PTEP

Prog. Theor. Exp. Phys. **2018**, 043B01 (12 pages)
DOI: 10.1093/ptep/pty025

Orbifold Schur index and IR formula

Yosuke Imamura*

Department of Physics, Tokyo Institute of Technology, Tokyo 152-8551, Japan
*E-mail: imamura@phys.titech.ac.jp

Received November 21, 2017; Revised February 5, 2018; Accepted February 9, 2018; Published April 10, 2018

We discuss an orbifold version of the Schur index defined as the supersymmetric partition function in $\boldsymbol{S}^3/\mathbb{Z}_n \times \boldsymbol{S}^1$. We first give a general formula for Lagrangian theories obtained by the localization technique, and then suggest a generalization of the Cordova and Shao IR formula. We confirm that the generalized IR formula gives the correct answer for systems with free hypermultiplets if we tune the background fields so that they are invariant under the orbifold action. Unfortunately, we find disagreement for theories with dynamical vector multiplets.

Subject Index B10, B16

1. Introduction

Exactly calculable quantities in supersymmetric field theories play important roles in recent progress in quantum field theories. The Schur index of $\mathcal{N} = 2$ superconformal field theories is one such quantity. There are different ways to calculate the index.

The Schur index is a specialization of the superconformal index of $\mathcal{N} = 1$ superconformal field theories [1,2]. (See Ref. [3] for various limits of the superconformal index.) The superconformal index can be regarded as the supersymmetric partition function in $\boldsymbol{S}^3 \times \boldsymbol{S}^1$. For Lagrangian theories it is defined as the path integral in the background, and we can reduce it as a finite-dimensional integral by using the localization method. This is also available for a non-Lagrangian theory if we know a UV Lagrangian theory that flows to the theory.

Cordova and Shao [4] proposed an interesting formula for the Schur index which gives the index as the trace of a so-called quantum monodromy operator. With this formula we can calculate the index from the information of the BPS spectrum of the theory in the Coulomb branch. Although the BPS spectrum depends on the Coulomb moduli parameters and jumps on walls of marginal stability the quantum monodromy operator is wall-crossing invariant, and so is the index. The formula is generalized to decorated indices by introducing defect operators [5–7].

For class S theories, which are realized on M5-branes wrapped on Riemann surfaces, the Schur index is expressed as a correlation function of a two-dimensional topological field theory on the associated Riemann surface [8–14].

The Schur index receives contributions of a special class of gauge-invariant operators, which are called Schur operators. Beem et al. [15] show that the set of Schur operators form a chiral algebra of two-dimensional conformal field theory (CFT). Once we identify the chiral algebra associated with a four-dimensional theory the Schur index is obtained as the vacuum character of the chiral algebra. The characters of other modules are also important and are related to line and surface operator insertions [7,16]. For theories of class S there is a prescription to obtain the corresponding chiral algebra [17].

Funded by SCOAP[3]

The purpose of this paper is investigate a generalization of the Schur index defined by replacing the background manifold $\boldsymbol{S}^3 \times \boldsymbol{S}^1$ with its orbifold $\boldsymbol{S}^3/\mathbb{Z}_n \times \boldsymbol{S}^1$. Such an orbifold generalization for the superconformal index of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ theories has already been investigated in Refs. [18–22]. The index is defined in Ref. [18] by using $\mathbb{Z}_n$, which preserves a supercharge of a specific chirality. Because we need only one supercharge (and its Hermitian conjugate) for the definition of the superconformal index, this orbifolding is consistent with the definition of the index. However, in the case of the Schur index, we need to preserve two supercharges with opposite chirality (and their Hermitian conjugates). Therefore, we need to modify the definition of the orbifold index by introducing an extra $SU(2)_R \times U(1)_r$ twist. The application of the localization technique for the orbifold Schur index is straightforward and will be shown in Sect. 2. Then, we discuss a generalization of the IR formula to the orbifold case. We suggest a natural generalization of Cordova and Shao's formula based on a physical interpretation of the formula, and apply it to some simple examples. For systems consisting of free hypermultiplets we find agreement of the UV and IR formulae. Unfortunately, however, for more general systems including dynamical vector multiplets the suggested formula does not give the desired results.

2. UV formula

Let us consider an $\mathcal{N} = 2$ superconformal field theory defined in $\boldsymbol{S}^3 \times \mathbb{R}_t$. Let H, J, $\overline{J}$, R, and r be the Hamiltonian, the third component of the left-handed spin, that of the right-handed spin, the $SU(2)_R$ Cartan generator, and the $U(1)_r$ charge, respectively. The superconformal index is defined by

$$I(p,q,t,\vec{z}_F) = \mathrm{Tr}\left[e^{2\pi i(J+\overline{J})} x^{\mu_x} p^{\mu_p} q^{\mu_q} t^{\mu_t} \vec{z}_F^{\vec{T}_F}\right], \tag{1}$$

where the trace is taken over all gauge-invariant states in $\boldsymbol{S}^3$. We denote the set of Cartan generators of the flavor group F by $\vec{T}_F = (T_{F,1}, \ldots, T_{F,r_F})$. Other Cartan generators are defined by

$$\mu_x = \frac{1}{2}H - J - R + \frac{1}{4}r = \{Q, Q^\dagger\}, \tag{2}$$

$$\mu_p = \frac{1}{2}H - \overline{J} - R - \frac{1}{4}r = \{\overline{Q}, \overline{Q}^\dagger\}, \tag{3}$$

$$\mu_q = J + \overline{J} + R, \tag{4}$$

$$\mu_t = R + \frac{1}{2}r, \tag{5}$$

where Q and $\overline{Q}$ are supercharges with the following quantum numbers:

$$\begin{aligned} Q &: (H, J, \overline{J}, R, r) = (\tfrac{1}{2}, -\tfrac{1}{2}, 0, +\tfrac{1}{2}, -1), \\ \overline{Q} &: (H, J, \overline{J}, R, r) = (\tfrac{1}{2}, 0, -\tfrac{1}{2}, +\tfrac{1}{2}, +1). \end{aligned} \tag{6}$$

The definition in Eq. (1) respects Q. Namely, all the Cartan generators used in Eq. (1) commute with Q. Because μ_x is Q-exact, the index in Eq. (1) is independent of x.[1]

[1] Our convention is different from the standard one used, for example, in Ref. [3]. The standard one is obtained from ours by the replacement $t \to t/q$, and then the Schur limit is given by $t \to q$, while in our convention the Schur limit is $t \to 1$.

The superconformal index can be regarded as the supersymmetric partition function in $\boldsymbol{S}^3 \times \boldsymbol{S}^1$. If the theory has a Lagrangian description with gauge group G we can define the index as the path integral. By using the localization technique we can reduce the path integral to the finite-dimensional integral

$$I = \int d\mu \, \mathrm{Pexp}\, i, \tag{7}$$

where Pexp is the plethystic exponential defined by

$$\mathrm{Pexp} f(x_1, x_2, \ldots, x_k) = \exp \sum_{m=1}^{\infty} \frac{1}{m} f(x_1^m, x_2^m, \ldots, x_k^m). \tag{8}$$

i is the one-particle index defined by

$$i(p, q, t, \vec{z}) = \mathrm{tr}\left[e^{2\pi i (J + \bar{J})} x^{\mu_x} p^{\mu_p} q^{\mu_q} t^{\mu_t} \vec{z}^{\vec{T}} \right], \tag{9}$$

where the trace is taken over all one-particle states including gauge-non-invariant states. $\vec{T}$ includes both flavor and gauge Cartan generators, and $\vec{z}$ is the set of the corresponding fugacities. Namely, $\vec{z}^{\vec{T}} = \vec{z}_F^{\vec{T}_F} \vec{z}_G^{\vec{T}_G}$. $\int d\mu$ is the integral over the gauge fugacities,

$$\int d\mu = \frac{1}{|W_G|} \prod_{i=1}^{r_G} \oint \frac{dz_{G,i}}{2\pi i z_{G,i}}, \tag{10}$$

where $|W_G|$ is the size of the Weyl group of G.

The Schur index is obtained from the superconformal index by the specialization $t = 1$. Then μ_t disappears from Eq. (5), and all the remaining Cartan charges commute with not only Q but also $\overline{Q}$. Therefore, the Schur index receives contributions of operators that carry $\mu_x = \mu_p = 0$. Such operators are called Schur operators. As a result, the Schur index is a function of the superconformal fugacity q and the flavor fugacities $\vec{z}_F$.

There is another index that receives from the same set of operators as the Schur index. It is the Macdonald index defined from the superconformal index by taking the limit $p \to 0$. We do not consider the Macdonald index here because its definition does not respect the supercharges Q and $\overline{Q}$ that are important when we discuss the BPS configuration associated with the IR formula. Even so, the Macdonald index is closely related to the Schur index, and can be reproduced from the chiral algebra up to some ambiguity that can be fixed by using information from four-dimensional SCFT [23]. In the recent paper [24] an orbifold version of the Macdonald index is studied, and a close relation to the chiral algebra is observed. It will be important to study the relation between the orbifold Schur index defined below and the orbifold Macdonald index in Ref. [24].

The one-particle Schur index is given by

$$i = r_G i_{U(1)} + i_V + i_H, \tag{11}$$

where $i_{U(1)}$, i_V, and i_H represent the contributions of a single $U(1)$ vector multiplet, charged vector multiplets, and half-hypermultiplets, respectively. They are explicitly given as

$$i_{U(1)} = \frac{-2q}{1-q}, \quad i_V = -\frac{1+q}{1-q} \chi'_{\mathrm{adj}}(\vec{z}_G), \quad i_H = \frac{q^{\frac{1}{2}}}{1-q} \chi_{R_H}(\vec{z}_F, \vec{z}_G), \tag{12}$$

where $\chi'_{\rm adj} \equiv \chi_{\rm adj} - r_G$ is the character of the adjoint representation of G with the Cartan contribution subtracted, and χ_{R_H} is the character of the $G \times F$ representation of the half-hypermultiplets.[2] Because $i_{U(1)}$ does not depend on the gauge fugacities $\vec{z}_G$ we can factor out the Cartan contribution in Eq. (7) as λ^{r_G}, with λ defined by

$$\lambda = \mathrm{Pexp}\, i_{U(1)} = (q;q)_\infty^2, \tag{13}$$

where $(z;q)_\infty$ is the q-Pochhammer symbol defined by

$$(z;q)_\infty = \prod_{k=0}^{\infty}(1 - zq^k). \tag{14}$$

A generalization of the superconformal index to the orbifold background $\boldsymbol{S}^3/\mathbb{Z}_n \times \boldsymbol{S}^1$ was first investigated in Ref. [18]. They defined the orbifold with the discrete group $\mathbb{Z}_n$ generated by

$$\overline{g}_n = \omega_n^{2\overline{J}}, \quad \omega_n \equiv e^{\frac{2\pi i}{n}}. \tag{15}$$

This is consistent with the definition in Eq. (1) of the superconformal index in the sense that $\overline{g}_n$ commutes with the supercharge Q. If the theory has gauge and/or flavor symmetry we can turn on holonomies $\vec{h}$ by using $\overline{g}_n\omega_n^{\vec{h}\vec{T}}$ instead of $\overline{g}_n$ as the $\mathbb{Z}_n$ generator. Once we choose the $\mathbb{Z}_n$ generator, it is straightforward to generalize the formula in Eq. (7) to the orbifold case. What we have to do first is project away the contribution of $\mathbb{Z}_n$ non-invariant states from the one-particle index in Eq. (9). This projection is carried out by inserting the projection

$$\frac{1}{n}\sum_{k=0}^{n-1}(\overline{g}_n\omega_n^{\vec{h}\vec{T}})^k \tag{16}$$

into the trace in Eq. (9). Because $\overline{g}_n$ can be expressed in terms of the Cartan generators appearing in Eq. (1) as $\overline{g}_n = \omega_n^{2\overline{J}} = \omega_n^{\mu_x-\mu_p+\mu_q-\mu_t}$, insertion of $(\overline{g}_n\omega_n^{\vec{h}\vec{T}})^k$ is equivalent to the replacement of the fugacities

$$(x,p,q,t,\vec{z}) \to (\omega_n^k x, \omega_n^{-k}p, \omega_n^k q, \omega_n^{-k}t, \omega_n^{k\vec{h}}\vec{z}), \tag{17}$$

and the one-particle index for the $\mathbb{Z}_n$ orbifold is

$$i_n^{\vec{h}}(p,q,t,\vec{z}) = \frac{1}{n}\sum_{k=0}^{n-1} i(\omega_n^{-k}p, \omega_n^k q, \omega_n^{-k}t, \omega_n^{k\vec{h}}\vec{z}). \tag{18}$$

By using this orbifold one-particle index, the orbifold index is given by

$$I_n^{\vec{h}_F} = \sum_{\vec{h}_G} e^{\varepsilon(\vec{h})}\int d\mu\, \mathrm{Pexp}\, i_n^{\vec{h}}, \tag{19}$$

where the factor $e^{\varepsilon(\vec{h})}$ is the contribution of the zero-point energy. For the ordinary index this factor just gives an overall factor and is usually neglected. However, in the orbifold case this factor is

[2] The reason why i_V in Eq. (12) is not simply $i_{U(1)}\chi'_{\rm adj}$ but $(i_{U(1)} - 1)\chi'_{\rm adj}$ is that we include the measure factor (Vandermonde determinant) in i_V. In other words, "-1" is the contribution of the Faddeev–Popov ghost associated with constant gauge transformation over $\boldsymbol{S}^3$.

important because it depends on the gauge holonomy $\vec{h}_G$. The zero-point factor $e^{\varepsilon(\vec{h})}$ is easily obtained by taking the product of all contributions of one-particle states. For example, if the one-particle index is expanded as $i_n^{\vec{h}} = \sum_i c_i p^{a_i} q^{b_i} (\cdots)$ (we explicitly show p and q dependence for simplicity, and the dots include other fugacities) then the corresponding zero-point factor is given by $e^{\varepsilon(\vec{h})} = \prod_i (p^{\frac{a_i}{2}} q^{\frac{b_i}{2}} \cdots)^{c_i}$. Although this infinite product is usually divergent, we can obtain a finite result by using an appropriate regularization such as ζ-function regularization.

Now, let us apply the same orbifolding prescription to the Schur index. We cannot obtain the orbifold Schur index by the specialization $t \to 1$ from the orbifold superconformal index above because this contradicts the orbifold action in Eq. (17). For consistency with $t = 1$ we use

$$g_n = \overline{g}_n e^{\frac{2\pi i}{n}(R+\frac{1}{2}r)} = e^{\frac{2\pi i}{n}(\mu_x - \mu_p + \mu_q)} \tag{20}$$

instead of $\overline{g}_n$. Namely, we combine $\overline{g}_n$ with the additional $SU(2)_R \times U(1)$ twist to keep both Q and $\overline{Q}$ invariant. Again, we can turn on holonomies $\vec{h}$, and the insertion of $(g_n \omega_n^{\vec{h}\vec{T}})^k$ is equivalent to the replacement

$$(x, p, q, \vec{z}) \to (\omega_n^k x, \omega_n^{-k} p, \omega_n^k q, \omega_n^{k\vec{h}} \vec{z}). \tag{21}$$

Therefore, the orbifold one-particle index is

$$i_n^{\vec{h}}(q, \vec{z}) = \frac{1}{n} \sum_{k=0}^{n-1} i(\omega^k q, \omega^{k\vec{h}} \vec{z}). \tag{22}$$

With this one-particle index we can calculate the orbifold Schur index of Lagrangian theories by using Eq. (19), which we call the UV formula in the following.

Before discussing the IR formula for the orbifold Schur index, let us apply the UV formula to some simple systems. In the following examples the zero-point factors give overall factors independent of the holonomies and we omit them.

2.1. $U(1)$ *vector multiplet*

The $U(1)$ contribution in $\boldsymbol{S}^3$ is given by λ in Eq. (13). The $\mathbb{Z}_n$ action is simple phase rotation of variable q, and we obtain the orbifold one-particle index $i_n = -2q^n/(1-q^n)$. The orbifold Schur index is

$$\lambda_n = \mathrm{Pexp}\left(\frac{-2q^n}{1-q^n}\right) = (q^n; q^n)_\infty^2. \tag{23}$$

2.2. *Free hypermultiplet*

Let us consider the system of a single free hypermultiplet $(q, \widetilde{q})$. This has $SU(2)_F$ flavor symmetry. Let z be the fugacity for $U(1)_F \subset SU(2)_F$ and F be the corresponding generator such that q and $\widetilde{q}$ carry $F = +1$ and -1, respectively. The one-particle index before $\mathbb{Z}_n$ projection is

$$i(q, z) = \frac{q^{\frac{1}{2}}}{1-q}(z + z^{-1}). \tag{24}$$

The $\mathbb{Z}_n$ generator acts on the variables as

$$g_n \omega_n^{hF} : (q, z) \to (\omega_n q, \omega_n^h z). \tag{25}$$

By definition, the generator $g_n\omega_n^{hF}$ must satisfy $(g_n\omega_n^{hF})^n = 1$ (otherwise the fiber bundle associated with the fields is ill-defined). Due to the fractional $SU(2)_R$ charge $R = -1/2$ of q and $\widetilde{q}$, g_n^n acts on the hypermultiplet as -1. To compensate for this, we need to take the fractional holonomy $h \in \mathbb{Z}+\frac{1}{2}$. Then, after the projection we obtain

$$i_n^h(q,z) = \frac{q^{\frac{1}{2}}}{1-q^n}(q^{[-h-\frac{1}{2}]_n}z + q^{[h-\frac{1}{2}]_n}z^{-1}), \quad h = \frac{1}{2},\dots,n-\frac{1}{2}, \tag{26}$$

where $[x]_n$ for $x \in \mathbb{Z}$ is the minimum non-negative integer satisfying $[x]_n \equiv x \mod n$. In the $\mathbb{Z}_2$ case, the one-particle indices for the two holonomies $h = \pm 1/2$ are

$$i_2^{+\frac{1}{2}} = \frac{q^{\frac{1}{2}}}{1-q^2}(qz + z^{-1}), \quad i_2^{-\frac{1}{2}} = \frac{q^{\frac{1}{2}}}{1-q^2}(z + qz^{-1}). \tag{27}$$

2.3. QED

Let us consider $U(1)$ gauge theory with N_f hypermultiplets. This theory is not conformal, but we can obtain the index by applying the localization formula. When we consider the orbifold index, we should be careful about the fact that $U(1)_r$ is broken to the subgroup $\mathbb{Z}_{N_f} \subset U(1)_r$ by an anomaly. The orbifold action must be consistent with this unbroken symmetry. For example, in the case of the $\mathbb{Z}_2$ orbifold, N_f must be even. For $N_f = 2$ the $\mathbb{Z}_2$ orbifold index is given by

$$I_2^{\frac{1}{2}} = \lambda_2 \sum_{h=\pm\frac{1}{2}} \oint \frac{dz}{2\pi i z} \operatorname{Pexp}\left(2i_2^h\right) = 2(1 + 2q^2 + 8q^4 + \cdots), \tag{28}$$

where i_2^h is the one-particle index of Eq. (27) for a single hypermultiplet. We did not turn on the $SU(N_F)$ flavor symmetry for simplicity, and we omitted the zero-point factor which does not depend on $h = \pm 1/2$.

3. IR formula

The IR formula proposed by Cordova and Shao [4] gives the Schur index by using the information of the BPS spectrum in the Coulomb branch.

Let Γ be the charge lattice of flavor and gauge charges, and $\langle\gamma,\gamma'\rangle$ be the associated Dirac pairing. The central charge of a particle is determined by its charge $\gamma \in \Gamma$, and we denote it by Z_γ. The flavor sublattice $\Gamma_F \subset \Gamma$ is defined as the set of charges γ which have vanishing pairing $\langle\gamma,\gamma'\rangle = 0$ with arbitrary $\gamma' \in \Gamma$.

Let L be the set of primitive charges such that the charge of an arbitrary BPS particle is given by $n\gamma$ with $\gamma \in L$ and $n \in \mathbb{Z}_+$. The BPS spectrum at a point of the Coulomb branch is encoded in a set of functions $K_\gamma(z)$ defined for each $\gamma \in L$. These functions are called quantum Kontsevich–Soibelman factors. The functional form of $K_\gamma(z)$ depends on the helicity of the BPS particle γ. (By an abuse of notation we use γ as a label for sorts of particles.) In the following we deal with only BPS particles belonging to a half-hypermultiplet, and the function K_γ for such a particle is given by

$$K_\gamma(z) = E_q(z) = \operatorname{Pexp}\frac{q^{\frac{1}{2}}z}{1-q} = \prod_{k=0}^{\infty}\frac{1}{1-q^{\frac{1}{2}+k}z} = \sum_{k=0}^{\infty}\frac{q^{\frac{k}{2}}}{(q)_k}z^k, \tag{29}$$

where $(q)_k$ is the q-factorial defined by $(q)_k = \prod_{i=1}^k(1-q^i)$.

Cordova and Shao's IR formula [4] is

$$I = \lambda^r \operatorname{tr}\left(\prod_{\gamma\in L}^{\curvearrowleft} K_\gamma(X_\gamma)\right), \tag{30}$$

where $\prod_{\gamma\in L}^{\curvearrowleft}$ is the phase-ordered product according to $\arg Z_\gamma$; r is the rank of the theory, which is the number of massless $U(1)$ gauge fields at a generic point of the Coulomb branch. X_γ are operators satisfying the quantum torus algebra[3]

$$X_\gamma X_{\gamma'} = (-q^{\frac{1}{2}})^{\langle\gamma,\gamma'\rangle} X_{\gamma+\gamma'} = q^{\langle\gamma,\gamma'\rangle} X_{\gamma'} X_\gamma. \tag{31}$$

The trace $\operatorname{tr} X_\gamma$ is defined as an isomorphic map from Γ_F to $\mathbb{C}^*$. If $\gamma \notin \Gamma_F$, $\operatorname{tr} X_\gamma = 0$.

To physically interpret this formula, we should understand the structure of BPS configurations in the Coulomb branch [5]. In general, a BPS configuration contains massive BPS particles with different charges. For a configuration in $\boldsymbol{S}^3 \times \mathbb{R}_t$ to preserve the supercharges Q and $\overline{Q}$, which are necessary to define the Schur index, the particles must be aligned along a large circle in $\boldsymbol{S}^3$, and the position θ_γ on the circle is determined by the central charge Z_γ by $\theta_\gamma = \arg Z_\gamma$. Thus we can specify the particle distribution in a BPS configuration by giving a set of occupation numbers n_γ for $\gamma \in L$. We denote this set by $\{n_\gamma\}_{\gamma\in L}$.

If we expand the function K_γ as

$$K_\gamma(z) = \sum_{n=0}^{\infty} C_\gamma(n) z^n, \tag{32}$$

then the index is given as the summation over the occupation numbers:

$$I = \sum_{\{n_\gamma\}_{\gamma\in L}} \lambda^r \left(\prod_{\gamma\in L} C_\gamma(n_\gamma)\right) \operatorname{tr}\left(\prod_{\gamma\in L}^{\curvearrowleft} X_\gamma^{n_\gamma}\right). \tag{33}$$

The summand gives the contribution of a specific set of occupation numbers, and consists of the following three factors:

- The factor λ^r is the one-loop contribution of the massless vector multiplets.
- The factor $\prod C$ is the contribution associated with internal degrees of freedom of BPS particles.
- The trace factor can be identified with the classical contribution of massless vector multiplets. The angular momentum induced by the Poynting vector due to the existence of mutually non-local charges contributes to the index by the factor $e^{2\pi i(J+\overline{J})} q^{J+\overline{J}}$, and the q-dependence of the trace factor gives this factor. For example, let us consider two adjacent charges γ and γ' on the large circle. If they carry mutually non-local charges, the induced massless gauge fields contribute to the angular momentum $J+\overline{J}$ by $\pm\frac{1}{2}\langle\gamma,\gamma'\rangle$. The angular momentum is independent of the distance between charges while the signature depends on the order of the charges along the circle. This order dependence is correctly reproduced by the algebra of Eq. (31). This factor also provides the dependence on the flavor fugacities.

Now we suggest an IR formula for the orbifold as a natural generalization of the original one. $\mathbb{Z}_n$ acts on the large circle as a shift by $2\pi/n$ and the orbifolding makes it a circle with circumference

[3] In the literature, q' defined by $q'^{1/2} = -q^{1/2}$ is often used and then the minus sign does not appear.

$2\pi/n$. In other words, the large circle in the covering space consists of n fundamental regions, and only BPS particles in one of the fundamental regions are independent. Let L_n be the set of primitive charges associated with a specific fundamental region. The charge distribution of a BPS configuration in the orbifold is specified by $\{n_\gamma\}_{\gamma\in L_n}$. Therefore, the index should be given as the summation over $\{n_\gamma\}_{\gamma\in L_n}$.

The first factor in the summand should be replaced by the $U(1)$ factor λ_n^r, with λ_n given in Eq. (23). The second factor associated with the internal degrees of freedom of BPS particles should be the product of $C_\gamma(n_\gamma)$ for $\gamma \in L_n$. The third factor, as we mentioned above, can be regarded as the classical contribution to the angular momentum and the flavor charges. The angular momentum can be obtained by integrating the contribution of the Poynting vector over the orbifold. The same result is obtained by first calculating the angular momentum for the covering space $\boldsymbol{S}^3$, and dividing the result by n. This is also the case for the flavor charges. Correspondingly, the trace factor should be replaced by the nth root of the trace for the configuration in the covering space. Combining these, we obtain

$$I_n = \sum_{\{n_\gamma\}_{L_n}} \lambda_n^r \left(\prod_{\gamma\in L_n} C_\gamma(n_\gamma)\right)\left[\mathrm{tr}\left(\overset{\curvearrowleft}{\prod_{\gamma\in L}} X_\gamma^{n_\gamma}\right)\right]^{1/n}. \tag{34}$$

This is the IR formula we want to discuss in the next section.

An additional comment on the trace factor would be in order. As we mentioned above, only charge distribution in a specific fundamental region is independent, and the distribution in other regions should be determined by the $\mathbb{Z}_n$ symmetry. The phase-ordered trance in Eq. (34) must be taken over all charges in the covering space. It is important that the $\mathbb{Z}_n$ acts on charges non-trivially, and the product is not the simple nth power of the product in the fundamental region. In the $n = 2$ case, for example, the $\mathbb{Z}_2$ action flips the signs of charges, as we will see explicitly in the next section, and the trace factor takes the form $\left[\mathrm{tr}(X_{\gamma_1}^{n_1}X_{\gamma_2}^{n_2}\cdots X_{-\gamma_1}^{n_1}X_{-\gamma_2}^{n_2}\cdots)\right]^{1/2}$.

4. Comparison

Let us apply the IR formula in Eq. (34) to a few simple examples and compare the results with those obtained by the UV formula.

We first consider the system of a free hypermultiplet $\vec{q} = (q, \widetilde{q})$. Before computing the index by the UV and IR formulae it is important to check the consistency between BPS configurations in the Coulomb branch and the orbifold action.

In the system of a single hypermultiplet, we have only two sorts of particles with flavor charge $\pm\gamma$. In a BPS configuration there are particles with charge $+\gamma$ at $\theta_\gamma = \arg Z_\gamma$ and anti-particles with charge $-\gamma$ at $\theta_\gamma + \pi$. This cannot be invariant under $\mathbb{Z}_n$ action with $n \geq 3$. Even for $n = 2$, $\mathbb{Z}_2$ exchanges particles and anti-particles, and the orbifolding seems to contradict the BPS configuration in the Coulomb branch. When $n = 2$, actually, we can take it back to the original configuration by performing additional charge conjugation. This additional transformation is also necessary to keep the mass term in the Lagrangian $\mathbb{Z}_2$ invariant. To make the hypermultiplet massive we need to turn on the vacuum expectation value (VEV) $\langle\phi\rangle = m$ of the scalar component ϕ of the non-dynamical background vector multiplet coupling to the $U(1)_F$ current. Because ϕ carries a $U(1)_r$ charge of $+2$, the g_2 action changes its sign. This is compensated by the additional charge conjugation, which flips the sign of the background vector multiplet.

The charge conjugation exchanging q and $\widetilde{q}$ is realized by the $SU(2)_F$ transformation $\vec{q} \to U\vec{q}$ with

$$U = i\sigma_x. \tag{35}$$

This anti-commutes with the $U(1)_F$ generator $i\sigma_z$, and works as the charge conjugation. For consistency, we should set the $U(1)_F$ Wilson line to be $z = \pm 1$. (A generic Wilson line is not allowed because the operator z^F does not commute with $i\sigma_x$, and is incompatible with the orbifold action.)

Now we have chosen a $\mathbb{Z}_2$ generator $g_2 U$ consistent with the Coulomb branch VEV. Let us calculate the index by using the UV and IR formulae. When $z = \pm 1$, the orbifold with the charge conjugation twist is in fact equivalent to the $\mathbb{Z}_2$ orbifold with fractional holonomies studied in the previous section up to $SU(2)_F$ rotation, because the fractional holonomy $\omega_2^{hF} = \pm i\sigma_z$ is $SU(2)_F$ conjugate to the charge conjugation $U = i\sigma_x$. Therefore, the index is still given by Eq. (27) with $z = \pm 1$. By setting $z = 1$, the one-particle index becomes

$$i_2^{\pm\frac{1}{2}} = \frac{q^{\frac{1}{2}}}{1-q}, \tag{36}$$

and the Schur index is

$$I_2 = \mathrm{Pexp}\, \frac{q^{\frac{1}{2}}}{1-q} = E_q(1). \tag{37}$$

On the IR side, we have only one charge γ in L_2. The suggested formula in Eq. (34) gives

$$I_2 = \sum_{n=0}^{\infty} C_\gamma(n) = E_q(1), \tag{38}$$

and this agrees with Eq. (37).

By introducing multiple hypermultiplets we can realize more general orbifolds. Let us consider a system of k free hypermultiplets. There are $2k$ sorts of particles. For $\mathbb{Z}_{2k}$ invariance, we need to tune the Coulomb branch VEV ϕ and introduce an appropriate twist U_k. Let $\phi = \mathrm{diag}(a_1, \ldots, a_k) \otimes \sigma_z \in sp(k)$ be the Coulomb branch VEV (we use the basis in which the $Sp(k)$ invariant tensor is given by $J = \mathbf{1}_k \otimes \epsilon$). We tune a_i so that the central charges are $\mathbb{Z}_{2k}$ symmetric and given by

$$Z(q_i) = a_i = \omega_{2k}^i a, \quad Z(\widetilde{q}_i) = -a_i = \omega_{2k}^{k+i} a. \tag{39}$$

The action of g_{2k} shifts a_i to a_{i+1} for $i = 1 \sim k-1$ and a_k to $-a_1$. To keep the background unchanged, we need a twist U_k such that the transformation $\phi' = U_k \phi U_k^\dagger$ acts on a_i as

$$a_1' = -a_k, \quad a_i' = a_{i-1} \quad (i = 2 \sim k). \tag{40}$$

The following U_k realize this transformation:

$$U_k = \begin{pmatrix} & \mathbf{1}_2 & & \\ & & \ddots & \\ & & & \mathbf{1}_2 \\ i\sigma_x & & & \end{pmatrix} \in Sp(k). \tag{41}$$

(This is a generalization of the charge conjugation in Eq. (35) in the $k = 1$ case.) For the same reason as the $\mathbb{Z}_2$ case we set the Wilson line to vanish. The diagonalization of U_k gives the eigenvalues $\pm\alpha_m$ with

$$\alpha_m = \exp\left(\frac{2\pi i}{2k}\left(m + \frac{1}{2}\right)\right), \quad m = 0, \ldots, k-1. \tag{42}$$

For each pair $(\alpha_m, -\alpha_m)$ of eigenvalues this is the same as the action of the fractional holonomy $h = m + 1/2$ on a single hypermultiplet, and the one-particle index of the whole system is the sum of Eq. (26) over $h = 1/2, \ldots, k - 1/2$:

$$i_n = \frac{q^{\frac{1}{2}}}{1-q^{2k}}(1 + q + \cdots + q^{2k-1}) = \frac{q^{\frac{1}{2}}}{1-q}. \tag{43}$$

The Schur index is $I_n = \mathrm{Pexp}\, i_n = E_q(1)$, and this is the same as what we obtain from the IR formula. Each fundamental region of the large circle contains one sort of particle and the IR formula in Eq. (34) gives $I = E_q(1)$.

We can consider a more general case in which n is not $2k$ but its divisor, $n = 2k/d$. Such an orbifold is defined by using $(g_{2k}U_k)^d$ as the $\mathbb{Z}_n$ generator, and the one-particle index becomes

$$i_n = \frac{q^{\frac{1}{2}}}{1-q^k}\sum_{m=0}^{2k-1} q^{[m]_n} = d\frac{q^{\frac{1}{2}}}{1-q}; \tag{44}$$

the Schur index is $I_n = \mathrm{Pexp}\, i_n = E_q(1)^d$. Again, this is reproduced by the IR formula of Eq. (34) by taking account of the d sorts of particles in a fundamental region.

Up to here we have found good agreement between the UV and IR results. Let us move on to a more complicated system, QED with N_f hypermultiplets. Although this system is not conformal, it is known that the two formulae give the same answer for the ordinary Schur index [4], and it is natural to expect this to hold for orbifolds, too. However, disappointingly, we find a discrepancy for the orbifold index in this case.

For example, let us consider the $\mathbb{Z}_2$ orbifold of two-flavored QED. The UV formula gives the index in Eq. (28). When we use the IR formula we need to take account of the existence of two sorts of particles in the fundamental region. Let γ_1 and γ_2 be their charges and n_1 and n_2 be the corresponding occupation numbers. In the covering space the charges of particles in a fundamental region are always canceled by the mirror images in the other fundamental region, and the trace factor in Eq. (34) does not impose any constraints on n_1 and n_2. If we assume the trivial flavor holonomy, the trace factor simply gives 1. This means that the index is essentially the same as that for two half-hypermultiplets in $\boldsymbol{S}^3$, and given by

$$I_2 = \lambda_2 \sum_{n_1,n_2=0}^{\infty} C_\gamma(n_1)C_\gamma(n_2) = \lambda_2 E_q^2(1) = 1 + 2q^{\frac{1}{2}} + 3q + 6q^{\frac{3}{2}} + \cdots. \tag{45}$$

This is obviously different from the result of the UV formula in Eq. (28).

5. Discussions

In this paper we generalized the Schur index to the orbifold $\boldsymbol{S}^3/\mathbb{Z}_n$ in such a way that the $\mathbb{Z}_n$ action preserves the two supersymmetries respected by the definition of the Schur index. We also naively generalized Cordova and Shao's IR formula for the Schur index to the orbifold case. This reproduces

the correct index for a system of free hypermultiplets when the fugacities and holonomies are chosen in a $\mathbb{Z}_n$-invariant way. However, it is far from satisfactory, with the following deficiencies:

- Although any SCFT admits a $\mathbb{Z}_n$ orbifold with an arbitrary $n = 2, 3, \ldots$ in the UV description, the IR formula works only for special values of n depending on the theory.
- Even for a system of free hypermultiplets it is not possible to turn on generic Wilson lines.
- For systems with dynamical vector multiplets the formula does not reproduce the correct answer.

The second and third deficiencies may be related to each other. The incompatibility of the generic Wilson lines seems to prevent us from performing path integrals over all gauge field configurations. At present, we cannot claim anything definite for this point, and more investigation is desired. It is important to study more general systems to clarify to what extent our formula works and (if possible) how we should improve it to make it applicable to general systems. In particular, it would be interesting and important to study the orbifold Schur index of non-Lagrangian theories such as Argyres–Douglas theories. Lagrangian theories that flow to a class of Argyres–Douglas theories have been proposed in Refs. [25–31], and this enables us to apply the UV formula of the orbifold index to the non-Lagrangian theories.

It is also interesting to analyze the relation between the orbifold index and chiral algebra. In the unorbifolded case, the chiral algebra is realized in a complex plane, which is identified by Weyl rescaling with $\boldsymbol{S}^1 \times \mathbb{R}_t \subset \boldsymbol{S}^3 \times \mathbb{R}_t$, where $\boldsymbol{S}^1$ is the large circle in $\boldsymbol{S}^3$ on which BPS particles are aligned. The Virasoro generators L_0 and $\overline{L}_0$ on the plane are related to the superconformal generators by

$$\begin{aligned} L_0 &= \frac{1}{2}(H + J + \overline{J}) = \frac{1}{2}(\mu_x + \mu_p) + \mu_q, \\ \overline{L}_0 &= \frac{1}{2}(H - J - \overline{J} - 2R) = \frac{1}{2}(\mu_x + \mu_p). \end{aligned} \tag{46}$$

For Schur operators with $\mu_x = \mu_p = 0$, $\overline{L}_0 = 0$ and the $\mathbb{Z}_n$ generator $g_n = \omega_n^{\mu_x - \mu_p + \mu_q}$ is given by

$$g_n = \omega_n^{L_0}. \tag{47}$$

Namely, this is the $\mathbb{Z}_n$ orbifold acting on the worldsheet. Therefore, in the context of the chiral algebra, the orbifolding should be regarded as the insertion of a twist operator at the origin. Such a relation is studied for an orbifold version of the Macdonald index in Ref. [24]. It would be interesting to do a similar analysis for the orbifold Schur index.

Acknowledgements

I would like to thank Hirotaka Kato for valuable discussions. This work was partially supported by Grant-in-Aid for Scientific Research (C) (No. 15K05044), Ministry of Education, Science and Culture, Japan.

Funding

Open Access funding: SCOAP[3].

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