pmatrix} V^{-1} & 0 \\ 0 & V^{-1} \end{pmatrix} = \begin{pmatrix} A' & -\mu'_{\mathbf{11}} B' \\ -\mu'_{\mathbf{11}} B' & A' \end{pmatrix},
$$

$$V = V^{-1} = \begin{pmatrix} 2^{-1/2} & 2^{-1/2} & 0 \\ 2^{-1/2} & -2^{-1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A' = \begin{pmatrix} S_R^2 & 0 & 0 \\ 0 & \cos\theta_H S_R^2 & \sin\theta_H C_R^2 \\ 0 & -\sin\theta_H C_L^2 & \cos\theta_H S_L^2 \end{pmatrix},$$

$$B' = \begin{pmatrix} C_L^2 & 0 & 0 \\ 0 & \cos\theta_H C_L^2 & \sin\theta_H S_L^2 \\ 0 & -\sin\theta_H S_R^2 & \cos\theta_H C_R^2 \end{pmatrix}. \tag{4.61}$$

From $\det K_{\nu_2} = \det K'_{\nu_2} = 0$ the mass spectra for the neutrino sector-2 are found to be

$$\begin{aligned} \det K_{\nu_2} = \big(S_R^2 S_R^2 - \mu_{\mathbf{11}}'^2 C_L^2 C_L^2\big) \Big\{ & \big(S_L^2 S_R^2 + \sin^2\theta_H\big)^2 \\ & + \mu_{\mathbf{11}}'^2 \big(2\sin^2\theta_H \cos^2\theta_H - C_R^2 C_R^2 S_R^2 S_R^2 - C_L^2 C_L^2 S_L^2 S_L^2\big) \\ & + \mu_{\mathbf{11}}'^4 \big(S_L^2 S_R^2 + \cos^2\theta_H\big)^2 \Big\} = 0. \end{aligned} \tag{4.62}$$

In the $\mu'_{\mathbf{11}} = 0$ limit, or in the absence of the brane interactions, Eq. (4.62) becomes

$$\big(S_R^2\big)^2 \big(S_L^2 S_R^2 + \sin^2\theta_H\big)^2 = 0. \tag{4.63}$$

For large $\mu'_{\mathbf{11}}$, it becomes

$$\big(C_L^2\big)^2 \big(S_L^2 S_R^2 + \cos^2\theta_H\big)^2 \simeq 0. \tag{4.64}$$

For $\tilde{\mu}_2 \neq 0$ the neutrino sector-1 and sector-2 mix through the boundary conditions. One needs to solve

$$K \big(\alpha_\nu \; \alpha_{\nu'} \; \alpha_\eta \; \alpha_{\hat{N}'_+} \; \alpha_{N'_+} \; \alpha_{S'_+} \; \alpha_{\hat{N}'_-} \; \alpha_{N'_-} \; \alpha_{S'_-}\big)^T = 0\,,$$

$$K = \begin{pmatrix} K_{\nu_1} & 0 & -i\tilde{\mu}_2 D \\ 0 & A & -\mu'_{\mathbf{11}} B \\ i\tilde{\mu}_2 C & -\mu'_{\mathbf{11}} B & A \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 \\ \cos\frac{1}{2}\theta_H C_L^0 & \sin\frac{1}{2}\theta_H S_L^0 & 0 \\ -\dfrac{\sin\frac{1}{2}\theta_H}{\sqrt{2}} S_R^0 & \dfrac{\cos\frac{1}{2}\theta_H}{\sqrt{2}} C_R^0 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} \frac{1-\cos\theta_H}{2} C_L^2 & \frac{1+\cos\theta_H}{2} C_L^2 & -\frac{\sin\theta_H}{\sqrt{2}} S_L^2 \\ -\frac{\sin\theta_H}{2} S_R^2 & \frac{\sin\theta_H}{2} S_R^2 & \frac{\cos\theta_H}{\sqrt{2}} C_R^2 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.65}$$

$\det K$ is evaluated to be

$$\begin{aligned}
\det K &= F_0 + F_2\,(\widetilde{\mu}_2)^2 + F_4\,(\widetilde{\mu}_2)^4\,, \\
F_0 &= \det K_{\nu_1} \cdot \det K_{\nu_2}\,, \\
F_2 &= \left\{ \frac{m_B}{4k}(S_L^0 S_R^0 + \cos^2 \tfrac{1}{2}\theta_H) + \frac{k\lambda + M}{2m_B} C_L^0 S_L^0 \right\} \\
&\quad \times \Big\{ C_L^2 S_R^2 (2X + \sin^2\theta_H)(X + \sin^2\theta_H) \\
&\quad - \mu_{11}'^2 \Big[(C_L^2)^3 S_L^2 (2X + \sin^2\theta_H) + C_R^2 (S_R^2)^3 (2X + 1 + \cos^2\theta_H) \\
&\quad + C_L^2 S_R^2 (X^2 - 2) \sin^2\theta_H \cos^2\theta_H \Big] + \mu_{11}'^4 C_L^2 S_R^2 (X + \cos^2\theta_H)(2X + 1 + \cos^2\theta_H) \Big\} \\
&\quad + \frac{k\lambda + M}{2m_B} \{ (S_R^2)^2 - \mu_{11}'^2 (C_L^2)^2 \} \Big\{ \Big[(1 + \mu_{11}'^2) X + \sin^2\theta_H + \mu_{11}'^2 \cos^2\theta_H \Big] \cos\theta_H \sin^2\theta_H \\
&\quad + C_R^0 S_R^0 \Big[C_R^2 S_R^2 (X + \sin^2\theta_H) - \mu_{11}'^2 C_L^2 S_L^2 (X + \cos^2\theta_H) \Big] \Big\}, \\
F_4 &= -\frac{k\lambda + M}{32 m_B} (S_L^0 S_R^0 + \cos^2 \tfrac{1}{2}\theta_H) \\
&\quad \times \Big[(S_R^2)^2 \{ 16 (S_L^2 S_R^2)^2 + (20 - 4\cos 2\theta_H) S_L^2 S_R^2 + 5 - 4\cos 2\theta_H - \cos 4\theta_H \} \\
&\quad - \mu_{11}'^2 (C_L^2)^2 \{ 16 (S_L^2 S_R^2)^2 + (12 + 4\cos 2\theta_H) S_L^2 S_R^2 - \cos 4\theta_H \} \Big],
\end{aligned} \tag{4.66}$$

where $X = S_L^2 S_R^2$. In evaluating the effective potential $V_{\rm eff}(\theta_H)$, we shall use the approximate formula $\det K \sim \det K_{\nu_1} \det K_{\nu_2}$ for small $\widetilde{\mu}_2^2$.

4.5. Dark fermion

(v) $\Psi'_{\mathbf{32}} = \Psi_{\mathbf{32}}^{\alpha=4}$

The 6D $SO(11)$ **32** Weyl fermion $\Psi'_{\mathbf{32}}$, which may be called a dark fermion multiplet, has sixth-dimensional $n = 0$ KK modes. For $\Psi'_{\mathbf{32}}$ one need not introduce brane interactions. We consider the action in Eq. (2.19) for the dark fermion sector $\Psi'_{\mathbf{32}}$. From the parity assignment shown in Table 2, the mass formula for the dark fermions is found to be

$$S_L^{0'} S_R^{0'} + \cos^2 \frac{\theta_H}{2} = 0. \tag{4.67}$$

For $\lambda z_L \ll 1$, the dark fermion mass is given by

$$m_{\rm Dark} \simeq \begin{cases} k z_L^{-1}\sqrt{1-4c_0'^2}\,\cos\dfrac{\theta_H}{2} & \text{for } c_0' < 1/2, \\ k z_L^{-1/2-c_0'}\sqrt{4c_0'^2-1}\,\cos\dfrac{\theta_H}{2} & \text{for } c_0' > 1/2. \end{cases} \tag{4.68}$$

As in the case of the 5D $SO(11)$ GHGUT discussed in Ref. [35], the bulk mass parameter of the dark fermions, c_0', must be relatively small not to become light exotic particles.

5. Effective potential

We evaluate the Higgs effective potential $V_{\rm eff}(\theta_H)$ by using the mass spectrum formulas of the sixth-dimensional $n = 0$ KK modes of the $SO(11)$ gauge bosons and fermions. The evaluation is done in the same manner as in Ref. [35].

One-loop effective potential from each KK tower is given by [9,45,46]:

$$V_{\rm eff}(\theta_H) = \pm\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\sum_n \ln\left(p^2 + m_n(\theta_H)^2\right), \tag{5.1}$$

where $m_n(\theta_H)$ is the mass spectrum of the KK tower and we take the $+$ and $-$ sign for bosons and fermions, respectively. When the mass spectrum $\{m_n = k\lambda_n\}$ is determined by $1 + \tilde{Q}(\lambda_n)f(\theta_H) = 0$, Eq. (5.1) can be written as

$$V_{\rm eff}(\theta_H) = \pm I\left[Q(q); f(\theta_H)\right]$$

$$I\left[Q(q); f(\theta_H)\right] = \frac{(kz_L^{-1})^4}{(4\pi)^2}\int_0^\infty dq\, q^3 \ln[1 + Q(q)f(\theta_H)], \tag{5.2}$$

where $Q(q) = \tilde{Q}(iqz_L^{-1})$. We utilize the mass spectra $m_n(\theta_H)$ of the $SO(11)$ bulk gauge and fermions obtained in Sects. 3 and 4. Note that only θ_H-dependent mass spectra contribute to the θ_H-dependent part of $V_{\rm eff}(\theta_H)$. It contains the $SO(11)$ bulk gauge field, $SO(11)$ spinor, and vector fermion fields.

We summarize those mass spectra. The $SO(11)$ gauge boson contribution is almost the same as in the 5D $SO(11)$ GHGUT. The difference from the 5D case is that the Y boson contributions can be neglected in the current scheme as they have masses of $O(m_{{\rm KK}_6})$. The relevant mass spectra of $SO(11)$ gauge fields A_μ and A_z are given, from Eqs. (3.7), (3.12), and (3.18), by

$$W^\pm \text{ tower}: \ 1 + \frac{\lambda}{2C'(1;\lambda)S(1;\lambda)}\sin^2\theta_H = 0,$$

$$Z \text{ tower}: \ 1 + \frac{\lambda}{2\cos^2\theta_W\, C'(1;\lambda)S(1;\lambda)}\sin^2\theta_H = 0,$$

$$A_z^{a4}, A_z^{a\,11}\ (a = 1,2,3): \ 1 + \frac{\lambda}{C'(1;\lambda)S(1;\lambda)}\sin^2\theta_H = 0. \tag{5.3}$$

The mass spectra of up- and down-type quarks, charged leptons, neutrinos, and dark fermions are given, from Eqs. (4.22), (4.30), (4.39), (4.55), (4.62), and (4.67), by

$$\text{(i)}: \ Q_{\rm EM} = +\frac{2}{3}: \ \ 1 + \frac{\sin^2\frac{1}{2}\theta_H}{S_L^0 S_R^0} = 0,$$

$$(\text{ii}):\ Q_{\rm EM}=-\frac{1}{3}:\quad 1+\frac{\sin^2\frac{1}{2}\theta_H}{S_L^0S_R^0+\dfrac{\mu_1^2S_R^0C_R^0S_L^1C_R^1}{\mu_{\bf 11}^2(C_R^1)^2-(S_L^1)^2}}=0,$$

$$(\text{iii}):\ Q_{\rm EM}=-1:\quad 1+\frac{\sin^2\frac{1}{2}\theta_H}{S_L^0S_R^0+\dfrac{\widetilde{\mu}_2^2S_L^0C_L^0S_R^2C_L^2}{\mu_{\bf 11}'^2(C_L^2)^2-(S_R^2)^2}}=0,$$

$$(\text{iv-1}):\ Q_{\rm EM}=0:\quad 1+\frac{\sin^2\frac{1}{2}\theta_H}{S_L^0S_R^0+\dfrac{m_B^2/k}{2(k\lambda+M)}S_R^0C_R^0}=0,$$

$$(\text{iv-2}):\ Q_{\rm EM}=0:\quad 1+\frac{f_1(\theta_H)}{f_2}=0,$$

$$f_1(\theta_H)=(1-\mu_{\bf 11}'^2)\big\{2(1+\mu_{\bf 11}'^2)S_L^2S_R^2+2\mu_{\bf 11}'^2+(1-\mu_{\bf 11}'^2)\sin^2\theta_H\big\}\sin^2\theta_H,$$

$$f_2=\mu_{\bf 11}'^4+2\mu_{\bf 11}'^4S_L^2S_R^2+(1+\mu_{\bf 11}'^4)(S_L^2S_R^2)^2-\mu_{\bf 11}'^2\left\{(C_R^2S_R^2)^2+(C_L^2S_L^2)^2\right\},$$

$$(\text{v}):\ Q_{\rm EM}=+\frac{2}{3},-\frac{2}{3},-1,0:\quad 1+\frac{\cos^2\frac{1}{2}\theta_H}{S_L^{0'}S_R^{0'}}=0, \tag{5.4}$$

where $m_B^2/k^2, m_BM/k^2, \mu_{\bf 11}'\gg\widetilde{\mu}_2$ has been assumed for (iv-1) and (iv-2). (v) is the dark fermion mass spectrum. c_0' stands for the bulk mass of the 6D $SO(11)$ spinor bulk Weyl fermion $\Psi_{\bf 32}^{\alpha=4}$.

We evaluate the effective potential $V_{\rm eff}(\theta_H)=V_{\rm eff}^{\rm gauge}(\theta_H)+V_{\rm eff}^{\rm fermion}(\theta_H)$. In the R_ξ gauge $V_{\rm eff}^{\rm gauge}(\theta_H)$ is decomposed into

$$\begin{aligned}
V_{\rm eff}^{\rm gauge}(\theta_H)&=V_{\rm eff}^{W^\pm}+V_{\rm eff}^{Z}+V_{\rm eff}^{A_z^{a4},A_z^{a,11}},\\
V_{\rm eff}^{W^\pm}(\theta_H)&=2(3-\xi^2)I\left[\tfrac{1}{2}Q_0\,(q,1)\,;\sin^2\theta_H\right],\\
V_{\rm eff}^{Z}(\theta_H)&=(3-\xi^2)I\left[\frac{Q_0\,(q,1)}{2\cos^2\theta_W};\sin^2\theta_H\right],\\
V_{\rm eff}^{A_z^{a4},A_z^{a\,11}}(\theta_H)&=3\xi^2I\left[Q_0\,(q,1)\,;\sin^2\theta_H\right].
\end{aligned} \tag{5.5}$$

Here,

$$\begin{aligned}
Q_0\,(q,c)&=\frac{z_L}{q^2}\frac{1}{\hat{F}_c^{++}(q)\hat{F}_c^{--}(q)},\\
\hat{F}_c^{\pm\pm}(q)&=\hat{F}_{c\pm\frac{1}{2},c\pm\frac{1}{2}}(qz_L^{-1},q),\\
\hat{F}_{\alpha,\beta}(u,v)&=I_\alpha(u)K_\beta(v)-e^{-i(\alpha-\beta)\pi}K_\alpha(u)I_\beta(v),
\end{aligned} \tag{5.6}$$

where $I_\alpha(u)$ and $K_\beta(u)$ are modified Bessel functions.

The fermion part $V_{\rm eff}^{\rm fermion}(\theta_H)$ is evaluated in a similar manner. Following the decomposition in Eq. (5.4), and taking into account four degrees of freedom for each Dirac fermion and the color factor 3 for quarks, one can write:

$$
\begin{aligned}
V_{\rm eff}^{\rm fermion}(\theta_H) &= V_{\rm eff}^{\rm (i)} + V_{\rm eff}^{\rm (ii)} + V_{\rm eff}^{\rm (iii)} + V_{\rm eff}^{\rm (iv-1)} + V_{\rm eff}^{\rm (iv-2)} + V_{\rm eff}^{\rm (v)},\\
V_{\rm eff}^{\rm (i)}(\theta_H) &= -12 I\left[Q_0\left(q, c_0\right); \sin^2 \tfrac{1}{2}\theta_H\right],\\
V_{\rm eff}^{\rm (ii)}(\theta_H) &= -12 I\left[Q_{\rm (ii)}(q, c_0, c_1, \mu_1, \mu_{\mathbf{11}}); \sin^2 \tfrac{1}{2}\theta_H\right],\\
V_{\rm eff}^{\rm (iii)}(\theta_H) &= -4 I\left[Q_{\rm (iii)}(q, c_0, c_2, \widetilde{\mu}_2, \mu'_{\mathbf{11}}); \sin^2 \tfrac{1}{2}\theta_H\right],\\
V_{\rm eff}^{\rm (iv-1)}(\theta_H) &= -4\cdot\frac{1}{2} I\left[\widetilde{Q}_{\rm (iv-1)}(q, c_0, m_B, M; \theta_H); 1\right],\\
V_{\rm eff}^{\rm (iv-2)}(\theta_H) &= -4 I\left[\widetilde{Q}_{\rm (iv-2)}(q, c_2, \mu'_{\mathbf{11}}; \theta_H); 1\right],\\
V_{\rm eff}^{\rm (v)}(\theta_H) &= -32 I\left[Q_0\left(q, c'_0\right); \cos^2 \tfrac{1}{2}\theta_H\right],
\end{aligned}
\tag{5.7}
$$

where

$$
\begin{aligned}
Q_{\rm (ii)}(q) &= \frac{z_L}{q^2}\left\{\hat{F}_{c_0}^{++}\hat{F}_{c_0}^{--} + \frac{\mu_1^2 \hat{F}_{c_0}^{--}\hat{F}_{c_0}^{-+}\hat{F}_{c_1}^{++}\hat{F}_{c_1}^{-+}}{\mu_{\mathbf{11}}^2(\hat{F}_{c_1}^{-+})^2 + (\hat{F}_{c_1}^{++})^2}\right\}^{-1},\\
Q_{\rm (iii)}(q) &= \frac{z_L}{q^2}\left\{\hat{F}_{c_0}^{++}\hat{F}_{c_0}^{--} + \frac{\widetilde{\mu}_2^2 \hat{F}_{c_0}^{++}\hat{F}_{c_0}^{+-}\hat{F}_{c_2}^{--}\hat{F}_{c_2}^{+-}}{\mu_{\mathbf{11}}^{\prime 2}(\hat{F}_{c_2}^{+-})^2 + (\hat{F}_{c_2}^{--})^2}\right\}^{-1},\\
Q_{\rm (iv-1)}(q) &= \frac{z_L}{q^2}\left\{\hat{F}_{c_0}^{++}\hat{F}_{c_0}^{--} - \frac{i m_B^2/k^2}{2(iqz_L^{-1} + M/k)}\hat{F}_{c_0}^{-+}\hat{F}_{c_0}^{--}\right\}^{-1},\\
\widetilde{Q}_{\rm (iv-1)}(q;\theta_H) &= (Q_{\rm (iv-1)}(q) + Q_{\rm (iv-1)}(q)^*)\sin^2\tfrac{1}{2}\theta_H\\
&\qquad + (Q_{\rm (iv-1)}(q) Q_{\rm (iv-1)}(q)^*)\sin^4\tfrac{1}{2}\theta_H,\\
\widetilde{Q}_{\rm (iv-2)}(q,\theta_H) &= \frac{z_L^2}{q^4}\frac{h_1(q,\theta_H)}{h_2(q)},\\
h_1 &= (1-\mu_{\mathbf{11}}^{\prime 2})\left\{2\frac{q^2}{z_L}(1+\mu_{\mathbf{11}}^{\prime 2})\hat{F}_{c_2}^{++}\hat{F}_{c_2}^{--} + 2\mu_{\mathbf{11}}^{\prime 2} + (1-\mu_{\mathbf{11}}^{\prime 2})\sin^2\theta_H\right\}\sin^2\theta_H,\\
h_2 &= \frac{z_L^2}{q^4}\mu_{\mathbf{11}}^{\prime 4} + 2\mu_{\mathbf{11}}^{\prime 4}\frac{z_L}{q^2}\hat{F}_{c_2}^{++}\hat{F}_{c_2}^{--} + (1+\mu_{\mathbf{11}}^{\prime 4})(\hat{F}_{c_2}^{++}\hat{F}_{c_2}^{--})^2\\
&\qquad + \mu_{\mathbf{11}}^{\prime 2}\left\{(\hat{F}_{c_2}^{-+}\hat{F}_{c_2}^{--})^2 + (\hat{F}_{c_2}^{+-}\hat{F}_{c_2}^{++})^2\right\}.
\end{aligned}
\tag{5.8}
$$

As $Q_{\rm (iv-1)}(q)$ is not a real function, $V_{\rm eff}^{\rm (iv-1)}(\theta_H)$ has been converted to the integral involving $\widetilde{Q}_{\rm (iv-1)}(q;\theta_H)$.

The Higgs mass m_H is determined, at the minimum $\theta_H = \theta_H^{\rm min}$ of $V_{\rm eff}(\theta_H)$, by

$$
\begin{aligned}
m_H^2 &= \frac{1}{f_H^2} \frac{d^2 V_{\rm eff}(\theta_H)}{d\theta_H^2}\bigg|_{\theta_H=\theta_H^{\rm min}}, \\
f_H &= \frac{\sqrt{6}}{g}\sqrt{\frac{2\pi R_6\, k}{z_L^3-1}} = \frac{\sqrt{6}}{g_w}\frac{k}{\sqrt{(1-z_L^{-1})(z_L^3-1)}} \\
g_w &= \frac{e}{\sin\theta_W} = g\sqrt{\frac{k}{2\pi R_6(1-z_L^{-1})}}\,.
\end{aligned}
\tag{5.9}
$$

Note that the 4D neutral Higgs field $H(x)$ is related to $\phi_H(x)$ in Eq. (2.33) by

$$
\phi_H(x) = \theta_H f_H + H(x). \tag{5.10}
$$

We give results for the effective potential and the Higgs mass for typical sample values of the parameters. In the following calculation, we take into account the sixth-dimensional $n=0$ modes of 6D bulk gauge bosons A_M, the third-generation $SO(11)$ spinor and vector fermions $\Psi_{\mathbf{32}}^{\alpha=3}$, $\Psi_{\mathbf{11}}^{\beta=3}$, $\Psi_{\mathbf{11}}^{\prime\beta=3}$, and dark fermions $\Psi_{\mathbf{32}}^{\alpha=4}$. Contributions coming from the first and second generations of quarks and leptons are numerically negligible. Some of the parameters in the fermion sector are fixed by the observed masses of quarks and leptons in their approximate forms given in Eqs. (4.23), (4.31), (4.40), and (4.56).

The calculation algorithm is almost the same as in 5D $SO(11)$ GHGUT given in Sect. 5 of Ref. [35]. We are interested in the effective potential $V_{\rm eff}(\theta_H)$ at the electroweak scale. Some of the relations derived in this paper are those at the GUT scale. The value of $\sin^2\theta_W$ evolves from $\frac{3}{8}$ at the GUT scale to the observed value at the electroweak scale by the renormalization group equation (RGE). The RGE analysis of the coupling constants is beyond the scope of the current paper. In this paper we content ourselves with inserting the observed value of $\sin^2\theta_W$ into the formulas for m_Z and $V_{\rm eff}(\theta_H)$. As input parameters we take $\alpha_{\rm EM}^{-1} = 127.916$, $\sin^2\theta_W = 0.2312$, $m_Z = 91.1876$ GeV, $m_t = 171.17$ GeV, $m_b = 4.18$ GeV, $m_\tau = 1.776$ GeV, and $m_{\nu_\tau} = 0.1$ eV. The evaluation is carried out in the following steps:

(1) We first pick the values for z_L and θ_H. So far consistent sets of parameters have been found only for $30 \lesssim z_L \lesssim 40$.
(2) From m_Z, Eq. (3.12), k and $m_{{\rm KK}_5}$ are determined.
(3) c_0 is fixed from m_t by Eq. (4.23).
(4) Some of the parameters in the brane interactions on the UV brane remain free. We take $c_1 = 0$, $c_2 = -0.7$, and $M = -10^7$ GeV. These three parameters are kept fixed in the evaluation below. We temporarily assign a value for $\mu_{\mathbf{11}} = \mu'_{\mathbf{11}}$. Then, μ_1, $\tilde{\mu}_2$, and m_B are determined by Eqs. (4.32), (4.41), and (4.56), or by

$$
\mu_1 \simeq \sqrt{\frac{1+2c_1}{1+2c_0}}\, z_L^{c_0-c_1}\frac{m_t}{m_b}\mu_{\mathbf{11}},
$$

$$
\tilde{\mu}_2 \simeq \sqrt{\frac{1-2c_2}{1-2c_0}}\, z_L^{c_2-c_0}\frac{m_t}{m_\tau}\mu'_{\mathbf{11}},
$$

$$m_B = \sqrt{-\frac{2Mm_t^2}{m_{\nu_\tau}} \frac{z_L^{2c_0+1}}{1+2c_0}}. \tag{5.11}$$

(5) Given the value of the bulk mass parameter c_0' of the dark fermion multiplet, the effective potential $V_{\text{eff}}(\theta_H)$ can be evaluated. We adjust the value of c_0' such that the minimum of $V_{\text{eff}}(\theta_H)$ is located at the initial value of θ_H. In this manner a consistent parameter set has been obtained.

(6) The Higgs boson mass m_H is determined from (5.9), which can be viewed as a function of z_L, θ_H and $\mu'_{\mathbf{11}}$, namely $m_H(\theta_H, z_L, \mu'_{\mathbf{11}})$.

(7) By demanding that the resulting m_H is the observed value, namely $m_H(z_L, \theta_H) = 125.09 \pm 0.24$ GeV, $\mu'_{\mathbf{11}}$ is determined with the given θ_H and z_L.

(8) Take a different θ_H. By repeating the above procedure one finds a new value for $\mu'_{\mathbf{11}}$.

The physical value of θ_H needs to be determined, for instance, from observation of the Z' bosons (the first KK bosons $\gamma^{(1)}, Z^{(1)}, Z_R^{(1)}$) and their masses. In the $SO(5) \times U(1)$ gauge–Higgs electroweak unification many of the physical quantities depend on the value of θ_H, but not on the details of the parameters in the theory. It has been shown that $m_{Z^{(1)}} \gtrsim 7$ TeV ($m_{\text{KK}_5} \gtrsim 8.5$ TeV) to be consistent with the current LHC data [15,16].

The results for the effective potential $V_{\text{eff}}(\theta_H)$ in the $R_{\xi=0}$ gauge are depicted in Fig. 1, for which $z_L = 35$, $\theta_H = 0.15$, and $m_H = 125.1$ GeV. The bulk and brane mass parameters are chosen as

$$\begin{aligned}
&z_L = 35, \;\; \theta_H = 0.15, \;\; m_H = 125.12 \text{ GeV}, \\
&c_0 = 0.3325, \;\; c_1 = 0, \;\; c_2 = -0.7, \;\; c_0' = 0.5224, \\
&\mu_1 \simeq 11.18, \;\; \tilde{\mu}_2 \simeq 0.7091, \;\; \mu_{\mathbf{11}} = \mu'_{\mathbf{11}} = 0.108, \\
&m_B = 1.145 \times 10^{12} \text{ GeV}, \;\; M = -10^7 \text{ GeV}.
\end{aligned} \tag{5.12}$$

The resultant $m_{\text{KK}_5} = \pi k z_L^{-1}$, k, and f_H are given by

$$m_{\text{KK}_5} \simeq 8.236 \text{ TeV}, \;\; k \simeq 8.913 \times 10^4 \text{ GeV}, \;\; f_H \simeq 1.642 \times 10^3 \text{ GeV}. \tag{5.13}$$

In Table 6, we have tabulated various sets of the parameters which yield $m_H = 125.1 \pm 0.1$ GeV. As θ_H increases, m_{KK} decreases. One sees $\theta_H \lesssim 0.15$ in order for $m_{\text{KK}} > 8$ TeV to be consistent with the current LHC data. We have observed that θ_H cannot be too small in order to reproduce $m_H \sim 125$ GeV.

In Figure 2, m_{KK_5} is plotted as a function of θ_H. The data points in Table 6 are fitted by

$$m_{\text{KK}_5} \sim \frac{\alpha}{(\sin\theta_H)^\beta}, \;\; \alpha = 1.230 \text{ TeV}, \;\; \beta = 1.000 \tag{5.14}$$

in the range $0.05 < \theta_H < 0.30$. A similar relation has been found in the $SO(5) \times U(1)$ gauge–Higgs EW unification where $\alpha = 1.352$ TeV and $\beta = 0.786$ [11,12]. The difference between the two is expected to originate from the different gauge group structure of the models and the different matter content. So far we have found consistent parameter sets only for $c_1 = 0$ and $c_2 \leq 0$.

6. Summary and discussions

In this paper, we discussed 6D $SO(11)$ GHGUT in the 6D hybrid warped space. The GUT gauge symmetry $SO(11)$ is reduced to the Pati–Salam symmetry G_{PS} ($= SU(2)_L \times SU(2)_R \times SU(4)_C$) by the orbifold BCs, and the symmetry G_{PS} is spontaneously broken to the SM gauge symmetry

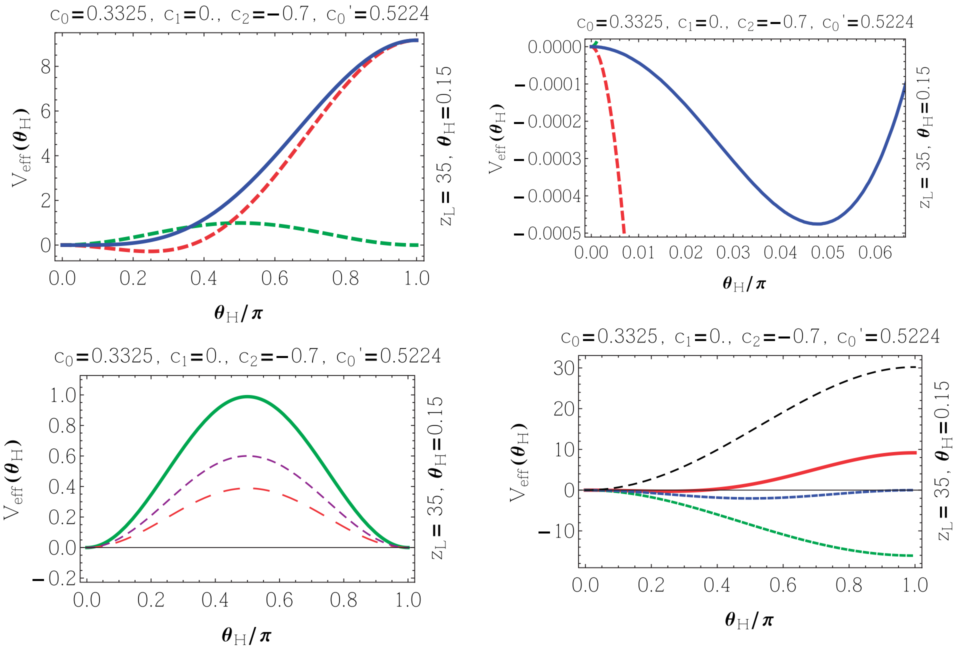


Fig. 1. The effective potential $V_{\rm eff}(\theta_H)$ in the $R_{\xi=0}$ gauge in which $z_L = 35$ and the minimum of $V_{\rm eff}(\theta_H)$ is located at $\theta_H = 0.15$. For the top two figures, the blue solid line shows the effective potential with both gauge and fermion effects, and the red and green dashed lines represent the contributions of fermions and gauge fields, respectively. For the bottom left figure, the green solid line represents the total contributions of gauge fields, and the red and purple dashed lines represent Z and W boson contributions. In the bottom right figure the red solid line represents the total contribution of all fermions, the green dashed lines represent contributions of the top quark, the blue dashed line represents that of the neutrino sector-2, and the black dashed line represents the contribution of dark fermions. Contributions coming from other fermions are negligible.

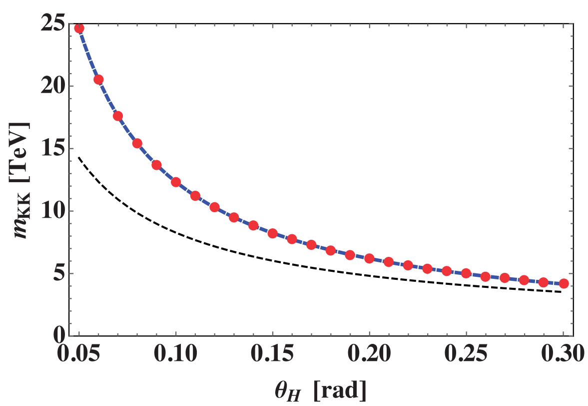


Fig. 2. The Kaluza–Klein mass scale $m_{\rm KK} = m_{{\rm KK}_5}$ as a function of θ_H. The red dots represent points in Table 6 which reproduce $m_H = 125.1 \pm 0.1$ GeV. The blue thick dashed line represents the fitting curve given in Eq. (5.14). The black thin dashed line indicates $m_{\rm KK}(\theta_H)$ found in the $SO(5) \times U(1)$ gauge–Higgs EW unification [11,12].

$G_{\rm SM}$ $(= SU(3)_C \times SU(2)_L \times U(1)_Y)$ by the non-vanishing VEV of the 5D brane scalar field $\Phi_{\bf 32}$ on the UV brane. Finally, the EW gauge symmetry $SU(2)_L \times U(1)_Y$ is broken to $U(1)_{\rm EM}$ by the Hosotani mechanism. The SM Higgs boson appears as a zero mode of the fifth-dimensional components of the $SO(11)$ gauge field.

Table 6. Parameter sets which give dynamical EW symmetry breaking with $m_H = 125.1 \pm 0.1$ GeV. Here $z_L = 35$, $c_1 = 0.0, c_2 = -0.7$, and $M = -10^7$ GeV. c_2 can be varied without affecting m_{KK_5} and c_0. For $\theta_H = 0.05$, for instance, a set of $c_2 = -0.9$ and $\mu_{\mathbf{11}} = \mu^{\ell}_{\mathbf{11}} = 0.1174$ yields $m_H = 125.05$ GeV, $c_0' = 0.6008$, and $\widetilde{\mu}_2 = 0.4084$.

θ_H	$\mu_{\mathbf{11}} = \mu^{l}_{\mathbf{11}}$	m_{KK_5} [TeV]	c_0	c_0'	m_H [GeV]	$\widetilde{\mu}_2$
0.05	0.1730	24.63	0.3289	0.5960	125.17	1.138
0.06	0.1590	20.53	0.3292	0.5828	125.07	1.046
0.07	0.1480	17.60	0.3294	0.5714	125.13	0.9736
0.08	0.1400	15.40	0.3297	0.5625	125.03	0.9208
0.09	0.1330	13.70	0.3300	0.5544	125.08	0.8746
0.10	0.1270	12.33	0.3303	0.5471	125.17	0.8350
0.11	0.1225	11.21	0.3307	0.5414	125.05	0.8052
0.12	0.1180	10.28	0.3311	0.5356	125.14	0.7755
0.13	0.1145	9.495	0.3315	0.5311	125.06	0.7523
0.14	0.1110	8.821	0.3320	0.5264	125.12	0.7291
0.15	0.1080	8.236	0.3325	0.5224	125.12	0.7091
0.16	0.1055	7.725	0.3331	0.5192	125.03	0.6925
0.17	0.1030	7.275	0.3337	0.5158	125.02	0.6759
0.18	0.1005	6.875	0.3343	0.5125	125.11	0.6593
0.19	0.0985	6.517	0.3349	0.5099	125.05	0.6459
0.20	0.0965	6.195	0.3356	0.5073	125.05	0.6326
0.21	0.0945	5.903	0.3363	0.5047	125.21	0.6192
0.22	0.0930	5.639	0.3371	0.5030	125.00	0.6092
0.23	0.0910	5.398	0.3379	0.5003	125.17	0.5959
0.24	0.0895	5.177	0.3387	0.4986	125.14	0.5858
0.25	0.0880	4.974	0.3395	0.4969	125.16	0.5758
0.26	0.0867	4.786	0.3404	0.4956	125.11	0.5671
0.27	0.0855	4.613	0.3413	0.4945	125.04	0.5590
0.28	0.0840	4.452	0.3423	0.4928	125.16	0.5490
0.29	0.0830	4.302	0.3432	0.4921	125.05	0.5423
0.30	0.0818	4.163	0.3442	0.4911	125.07	0.5342

The SM fermions are contained in $\Psi_{\mathbf{32}}$, $\Psi_{\mathbf{11}}$, $\Psi'_{\mathbf{11}}$, $\chi_{\mathbf{1}}$. The SM fermions as well as the W and Z bosons acquire masses via the Hosotani mechanism. In Sect. 4, we showed that the quark and lepton mass spectra can be reproduced by choosing the parameters of the bulk masses and the brane interactions. In the 5D GHGUT in Ref. [35], there necessarily have arisen light exotic vector-like fermions accompanied by SM fermions, which is experimentally unacceptable. In contrast to the case of the 5D GHGUT, the non-SM fermions have masses of either $O(m_{\mathrm{KK}_6})$ or $O(m_{\mathrm{KK}_5})$ in the current 6D model. Thus the problem of light exotic fermions has been solved.

From the analysis of the effective potential $V_{\mathrm{eff}}(\theta_H)$ in Sect. 5, we have seen that the EW symmetry is dynamically broken by the AB phase θ_H. The Higgs boson mass $m_H = 125.1$ GeV can be obtained for $0.05 \lesssim \theta_H \lesssim 0.30$, namely for 25 TeV $\gtrsim m_{\mathrm{KK}_5} \gtrsim 4.1$ TeV.

In the present paper we have treated each generation of quarks and leptons independently. It is necessary to incorporate the flavor mixing in the gauge–Higgs grand unification. The mixing arises from the brane interactions. For quarks and charged leptons the mixing would result from the matrix structure of various κs and μs in the brane interactions in Eq. (2.27) of $\Psi_{\mathbf{32}}$, $\Psi_{\mathbf{11}}$, and $\Phi_{\mathbf{32}}$. On the other hand, the dominant mixing in the neutrino sector would arise from the matrix structure in the mass $M^{\beta\beta'}$ associated with the symplectic Majorana fields $\chi_{\mathbf{1}}$ in Eq. (2.26). The large mixing angles observed in the neutrino sector might be related to this special circumstance.

We would like to add a comment on the effective potential $V_{\rm eff}^6(\theta_H^6)$ of the AB phase θ_H^6 along the sixth dimension. From Table 5, not only the $SO(5)/SO(4)$ components of the fifth-dimensional gauge field A_y but also those of the sixth-dimensional gauge field A_v have zero modes in the absence of brane interactions. As in the effective potential $V_{\rm eff}(\theta_H)$ for the AB phase θ_H along the fifth dimension, the $SO(11)$ bulk gauge bosons A_M and the $SO(11)$ bulk fermions $\Psi_{\mathbf{32}}^{\alpha}$, $\Psi_{\mathbf{11}}^{\beta}$, and $\Psi_{\mathbf{11}}^{\prime\beta}$, contribute to $V_{\rm eff}^6(\theta_H^6)$. If there were only gauge fields, it can be shown that the minimum of $V_{\rm eff}^6(\theta_H^6)$ would be located at $\theta_H^6 = \frac{1}{2}\pi \pmod{\pi}$. In other words, the EW symmetry would be broken by the sixth-dimensional AB phase θ_6 and the W and Z bosons would acquire masses of $O(m_{{\rm KK}_6})$. The situation is similar to that in the effective potential $V_{\rm eff}(\theta_H)$ in the 5D $SO(11)$ GHGUT [32,35]. Further, the contribution from the $SO(11)$ bulk spinor fermions $\Psi_{\mathbf{32}}^{\alpha}$ to $V_{\rm eff}^6(\theta_H^6)$ has little θ_H^6 dependence, as is inferred from Table 2. The θ_H^6-dependent contributions coming from a set of $SO(11)$ bulk vector fermions $\Psi_{\mathbf{11}}^{\beta}$ and $\Psi_{\mathbf{11}}^{\prime\beta}$ almost cancel each other, as can be deduced from Table 3. Hence, with the minimal fermion content presented in this paper the minimum of $V_{\rm eff}^6(\theta_H^6)$ would be located at around $\theta_H^6 = \pi/2 \pmod{\pi}$. This undesirable result can be avoided by introducing additional $SO(11)$ bulk vector fermions $\Psi_{\mathbf{11}}^{\beta}$ with appropriate boundary conditions.

The phase θ_H plays a crucial role in the $SO(11)$ gauge–Higgs grand unification. It is found that the 5D KK mass scale $m_{{\rm KK}_5}$ is determined by θ_H in a very good approximation by the formula in Eq. (5.14). Similar relations are expected for the 4D Higgs cubic and quartic couplings as in the $SO(5)\times U(1)$ gauge–Higgs EW unification. For $m_{{\rm KK}_5}(\theta_H)$ we have recognized the difference between the gauge–Higgs EW and grand unification. We need experimental data to find which one is preferred.

We will come back to these issues in the near future.

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Appendix A. Basics for 6D Kaluza–Klein expansion

We present KK mode expansions in 6D hybrid warped space with the metric given in Eq. (2.1).

A.1. 6D gauge fields

The equations of motion for free gauge fields in the 6D hybrid warped space are given by

$$\Big\{\hat{\eta}^{\lambda\rho}\big(\Box + k^2\mathcal{P}_5 + \partial_v^2\big) - \Big(1-\frac{1}{\xi}\Big)\partial^\lambda\partial^\rho\Big\}A_\rho = 0\,,$$
$$\big(\Box + \xi k^2\mathcal{P}_z + \partial_v^2\big)A_z = 0, \tag{A.1}$$

where $\mathcal{P}_5$ and $\mathcal{P}_z$ are defined in Eq. (2.32). These $\mathcal{P}_5$ and $\mathcal{P}_z$ differ from $\mathcal{P}_4$ and $\mathcal{P}_z$ in the 5d case. Basis mode functions in the z-coordinate in the 6d case are given, except for zero modes, by

$$
\begin{aligned}
C(z;\lambda) &= +\frac{\pi}{2}\lambda z^{3/2} z_L^{1/2} F_{\frac{3}{2},\frac{1}{2}}(\lambda z, \lambda z_L)\,, \\
S(z;\lambda) &= -\frac{\pi}{2}\lambda z^{3/2} z_L^{1/2} F_{\frac{3}{2},\frac{3}{2}}(\lambda z, \lambda z_L)\,, \\
C'(z;\lambda) &= +\frac{\pi}{2}\lambda^2 z^{3/2} z_L^{1/2} F_{\frac{1}{2},\frac{1}{2}}(\lambda z, \lambda z_L)\,, \\
S'(z;\lambda) &= -\frac{\pi}{2}\lambda^2 z^{3/2} z_L^{1/2} F_{\frac{1}{2},\frac{3}{2}}(\lambda z, \lambda z_L)\,,
\end{aligned} \tag{A.2}
$$

where $F_{\alpha,\beta}(u,v) = J_\alpha(u)Y_\beta(v) - Y_\alpha(u)J_\beta(v)$. They can be expressed as

$$
\begin{aligned}
C(z;\lambda) &= z\cos\lambda(z-z_L) - \frac{1}{\lambda}\sin\lambda(z-z_L)\,, \\
S(z;\lambda) &= \Big(z + \frac{1}{\lambda^2 z_L}\Big)\sin\lambda(z-z_L) + \Big(\frac{1}{\lambda} - \frac{z}{\lambda z_L}\Big)\cos\lambda(z-z_L)\,, \\
C'(z;\lambda) &= -\lambda z \sin\lambda(z-z_L)\,, \\
S'(z;\lambda) &= \lambda z\cos\lambda(z-z_L) + \frac{z}{z_L}\sin\lambda(z-z_L)\,,
\end{aligned} \tag{A.3}
$$

and satisfy

$$
\begin{aligned}
&\mathcal{P}_5 \begin{pmatrix} C \\ S \end{pmatrix} = z^2 \frac{d}{dz}\frac{1}{z^2}\frac{d}{dz} \begin{pmatrix} C \\ S \end{pmatrix} = -\lambda^2 \begin{pmatrix} C \\ S \end{pmatrix}, \\
&\mathcal{P}_z \begin{pmatrix} C' \\ S' \end{pmatrix} = \frac{d}{dz} z^2 \frac{d}{dz}\frac{1}{z^2} \begin{pmatrix} C' \\ S' \end{pmatrix} = -\lambda^2 \begin{pmatrix} C' \\ S' \end{pmatrix}, \\
&C(z_L;\lambda) = z_L,\ \ C'(z_L;\lambda) = 0,\ \ S(z_L;\lambda) = 0,\ \ S'(z_L;\lambda) = \lambda z_L, \\
&CS' - SC' = \lambda z^2.
\end{aligned} \tag{A.4}
$$

These basis functions have been employed in the text. We note that the basis functions for fermions, $C_{L/R}(z;\lambda,c)$ and $S_{L/R}(z;\lambda,c)$, remain the same as in the 5d case defined in Ref. [35].

A.2. 6D scalar fields

The action of a massless 6D scalar field $\Phi(x,y,v)$ is given by

$$
S_{\text{bulk}}^{\text{scalar}} = \int d^6x \sqrt{-\det G}\, G^{MN}\, \partial_M \Phi(x,y,v)^\dagger \partial_N \Phi(x,y,v), \tag{A.5}
$$

where $M,N = 0,1,2,3,5,6$. The equation of motion for the scalar field $\Phi(x,z,v)$ is given by

$$
z^2 \left\{ \Box_4 + \partial_v^2 + k^2 z^4 \frac{\partial}{\partial z}\frac{1}{z^4}\frac{\partial}{\partial z} \right\} \Phi(x,z,v) = 0. \tag{A.6}
$$

It follows that $\phi(x,z,v) = z^{-\nu}\Phi(x,z,v)$ ($\nu = \frac{5}{2}$) satisfies

$$
\left\{ \Box_4 + \partial_v^2 + k^2 \left(\frac{\partial^2}{\partial z^2} + \frac{1}{z}\frac{\partial}{\partial z} - \frac{\nu^2}{z^2} \right) \right\} \phi(x,z,v) = 0. \tag{A.7}
$$

The Bessel equation is given by

$$\left(\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}+1-\frac{\nu^2}{z^2}\right)u(z)=0,$$

whose solutions are given by the Bessel functions $J_\nu(z)$ and $Y_\nu(z)$. Hence, a mode function can be written as $\phi(x,z,v)=[\alpha J_\nu(\lambda z)+\beta Y_\nu(\lambda z)]\phi_\lambda(x,v)$, where $\phi_\lambda(x,v)$ satisfies

$$\left\{\Box_4+\partial_v^2-k^2\lambda^2\right\}\phi_\lambda(x,v)=0. \tag{A.8}$$

The orbifold boundary condition for the scalar field ϕ is given by

$$\Phi(x,y_j-y,v_j-v)=P_j\,\Phi(x,y_j+y,v_j+v). \tag{A.9}$$

In the y-coordinate, $\phi(x,y,v)=e^{-\nu\sigma(y)}\Phi(x,y,v)$ satisfies the same boundary condition as Eq. (A.9), and Eq. (A.6) becomes

$$\left\{\Box_4+\partial_v^2+e^{\frac{1}{2}\sigma}\left(\partial_y^2+\nu\sigma''-\nu^2\sigma'^2\right)\right\}\phi=0. \tag{A.10}$$

Due caution must be taken as $\sigma''(y)=2k\{\delta_{2L_5}(y)-\delta_{2L_5}(y-L_5)\}$. If ϕ is parity even in the y-coordinate, the Neumann (N) condition becomes, in the z-coordinate ($1\le z\le z_L$),

$$N:\ \left(\frac{\partial}{\partial z}+\frac{\nu}{z}\right)\phi=0\quad \text{at } z=1^+ \text{ or } z_L^-, \tag{A.11}$$

as can be confirmed by integrating Eq. (A.10) over y from $-\varepsilon$ to $+\varepsilon$ (or from $L_5-\varepsilon$ to $L_5+\varepsilon$). If ϕ is parity odd in the y-coordinate, it satisfies the Dirichlet (D) condition $\phi=0$ at $z=1$ or z_L.

(i) $(P_0=P_1,P_2=P_3)=(+,+)$
The 6D scalar field $\Phi(x,y,v)$ is expanded as

$$\Phi(x,y,v)=e^{\nu\sigma(y)}\left\{\sum_{n=0}^{\infty}\widetilde{\phi}_n^C(x,y)f_n^C(v)+\sum_{n=1}^{\infty}\widetilde{\phi}_n^S(x,y)f_n^S(v)\right\}, \tag{A.12}$$

where $\widetilde{\phi}_n^C(x,y)$ and $\widetilde{\phi}_n^S(x,y)$ satisfy

$$\begin{cases}\widetilde{\phi}_n^C(x,-y)=\widetilde{\phi}_n^C(x,y),\\ \widetilde{\phi}_n^C(x,L_5-y)=\widetilde{\phi}_n^C(x,L_5+y),\end{cases}$$
$$\begin{cases}\widetilde{\phi}_n^S(x,-y)=-\widetilde{\phi}_n^S(x,y),\\ \widetilde{\phi}_n^S(x,L_5-y)=-\widetilde{\phi}_n^S(x,L_5+y),\end{cases} \tag{A.13}$$

and Fourier modes $f_n^C(v)$ and $f_n^S(v)$ are given by

$$f_n^C(v)=\begin{cases}\dfrac{1}{\sqrt{2\pi R_6}} & (n=0),\\ \dfrac{1}{\sqrt{\pi R_6}}\cos\dfrac{nv}{R_6} & (n\ge 1),\end{cases}$$
$$f_n^S(v)=\frac{1}{\sqrt{\pi R_6}}\sin\frac{nv}{R_6}\quad (n\ge 1). \tag{A.14}$$

$\widetilde{\phi}_n^C$ and $\widetilde{\phi}_n^S$ satisfy the Neumann and Dirichlet conditions at $z = 1, z_L$, respectively. There is a zero mode for $\widetilde{\phi}_n^C$:

$$\lambda_0^C = 0, \ \ \widetilde{\phi}_n^C(x,z) = \phi_{n,0}^C(x) z^{-\nu}, \tag{A.15}$$

which corresponds to a mode constant in the fifth dimension. Other modes can be written as

$$\begin{aligned} &\widetilde{\phi}_n^{C,S}(x,z) = Z_\nu(\lambda z)\phi_{m,\lambda}^{C,S}(x), \\ &Z_\nu(\lambda z) = \alpha_{n,\lambda}^{C,S} J_\nu(\lambda z) + \beta_{n,\lambda}^{C,S} Y_\nu(\lambda z). \end{aligned} \tag{A.16}$$

Eigenvalues $\lambda > 0$ and the relative coefficient α/β are determined by the boundary conditions. Noting a recursion relation

$$\left(\frac{d}{dz} + \frac{\nu}{z}\right) Z_\nu(\lambda z) = \lambda Z_{\nu-1}(\lambda z),$$

one finds eigenvalues $\{\lambda_\ell^{C,S}\}$ $(\ell \geq 1)$ from

$$\begin{aligned} &\widetilde{\phi}_n^C : \ Z_{\nu-1}(\lambda_\ell^C) = Z_{\nu-1}(\lambda_\ell^C z_L) = 0, \\ &\widetilde{\phi}_n^S : \ Z_\nu(\lambda_\ell^S) = Z_\nu(\lambda_\ell^S z_L) = 0. \end{aligned} \tag{A.17}$$

Thus Φ is expanded as

$$\begin{aligned} \Phi(x,z,v) = e^{\nu\sigma(y)} \Bigg\{ &\sum_{n=0}^\infty \sum_{\ell=0}^\infty \phi_{n,\ell}^C(x) Z_\nu(\lambda_\ell^C z) f_n^C(v) \\ &+ \sum_{n=1}^\infty \sum_{\ell=1}^\infty \phi_{n,\ell}^S(x) Z_\nu(\lambda_\ell^S z) f_n^S(v) \Bigg\}, \end{aligned} \tag{A.18}$$

where $\widetilde{\phi}_{n,\ell}^{C,S}(x)$ satisfies

$$\begin{aligned} &\left\{\Box_4 - \left(m_{n,\ell}^{C,S}\right)^2\right\} \phi_{n,\ell}^{C,S}(x) = 0, \\ &\left(m_{n,\ell}^{C,S}\right)^2 = \frac{n^2}{R_6^2} + \left(k\lambda_\ell^{C,S}\right)^2. \end{aligned} \tag{A.19}$$

There is one massless mode $\phi_{0,0}^C(x)$ with $m_{0,0}^C = 0$.

(ii) $(P_0 = P_1, P_2 = P_3) = (-,-)$

The expansion is given by

$$\begin{aligned} \Phi(x,z,v) = e^{\nu\sigma(y)} \Bigg\{ &\sum_{n=0}^\infty \sum_{\ell=1}^\infty \phi_{n,\ell}^S(x) Z_\nu(\lambda_\ell^S z) f_n^C(v) \\ &+ \sum_{n=1}^\infty \sum_{\ell=0}^\infty \phi_{n,\ell}^C(x) Z_\nu(\lambda_\ell^C z) f_n^S(v) \Bigg\}. \end{aligned} \tag{A.20}$$

There is no massless mode.

(iii) $(P_0 = P_1, P_2 = P_3) = (+,-)$

$$\Phi(x,z,v) = e^{\nu\sigma(y)} \left\{ \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \phi^C_{n,\ell}(x) Z_\nu(\lambda^C_\ell z) g^C_{n+\frac{1}{2}}(v) \right.$$
$$\left. + \sum_{n=0}^{\infty} \sum_{\ell=1}^{\infty} \phi^S_{n,\ell}(x) Z_\nu(\lambda^S_\ell z) g^S_{n+\frac{1}{2}}(v) \right\}, \tag{A.21}$$

where

$$g^C_{n+\frac{1}{2}}(v) = \frac{1}{\sqrt{\pi R_6}} \cos\frac{(n+\frac{1}{2})v}{R_6}, \;\; g^S_{n+\frac{1}{2}}(v) = \frac{1}{\sqrt{\pi R_6}} \sin\frac{(n+\frac{1}{2})v}{R_6}. \tag{A.22}$$

Note that the mass spectrum is given by

$$\left(m^{C,S}_{n,\ell}\right)^2 = \frac{(n+\frac{1}{2})^2}{R_6^2} + \left(k\lambda^{C,S}_\ell\right)^2 \geq \left(\frac{1}{2R_6}\right)^2. \tag{A.23}$$

(iv) $(P_0 = P_1, P_2 = P_3) = (-,+)$

$$\Phi(x,z,v) = e^{\nu\sigma(y)} \left\{ \sum_{n=0}^{\infty} \sum_{\ell=1}^{\infty} \phi^S_{n,\ell}(x) Z_\nu(\lambda^S_\ell z) g^C_{n+\frac{1}{2}}(v) \right.$$
$$\left. + \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \phi^C_{n,\ell}(x) Z_\nu(\lambda^C_\ell z) g^S_{n+\frac{1}{2}}(v) \right\}. \tag{A.24}$$

A.3. 6D fermion field

Let us consider a free 6D Weyl fermion $\Psi(x,y,v)$ with $\gamma^7_{6D} = +1$:

$$\Psi(x,y,v) = e^{\frac{5}{2}\sigma(y)}\, \check{\Psi}, \;\; \check{\Psi}(x,z,v) = \begin{bmatrix} \xi(x,z,v) \\ \eta(x,z,v) \\ 0 \\ 0 \end{bmatrix},$$

$$\gamma^7_{6D} = \begin{pmatrix} I_4 & \\ & -I_4 \end{pmatrix} = \gamma^5_{4D} \cdot (-i\gamma^5\gamma^6), \;\; \gamma^5_{4D} = \begin{pmatrix} I_2 &&& \\ & -I_2 && \\ && I_2 & \\ &&& -I_2 \end{pmatrix}. \tag{A.25}$$

Its action is given by

$$I = \int d^6x \sqrt{-\det G}\, \overline{\Psi} \left\{ \gamma^A E_A{}^M \mathcal{D}_M + ick\gamma^6 \right\} \Psi$$
$$= \int d^4x \int_0^{2\pi R_6} dv \int_1^{z_L} \frac{dz}{k} i[-\eta^\dagger, \xi^\dagger] \begin{bmatrix} -kD_-(c) + i\partial_v & \sigma^\mu\partial_\mu \\ \overline{\sigma}^\mu\partial_\mu & -kD_+(c) + i\partial_v \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \tag{A.26}$$

where $D_\pm(c)$ is defined in Eq. (4.5). As an example we consider the case in which the BCs are given by

$$\begin{aligned}\Psi(x, y_j - y, v_j - v) &= -i\gamma^5\gamma^6 P_j \Psi(x, y_j + y, v_j + v)\\ &= \gamma^7_{6D}\gamma^5_{4D} P_j \Psi(x, y_j + y, v_j + v).\end{aligned} \tag{A.27}$$

The equations of motion are

$$\begin{aligned}&(-kD_-(c) + i\partial_v)\xi + \sigma^\mu\partial_\mu\eta = 0,\\ &\overline{\sigma}^\mu\partial_\mu\xi + (-kD_+(c) + i\partial_v)\eta = 0.\end{aligned} \tag{A.28}$$

To find the eigenmodes, we separate 4D and the fifth- and sixth-dimensional parts as

$$\begin{aligned}\xi(x, z, v) &= \tilde{\xi}(z, v) f_R(x), \ \ \overline{\sigma}^\mu\partial_\mu f_R(x) = m f_L(x),\\ \eta(x, z, v) &= \tilde{\eta}(z, v) f_L(x), \ \ \sigma^\mu\partial_\mu f_L(x) = m f_R(x).\end{aligned} \tag{A.29}$$

It follows that

$$\begin{aligned}&(-kD_-(c) + i\partial_v)\tilde{\xi} + m\tilde{\eta} = 0,\\ &m\tilde{\xi} + (-kD_+(c) + i\partial_v)\tilde{\eta} = 0.\end{aligned} \tag{A.30}$$

In the y-coordinate, ξ and η satisfy the BCs

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix}(x, y_j - y, v_j - v) = P_j \begin{pmatrix}\xi\\ -\eta\end{pmatrix}(x, y_j + y, v_j + v). \tag{A.31}$$

(i) $(P_0 = P_1, P_2 = P_3) = (+, +)$
In this case,

$$\begin{aligned}\xi(x, y_j - y, v_j - v) &= +\xi(x, y_j + y, v_j + v),\\ \eta(x, y_j - y, v_j - v) &= -\eta(x, y_j + y, v_j + v).\end{aligned} \tag{A.32}$$

The mode functions can be written as

$$\begin{aligned}\xi(x, y, v) &= \left\{\sum_{n=0}^{\infty}\tilde{\xi}^C_n(y) f^C_n(v) + \sum_{n=1}^{\infty}\tilde{\xi}^S_n(y) f^S_n(v)\right\} f_R(x),\\ \eta(x, y, v) &= \left\{\sum_{n=0}^{\infty}\tilde{\eta}^S_n(y) f^C_n(v) + \sum_{n=1}^{\infty}\tilde{\eta}^C_n(y) f^S_n(v)\right\} f_L(x),\end{aligned} \tag{A.33}$$

where the $f^C_n(v), f^S_n(v)$ are defined in Eq. (A.14). $\tilde{\xi}^{C,S}_n$ and $\tilde{\eta}^{C,S}_n$ satisfy the BCs

$$\begin{aligned}\tilde{\xi}^C_n(y_j - y) &= +\tilde{\xi}^C_n(y_j + y), \ \ \tilde{\xi}^S_n(y_j - y) = -\tilde{\xi}^S_n(y_j + y),\\ \tilde{\eta}^C_n(y_j - y) &= +\tilde{\eta}^C_n(y_j + y), \ \ \tilde{\eta}^S_n(y_j - y) = -\tilde{\eta}^S_n(y_j + y).\end{aligned} \tag{A.34}$$

By inserting Eq. (A.33) into Eq. (A.28), one finds equations for $\tilde{\xi}_n^{C,S}, \tilde{\eta}_n^{C,S}$ in the z-coordinate ($1 \le z \le z_L$):

$$\begin{pmatrix} -kD_- & m \\ m & -kD_+ \end{pmatrix} \begin{pmatrix} \tilde{\xi}_0^C \\ \tilde{\eta}_0^S \end{pmatrix} = 0,$$

$$\begin{pmatrix} -kD_- & i\frac{n}{R_6} & m & 0 \\ -i\frac{n}{R_6} & -kD_- & 0 & m \\ m & 0 & -kD_+ & i\frac{n}{R_6} \\ 0 & m & -i\frac{n}{R_6} & -kD_+ \end{pmatrix} \begin{pmatrix} \tilde{\xi}_n^C \\ \tilde{\xi}_n^S \\ \tilde{\eta}_n^S \\ \tilde{\eta}_n^C \end{pmatrix} = 0 \quad (n \ge 1). \tag{A.35}$$

It follows that

$$(k^2 D_+ D_- - m^2)\, \tilde{\xi}_0^C = 0,$$

$$(k^2 D_- D_+ - m^2)\, \tilde{\eta}_0^S = 0,$$

$$\left(k^2 D_+ D_- \mp \frac{2kcn}{R_6}\frac{1}{z} + \frac{n^2}{R_6^2} - m^2 \right) \left(\tilde{\xi}_n^C \pm i\tilde{\xi}_n^S \right) = 0,$$

$$\left(k^2 D_- D_+ \mp \frac{2kcn}{R_6}\frac{1}{z} + \frac{n^2}{R_6^2} - m^2 \right) \left(\tilde{\eta}_n^S \pm i\tilde{\eta}_n^C \right) = 0 \quad (n \ge 1). \tag{A.36}$$

The equations for the zero ($n = 0$) modes in the sixth dimension, $\tilde{\xi}_0^C$ and $\tilde{\eta}_0^S$, are reduced to the Bessel equation. In terms of the basis functions $C_{R/L}(z; \lambda, c)$, $S_{R/L}(z; \lambda, c)$ given in Appendix B of Ref. [35], solutions satisfying the BCs are given by

$$\begin{aligned} &\tilde{\xi}_{0,0}^C = a_0 z^c, \;\; \tilde{\eta}_{0,0}^S = 0, \;\; m_0 = 0, \\ &\tilde{\xi}_{0,\ell}^C = a_\ell C_R(z; \lambda_\ell, c), \;\; \tilde{\eta}_{0,\ell}^S = a_\ell S_L(z; \lambda_\ell, c), \\ &S_L(1; \lambda_\ell, c) = 0, \;\; m_\ell = k\lambda_\ell > 0, \end{aligned} \tag{A.37}$$

where a_ℓ is a normalization constant. There is one massless mode for the right-handed component. For the $n \ne 0$ modes the solutions are more involved.

(ii) $(P_0 = P_1, P_2 = P_3) = (-, -)$
In this case the mode functions of ξ and η can be written as

$$\xi(x, y, v) = \left\{ \sum_{n=0}^{\infty} \tilde{\xi}_n^S(y) f_n^C(v) + \sum_{n=1}^{\infty} \tilde{\xi}_n^C(y) f_n^S(v) \right\} f_R(x),$$

$$\eta(x, y, v) = \left\{ \sum_{n=0}^{\infty} \tilde{\eta}_n^C(y) f_n^C(v) + \sum_{n=1}^{\infty} \tilde{\eta}_n^S(y) f_n^S(v) \right\} f_L(x). \tag{A.38}$$

The equations for the zero ($n = 0$) modes in the sixth dimension, $\tilde{\xi}_0^S$ and $\tilde{\eta}_0^C$, are

$$\begin{pmatrix} -kD_- & m \\ m & -kD_+ \end{pmatrix} \begin{pmatrix} \tilde{\xi}_0^S \\ \tilde{\eta}_0^C \end{pmatrix} = 0, \tag{A.39}$$

and solutions satisfying the BCs are given by

$$\begin{aligned}
&\tilde{\xi}^S_{0,0} = 0, \ \ \tilde{\eta}^C_{0,0} = a_0 z^{-c}, \ \ m_0 = 0, \\
&\tilde{\xi}^S_{0,\ell} = a_\ell S_R(z;\lambda_\ell, c), \ \ \tilde{\eta}^C_{0,\ell} = a_\ell C_L(z;\lambda_\ell, c), \\
&S_R(1;\lambda_\ell, c) = 0 \ , \ \ m_\ell = k\lambda_\ell > 0,
\end{aligned} \tag{A.40}$$

where a_ℓ is a normalization constant. There is one massless mode for the left-handed component.

(iii) $(P_0 = P_1, P_2 = P_3) = (+, -)$
In this case the mode functions can be written as

$$\begin{aligned}
\xi(x, y, v) &= \left\{ \sum_{n=0}^{\infty} \tilde{\xi}^C_n(y) g^C_{n+\frac{1}{2}}(v) + \sum_{n=0}^{\infty} \tilde{\xi}^S_n(y) g^S_{n+\frac{1}{2}}(v) \right\} f_R(x), \\
\eta(x, y, v) &= \left\{ \sum_{n=0}^{\infty} \tilde{\eta}^S_n(y) g^C_{n+\frac{1}{2}}(v) + \sum_{n=1}^{\infty} \tilde{\eta}^C_n(y) g^S_{n+\frac{1}{2}}(v) \right\} f_L(x),
\end{aligned} \tag{A.41}$$

where $g^{C,S}_{n+\frac{1}{2}}(v)$ are defined in Eq. (A.22). There are no zero modes in the sixth dimension.

(iv) $(P_0 = P_1, P_2 = P_3) = (-, +)$
In this case the mode functions can be written as

$$\begin{aligned}
\xi(x, y, v) &= \left\{ \sum_{n=0}^{\infty} \tilde{\xi}^S_n(y) g^C_{n+\frac{1}{2}}(v) + \sum_{n=0}^{\infty} \tilde{\xi}^C_n(y) g^S_{n+\frac{1}{2}}(v) \right\} f_R(x), \\
\eta(x, y, v) &= \left\{ \sum_{n=0}^{\infty} \tilde{\eta}^C_n(y) g^C_{n+\frac{1}{2}}(v) + \sum_{n=1}^{\infty} \tilde{\eta}^S_n(y) g^S_{n+\frac{1}{2}}(v) \right\} f_L(x).
\end{aligned} \tag{A.42}$$

There are no zero modes in the sixth dimension.

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