

Current matrix element in HAL QCD's wavefunction-equivalent potential method

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We give a formula to calculate a matrix element of a conserved current in the effective quantum mechanics defined by the wavefunction-equivalent potentials proposed by the HAL QCD collaboration. As a first step, a non-relativistic field theory with two-channel coupling is considered as the original theory, with which a wavefunction-equivalent HAL QCD potential is obtained in a closed analytic form. The external field method is used to derive the formula by demanding that the result should agree with the original theory. With this formula, the matrix element is obtained by sandwiching the effective current operator between the left and right eigenfunctions of the effective Hamiltonian associated with the HAL QCD potential. In addition to the naive one-body current, the effective current operator contains an additional two-body term emerging from the degrees of freedom which has been integrated out.
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1. Introduction

The nuclear force plays a key role in understanding various properties of atomic nuclei. It is important not only for nuclear physics but also for astrophysics, for example in the explosion of a supernova and the structure of neutron stars. Enormous efforts have been devoted to investigations of the nuclear force, including phenomenological studies such as meson exchange interactions [1], the chiral effective field theory [2], and the recently developed QCD-based studies [3]. Since the 1990s, high-precision phase-equivalent NN potentials have been available [4–6]. These high-precision potentials are so determined as to reproduce a wide range of nucleon–nucleon scattering data together with the deuteron properties.

A lattice QCD (LQCD) method to determine the nuclear force has recently been developed by the HAL QCD collaboration [3,7–9], which we will refer to as the HAL QCD method. It has been applied to many systems [10,11]. With this method, LQCD is used to generate equal-time Nambu–Bethe–Salpeter (NBS) wavefunctions for NN system in the center-of-mass frame. By regarding these NBS wavefunctions as NN wavefunctions, the NN potentials are defined by demanding that these NBS wavefunctions should be reproduced by the Schrödinger equation below the inelastic threshold. We will refer to the potential thus defined as the HAL QCD potential or wavefunction-equivalent potential. Note that, by using LSZ reduction formula, it is shown that these NBS wavefunctions have asymptotic long-distance behavior that is parameterized by the scattering phase shift δ in exactly the

same way as that of non-relativistic quantum mechanics [7,12,13]. For instance, for the s-wave,

$$\langle 0 | N(\mathbf{x})N(\mathbf{y}) | N(\mathbf{p})N(-\mathbf{p}), in \rangle \sim Z e^{i\delta(\mathbf{p})} \frac{\sin(|\mathbf{p}||\mathbf{x} - \mathbf{y}| + \delta(\mathbf{p}))}{|\mathbf{p}||\mathbf{x} - \mathbf{y}|}. \quad (1)$$

This implies that HAL QCD potentials reproduce the scattering phase shifts together with the NBS wavefunctions. It is therefore a phase-equivalent potential as well as a wavefunction-equivalent potential.

These phase-equivalent potentials can be used to obtain an effective NN quantum mechanics. However, although the scattering phase shifts are guaranteed to be reproduced by these effective NN quantum mechanics, it is not straightforward to calculate the matrix elements. Note that phase-equivalent potentials generate wavefunctions whose long-distance behaviors are constrained by the scattering phase shift. However, there are no constraints imposed on their short- and medium-distance behaviors. As a result, with the naive formula of matrix elements, the result depends on the choice of phase-equivalent potentials, which suggests the existence of an additional contribution to absorb the difference. In fact, it is known that the electromagnetic current of the two-nucleon system has such an additional two-body contribution, i.e., $J_\mu(\mathbf{x}) = J_\mu^{(1)}(\mathbf{x}) + J_\mu^{(2)}(\mathbf{x})$, where $J_\mu^{(1)}(\mathbf{x})$ denotes the naive one-body nucleon current while $J_\mu^{(2)}(\mathbf{x})$ denotes the additional two-body current, which is referred to as the exchange current. The exchange current is known to emerge from the charged mesons exchanged between the two nucleons. Because these charged mesons are “frozen” in the instantaneous potentials, their contribution must appear as an additional two-body operator in the effective NN quantum mechanics. It is known that the exchange current gives a dominant contribution in the $d\gamma \rightarrow np$ reaction [14].

The current conservation imposes a constraint on the exchange current $J_\mu^{(2)}(\mathbf{x})$ as

$$\nabla \cdot \mathbf{J}^{(2)}(\mathbf{x}) = -i \left[V, \sum_{i=1,2} e_i \delta^3(\mathbf{x} - \mathbf{r}_i) \right], \quad (2)$$

where $J_0(\mathbf{x}) \simeq \sum_{i=1,2} e_i \delta^3(\mathbf{x} - \mathbf{r}_i)$ is assumed with $\mathbf{r}_i$ and $e_i \equiv \frac{e}{2}(1 + \tau^3)_i$ being the position and the charge operator acting on the isospin space of the i th nucleon, respectively [15,16]. In particular, this constraint implies that (1) $\mathbf{J}^{(2)}(\mathbf{x})$ does not vanish, if the potential V is either isospin dependent or is non-local; and (2) an explicit form of $\mathbf{J}^{(2)}(\mathbf{x})$ depends on the particular choice of the potential V . Although it is rather a strong constraint, it is not strong enough to determine the complete form of $J_\mu^{(2)}(\mathbf{x})$. In fact, for the one pion exchange potential (OPEP) $V = V_{\text{OPEP}}$, the constraint of Eq. (2) is satisfied by two different currents $\mathbf{J}^{(2)}(\mathbf{x})$: (1) the Sachs current [17], and (2) the one pion exchange current (OPEC) [18]. OPEC is considered to be reasonable, because it is obtained by going back to the original theory (a relativistic pion–nucleon coupling model) with the demand that the Bremsstrahlung amplitude of the original theory should be reproduced in the effective quantum mechanics.

The same strategy does not work for the phenomenologically constructed phase-equivalent potentials, because their connection to the original theory (QCD) is unclear. (For those potentials whose relation to QCD is unknown, a prescription to introduce a current operator that is conserved has been proposed [15,16].) In contrast, since HAL QCD potentials are constructed in LQCD, it may be possible to derive an explicit form of the exchange current operator for the HAL QCD method. Note that once such a formula is established for the HAL QCD method, it enables us to consider QCD matrix elements by the nuclear physics with the conventional nucleon degrees of freedom. There are many applications. In addition to the standard calculation of form factors, it can be applied to

the nuclear electric dipole moments for physics beyond the standard model [19], $np \rightarrow d\gamma$ in a big bang nucleosynthesis [20], etc.

In this paper, we consider a method to calculate a matrix element in an effective quantum mechanics associated with HAL QCD potentials. As a first step, we restrict ourselves to the matrix element of a conserved current. Instead of Lorentz-covariant QCD, we employ a non-relativistic Galilei-covariant (field-theoretical) coupled-channel model [21,22] as an original theory to present a formula to calculate the matrix element of a conserved current. This non-relativistic model enables us to obtain a HAL QCD potential in a closed analytic form, which is used to define a non-relativistic effective quantum mechanics. We use a non-relativistic original theory because, at this initial stage, we prefer to have a formula that involves as little approximation as possible. To obtain the formula to calculate a matrix element we use the external field method, which was conveniently used in the Bethe–Salpeter framework [23–25]. The external field method enables us to obtain the formula in the effective quantum mechanics that agrees with the calculation in the original theory.

The paper is organized as follows. In Sect. 2, a second quantized non-relativistic coupled-channel model is introduced, which mimics an np – np^* coupling system. In Sect. 3, the non-relativistic model is used as the original theory to obtain an effective quantum mechanics of an np system below the np^* threshold by integrating out the closed np^* channel. This is done by using the HAL QCD method. We will give a wavefunction-equivalent HAL QCD potential in a closed analytic form that reproduces the equal-time Nambu–Bethe–Salpeter (NBS) wavefunctions of the open np channel. In Sect. 4, we introduce an external gauge field and extend the HAL QCD potential in the external field. In Sect. 5.1, the external field method is used to derive a formula to calculate the current matrix element in the effective quantum mechanics so that the result agrees with the original theory.

Readers may wonder why we employ the np – np^* coupling model to consider the “exchange current” instead of those models where a potential is obtained by integrating out the exchanged mesons between two nucleons. This is because we are determined to stick to a non-relativistic original theory to obtain an analytic expression of the formula of the matrix element. Note that, even with np – np^* coupling model, the two-body current will appear due to the np^* degrees of freedom which is integrated out to obtain the effective np potential below the np^* threshold.

2. The original theory

2.1. Hamiltonian

We consider a second quantized non-relativistic Hamiltonian,

$$\hat{H} \equiv \hat{T} + \hat{V}, \quad (3)$$

where $\hat{T} \equiv \hat{T}_1 + \hat{T}_2 + \hat{T}_3$ denotes the kinetic term with

$$\begin{aligned} \hat{T}_0 &\equiv \int d^3x \hat{\phi}_0^\dagger(\mathbf{x}) \left(-\frac{\partial^2}{2m} \right) \hat{\phi}_0(\mathbf{x}), \\ \hat{T}_1 &\equiv \int d^3x \hat{\phi}_1^\dagger(\mathbf{x}) \left(-\frac{\partial^2}{2m} \right) \hat{\phi}_1(\mathbf{x}), \\ \hat{T}_2 &\equiv \int d^3x \hat{\phi}_2^\dagger(\mathbf{x}) \left(-\frac{\partial^2}{2m} + \Delta \right) \hat{\phi}_2(\mathbf{x}). \end{aligned} \quad (4)$$

$\hat{V} \equiv \sum_{\alpha,\beta=1,2} \hat{V}_{\alpha\beta}$ denotes the interaction term with

$$\hat{V}_{\alpha\beta} \equiv \int d^3x d^3y \hat{\phi}_0^\dagger(\mathbf{x}) \hat{\phi}_\alpha^\dagger(\mathbf{y}) V_{\alpha\beta}(\mathbf{x} - \mathbf{y}) \hat{\phi}_\beta(\mathbf{y}) \hat{\phi}_0(\mathbf{x}), \quad (5)$$

which is used to mimic the np–np* coupling system [21,22], where $\hat{\phi}_0(\mathbf{x})$, $\hat{\phi}_1(\mathbf{x})$, and $\hat{\phi}_2(\mathbf{x})$ correspond to the neutron (n), the proton (p), and an excited proton (p*) with excitation energy Δ , respectively. (We use a bold font for three-dimensional vectors. A variable with a hat “*” is used to indicate that it is a field operator acting on the Fock space.) We employ the same non-relativistic mass m for all these three fields for Galilei covariance. Since we do not stick to the detail of the np–np* coupling system, we consider the scalar boson fields for simplicity, which satisfy the equal-time commutation relation

$$[\hat{\phi}_\alpha(\mathbf{x}), \hat{\phi}_\beta^\dagger(\mathbf{y})] = \delta_{\alpha\beta} \delta^3(\mathbf{x} - \mathbf{y}). \quad (6)$$

All the other combinations vanish.

We consider eigenvalues and eigenvectors of $\hat{H}$ as

$$\hat{H}|n, \mathbf{P}\rangle = E_n(\mathbf{P}^2)|n, \mathbf{P}\rangle, \quad (7)$$

where $|n, \mathbf{P}\rangle$ denotes an energy eigenstate with the normalization $\langle m, \mathbf{Q}|n, \mathbf{P}\rangle = \delta_{mn} \cdot (2\pi)^3 \delta^3(\mathbf{Q} - \mathbf{P})$. $\mathbf{P}$ denotes the total spatial momentum, while n labels an “intrinsic excitation” in the center-of-mass frame. Due to the Galilei covariance, the energy eigenvalue $E_n(\mathbf{P}^2)$ decomposes as

$$E_n(\mathbf{P}^2) = \tilde{E}_n + \frac{1}{4m} \mathbf{P}^2, \quad (8)$$

where $\tilde{E}_n$ denotes the energy of the “intrinsic excitation” in the center-of-mass frame. We will refer to $\tilde{E}_n$ as the “reduced” energy. (We use a variable with a tilde “*” for “reduced” objects that have something to do with the center-of-mass frame, such as the reduced NBS wavefunctions $\tilde{\psi}(\mathbf{r})$ that will be introduced later.)

2.2. Conserved currents and conserved charges

The Hamiltonian $\hat{H}$ has two $U(1)$ symmetries, $U_n(1)$ and $U_p(1)$. They are generated by the conserved charges

$$\hat{Q}_n \equiv \int d^3x \hat{\phi}_0^\dagger(\mathbf{x}) \hat{\phi}_0(\mathbf{x}), \quad (9)$$

$$\hat{Q}_p \equiv \int d^3x \left(\hat{\phi}_1^\dagger(\mathbf{x}) \hat{\phi}_1(\mathbf{x}) + \hat{\phi}_2^\dagger(\mathbf{x}) \hat{\phi}_2(\mathbf{x}) \right),$$

respectively. $\hat{Q}_n$ corresponds to the conservation of n-number, whereas $\hat{Q}_p$ corresponds to the conservation of p-number, which also counts p*. These charges are associated with the non-relativistic conserved currents

$$\hat{j}_n^\mu(\mathbf{x}) \equiv \hat{j}_0^\mu(\mathbf{x}), \quad (10)$$

$$\hat{j}_p^\mu(\mathbf{x}) \equiv \hat{j}_1^\mu(\mathbf{x}) + \hat{j}_2^\mu(\mathbf{x}), \quad (11)$$

respectively, with

$$\begin{aligned} \hat{j}_\alpha^0(x) &\equiv \hat{\phi}_\alpha^\dagger(x) \hat{\phi}_\alpha(x), \\ \hat{j}_\alpha^i(x) &\equiv \frac{1}{2mi} \left\{ \hat{\phi}_\alpha^\dagger(x) \left(\partial^i \hat{\phi}_\alpha(x) \right) - \left(\partial^i \hat{\phi}_\alpha^\dagger(x) \right) \hat{\phi}_\alpha(x) \right\}, \end{aligned} \quad (12)$$

where $\hat{\phi}_\alpha(x) \equiv e^{i\hat{H}x_0} \hat{\phi}_\alpha(\mathbf{x}) e^{-i\hat{H}x_0}$ denotes the Heisenberg operators for $\alpha = 0, 1, 2$. By using Heisenberg's equation $i\partial_0 \hat{\phi}_\alpha(x) = [\hat{\phi}_\alpha(x), \hat{H}]$, it is straightforward to see that these two currents $\hat{j}_n^\mu(x)$ and $\hat{j}_p^\mu(x)$ conserve.

2.3. The two-particle subspaces

We restrict ourselves to the two-particle subspace with $(Q_n, Q_p) = (1, 1)$, which we will refer to as $\mathcal{F}$. The subspace $\mathcal{F}$ is spanned by all the state vectors of the form

$$|\psi\rangle \equiv \int d^3x d^3y \left(\phi_0^\dagger(\mathbf{x}) \phi_1^\dagger(\mathbf{y}) |0\rangle \psi_1(\mathbf{x}, \mathbf{y}) + \phi_0^\dagger(\mathbf{x}) \phi_2^\dagger(\mathbf{y}) |0\rangle \psi_2(\mathbf{x}, \mathbf{y}) \right), \quad (13)$$

where $|0\rangle$ denotes the non-relativistic vacuum defined by the relation

$$\hat{\phi}_\alpha(\mathbf{x})|0\rangle = 0, \quad (14)$$

for $\alpha = 0, 1, 2$ and all $\mathbf{x} \in \mathbb{R}^3$ with $\langle 0|0\rangle = 1$.

We introduce a cutoff by using a projection operator

$$\hat{\mathbb{P}}_\Lambda \equiv |0\rangle\langle 0| + \sum_n^{\tilde{E}_n < \Lambda} \int \frac{d^3P}{(2\pi)^3} |n, \mathbf{P}\rangle \langle n, \mathbf{P}|. \quad (15)$$

Note that this cutoff is Galilei covariant. We use it to define a truncated subspace $\mathcal{F}_\Delta$ by

$$\mathcal{F}_\Delta \equiv \hat{\mathbb{P}}_\Delta \cdot \mathcal{F}. \quad (16)$$

The truncated subspace $\mathcal{F}_\Delta$ consists of all states in $\mathcal{F}$ which exist below the np^* threshold. From Sect. 3, we will use the HAL QCD method to construct an effective quantum mechanics for states in $\mathcal{F}_\Delta$ by integrating out all the states $|n, \mathbf{P}\rangle$ with $\tilde{E}_n > \Delta$.

2.4. The NBS wavefunction

We define the (equal-time) NBS wavefunction as

$$\psi_\alpha(\mathbf{x}, \mathbf{y}; t|n, \mathbf{P}) \equiv \left\langle 0 \left| \hat{\phi}_0(\mathbf{x}, t) \hat{\phi}_\alpha(\mathbf{y}, t) \right| n, \mathbf{P} \right\rangle, \quad (17)$$

for $\alpha = 1, 2$ and $|n, \mathbf{P}\rangle \in \mathcal{F}$. (If $t = 0$, we simply omit to write t , i.e., $\psi_\alpha(\mathbf{x}, \mathbf{y}|n, \mathbf{P}) \equiv \langle 0 | \hat{\phi}_0(\mathbf{x}) \hat{\phi}_\alpha(\mathbf{y}) | n, \mathbf{P} \rangle$.) NBS wavefunctions satisfy the coupled-channel Schrödinger equation

$$\begin{aligned} &\left(E_n(\mathbf{P}^2) + \frac{\partial_x^2}{2m} + \frac{\partial_y^2}{2m} \right) \psi_1(\mathbf{x}, \mathbf{y}|n, \mathbf{P}) \\ &= V_{11}(\mathbf{x} - \mathbf{y}) \psi_1(\mathbf{x}, \mathbf{y}|n, \mathbf{P}) + V_{12}(\mathbf{x} - \mathbf{y}) \psi_2(\mathbf{x}, \mathbf{y}|n, \mathbf{P}), \end{aligned}$$

$$\begin{aligned} & \left(E_n(\mathbf{P}^2) + \frac{\partial_x^2}{2m} + \frac{\partial_y^2}{2m} - \Delta \right) \psi_2(\mathbf{x}, \mathbf{y} | n, \mathbf{P}) \\ &= V_{21}(\mathbf{x} - \mathbf{y}) \psi_1(\mathbf{x}, \mathbf{y} | n, \mathbf{P}) + V_{22}(\mathbf{x} - \mathbf{y}) \psi_2(\mathbf{x}, \mathbf{y} | n, \mathbf{P}), \end{aligned} \quad (18)$$

which can be verified by sandwiching $[\hat{\phi}_0(\mathbf{x}, t) \hat{\phi}_\alpha(\mathbf{y}, t), \hat{H}]$ between $\langle 0 |$ and $|n, \mathbf{P}\rangle$.

Due to the Galilei covariance, NBS wavefunctions factorize as

$$\psi_\alpha(\mathbf{x}, \mathbf{y}, t | n, \mathbf{P}) = \tilde{\psi}_\alpha(\mathbf{x} - \mathbf{y}, t | n) \exp\left(i\mathbf{P} \cdot \frac{\mathbf{x} + \mathbf{y}}{2}\right) \exp\left(-i\frac{1}{4m}\mathbf{P}^2 t\right), \quad (19)$$

where $\tilde{\psi}_\alpha(\mathbf{r}, t | n)$ denotes the NBS wavefunction in the center-of-mass frame,

$$\tilde{\psi}_\alpha(\mathbf{r}, t | n) \equiv \langle 0 | \hat{\phi}_0(\mathbf{r}/2, t) \hat{\phi}_\alpha(-\mathbf{r}/2, t) | n, \mathbf{P} = \mathbf{0} \rangle, \quad (20)$$

which will be referred to as the reduced NBS wavefunction. (Again, if $t = 0$, we simply omit to write t , i.e., $\tilde{\psi}_\alpha(\mathbf{r} | n) \equiv \langle 0 | \hat{\phi}_0(\mathbf{r}/2) \hat{\phi}_\alpha(-\mathbf{r}/2) | n, \mathbf{P} = \mathbf{0} \rangle$.) Reduced NBS wavefunctions satisfy the coupled-channel Schrödinger equation

$$\begin{aligned} & \left(\tilde{E}_n + \frac{\partial^2}{2\tilde{m}} \right) \tilde{\psi}_1(\mathbf{r} | n) = V_{11}(\mathbf{r}) \tilde{\psi}_1(\mathbf{r} | n) + V_{12}(\mathbf{r}) \tilde{\psi}_2(\mathbf{r} | n), \\ & \left(\tilde{E}_n + \frac{\partial^2}{2\tilde{m}} - \Delta \right) \tilde{\psi}_2(\mathbf{r} | n) = V_{21}(\mathbf{r}) \tilde{\psi}_1(\mathbf{r} | n) + V_{22}(\mathbf{r}) \tilde{\psi}_2(\mathbf{r} | n), \end{aligned} \quad (21)$$

where $\tilde{m} \equiv m/2$ denotes the reduced mass.

3. The HAL QCD potential and the effective quantum mechanics

3.1. The HAL QCD potentials

We use the HAL QCD method to obtain an effective np potential (wavefunction-equivalent HAL QCD potential). This is carried out in two steps. We first construct the (reduced) HAL QCD potential in the center-of-mass frame. To do this, we require the Schrödinger equation to reproduce the reduced NBS wavefunctions of the np channel below the np* threshold. We then use Galilei covariance to generalize the reduced potential for a general Galilei frame, with which (full) NBS wavefunctions satisfy the Schrödinger equation.

To obtain the reduced HAL QCD potential $\tilde{\mathcal{V}}(\mathbf{r}, \mathbf{r}')$, we demand that, for any states $|n, \mathbf{P} = 0\rangle \in \mathcal{F}_\Delta$, the reduced NBS wavefunction of the np channel, $\tilde{\psi}_1(\mathbf{r} | n) \equiv \langle 0 | \hat{\phi}_0(\mathbf{r}/2) \phi_1(-\mathbf{r}/2) | n, \mathbf{P} = \mathbf{0} \rangle$, satisfies the Schrödinger equation

$$\left(\tilde{E}_n + \frac{1}{2\tilde{m}} \partial^2 \right) \tilde{\psi}_1(\mathbf{r} | n) = \int d^3 r' \tilde{\mathcal{V}}(\mathbf{r}, \mathbf{r}') \tilde{\psi}_1(\mathbf{r}' | n), \quad (22)$$

where $\tilde{E}_n$ denotes the energy eigenvalue of $|n, \mathbf{P} = 0\rangle$. This demand is satisfied by the following energy-independent non-local potential $\tilde{\mathcal{V}}(\mathbf{r}, \mathbf{r}')$:

$$\tilde{\mathcal{V}}(\mathbf{r}, \mathbf{r}') \equiv \sum_m^{\tilde{E}_m < \Delta} (V_{11}(\mathbf{r}) \tilde{\psi}_1(\mathbf{r} | m) + V_{12}(\mathbf{r}) \tilde{\psi}_2(\mathbf{r} | m)) \tilde{\psi}_1^\vee(\mathbf{r}' | m), \quad (23)$$

where $\tilde{\psi}_2(\mathbf{r}|m) \equiv \langle 0|\hat{\phi}_0(\mathbf{r}/2)\hat{\phi}_2(-\mathbf{r}/2)|m, \mathbf{P} = 0\rangle$. $\tilde{\psi}_1^\vee(\mathbf{r}|m)$ denotes a dual basis associated with a linearly independent set of reduced NBS wavefunctions $\{\tilde{\psi}_1(\mathbf{r}|m) | m, \mathbf{P} = \mathbf{0}\} \in \mathcal{F}_\Delta$. The dual basis satisfies the orthogonality relation

$$\int d^3r \tilde{\psi}_1^\vee(\mathbf{r}|m) \tilde{\psi}_1(\mathbf{r}|n) = \delta_{mn}, \quad (24)$$

for m, n with $\tilde{E}_m, \tilde{E}_n < \Delta$. An explicit form of dual basis is given, for instance, by

$$\tilde{\psi}_1^\vee(\mathbf{r}|n) \equiv \sum_{m, \tilde{E}_m < \Delta} (\tilde{\mathcal{N}}^{-1})_{nm} \tilde{\psi}_1^*(\mathbf{r}|m), \quad (25)$$

where $\tilde{\mathcal{N}}_{nm} \equiv \int d^3r \tilde{\psi}_1^*(\mathbf{r}|n) \tilde{\psi}_1(\mathbf{r}|m)$ denotes the norm kernel.

It is straightforward to see that Eq. (22), i.e., the Schrödinger equation with the reduced HAL QCD potential $\tilde{\mathcal{V}}(\mathbf{r}, \mathbf{r}')$, is satisfied by the reduced NBS wavefunctions. For this purpose, we insert Eq. (23) into Eq. (22) and perform the integration over $\mathbf{r}'$:

$$\begin{aligned} \text{r.h.s. of Eq. (22)} &= \int d^3r' \sum_{m, \tilde{E}_m < \Delta} (V_{11}(\mathbf{r}) \tilde{\psi}_1(\mathbf{r}|m) + V_{12}(\mathbf{r}) \tilde{\psi}_2(\mathbf{r}|m)) \tilde{\psi}_1^\vee(\mathbf{r}'|m) \tilde{\psi}_1(\mathbf{r}'|n) \\ &= V_{11}(\mathbf{r}) \tilde{\psi}_1(\mathbf{r}|n) + V_{12}(\mathbf{r}) \tilde{\psi}_2(\mathbf{r}|n), \end{aligned} \quad (26)$$

where the orthogonality relation Eq. (24) is used. Thus the Schrödinger equation in Eq. (22) reduces to the coupled-channel equation of Eq. (21).

We use Galilei covariance to generalize the reduced HAL QCD potential $\tilde{\mathcal{V}}(\mathbf{r}, \mathbf{r}')$ to the (full) HAL QCD potential $\mathcal{V}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}')$ by

$$\mathcal{V}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}') \equiv \tilde{\mathcal{V}}(\mathbf{x} - \mathbf{y}; \mathbf{x}' - \mathbf{y}') \delta^3 \left(\frac{\mathbf{x} + \mathbf{y}}{2} - \frac{\mathbf{x}' + \mathbf{y}'}{2} \right). \quad (27)$$

Note that, for any states $|n, \mathbf{P}\rangle \in \mathcal{F}_\Delta$, (full) NBS wavefunctions of the np channel $\psi_1(\mathbf{x}, \mathbf{y}|n, \mathbf{P}) \equiv \langle 0|\hat{\phi}_0(\mathbf{x})\hat{\phi}_1(\mathbf{y})|n, \mathbf{P}\rangle$ satisfy the Schrödinger equation with this potential, as

$$\left(E_n(\mathbf{P}^2) + \frac{1}{2m} \partial_x^2 + \frac{1}{2m} \partial_y^2 \right) \psi_1(\mathbf{x}, \mathbf{y}|n, \mathbf{P}) = \int d^3x' d^3y' \mathcal{V}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}') \psi_1(\mathbf{x}', \mathbf{y}'|n, \mathbf{P}). \quad (28)$$

To see this, we insert Eq. (27) into the right-hand side and use the factorization formula in Eq. (19). Then the Schrödinger equation reduces to Eq. (18) as

$$\begin{aligned} \text{r.h.s. of Eq. (28)} &= \int d^3r' \tilde{\mathcal{V}}(\mathbf{x} - \mathbf{y}, \mathbf{r}') \tilde{\psi}_1(\mathbf{r}'|n) \exp \left(i\mathbf{P} \cdot \frac{1}{2}(\mathbf{x} + \mathbf{y}) \right) \\ &= V_{11}(\mathbf{x} - \mathbf{y}) \psi_1(\mathbf{x}, \mathbf{y}|n, \mathbf{P}) + V_{12}(\mathbf{x} - \mathbf{y}) \psi_2(\mathbf{x}, \mathbf{y}|n, \mathbf{P}), \end{aligned} \quad (29)$$

where the last line is obtained by using Eq. (26).

3.2. The effective quantum mechanics

The HAL QCD potentials Eq. (23) and Eq. (27) are used to define an effective quantum mechanics. For later convenience, we give a summary of the eigenvalue property of the effective quantum mechanics.

We begin with the reduced system in the center-of-mass frame. The eigenvalue relations are given as

$$\begin{aligned}\tilde{\mathcal{H}} \tilde{\chi}_n^{\text{R}}(\mathbf{r}) &= \tilde{\mathcal{E}}_n \tilde{\chi}_n^{\text{R}}(\mathbf{r}), \\ \tilde{\chi}_n^{\text{L}}(\mathbf{r}) \tilde{\mathcal{H}} &= \tilde{\mathcal{E}}_n \tilde{\chi}_n^{\text{L}}(\mathbf{r}),\end{aligned}\quad (30)$$

where $\tilde{\mathcal{H}}$ denotes the effective Hamiltonian, whose actions on the left and right wavefunctions are defined as

$$\begin{aligned}\tilde{\mathcal{H}} \tilde{\chi}_n^{\text{R}}(\mathbf{r}) &\equiv -\frac{\partial^2}{2\tilde{m}} \tilde{\chi}_n^{\text{R}}(\mathbf{r}) + \int d^3 r' \tilde{\mathcal{V}}(\mathbf{r}, \mathbf{r}') \tilde{\chi}_n^{\text{R}}(\mathbf{r}'), \\ \tilde{\chi}_n^{\text{L}}(\mathbf{r}) \tilde{\mathcal{H}} &\equiv -\frac{\partial^2}{2\tilde{m}} \tilde{\chi}_n^{\text{L}}(\mathbf{r}) + \int d^3 r' \tilde{\chi}_n^{\text{L}}(\mathbf{r}') \tilde{\mathcal{V}}(\mathbf{r}', \mathbf{r}).\end{aligned}\quad (31)$$

$\tilde{\chi}_n^{\text{L}}(\mathbf{r})$ and $\tilde{\chi}_n^{\text{R}}(\mathbf{r})$ denote the left and right eigenfunctions associated with the energy eigenvalue $\tilde{\mathcal{E}}_n$. They satisfy the orthogonality relations

$$\begin{aligned}\sum_{n=0}^{\infty} \tilde{\chi}_n^{\text{R}}(\mathbf{r}) \tilde{\chi}_n^{\text{L}}(\mathbf{r}') &= \delta^3(\mathbf{r} - \mathbf{r}'), \\ \int d^3 r \tilde{\chi}_n^{\text{L}}(\mathbf{r}) \tilde{\chi}_m^{\text{R}}(\mathbf{r}) &= \delta_{nm}.\end{aligned}\quad (32)$$

Note that, in the upper relation, the summation is not restricted to $\tilde{E}_n < \Delta$. Since $\tilde{\mathcal{V}}(\mathbf{r}, \mathbf{r}')$ is not Hermitian in general, $\tilde{\chi}_n^{\text{L}}(\mathbf{r})$ is not a complex conjugate of $\tilde{\chi}_n^{\text{R}}(\mathbf{r})$.

We note that, since $\tilde{\mathcal{V}}$ is defined to reproduce the NBS wavefunctions in the elastic region, we have

$$\begin{aligned}\tilde{\chi}_n^{\text{R}}(\mathbf{x} - \mathbf{y}) &= \left\langle 0 \left| \hat{\phi}_0(\mathbf{x}) \hat{\phi}_1(\mathbf{y}) \right| n, \mathbf{P} = \mathbf{0} \right\rangle, \\ \tilde{\mathcal{E}}_n &= \tilde{E}_n\end{aligned}\quad (33)$$

for the states $|n, \mathbf{P} = \mathbf{0}\rangle \in \mathcal{F}_\Delta$. Due to the upper relation, the scattering phase shift of the original theory is reproduced by the effective quantum mechanics through the right eigenfunction $\tilde{\chi}_n^{\text{R}}(\mathbf{r})$ for the energy region $\tilde{E}_n < \Delta$.

We then use Galilei covariance to generalize these relations to arbitrary Galilei frames. By defining

$$\begin{aligned}\chi_{n,\mathbf{P}}^{\text{R}}(\mathbf{x}, \mathbf{y}) &\equiv \tilde{\chi}_n^{\text{R}}(\mathbf{x} - \mathbf{y}) \cdot \exp\left(i\mathbf{P} \cdot \frac{1}{2}(\mathbf{x} + \mathbf{y})\right), \\ \chi_{n,\mathbf{P}}^{\text{L}}(\mathbf{x}, \mathbf{y}) &\equiv \tilde{\chi}_n^{\text{L}}(\mathbf{x} - \mathbf{y}) \cdot \exp\left(-i\mathbf{P} \cdot \frac{1}{2}(\mathbf{x} + \mathbf{y})\right),\end{aligned}\quad (34)$$

the eigenvalue relations are given as

$$\begin{aligned}\mathcal{H} \chi_{n,\mathbf{P}}^{\text{R}}(\mathbf{x}, \mathbf{y}) &= \mathcal{E}_n(\mathbf{P}^2) \chi_{n,\mathbf{P}}^{\text{R}}(\mathbf{x}, \mathbf{y}), \\ \chi_{n,\mathbf{P}}^{\text{L}}(\mathbf{x}, \mathbf{y}) \mathcal{H} &= \mathcal{E}_n(\mathbf{P}^2) \chi_{n,\mathbf{P}}^{\text{L}}(\mathbf{x}, \mathbf{y}),\end{aligned}\quad (35)$$

where $\mathcal{H}$ denotes the effective Hamiltonian, whose actions on the left and right wavefunctions are defined as

$$\begin{aligned}\mathcal{H} \chi_{n,\mathbf{P}}^{\text{R}}(\mathbf{x}, \mathbf{y}) &\equiv \left(-\frac{\partial_{\mathbf{x}}^2}{2m} - \frac{\partial_{\mathbf{y}}^2}{2m} \right) \chi_{n,\mathbf{P}}^{\text{R}}(\mathbf{x}, \mathbf{y}) + \int d^3x' d^3y' \mathcal{V}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}') \chi_{n,\mathbf{P}}^{\text{R}}(\mathbf{x}', \mathbf{y}'), \\ \chi_{n,\mathbf{P}}^{\text{L}}(\mathbf{x}, \mathbf{y}) \mathcal{H} &\equiv \left(-\frac{\partial_{\mathbf{x}}^2}{2m} - \frac{\partial_{\mathbf{y}}^2}{2m} \right) \chi_{n,\mathbf{P}}^{\text{L}}(\mathbf{x}, \mathbf{y}) + \int d^3x' d^3y' \chi_{n,\mathbf{P}}^{\text{L}}(\mathbf{x}', \mathbf{y}') \mathcal{V}(\mathbf{x}', \mathbf{y}'; \mathbf{x}, \mathbf{y}).\end{aligned}\quad (36)$$

$\chi_n^{\text{L}}(\mathbf{x}, \mathbf{y})$ and $\chi_n^{\text{R}}(\mathbf{x}, \mathbf{y})$ are the left and right eigenfunctions, respectively, associated with the energy eigenvalue $\mathcal{E}_n(\mathbf{P}^2) \equiv \tilde{\mathcal{E}}_n + \frac{1}{4m}\mathbf{P}^2$. They satisfy the orthogonality relations

$$\sum_{n=0}^{\infty} \int \frac{d^3P}{(2\pi)^3} \chi_{n,\mathbf{P}}^{\text{R}}(\mathbf{x}', \mathbf{y}') \chi_{n,\mathbf{P}}^{\text{L}}(\mathbf{x}, \mathbf{y}) = \delta^3(\mathbf{x}' - \mathbf{x}) \delta^3(\mathbf{y}' - \mathbf{y}) \quad (37)$$

and

$$\int d^3x d^3y \chi_{n',\mathbf{P}'}^{\text{L}}(\mathbf{x}, \mathbf{y}) \chi_{n,\mathbf{P}}^{\text{R}}(\mathbf{x}, \mathbf{y}) = \delta_{n'n} \cdot (2\pi)^3 \delta^3(\mathbf{P}' - \mathbf{P}). \quad (38)$$

Needless to say, for the states $|n, \mathbf{P}\rangle \in \mathcal{F}_{\Delta}$, the right eigenfunctions agree with the NBS wavefunctions as

$$\begin{aligned}\chi_{n,\mathbf{P}}^{\text{R}}(\mathbf{x}, \mathbf{y}) &= \left\langle 0 \left| \hat{\phi}_0(\mathbf{x}) \hat{\phi}_1(\mathbf{y}) \right| n, \mathbf{P} \right\rangle, \\ \mathcal{E}_n(\mathbf{P}^2) &= E_n(\mathbf{P}^2).\end{aligned}\quad (39)$$

4. The external field

In order to consider the matrix element of the conserved $U_{\text{p}}(1)$ current in the effective quantum mechanics, we employ the external field method. We introduce an external $U_{\text{p}}(1)$ gauge field $A_{\mu}(\mathbf{x}, t)$ ($\mu = 0, 1, 2, 3$) and demand that the response of the effective quantum mechanics to the external field should be the same as the original theory. To do this, we construct a HAL QCD potential in the external field. To avoid unnecessary complexity, we restrict ourselves to those external fields $A_{\mu}(\mathbf{x}, t)$ that are non-zero only for the time region $t > 0$.

4.1. Hamiltonian in the external field

We consider the coupling of the original theory to the external field. In order for the coupling to be consistent with the conserved $U_{\text{p}}(1)$ current of Eq. (11), the Hamiltonian should be

$$\hat{H}[A_t] \equiv \hat{T}[A_t] + \hat{V}, \quad (40)$$

where the kinetic term $\hat{T} \equiv \hat{T}_0 + \hat{T}_1[A_t] + \hat{T}_2[A_t]$ couples to the external fields as

$$\begin{aligned}\hat{T}_0 &\equiv \int d^3x \hat{\phi}_0^{\dagger}(\mathbf{x}) \left(-\frac{\partial^2}{2m} \right) \hat{\phi}_0(\mathbf{x}), \\ \hat{T}_1[A_t] &\equiv \int d^3x \hat{\phi}_1^{\dagger}(\mathbf{x}) \left(-\frac{(\partial - i\mathbf{A}(\mathbf{x}, t))^2}{2m} - A_0(\mathbf{x}, t) \right) \hat{\phi}_1(\mathbf{x}), \\ \hat{T}_2[A_t] &\equiv \int d^3x \hat{\phi}_2^{\dagger}(\mathbf{x}) \left(-\frac{(\partial - i\mathbf{A}(\mathbf{x}, t))^2}{2m} - A_0(\mathbf{x}, t) + \Delta \right) \hat{\phi}_2(\mathbf{x}),\end{aligned}\quad (41)$$

whereas the interaction term $\hat{V} \equiv \sum_{\alpha,\beta=1,2} \hat{V}_{\alpha\beta}$ does not couple to the external fields, which is the setup of our np–np* coupling model. Note that this is consistent with the conserved $U_p(1)$ current in Eq. (11) of the np–np* coupling model. The subscript “ t ” of the external field A_t is used to indicate that $\hat{H}[A_t]$ depends on $A(\mathbf{x}, t)$ of time-slice t . The Schrödinger equation is given as

$$i \frac{\partial}{\partial t} |\psi, A; t\rangle = \hat{H}[A_t] |\psi, A; t\rangle. \quad (42)$$

4.2. Truncated Hamiltonian and truncated time evolution in the external field

The time dependence of the external field causes an unwanted transition to np* above the inelastic threshold, which is harmful in constructing a low-energy effective quantum mechanics below the np* threshold. In order to suppress such an unwanted transition, we insert the projection operator $\hat{\mathbb{P}}_{\Delta}$ at every step of the time evolution. This is done by replacing the Hamiltonian with the truncated Hamiltonian

$$\hat{H}_{\Delta}[A_t] \equiv \hat{\mathbb{P}}_{\Delta} \hat{H}[A_t] \hat{\mathbb{P}}_{\Delta}. \quad (43)$$

$\hat{H}_{\Delta}[A_t]$ generates a time evolution, which will be referred to as the truncated time evolution. The truncated time evolution is denoted by $\hat{U}_{\Delta}(t, s; A)$, which is explicitly expressed as a time-ordered product as

$$\hat{U}_{\Delta}(t, s; A) \equiv \sum_{n=0}^{\infty} (-i)^n \int_s^t dt_n \int_s^{t_n} dt_{n-1} \cdots \int_s^{t_2} dt_1 \hat{H}_{\Delta}[A_{t_n}] \hat{H}_{\Delta}[A_{t_{n-1}}] \cdots \hat{H}_{\Delta}[A_{t_1}]. \quad (44)$$

Note that $U_{\Delta}(t, s; A)$ is a solution of the initial value problems

$$\begin{aligned} i \frac{\partial}{\partial t} \hat{U}_{\Delta}(t, s; A) &= \hat{H}_{\Delta}[A_t] \hat{U}_{\Delta}(t, s; A), \\ i \frac{\partial}{\partial s} \hat{U}_{\Delta}(t, s; A) &= -\hat{U}_{\Delta}(t, s; A) \hat{H}_{\Delta}[A_s], \end{aligned} \quad (45)$$

with $\hat{U}_{\Delta}(t = s, s; A) = \hat{\mathbb{I}}$.

4.3. The truncated NBS wavefunctions in the external field

The equal-time NBS wavefunction in the external field with the truncated time evolution is defined as

$$\psi_{\alpha}^{(\Delta)}(\mathbf{x}, \mathbf{y}, t; A | n, \mathbf{P}) \equiv \left\langle 0 \left| \hat{\phi}_0^{(\Delta)}(\mathbf{x}, t; A) \hat{\phi}_{\alpha}^{(\Delta)}(\mathbf{y}, t; A) \right| n, \mathbf{P} \right\rangle \quad (46)$$

for $\alpha = 1, 2$, where $\hat{\phi}_{\alpha}^{(\Delta)}(\mathbf{x}, t; A)$ denotes the Heisenberg operator with the truncated time evolution as

$$\hat{\phi}_{\alpha}^{(\Delta)}(\mathbf{x}, t; A) \equiv \hat{U}_{\Delta}(0, t; A) \hat{\phi}_{\alpha}(\mathbf{x}) \hat{U}_{\Delta}(t, 0; A). \quad (47)$$

We will refer to Eq. (46) as the truncated NBS wavefunction. Note that, unlike Eq. (19), $\psi_{\alpha}^{(\Delta)}(\mathbf{x}, \mathbf{y}, t; A | n, \mathbf{P})$ does not factorize any more, since the external field breaks the Galilei covariance.

As will be shown in Appendix A, the truncated NBS wavefunctions satisfy the coupled-channel Schrödinger equations

$$\begin{aligned}
 & \left(i\partial_t + \frac{\partial_x^2}{2m} + \frac{\mathbf{D}_y^2}{2m} + A_0(\mathbf{y}, t) \right) \psi_1^{(\Delta)}(\mathbf{x}, \mathbf{y}, t; A|n, \mathbf{P}) \\
 &= \int d^3x' d^3y' \\
 & \quad \times \left\{ V_{11}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; A_t) \psi_1^{(\Delta)}(\mathbf{x}', \mathbf{y}', t; A|n, \mathbf{P}) + V_{12}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; A_t) \psi_2^{(\Delta)}(\mathbf{x}', \mathbf{y}', t; A|n, \mathbf{P}) \right\}, \\
 & \left(i\partial_t + \frac{\partial_x^2}{2m} + \frac{\mathbf{D}_y^2}{2m} + A_0(\mathbf{y}, t) - \Delta \right) \psi_2^{(\Delta)}(\mathbf{x}, \mathbf{y}, t; A|n, \mathbf{P}) \\
 &= \int d^3x' d^3y' \\
 & \quad \times \left\{ V_{21}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; A_t) \psi_1^{(\Delta)}(\mathbf{x}', \mathbf{y}', t; A|n, \mathbf{P}) + V_{22}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; A_t) \psi_2^{(\Delta)}(\mathbf{x}', \mathbf{y}', t; A|n, \mathbf{P}) \right\},
 \end{aligned} \tag{48}$$

where $\mathbf{D}_y \equiv \partial_y - iA(\mathbf{y}, t)$ denotes the covariant derivative and $V_{\alpha\beta}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; A_t)$ ($\alpha, \beta = 1, 2$) is defined as

$$V_{\alpha\beta}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; A_t) \equiv V_{\alpha\beta}(\mathbf{x} - \mathbf{y}) \delta^3(\mathbf{x} - \mathbf{x}') \delta^3(\mathbf{y} - \mathbf{y}') + \Delta V_{\alpha\beta}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; A_t), \tag{49}$$

with

$$\Delta V_{\alpha\beta}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; A_t) \equiv - \left\langle 0 \left| \hat{\phi}_0(\mathbf{x}) \hat{\phi}_\alpha(\mathbf{y}) \left(\hat{\mathbb{I}} - \hat{\mathbb{P}}_\Delta \right) \hat{H}[A_t] \hat{\mathbb{P}}_\Delta \hat{\phi}_0^\dagger(\mathbf{x}') \hat{\phi}_\beta^\dagger(\mathbf{y}') \right| 0 \right\rangle. \tag{50}$$

We give several comments:

- (1) Since $\hat{\phi}_0(\mathbf{x}, t)$ does not have Q_p charge, the derivative for $\mathbf{x}$ remains an ordinary one, i.e., ∂_x in Eq. (48).
- (2) The additional term $\Delta V_{\alpha\beta}(\cdots)$ originates from the cutoff $\hat{\mathbb{P}}_\Delta$ in $\hat{H}_\Delta[A_t]$.
- (3) $\Delta V_{\alpha\beta}(\cdots)$ vanishes for $A(\mathbf{x}, t) = 0$ due to the factor $(\hat{\mathbb{I}} - \hat{\mathbb{P}}_\Delta) \hat{H}[A_t] \hat{\mathbb{P}}_\Delta$.
- (4) $\Delta V_{\alpha\beta}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; A_t)$ depends on $A_\mu(\mathbf{x}, t)$ of time-slice t .

4.4. The HAL QCD potential in the external field

We define the HAL QCD potential in the presence of the external field by demanding that, for any states $|n, \mathbf{P}\rangle \in \mathcal{F}_\Delta$, the time-dependent Schrödinger equation should reproduce the truncated NBS wavefunctions for the np channel as

$$\begin{aligned}
 & \left(i\partial_t + \frac{\partial_x^2}{2m} + \frac{\mathbf{D}_y^2}{2m} + A_0(\mathbf{y}, t) \right) \psi_1^{(\Delta)}(\mathbf{x}, \mathbf{y}, t; A|n, \mathbf{P}) \\
 &= \int d^3x' d^3y' \mathcal{V}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; A_t) \psi_1^{(\Delta)}(\mathbf{x}', \mathbf{y}', t; A|n, \mathbf{P}).
 \end{aligned} \tag{51}$$

We note that, due to the time dependence of the external field, we need to use the time-dependent Schrödinger equation to define the HAL QCD potential. The demand is satisfied by

$$\begin{aligned}
& \mathcal{V}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; A_t) \\
& \equiv \sum_{m=0}^{\tilde{E}_m < \Delta} \int d^3 x'' d^3 y'' \\
& \quad \{ V_{11}(\mathbf{x}, \mathbf{y}; \mathbf{x}'', \mathbf{y}''; A_t) \tilde{\psi}_1(\mathbf{x}'' - \mathbf{y}'' | m) + V_{12}(\mathbf{x}, \mathbf{y}; \mathbf{x}'', \mathbf{y}''; A_t) \tilde{\psi}_2(\mathbf{x}'' - \mathbf{y}'' | m) \} \\
& \quad \times \tilde{\psi}_1^\vee(\mathbf{x}' - \mathbf{y}' | m) \delta^3 \left(\frac{\mathbf{x}'' + \mathbf{y}''}{2} - \frac{\mathbf{x}' + \mathbf{y}'}{2} \right), \tag{52}
\end{aligned}$$

where $\tilde{\psi}_\alpha(\mathbf{x}'' - \mathbf{y}'' | m)$ and $\tilde{\psi}_1^\vee(\mathbf{x}' - \mathbf{y}' | m)$ denote the reduced NBS wavefunction and the dual basis defined at Eqs. (20) and (25), respectively, in the absence of the external field. The proof is straightforward, and is given in Appendix B.

5. The current matrix element in the effective quantum mechanics

In Eq. (51) in Sect. 4, we derived the Schrödinger equation in the external field that is satisfied by the truncated NBS wavefunctions. In this section, we use this Schrödinger equation to derive a formula to calculate a current matrix element in the effective quantum mechanics associated with HAL QCD potentials. In Sect. 5.1, we will just give the formula together with several remarks. The derivation of the formula will be given in Sect. 5.2.

5.1. The formula to calculate a matrix element

Suppose that we have the potential $\mathcal{V}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; A_t)$ with which the Schrödinger equation in the external field is satisfied by the truncated NBS wavefunction of the np channel for any states $|n, \mathbf{P}\rangle \in \mathcal{F}_\Delta$ in the form of Eq. (51). Then the matrix element of the current $\hat{j}_p^\mu(\mathbf{z})$ is calculated for any states $|m, \mathbf{Q}\rangle, |n, \mathbf{P}\rangle \in \mathcal{F}_\Delta$ in the effective np quantum mechanics by the formula

$$\begin{aligned}
& \langle m, \mathbf{Q} | \hat{j}_p^\mu(\mathbf{z}) | n, \mathbf{P} \rangle \\
& = \int d^3 x d^3 y \int d^3 x' d^3 y' \chi_{m, \mathbf{Q}}^L(\mathbf{x}, \mathbf{y}) K^\mu(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; \mathbf{z}) \chi_{n, \mathbf{P}}^R(\mathbf{x}', \mathbf{y}'), \tag{53}
\end{aligned}$$

where $\chi_{m, \mathbf{Q}}^L(\mathbf{x}, \mathbf{y})$ and $\chi_{n, \mathbf{P}}^R(\mathbf{x}, \mathbf{y})$ denote the left and the right eigenfunctions of the effective Hamiltonian $\mathcal{H}$ in the absence of the external field [see Eq. (35)]. $K^\mu(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; \mathbf{z})$ denotes the effective current operator, which is defined by

$$\begin{aligned}
& K^0(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; \mathbf{z}) \delta(t - z_0) \\
& \equiv -\delta^3(\mathbf{z} - \mathbf{y}) \delta^3(\mathbf{x} - \mathbf{x}') \delta^3(\mathbf{y} - \mathbf{y}') \delta(t - z_0) + \left. \frac{\delta \mathcal{V}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; A_t)}{\delta A_0(\mathbf{z}, z_0)} \right|_{A \equiv 0}, \tag{54} \\
& K^i(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; \mathbf{z}) \delta(t - z_0) \\
& \equiv -\frac{\overleftrightarrow{\partial}_z^i}{2mi} \delta^3(\mathbf{z} - \mathbf{y}) \delta^3(\mathbf{x} - \mathbf{x}') \delta^3(\mathbf{y} - \mathbf{y}') \delta(t - z_0) + \left. \frac{\delta \mathcal{V}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; A_t)}{\delta A_i(\mathbf{z}, z_0)} \right|_{A \equiv 0},
\end{aligned}$$

with $\overleftrightarrow{\partial}_z \equiv \overrightarrow{\partial}_z - \overleftarrow{\partial}_z$. The derivation of Eq. (53) is given in Sect. 5.2.

We give several remarks:

- (1) In the conventional quantum mechanics, matrix elements are obtained by sandwiching an operator with a state vector and its Hermitian conjugate. In contrast, in our effective quantum mechanics associated with the HAL QCD potential, an operator is sandwiched by the left and right eigenfunctions of the effective Hamiltonian $\mathcal{H}$. Since HAL QCD potentials are not Hermitian in general, this could be a natural generalization.
- (2) The first terms of the effective current operator $K^\mu(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; \mathbf{z})$ correspond to the naive one-body current carried by a single proton as

$$\begin{aligned} \langle m, \mathbf{Q} | \hat{j}_p^0(\mathbf{z}) | n, \mathbf{P} \rangle_{\text{naive}} &= - \int d^3x \chi_{m, \mathbf{Q}}^L(\mathbf{x}, \mathbf{z}) \chi_{n, \mathbf{P}}^R(\mathbf{x}, \mathbf{z}), \\ \langle m, \mathbf{Q} | \hat{j}_p^i(\mathbf{z}) | n, \mathbf{P} \rangle_{\text{naive}} &= - \frac{1}{2mi} \int d^3x \chi_{m, \mathbf{Q}}^L(\mathbf{x}, \mathbf{z}) \overleftrightarrow{\partial}_z^i \chi_{n, \mathbf{P}}^R(\mathbf{x}, \mathbf{z}). \end{aligned} \quad (55)$$

In contrast, the second terms are the two-body current, which corresponds to the “exchange current.” The two-body currents originate from the states above the np^* threshold that have been integrated out during the construction of the HAL QCD potential.

- (3) In a realistic situation, HAL QCD potentials are constructed not by the method employed in the previous sections but by derivative expansion using the NBS wavefunctions as inputs [7,22]. However, as we shall see in Sect. 5.2, the derivation of the formula in Eq. (53) given in Sect. 5.2 does not depend on how the HAL QCD potential in the external field is constructed. All we need to use the formula is that there is a potential with which the Schrödinger equation in the external field is satisfied by the truncated NBS wavefunctions.
- (4) It would be interesting to discuss the gauge covariance property as was done in Ref. [16]. However, the cutoff that we have introduced in the Hamiltonian makes the situation complicated. Since the same matrix elements as the original theory can be reproduced, we do not stick too much to this point in this paper.
- (5) The application of two functional derivatives $\delta/\delta A_\mu$ to Eq. (56) in Sect. 5.2 does not lead to $\langle m, \mathbf{Q} | j_p^\mu(\mathbf{z}_1) j_p^\nu(\mathbf{z}_2) | n, \mathbf{P} \rangle$ but to $\langle m, \mathbf{Q} | j_p^\mu(\mathbf{z}_1) \mathbb{P}_\Delta j_p^\nu(\mathbf{z}_2) | n, \mathbf{P} \rangle$. To calculate the former matrix element, an additional consideration is needed.

5.2. The derivation

For notational convenience, we arrange Eq. (51) as

$$(i\partial_t - \mathcal{H}[A_t]) \psi_1^{(\Delta)}(\mathbf{x}, \mathbf{y}, t; A | n, \mathbf{P}) = 0, \quad (56)$$

where the effective Hamiltonian $\mathcal{H}[A_t]$ acts on the truncated NBS wavefunction as

$$\begin{aligned} \mathcal{H}[A_t] \psi_1^{(\Delta)}(\mathbf{x}, \mathbf{y}, t; A | n, \mathbf{P}) &\equiv - \left(\frac{1}{2m} \partial_x^2 + \frac{1}{2m} \mathbf{D}_y^2 + A_0(\mathbf{y}, t) \right) \psi_1^{(\Delta)}(\mathbf{x}, \mathbf{y}, t; A | n, \mathbf{P}) \\ &\quad + \int d^3x' d^3y' \mathcal{V}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; A_t) \psi_1^{(\Delta)}(\mathbf{x}', \mathbf{y}', t; A | n, \mathbf{P}). \end{aligned} \quad (57)$$

We apply the functional derivative $\delta/\delta A_\mu(\mathbf{z}, z_0)$ to both sides of Eq. (56) and then set the external field $A_\mu \equiv 0$ to have

$$\begin{aligned} & (i\partial_t - \mathcal{H}) \frac{\delta\psi_1^{(\Delta)}(\mathbf{x}, \mathbf{y}, t; A|n, \mathbf{P})}{\delta A_\mu(\mathbf{z}, z_0)} \Big|_{A \equiv 0} \\ &= \int d^3x' d^3y' K^\mu(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; \mathbf{z}) \delta(t - z_0) \psi_1(\mathbf{x}', \mathbf{y}', t|n, \mathbf{P}), \end{aligned} \quad (58)$$

where

$$\begin{aligned} & K^\mu(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; \mathbf{z}) \delta(t - z_0) \\ & \equiv \frac{\delta}{\delta A_\mu(\mathbf{z}, z_0)} \left[\left(-\frac{1}{2m} (\partial_y - iA(\mathbf{y}, t))^2 - A_0(\mathbf{y}, t) \right) \delta^3(\mathbf{x} - \mathbf{x}') \delta^3(\mathbf{y} - \mathbf{y}') + \mathcal{V}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; A_t) \right] \Big|_{A \equiv 0}, \end{aligned} \quad (59)$$

which reduces to the explicit expression of K^μ given in Eq. (54).

In the original theory, $\delta\psi^{(\Delta)}/\delta A_\mu(z)$ is expressed as

$$\begin{aligned} & \frac{\delta\psi_1^{(\Delta)}(\mathbf{x}, \mathbf{y}, t; A|n, \mathbf{P})}{\delta A_\mu(z)} \Big|_{A \equiv 0} \\ &= i\theta(z_0)\theta(t - z_0) \left\langle 0 \left| \hat{\phi}_0(\mathbf{x}) \hat{\phi}_1(\mathbf{y}) e^{i(t-z_0)\hat{H}} \hat{\mathbb{P}}_\Delta j_p^\mu(\mathbf{z}) \hat{\mathbb{P}}_\Delta e^{iz_0\hat{H}} \right| n, \mathbf{P} \right\rangle \\ &= i\theta(z_0)\theta(t - z_0) \sum_m^{\tilde{E}_m < \Delta} \int \frac{d^3Q}{(2\pi)^3} \chi_{m, \mathbf{Q}}^R(\mathbf{x}, \mathbf{y}) e^{i(t-z_0)E_n(\mathbf{Q}^2)} \left\langle m, \mathbf{Q} \left| \hat{j}_p^\mu(\mathbf{z}) \right| n, \mathbf{P} \right\rangle e^{iz_0E_n(\mathbf{P}^2)}, \end{aligned} \quad (60)$$

where we used Eq. (C.1) to derive the second line. The existence of $\theta(t - z_0)$ indicates that

$$\lim_{t \rightarrow -\infty} \frac{\delta\psi_1^{(\Delta)}(\mathbf{x}, \mathbf{y}, t; A|n, \mathbf{P})}{\delta A_\mu(z)} \Big|_{A \equiv 0} = 0. \quad (61)$$

To express $\delta\psi_1^{(\Delta)}/\delta A_\mu(z)$ in the effective quantum mechanics, we solve Eq. (58) with the initial value of Eq. (61) by using the retarded Green's function of $\mathcal{H}$ given in Eq. (D.3), to have

$$\begin{aligned} & \frac{\delta\psi_1^{(\Delta)}(\mathbf{x}, \mathbf{y}, t; A|n, \mathbf{P})}{\delta A_\mu(z)} \Big|_{A \equiv 0} \\ &= \int dt'' \int d^3x'' d^3y'' \int d^3x' d^3y' \\ & \quad \times G(\mathbf{x}, \mathbf{y}, t; \mathbf{x}'', \mathbf{y}'', t'') K^\mu(\mathbf{x}'', \mathbf{y}''; \mathbf{x}', \mathbf{y}'; \mathbf{z}) \delta(t'' - z_0) \psi_1(\mathbf{x}', \mathbf{y}', t''|n, \mathbf{P}) \\ &= -i\theta(t - z_0) \sum_m^\infty \int \frac{d^3Q}{(2\pi)^3} \chi_{m, \mathbf{Q}}^R(\mathbf{x}, \mathbf{y}) e^{-iE_m(\mathbf{Q}^2)(t-z_0)} \\ & \quad \times \int d^3x'' d^3y'' \int d^3x' d^3y' \chi_{m, \mathbf{Q}}^L(\mathbf{x}'', \mathbf{y}'') K^\mu(\mathbf{x}'', \mathbf{y}''; \mathbf{x}', \mathbf{y}'; \mathbf{z}) \chi_{n, \mathbf{P}}^R(\mathbf{x}', \mathbf{y}') e^{-iE_n(\mathbf{P}^2)z_0}. \end{aligned} \quad (62)$$

By comparing Eqs. (60) and (62), we arrive at the formula in Eq. (53).

6. Summary and conclusion

We have considered how to deal with a matrix element in HAL QCD's potential method. The HAL QCD method is a lattice QCD (LQCD) method to obtain a potential (HAL QCD potential) which is faithful to the scattering phase shift. This is supported by the fact that the HAL QCD potential is defined by demanding that the Schrödinger equation should reproduce the equal-time NBS wavefunctions, and the NBS wavefunctions contain the scattering phase shift in its long-distance part in exactly the same way as that of the scattering wavefunctions of the non-relativistic quantum mechanics. Therefore, the effective NN quantum mechanics associated with the HAL QCD potential is supported to reproduce the scattering phase shift. However, an additional consideration is needed to calculate the matrix elements. In fact, there have been no discussions concerning the relation between the matrix elements calculated from the effective NN quantum mechanics associated with the HAL QCD potential and QCD, the original theory.

As a first step to considering a matrix element in the HAL QCD method, we have considered a simplified non-relativistic field-theoretical model instead of Lorentz-covariant QCD. We have employed a two-channel coupling model as the original theory, where np - np^* coupling is mimicked (the np - np^* coupling model). By integrating out the closed np^* channel with the HAL QCD method, we have obtained an effective np potential (the HAL QCD potential) that is used to define the effective np quantum mechanics. Due to the simplicity of our np - np^* coupling model, we have obtained the HAL QCD potential in a closed analytic form.

We have used the external field method and obtained a formula to calculate a matrix element of a conserved current in the effective np quantum mechanics by demanding that the response of the effective quantum mechanics to the external field is the same as that of the original theory. With our formula, the matrix element is calculated by sandwiching the effective current operator between the left and the right eigenfunctions of the effective np Hamiltonian. The effective current operator consists of two parts: (1) a naive one-body current, and (2) the two-body current that corresponds to the exchange current. In our np - np^* coupling model, the two-body current emerges from the states above the np^* threshold that have been integrated out when obtaining the effective np potential.

To extend the formula for QCD, several generalizations are needed. To use the formula in relativistic original theories, it is necessary to generalize the HAL QCD potential for a boosted Lorentz frame. Note that it is only in the center-of-mass frame where the asymptotic forms of NBS wavefunctions Eq. (1) agree with those of the scattering wavefunctions of the quantum mechanics. To consider a system of composite particles, we have to take into account the form factors. To use the formula in LQCD, we have to deal with the time evolution in an external field with a cutoff. To do these things, we may have to introduce several approximations.

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Appendix A. Derivation of Eq. (48)

We prove that the truncated NBS wavefunctions satisfy the coupled-channel Schrödinger equations of Eq. (48). By using Eq. (45), the time derivative of $\hat{\phi}_0^{(\Delta)}(\mathbf{x}, t; A)\hat{\phi}_\alpha^{(\Delta)}(\mathbf{y}, t; A)$ for $\alpha = 1, 2$ is given by

$$\begin{aligned} i\partial_t \left\{ \hat{\phi}_0^{(\Delta)}(\mathbf{x}, t; A)\hat{\phi}_\alpha^{(\Delta)}(\mathbf{y}, t; A), \right\} \\ = \hat{U}_\Delta(0, t; A) \left[\hat{\phi}_0(\mathbf{x})\hat{\phi}_\alpha(\mathbf{y}), \hat{H}_\Delta[A_t] \right] \hat{U}_\Delta(t, 0; A). \end{aligned} \quad (\text{A.1})$$

The commutator on the right-hand side is arranged as

$$\begin{aligned} & \left[\hat{\phi}_0(\mathbf{x})\hat{\phi}_\alpha(\mathbf{y}), \hat{\mathbb{P}}_\Delta \hat{H}[A_t] \hat{\mathbb{P}}_\Delta \right] \\ &= \hat{\mathbb{P}}_\Delta \hat{H}[A_t] \left[\hat{\phi}_0(\mathbf{x})\hat{\phi}_\alpha(\mathbf{y}), \hat{\mathbb{P}}_\Delta \right] \\ &+ \hat{\mathbb{P}}_\Delta \left[\hat{\phi}_0(\mathbf{x})\hat{\phi}_\alpha(\mathbf{y}), \hat{H}[A_t] \right] \hat{\mathbb{P}}_\Delta + \left[\hat{\phi}_0(\mathbf{x})\hat{\phi}_\alpha(\mathbf{y}), \hat{\mathbb{P}}_\Delta \right] \hat{H}[A_t] \hat{\mathbb{P}}_\Delta. \end{aligned} \quad (\text{A.2})$$

By noting that $\langle 0 | \hat{U}_\Delta(0, t; A) = \langle 0 |$ and $\langle 0 | \hat{H}[A_t] = 0$, we calculate a matrix element of both sides of Eq. (A.1) between $\langle 0 |$ and $|n, \mathbf{P}\rangle \in \mathcal{F}_\Delta$ to have

$$\begin{aligned} i\partial_t \psi_\alpha^{(\Delta)}(\mathbf{x}, \mathbf{y}, t; A | n, \mathbf{P}) &= \left\langle 0 \left| \left[\hat{\phi}_0(\mathbf{x})\hat{\phi}_\alpha(\mathbf{y}), \hat{H}[A_t] \right] \hat{U}_\Delta(t, 0; A) \right| n, \mathbf{P} \right\rangle \\ &+ \left\langle 0 \left| \left[\hat{\phi}_0(\mathbf{x})\hat{\phi}_\alpha(\mathbf{y}), \hat{\mathbb{P}}_\Delta \right] \hat{H}[A_t] \hat{\mathbb{P}}_\Delta \hat{U}_\Delta(t, 0; A) \right| n, \mathbf{P} \right\rangle. \end{aligned} \quad (\text{A.3})$$

The first term on the right-hand side reduces to

$$\begin{aligned} \text{The 1st term} &= \left(-\frac{\partial_x^2}{2m} - \frac{\mathbf{D}_y^2}{2m} - A_0(\mathbf{y}, t) + \Delta\delta_{\alpha 2} \right) \psi_\alpha^{(\Delta)}(\mathbf{x}, \mathbf{y}, t; A | n, \mathbf{P}) \\ &+ V_{\alpha 1}(\mathbf{x} - \mathbf{y}) \psi_1^{(\Delta)}(\mathbf{x}, \mathbf{y}, t; A | n, \mathbf{P}) + V_{\alpha 2}(\mathbf{x} - \mathbf{y}) \psi_2^{(\Delta)}(\mathbf{x}, \mathbf{y}, t; A | n, \mathbf{P}), \end{aligned} \quad (\text{A.4})$$

while the second term on the right-hand side is arranged as

$$\begin{aligned} \text{The 2nd term} &= - \left\langle 0 \left| \hat{\phi}_0(\mathbf{x})\hat{\phi}_\alpha(\mathbf{y}) \left(\hat{\mathbb{I}} - \hat{\mathbb{P}}_\Delta \right) \hat{H}[A_t] \hat{\mathbb{P}}_\Delta \hat{U}_\Delta(t, 0; A) \right| n, \mathbf{P} \right\rangle \\ &= - \int d^3x' d^3y' \sum_{\beta=1,2} \left\langle 0 \left| \hat{\phi}_0(\mathbf{x})\hat{\phi}_\alpha(\mathbf{y}) \left(\hat{\mathbb{I}} - \hat{\mathbb{P}}_\Delta \right) \hat{H}[A_t] \hat{\mathbb{P}}_\Delta \hat{\phi}_0^\dagger(\mathbf{x}')\hat{\phi}_\beta^\dagger(\mathbf{y}') \right| 0 \right\rangle \\ &\times \psi_\beta^{(\Delta)}(\mathbf{x}', \mathbf{y}', t; A | n, \mathbf{P}), \end{aligned} \quad (\text{A.5})$$

where the last line is obtained by using the following expression for the identity operator in the subspace $\mathcal{F}$:

$$\hat{\mathbb{I}}_\mathcal{F} = \sum_{\beta=1,2} \int d^3x' d^3y' \hat{\phi}_0^\dagger(\mathbf{x}')\hat{\phi}_\beta^\dagger(\mathbf{y}') |0\rangle \langle 0| \hat{\phi}_0(\mathbf{x}')\hat{\phi}_\beta(\mathbf{y}'). \quad (\text{A.6})$$

By inserting Eqs. (A.4) and (A.5) into Eq. (A.3), we arrive at Eq. (48).

Appendix B. Proof that $\mathcal{V}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; A_t)$ satisfies the Schrödinger equation in the external field

We give a proof of Eq. (51), i.e., that the Schrödinger equation in the external field with the potential $\mathcal{V}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}'; A_t)$ of Eq. (52) is satisfied by the truncated NBS wavefunctions. For this purpose, we first note that the truncated NBS wavefunction is rewritten as

$$\begin{aligned} \psi_\alpha^{(\Delta)}(\mathbf{x}, \mathbf{y}, t; A|n, \mathbf{P}) &= \sum_{n'}^{\tilde{E}_{n'} < \Delta} \int \frac{d^3 P'}{(2\pi)^3} \left\langle 0 \left| \hat{\phi}_0(\mathbf{x}) \hat{\phi}_\alpha(\mathbf{y}) \right| n', \mathbf{P}' \right\rangle \left\langle n', \mathbf{P}' \left| \hat{U}_\Delta(t, 0; A) \right| n, \mathbf{P} \right\rangle \\ &= \sum_{n'}^{\tilde{E}_{n'} < \Delta} \int \frac{d^3 P'}{(2\pi)^3} \tilde{\psi}_\alpha(\mathbf{x} - \mathbf{y}|n') \exp\left(i\mathbf{P}' \cdot \frac{\mathbf{x} + \mathbf{y}}{2}\right) \left\langle n', \mathbf{P}' \left| \hat{U}_\Delta(t, 0; A) \right| n, \mathbf{P} \right\rangle. \end{aligned} \quad (\text{B.1})$$

Equation (51) reduces to the coupled-channel equation of Eq. (48) in the following way:

r.h.s. of Eq. (51)

$$\begin{aligned} &= \int d^3 r' d^3 R' \int d^3 x'' d^3 y'' \sum_m^{\tilde{E}_m < \Delta} \\ &\quad \times \{V_{11}(\mathbf{x}, \mathbf{y}; \mathbf{x}'', \mathbf{y}''; A_t) \tilde{\psi}_1(\mathbf{x}'' - \mathbf{y}''|m) + V_{12}(\mathbf{x}, \mathbf{y}; \mathbf{x}'', \mathbf{y}''; A_t) \tilde{\psi}_2(\mathbf{x}'' - \mathbf{y}''|m)\} \\ &\quad \times \tilde{\psi}_1^\vee(\mathbf{r}'|m) \delta^3\left(\frac{\mathbf{x}'' + \mathbf{y}''}{2} - \mathbf{R}'\right) \\ &\quad \times \sum_{n'}^{\tilde{E}_{n'} < \Delta} \int \frac{d^3 P'}{(2\pi)^3} \tilde{\psi}_1(\mathbf{r}'|n') \exp(i\mathbf{P}' \cdot \mathbf{R}') \left\langle n', \mathbf{P}' \left| \hat{U}_\Delta(t, 0; A) \right| n, \mathbf{P} \right\rangle \\ &= \int d^3 x'' d^3 y'' \sum_m^{\tilde{E}_m < \Delta} \\ &\quad \times \{V_{11}(\mathbf{x}, \mathbf{y}; \mathbf{x}'', \mathbf{y}''; A_t) \tilde{\psi}_1(\mathbf{x}'' - \mathbf{y}''|m) + V_{12}(\mathbf{x}, \mathbf{y}; \mathbf{x}'', \mathbf{y}''; A_t) \tilde{\psi}_2(\mathbf{x}'' - \mathbf{y}''|m)\} \\ &\quad \times \int \frac{d^3 P'}{(2\pi)^3} \exp\left(i\mathbf{P}' \cdot \frac{\mathbf{x}'' + \mathbf{y}''}{2}\right) \left\langle m, \mathbf{P}' \left| \hat{U}_\Delta(t, 0; A) \right| n, \mathbf{P} \right\rangle \\ &= \int d^3 x'' d^3 y'' \left\{ V_{11}(\mathbf{x}, \mathbf{y}; \mathbf{x}'', \mathbf{y}''; A_t) \tilde{\psi}_1^{(\Delta)}(\mathbf{x}'', \mathbf{y}'', t; A|n, \mathbf{P}) \right. \\ &\quad \left. + V_{12}(\mathbf{x}, \mathbf{y}; \mathbf{x}'', \mathbf{y}''; A_t) \tilde{\psi}_2^{(\Delta)}(\mathbf{x}'', \mathbf{y}'', t; A|n, \mathbf{P}) \right\}, \end{aligned} \quad (\text{B.2})$$

where, to obtain the second line, we used Eq. (B.1) and introduced $\mathbf{r}' \equiv \mathbf{x}' - \mathbf{y}'$ and $\mathbf{R}' \equiv (\mathbf{x}' + \mathbf{y}')/2$. To obtain the third line, we completed the integration of $\mathbf{r}'$ and $\mathbf{R}'$ by using the orthogonality relation of the dual basis in Eq. (24). To obtain the last line, Eq. (B.1) was used again.

Appendix C. The functional derivative of the evolution operator by the external field

We derive the formula:

$$\left. \frac{\delta \hat{U}_\Delta(t, 0; A)}{\delta A_\mu(z)} \right|_{A=0} = ie^{(t-z_0)\hat{H}} \hat{\mathbb{P}}_\Delta \hat{j}_p^\mu(\mathbf{z}) \hat{\mathbb{P}}_\Delta e^{iz_0\hat{H}}. \quad (\text{C.1})$$

From Eq. (44), we have

$$\begin{aligned}
& \frac{\delta \hat{U}_\Delta(t, 0; A)}{\delta A_\mu(z)} \\
&= \sum_{n=1}^{\infty} (-i)^n \sum_{j=1}^n \int_0^t dt_n \cdots \int_0^{t_2} dt_1 \hat{H}_\Delta[A_{t_n}] \cdots \hat{H}_\Delta[A_{t_{j+1}}] \frac{\delta \hat{H}_\Delta[A_{t_j}]}{\delta A_\mu(z)} \hat{H}_\Delta[A_{t_{j-1}}] \cdots \hat{H}_\Delta[A_{t_1}] \\
&= -i \int_0^t dt' \hat{U}_\Delta(t, t'; A) \frac{\delta \hat{H}_\Delta[A_{t'}]}{\delta A_\mu(z)} \hat{U}_\Delta(t', 0; A).
\end{aligned} \tag{C.2}$$

By using

$$\left. \frac{\delta \hat{H}_\Delta[A_{t'}]}{\delta A_\mu(z)} \right|_{A=0} = -\hat{j}_p^\mu(z) \delta(t' - z_0) \tag{C.3}$$

and

$$U_\Delta(t, s; A = 0) \hat{\mathbb{P}}_\Delta = e^{i(t-s)\hat{H}} \hat{\mathbb{P}}_\Delta, \tag{C.4}$$

we are left with Eq. (C.1).

Appendix D. Green's function of the effective quantum mechanics

The retarded Green's function of the effective Hamiltonian $\mathcal{H}$ is defined as the solution to the differential equation

$$\left(i \frac{\partial}{\partial t} - \mathcal{H} \right) G(\mathbf{x}, \mathbf{y}, t; \mathbf{x}', \mathbf{y}', t') = \delta(t - t') \delta^3(\mathbf{x} - \mathbf{x}') \delta^3(\mathbf{y} - \mathbf{y}') \tag{D.1}$$

with the boundary condition

$$\lim_{t \rightarrow -\infty} G(\mathbf{x}, \mathbf{y}, t; \mathbf{x}', \mathbf{y}', t') = 0. \tag{D.2}$$

The solution is explicitly given as

$$\begin{aligned}
& G(\mathbf{x}, \mathbf{y}, t; \mathbf{x}', \mathbf{y}', t') \\
& \equiv -i\theta(t - t') \sum_{n=0}^{\infty} \int \frac{d^3 P}{(2\pi)^3} \chi_{n, \mathbf{P}}^R(\mathbf{x}, \mathbf{y}) \chi_{n, \mathbf{P}}^L(\mathbf{x}', \mathbf{y}') e^{-i\mathcal{E}_n(\mathbf{P}^2)(t-t')},
\end{aligned} \tag{D.3}$$

where $\chi_{n, \mathbf{P}}^L(\mathbf{x}, \mathbf{y})$ and $\chi_{n, \mathbf{P}}^R(\mathbf{x}, \mathbf{y})$ are the left and the right eigenfunctions of $\mathcal{H}$ associated with the eigenvalue $\mathcal{E}_n(\mathbf{P}^2)$ given in Eq. (35). The retarded Green's function is used to solve the differential equation

$$(i\partial_t - \mathcal{H}) F(\mathbf{x}, \mathbf{y}, t) = f(\mathbf{x}, \mathbf{y}, t) \tag{D.4}$$

with boundary condition $\lim_{t \rightarrow -\infty} F(\mathbf{x}, \mathbf{y}, t) = 0$. Its solution is given as

$$F(\mathbf{x}, \mathbf{y}, t) = \int dt' \int d^3 x' d^3 y' G(\mathbf{x}, \mathbf{y}, t; \mathbf{x}', \mathbf{y}', t') f(\mathbf{x}', \mathbf{y}', t'). \tag{D.5}$$

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