Prog. Theor. Exp. Phys. **2018**, 053B01 (16 pages)
DOI: 10.1093/ptep/pty045

Codimension-2 brane solutions of maximal supergravities in 9, 8, and 7 dimensions

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Received November 21, 2017; Revised March 13, 2018; Accepted March 25, 2018; Published May 8, 2018

We construct codimension-2 BPS brane solutions in $D = 9, 8, 7$ maximal supergravities by solving Killing spinor equations. We assume the Poincaré invariance along the worldvolume and vanishing gauge fields, and determine the metric and the scalar fields. The solution in $D = 9$ is essentially the same as the ten-dimensional one, which is specified by a holomorphic function in the transverse space. For $D = 8$, the solution is specified by two holomorphic functions, and regarded as $T^2 \times T^2$ compactification of F-theory. For $D = 7$, we find that the solution can be interpreted as M-theory on Calabi–Yau, and under an additional assumption a solution is specified by two holomorphic functions.

Subject Index B10, B11

1. Introduction

An important feature of codimension-2 branes is that they can have non-trivial monodromies [1]. Namely, when we move charged objects around such branes they may get transformed to dual objects. The element of the duality group specifying this duality transformation is called a monodromy associated with the branes. In the context of brane realization of field theories, such monodromy transformations are often interpreted as electric–magnetic duality, and in some cases the existence of branes with non-trivial monodromies causes the emergence of particles with mutually non-local charges. This is a common feature of some classes of non-Lagrangian theories such as Argyres–Douglas theories [2,3] and four-dimensional $\mathcal{N} = 3$ superconformal theories [4–6].[1] This fact motivates us to investigate codimension-2 branes.

In the context of string/M-theory a maximal supergravity is obtained by torus compactification of ten- or eleven-dimensional theory. The U-duality group is generated by geometric coordinate changes of the torus and duality transformations. Half BPS branes in eleven or ten dimensions descend to various sorts of codimension-2 branes in lower dimensions by dimensional reduction and duality transformations [7–9].

F- and M-theories are useful to describe codimension-2 branes. For example, 7-branes in type IIB string theory are described as purely geometric objects in the context of F-theory. We can also realize various branes in lower dimensions by considersing M- and F-theory in different purely geometric backgrounds in which fields other than the metric are vanishing or constant. However,

[1] Non-trivial monodromies arise not only in systems of codimension-2 branes but also in more general backgrounds with non-trivial fundamental groups. Indeed, the first example of $\mathcal{N} = 3$ theory in Ref. [5] is realized by using an orbifold $\mathbb{C}^3/\mathbb{Z}_k$, which has the fundamental group $\mathbb{Z}_k$. Another example of $\mathcal{N} = 3$ theories, in Ref. [6], can be regarded as a realization with codimension-2 branes.

Table 1. The global symmetry group G, the duality group $G_{\mathbb{Z}}$, and the local symmetry group H in maximal supergravities.

dim	G	$G_{\mathbb{Z}}$	H
10(A)	$SO(1,1)/\mathbb{Z}_2$	1	1
10(B)	$SL(2,\mathbb{R})$	$SL(2,\mathbb{Z})$	$SO(2)$
9	$SL(2,\mathbb{R}) \times O(1,1)$	$SL(2,\mathbb{Z}) \times \mathbb{Z}_2$	$SO(2)$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$	$SO(3) \times SO(2)$
7	$SL(5,\mathbb{R})$	$SL(5,\mathbb{Z})$	$SO(5)$
6	$O(5,5)$	$O(5,5;\mathbb{Z})$	$SO(5) \times SO(5)$
5	$E_{6(6)}$	$E_{6(6)}(\mathbb{Z})$	$USp(8)$
4	$E_{7(7)}$	$E_{7(7)}(\mathbb{Z})$	$SU(8)$

there may be branes that do not have a geometric description in M- or F-theory. Such branes have not been investigated in detail, and their realization and classification may give new insight for strongly coupled field theories. A purpose of this paper is to search for such branes in the case of codimension-2 BPS branes.

Actually, construction of such branes is quite restricted, as argued in Ref. [10]. Some examples given in Ref. [10] have non-compact dimensions less than four, and it seems difficult to give examples in higher dimensions. We show that this is actually the case for codimension-2 BPS brane solutions by explicitly solving Killing spinor equations. Namely, BPS solutions of codimension-2 branes always have geometric realization in M/F-theory for $D = 9, 8, 7$.

Maximal supergravities in various dimensions have common structure. The scalar manifolds of these theories have the form G/H, where G is the classical global symmetry and H is the local symmetry group, which is the maximal compact subgroup of G. See Table 1 for G and H in dimensions $D = 10, \ldots, 4$ [11]. The scalar fields are coordinates of this manifold, and represented as a matrix $L \in G$ with left action of G and right action of H. The U-duality group $G_{\mathbb{Z}}$ is the integral form of G. For the theories in seven or higher dimensions G are all SL type and H are all SO type, and they can be dealt with in similar ways. In this paper we investigate these theories; theories in $D \leq 6$ are left for future work.

A maximal supergravity contains

- the vielbein $e_M^{\widehat{M}}$,
- scalar fields $L^{\alpha}{}_i$,
- gravitino ψ_M,
- dilatino λ_i,

and anti-symmetric tensor fields of different ranks, which are not relevant to our analysis in this paper.

We use the following indices:

- $M, N, \ldots$: global coordinates
- $\widehat{M}, \widehat{N}, \ldots$: local Lorentz
- $\alpha, \beta, \ldots$: $SL(m)$ fundamental representation
- $i, j, \ldots$: $H = SO(n)$ vector.

The scalar fields appear in the action and the supersymmetry transformation laws through one-form fields P and Q, which are defined as the traceless symmetric and anti-symmetric parts of the

Table 2. Quantum numbers of scalar and spinor fields in type IIB supergravity.

	$G = SL(2,\mathbb{R})$	$H = SO(2)_R$
$L^\alpha{}_i$	**2**	± 1
ψ_M	**1**	$\pm\frac{1}{2}$
λ_i	**1**	$\pm\frac{3}{2}$
ϵ	**1**	$\pm\frac{1}{2}$

Maurer–Cartan form:

$$P_{ij} = (L^{-1}dL)_{(ij)}, \quad Q_{ij} = (L^{-1}dL)_{[ij]}. \tag{1.1}$$

Under H transformation, P transforms homogeneously as the symmetric matrix representation of H, while Q transforms inhomogeneously and plays the role of H-connection.

To obtain BPS solutions we solve the Killing spinor equations for the gravitino ψ_M and dilatino λ_i. In the next section we first look at the ten-dimensional case to explain the basic prescription for solving the Killing spinor equations and then we move on to lower-dimensional cases.

2. Solving Killing spinor equations

2.1. $D = 10$

Let us consider BPS solutions in type IIB supergravity. Such solutions have been well investigated [1] and 7-branes are classified by the Kodaira classification [12]. Various four-dimensional $\mathcal{N} = 2$ supersymmetric theories are realized on D3-branes probing these solutions [13–15]. A purpose of this subsection is to review how we can obtain BPS solutions in ten dimensions by solving the Killing spinor equations. The derivations in lower dimensions are parallel.

The classical global symmetry of type IIB supergravity is $G = SL(2,\mathbb{R})$ and the local R-symmetry group is $H = SO(2)_R$. Namely, the scalar manifold is locally the two-dimensional homogeneous space $SL(2,\mathbb{R})/SO(2)_R$. When we discuss the global structure, we also need to take account of the duality group $G_\mathbb{Z} = SL(2,\mathbb{Z})$. Quantum numbers of scalar and spinor fields in type IIB supergravity [16] are summarized in Table 2. The gravitino field ψ_M belongs to the spinor representation of H. Namely, ψ_M has the spacetime vector index M and an $SO(2)_R$ spinor index which is implicit. The dilatino field λ_i has the $SO(2)_R$ vector index i and an implicit $SO(2)_R$ spinor index. It satisfies the ρ-traceless condition

$$\rho_i\lambda^i = 0, \tag{2.1}$$

where ρ_i are Dirac matrices associated with the orthogonal group $H = SO(2)_R$. See the appendix for our notation. This condition removes components carrying $SO(2)_R$ charge $\pm 1/2$ from λ^i, and the remaining components in λ^i carry $SO(2)_R$ charge $\pm 3/2$, as shown in Table 2.

Due to the existence of the self-dual four-form field it is difficult to write down the full Lagrangian of the type IIB supergravity. However, it is easy to give the Lagrangian of the subsector which is relevant to us. If we assume vanishing anti-symmetric tensor fields, the equations of motion for the remaining fields are obtained from the Lagrangian

$$\begin{aligned}\mathcal{L} = {}& \frac{e}{4}R + \frac{e}{2}(\psi_M\Gamma^{MNP}D_N\psi_P)\\ &- \frac{e}{4}(P_M{}^{ij})^2 + \frac{e}{2}(\lambda_i\Gamma^N D_N\lambda_i) + \frac{e}{2}P_M{}^{ij}(\psi_N\Gamma^M\Gamma^N\Gamma_i\lambda_j),\end{aligned} \tag{2.2}$$

up to higher-order fermion terms.

The Killing spinor equations are

$$0 = \delta\psi_M = D_M\epsilon, \tag{2.3}$$

$$0 = \delta\lambda^i = P_M^{ij}\Gamma^M\rho_j\epsilon, \tag{2.4}$$

where D_M is the covariant derivative defined with the spin connection ω and the $SO(2)_R$ connection Q:

$$D_M\epsilon = \left(\partial_M + \frac{1}{4}\omega_{M\widehat{P}\widehat{Q}}\Gamma^{\widehat{P}\widehat{Q}} + \frac{1}{4}Q_{Mij}\rho^{ij}\right)\epsilon. \tag{2.5}$$

We are interested in codimension-2 brane solutions. Let us assume the solution has the eight-dimensional Poincaré invariance along the eight longitudinal directions. We use x^μ ($\mu = 0, 1, \ldots, 7$) and x^m ($m = 8, 9$) for longitudinal and transverse coordinates, respectively. We take the ansatz

$$ds^2 = f^2(x^m)\eta_{\mu\nu}dx^\mu dx^\nu + g^2(x^m)dx^m dx^m \tag{2.6}$$

for the metric and

$$L^\alpha{}_i = L^\alpha{}_i(x^m) \tag{2.7}$$

for the scalar fields. We introduce the local frame so that the vielbein has the diagonal components

$$e^{\widehat{\mu}} = f(x^m)\delta_\mu^{\widehat{\mu}}dx^\mu, \quad e^a = g(x^m)\delta_m^a dx^m. \tag{2.8}$$

Because we are interested in the rigid supersymmetry on the branes we assume that the supersymmetry parameter ϵ depends only on the transverse coordinates:

$$\epsilon = \epsilon(x^m). \tag{2.9}$$

Because $L^\alpha{}_i$ is independent of the longitudinal coordinates x^μ, the longitudinal components of Q vanish. For the longitudinal components of the Killing spinor equation of Eq. (2.3),

$$\delta\psi_\mu = D_\mu\epsilon = \left(\partial_\mu - \frac{1}{2g}(\partial_m f)\Gamma_{m\widehat{\mu}}\right)\epsilon = 0, \tag{2.10}$$

to have non-trivial solutions the function f must be constant, and without loss of generality we can set $f = 1$.

The covariant derivative in the transverse components of Eq. (2.3) include the connection of $SO(2)_{89}$, the rotation in the 8–9 plane, and that of $H = SO(2)_R$:

$$D_m\epsilon = \left(\partial_m + \frac{1}{2}\omega_{m89}\Gamma^{89} + \frac{1}{2}Q_{m12}\rho^{12}\right)\epsilon = 0. \tag{2.11}$$

For the existence of non-vanishing solutions, the action of two connections on some components of ϵ must be pure gauge. To study this condition, it is convenient to decompose the parameter ϵ into four parts $\epsilon_{s,r}$ according to $SO(2)_{89}$ and $SO(2)_R$ charges so that

$$\frac{1}{2}\Gamma_{89}\epsilon_{s,r} = is\epsilon_{s,r}, \quad \frac{1}{2}\rho_{12}\epsilon_{s,r} = ir\epsilon_{s,r}, \tag{2.12}$$

where both the indices s and r take values in $\{+\frac{1}{2}, -\frac{1}{2}\}$. We also decompose λ^i in the same way into λ^i_{sr}. For distinction we use $s = \{\uparrow, \downarrow\}$ for $SO(2)_{89}$ and $r = \{+, -\}$ for $SO(2)_R$. We also introduce

$i = \{\oplus, \ominus\}$ for the complex basis of $SO(2)_R$ vectors, which carry $SO(2)_R$ charge ± 1. See the appendix for details.

Let us require the solution to be half BPS. Without loss of generality we can assume that $\epsilon_{\uparrow +}$ and its Majorana conjugate $\epsilon_{\downarrow -}$ correspond to the unbroken supersymmetries. The other components are set to zero: $\epsilon_{\downarrow +} = \epsilon_{\uparrow -} = 0$. Then, the non-vanishing components of $\delta\lambda$ are

$$\delta\lambda^i_{\downarrow -} = P^{i\oplus}_{z^*}\epsilon_{\uparrow +}, \tag{2.13}$$

and its complex conjugate.

Before proceeding, it would be instructive to check the consistency of the quantum numbers in Eq. (2.13). Let us first consider the $SO(2)_{89}$ quantum numbers. The left-hand side has the lower index $\downarrow$. This means the component carries $SO(2)_{89}$ charge (spin) $-1/2$. On the right-hand side, the parameter ϵ has lower index $\uparrow$ which means $SO(2)_{89}$ spin $+1/2$. In addition, P has lower index z^* and this component carries $SO(2)_{89}$ spin -1. Therefore, both left- and right-hand sides carry the same $SO(2)_{89}$ spin $-1/2$. The coincidence of the $SO(2)_R$ charge can be confirmed in a similar way. The index i is common for the left- and right-hand sides and thus let us focus on the other indices. On the left-hand side we have the lower $-$ index and this means it carries $SO(2)_R$ charge $-1/2$. On the right-hand side there are the upper $\oplus$ index on P and the lower $+$ index on ϵ, which carry $SO(2)_R$ charges -1 and $+1/2$, respectively. Therefore, the left- and right-hand sides carry the same $SO(2)_R$ charge $-1/2$. The charge counting we have just explained is quite useful when we extract the condition imposed on P from Killing spinor equations associated with dilatino fields in different dimensions.

The vanishing of Eq. (2.13) means

$$P^{\oplus\oplus}_{z^*} = 0. \tag{2.14}$$

($P^{\ominus\oplus}_{z^*}$ is identically zero due to the traceless condition.) We want to solve this with respect to the scalar fields $L^\alpha{}_i$. For this purpose it is convenient to gauge fix the local $SO(2)_R$ symmetry so that the matrix L is given by

$$L = K(\tau), \tag{2.15}$$

where $K(\tau)$ for a complex number τ in the upper half-plane is the following 2×2 matrix:

$$K(\tau) = \frac{1}{\sqrt{\tau_2}}\begin{pmatrix} 1 & 0 \\ \tau_1 & \tau_2 \end{pmatrix}, \quad \tau \equiv \tau_1 + i\tau_2 \in H_+. \tag{2.16}$$

Then P and Q have the components

$$P^{ij} = \frac{1}{2\tau_2}\begin{pmatrix} -d\tau_2 & d\tau_1 \\ d\tau_1 & d\tau_2 \end{pmatrix}, \quad Q^{ij} = \frac{d\tau_1}{2\tau_2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.17}$$

In this gauge, Eq. (2.14) gives

$$P^{\oplus\oplus}_{z^*} = \frac{i}{2\tau_2}\partial_{z^*}\tau = 0. \tag{2.18}$$

Namely, τ must be a holomorphic function of z.

Now let us turn to the equation $\delta\psi_m = D_m\epsilon = 0$. The components including the non-vanishing parameters $\epsilon_{\uparrow+}$ and $\epsilon_{\downarrow-}$ are

$$D_m\epsilon_{\uparrow+} = \left(\partial_m + \frac{i}{2}(\omega_{m89} + Q_{m12})\right)\epsilon_{\uparrow+} = 0 \tag{2.19}$$

and its complex conjugation. For Eq. (2.19) to have solutions with $\epsilon_{\uparrow+} \neq 0$, the net connection $\omega_{89} + Q_{12}$ must be pure gauge, and we can take the gauge with $\omega_{89} + Q_{12} = 0$. The explicit forms of the spin connection and the $SO(2)_R$ connection are

$$\omega_{89} = i\frac{\partial g}{g}dz - i\frac{\overline{\partial} g}{g}dz^*, \quad Q_{12} = -\frac{d\tau_1}{2\tau_2} = -i\frac{\partial\tau_2}{2\tau_2}dz + i\frac{\overline{\partial}\tau_2}{2\tau_2}dz^*, \tag{2.20}$$

where we used holomorphy of τ in the last equality. From $\omega_{89} + Q_{12} = 0$ we obtain

$$\frac{dg}{g} = \frac{d\tau_2}{2\tau_2}, \tag{2.21}$$

and this is solved by

$$g = c\sqrt{\tau_2}, \tag{2.22}$$

where c is an arbitrary real positive constant, which can be absorbed by the coordinate change $cx^m \to x^m$.

The solution is summarized as follows:

$$L^\alpha{}_i = K(\tau), \tag{2.23}$$

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + \tau_2 dx^m dx^m, \tag{2.24}$$

$$\tau(z) = \tau_1 + i\tau_2, \quad \tau_2 > 0. \tag{2.25}$$

This solution is specified by the single holomorphic function $\tau(z)$.

The imaginary part of $\tau(z)$ must be positive, and no globally defined holomorphic function satisfies this condition unless $\tau(z)$ is a constant. For a non-trivial solution $\tau(z)$ must be given as a multi-valued solution with singularities. These singularities are regarded as branes, and the monodromies associated with the multi-valueness specify the charges of the branes. It is well known that these singularities are classified by the Kodaira classification, and we do not give a detailed explanation of this.

In the following we will construct solutions in lower dimensions, and find that they are also described by holomorphic functions with positive imaginary part. Because the classification of the singularity can be done in a similar way to the ten-dimensional case, and it has been well studied, we only focus on the local structure of solutions.

2.2. $D = 9$

Let us start the analysis in lower dimensions following the prescription in the last subsection. The scalar and spinor fields in the nine-dimensional $\mathcal{N} = 2$ supergravity [17] are summarized in Table 3.

Table 3. The quantum numbers of scalar and spinor fields in the nine-dimensional $\mathcal{N} = 2$ supergravity.

	$SL(2)$	$SO(2)$
$L^{\alpha}{}_{i}$	$\mathbf{2}$	± 1
φ	$\mathbf{1}$	0
ψ_M	$\mathbf{1}$	$\pm\frac{1}{2}$
λ_i	$\mathbf{1}$	$\pm\frac{3}{2}$
$\widetilde{\lambda}$	$\mathbf{1}$	$\pm\frac{1}{2}$
ϵ	$\mathbf{1}$	$\pm\frac{1}{2}$

The fields λ_i are subject to the gamma-traceless condition $\rho_i\lambda_i = 0$. The Lagrangian is

$$\begin{aligned}\mathcal{L} = &-\frac{e}{4}R - \frac{i}{2}e(\psi_L\Gamma^{LMN}D_M\psi_N)\\ &+\frac{e}{4}(P_{Mij})^2 + \frac{i}{2}e(\lambda_i\Gamma^M D_M\lambda_i) + \frac{i}{2}e(\psi_M\rho_i\Gamma^N\Gamma^M\lambda_j)P_{Nij}\\ &+\frac{e}{2}(\partial_M\varphi)^2 + \frac{i}{2}e(\widetilde{\lambda}\Gamma^M D_M\widetilde{\lambda}) + \frac{i}{\sqrt{2}}e(\psi_M\Gamma^N\Gamma^M\widetilde{\lambda}_j)\partial_N\varphi + \cdots,\end{aligned} \tag{2.26}$$

where the dots represent terms with gauge fields and four-fermion terms, which play no role in the following analysis. The supersymmetry transformation rules for the spinor fields are

$$\delta\psi_M = D_M\epsilon, \tag{2.27}$$

$$\delta\lambda_i = \frac{1}{2}P_{Mij}\Gamma^M\rho_j\epsilon, \tag{2.28}$$

$$\delta\widetilde{\lambda} = \frac{1}{\sqrt{2}}D_M\varphi\Gamma^M\epsilon. \tag{2.29}$$

We want to obtain codimension-2 brane solutions by solving the Killing spinor equations. We use x^μ ($\mu = 0, 1, \ldots, 6$) and x^m ($m = 7, 8$) for longitudinal and transverse coordinates, respectively. We take the ansatz

$$L^{\alpha}{}_{i} = L^{\alpha}{}_{i}(x^m), \quad \varphi = \varphi(x^m), \quad e^{\hat{\mu}} = \delta^{\hat{\mu}}_{\mu}dx^\mu, \quad e^a = g(x^m)\delta^a_m dx^m, \quad \epsilon = \epsilon(x^m). \tag{2.30}$$

In fact, the solution is almost the same as that of the type IIB case. Although we have extra fields φ and $\widetilde{\lambda}$ compared to the ten-dimensional case, the condition $\delta\widetilde{\lambda} = 0$ forces φ to be constant;

$$0 = \delta\widetilde{\lambda} = \frac{1}{\sqrt{2}}\partial_m\varphi\Gamma^m\epsilon \quad \to \quad \partial_m\varphi = 0. \tag{2.31}$$

Therefore, we can forget about $\widetilde{\lambda}$ and φ, and the remaining fields give a set of equations identical to the ten-dimensional case. After some gauge choices the general solution is given by

$$\varphi = \text{const}, \tag{2.32}$$

$$L^{\alpha}{}_{i} = K(\tau), \quad \tau = \tau_1 + i\tau_2 : \text{holomorphic function}, \tag{2.33}$$

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + \tau_2 dx^m dx^m. \tag{2.34}$$

A solution is specified by a single holomorphic function $\tau(z)$ and a constant vacuum expectation value of φ. Codimenison-2 brane solutions appear as singularities of the function $\tau(z)$.

Table 4. Quantum numbers of scalar and spinor fields in eight-dimensional maximal supergravity. The $SO(2)_R$ charge of each component of a spinor is proportional to the chirality.

	$SO(3)_R$	$SO(2)_R$
$\widetilde{L}^{\widetilde{\alpha}}{}_{\widetilde{i}}$	**3**	0
$L^{\alpha}{}_{i}$	**1**	± 1
ψ_M	**2**	$\frac{1}{2}\Gamma_9$
λ_i	**2**	$\frac{3}{2}\Gamma_9$
$\widetilde{\lambda}_{\widetilde{i}}$	**4**	$-\frac{1}{2}\Gamma_9$

2.3. $D = 8$

The scalar and fermion fields in eight-dimensional maximal supergravity [18] are shown in Table 4. Classical p-brane solutions with $p = 0, 1, 3, 4$ are given in Ref. [19]. Half BPS solutions of ten-dimensional supergravity given in Ref. [20] can be regarded as codimension-2 branes in eight-dimensional supergravity. In the following we construct general 5-brane solutions without assuming a ten-dimensional supergravity description.

The scalar manifold of the eight-dimensional maximal supergravity is the direct product of two homogeneous spaces: $SL(2,\mathbb{Z})/SO(2)_R \times SL(3,\mathbb{Z})/SO(3)_R$. Each factor can be interpreted geometrically in an appropriate duality frame. The $SL(2,\mathbb{Z})/SO(2)_R$ becomes manifest when we regard the theory as T^2 compactification of type IIB theory, while $SL(3,\mathbb{Z})/SO(3)_R$ can be regarded as the moduli space associated with T^3 compactification of M-theory. The S-duality group in the type IIB picture is a subgroup of $SL(3,\mathbb{Z})$.

For each factor of the R-symmetry group $SO(2)_R \times SO(3)_R$ there is an associated dilatino field. We denote fields associated with $SO(2)_R$ and $SO(3)_R$ by λ^i and $\widetilde{\lambda}^{\widetilde{i}}$, respectively. All fermion fields have implicit spinor indices for all $SO(1,7)$, $SO(2)_R$, and $SO(3)_R$. In addition, λ and $\widetilde{\lambda}$ have $SO(2)$ and $SO(3)$ vector indices, respectively, and they satisfy the traceless conditions $\rho_i\lambda^i = 0$ and $\widetilde{\rho}_{\widetilde{i}}\widetilde{\lambda}^{\widetilde{i}} = 0$. Namely, λ and $\widetilde{\lambda}$ belong to $2_{\pm\frac{3}{2}}$ and $4_{\pm\frac{1}{2}}$, respectively, of $SO(2)_R \times SO(3)_R$.

The Lagrangian is

$$\begin{aligned}\mathcal{L} = & \frac{e}{4}R + \frac{e}{2}(\psi_M\Gamma^{MNP}D_N\psi_P) \\ & - \frac{e}{4}(P_M{}^{ij})^2 + \frac{e}{2}(\lambda_i\Gamma^N D_N\lambda_i) + \frac{e}{2}P_M{}^{ij}(\psi_N\Gamma^M\Gamma^N\rho_i\lambda_j) \\ & - \frac{e}{4}(\widetilde{P}_M{}^{\widetilde{i}\widetilde{j}})^2 - \frac{e}{2}(\widetilde{\lambda}_{\widetilde{i}}\Gamma^N D_N\widetilde{\lambda}_{\widetilde{i}}) + i\frac{e}{2}\widetilde{P}_M{}^{\widetilde{i}\widetilde{j}}(\psi_N\Gamma^M\Gamma^N\Gamma_{\widetilde{i}}\widetilde{\lambda}_{\widetilde{j}}) + \cdots \end{aligned} \tag{2.35}$$

where the dots represent four-fermion terms and terms with gauge fields. The supersymmetry transformation laws of fermions are

$$\delta\psi_M = D_M\epsilon, \tag{2.36}$$

$$\delta\lambda^i = \frac{1}{2}P_M{}^{ij}\Gamma^M\rho_j\epsilon, \tag{2.37}$$

$$\delta\widetilde{\lambda}^{\widetilde{i}} = \frac{i}{2}\widetilde{P}_M{}^{\widetilde{i}\widetilde{j}}\Gamma^M\widetilde{\rho}_{\widetilde{j}}\epsilon. \tag{2.38}$$

We are interested in codimension-2 brane solutions and we use x^μ ($\mu = 0, 1, \ldots, 5$) and x^m ($m = 6, 7$) for longitudinal and transverse coordinates, respectively. We take the following ansatz:

$$L^\alpha{}_i = L^\alpha{}_i(x^m), \quad \widetilde{L}^{\widetilde{\alpha}}{}_{\widetilde{i}} = \widetilde{L}^{\widetilde{\alpha}}{}_{\widetilde{i}}(x^m), \quad e^{\hat{\mu}} = \delta^{\hat{\mu}}_\mu dx^\mu, \quad e^a = g(x^m)\delta^a_m dx^m, \quad \epsilon = \epsilon(x^m). \tag{2.39}$$

The covariant derivative $D_M\epsilon$ contains three connections, ω, Q, and $\widetilde{Q}$, corresponding to $SO(2)_{67}$, $SO(2)_R$, and $SO(3)_R$, respectively. For the existence of non-trivial solution to $\delta\psi_m = 0$, the actions of the three connections to some components of ϵ must be pure gauge. For this to be the case, non-vanishing components of $SO(3)_R$ connection $\widetilde{Q}$ should be in a certain $SO(2)$ subgroup of $SO(3)_R$. We can take the gauge such that it is rotation of the $\widetilde{12}$ plane and

$$\widetilde{Q}^{\widetilde{i}\widetilde{3}} = 0. \tag{2.40}$$

After taking this gauge, we have three $SO(2)$ connections, ω_{67}, Q_{12}, and $\widetilde{Q}_{\widetilde{12}}$. As in the ten-dimensional case it is convenient to divide the parameter ϵ into components $\epsilon_{sr\widetilde{r}}$ so that

$$\frac{1}{2}\Gamma_{67}\epsilon_{sr\widetilde{r}} = is\epsilon_{sr\widetilde{r}}, \quad \frac{1}{2}\rho_{12}\epsilon_{sr\widetilde{r}} = ir\epsilon_{sr\widetilde{r}}, \quad \frac{1}{2}\widetilde{\rho}_{12}\epsilon_{sr\widetilde{r}} = i\widetilde{r}\epsilon_{sr\widetilde{r}}, \tag{2.41}$$

where all of s, r, and $\widetilde{r}$ take values in $\{+\frac{1}{2}, -\frac{1}{2}\}$. For distinction we introduce the notation $s \in \{\uparrow, \downarrow\}$ for $SO(2)_{67}$, $r \in \{+, -\}$ for $SO(2)_R$, and $\widetilde{r} \in \{\widetilde{+}, \widetilde{-}\}$ for $SO(3)_R$. We also introduce $\{\oplus, \ominus\}$ for the complex basis of an $SO(2)_R$ vector and $\{\widetilde{\oplus}, \widetilde{\ominus}, \widetilde{3}\}$ for the basis of an $SO(3)_R$ vector that diagonalizes $SO(2)_{\widetilde{12}}$.

The six-dimensional chirality of $\epsilon_{sr\widetilde{r}}$ is given by s and $\widetilde{r}$ as

$$\gamma_7\epsilon_{sr\widetilde{r}} = \mathrm{sign}(s\widetilde{r})\epsilon_{sr\widetilde{r}}. \tag{2.42}$$

We want to consider a solution in which some of the $\epsilon_{sr\widetilde{r}}$ are preserved. Without loss of generality, we can suppose that $\epsilon_{\uparrow+\widetilde{+}}$ and its complex conjugate $\epsilon_{\downarrow-\widetilde{-}}$ are non-vanishing. Both of them have positive six-dimensional chirality Eq. (2.42), and they generate six-dimensional $\mathcal{N} = (1, 0)$ supersymmetry.

Let us consider the condition $\delta\lambda = 0$ first. The component of $\delta\lambda$ depending on $\epsilon_{\uparrow+\widetilde{+}}$ is

$$0 = \delta\lambda^{\oplus}_{\downarrow-\widetilde{+}} = P^{\oplus\oplus}_{z^*}\epsilon_{\uparrow+\widetilde{+}}. \tag{2.43}$$

For this to hold for $\epsilon_{\uparrow+\widetilde{+}} \neq 0$, $P^{\oplus\oplus}_{z^*}$ must vanish. This is the same as Eq. (2.14) in Sect. 2.1, and the solution is given by $L = K(\tau)$ where $K(\tau)$ is defined in Eq. (2.16) with a holomorphic function $\tau(z)$.

We can also obtain a similar condition for $\widetilde{L}$ from $\delta\widetilde{\lambda} = 0$. The components of $\delta\widetilde{\lambda}$ depending on $\epsilon_{\uparrow+\widetilde{+}}$ are $\delta\widetilde{\lambda}^{\widetilde{i}}_{\downarrow+\widetilde{\pm}}$, and we obtain the following Killing spinor equations:

$$0 = \delta\widetilde{\lambda}^{\widetilde{i}}_{\downarrow+\widetilde{+}} = \frac{i}{\sqrt{2}}\widetilde{P}^{\widetilde{i}\widetilde{3}}_{z^*}\epsilon_{\uparrow+\widetilde{+}}, \tag{2.44}$$

$$0 = \delta\widetilde{\lambda}^{\widetilde{i}}_{\downarrow+\widetilde{-}} = i\widetilde{P}^{\widetilde{i}\widetilde{\oplus}}_{z^*}\epsilon_{\uparrow+\widetilde{+}}. \tag{2.45}$$

Equation (2.44) requires $\widetilde{P}^{\widetilde{i}\widetilde{3}} = 0$, and combining this with Eq. (2.40) we conclude that L is essentially an $SL(2)$ element. Namely, in an appropriate choice of gauge it is given by

$$\widetilde{L}^{\widetilde{\alpha}}{}_{\widetilde{i}} = \widetilde{L}_0 \begin{pmatrix} K(\widetilde{\tau}) & 0 \\ 0 & 1 \end{pmatrix} \quad (\widetilde{\tau} \equiv \widetilde{\tau}_1 + i\widetilde{\tau}_2 \in H_+), \tag{2.46}$$

where $\widetilde{L}_0 \in SL(3, \mathbb{R})$ is a constant matrix. The condition $\widetilde{P}^{\widetilde{i}\widetilde{\oplus}}_{z^*} = 0$ obtained from Eq. (2.45) requires the function $\widetilde{\tau}$ to be a holomorphic function of z.

Finally, we can determine the function g by using $\delta\psi_m = D_m\epsilon = 0$. For this equation to hold for $\epsilon_{\uparrow+\tilde{+}} \neq 0$, the sum of three connections ω, Q, and $\widetilde{Q}$ must be pure gauge, and we can take the gauge in which

$$\omega_{m67} + Q_{m12} + \widetilde{Q}_{m\tilde{1}\tilde{2}} = 0. \tag{2.47}$$

This gives the differential equation

$$\frac{i}{g}\partial_z g = \frac{i}{2\tau_2}\partial_z\tau_2 + \frac{i}{2\widetilde{\tau}_2}\partial_z\widetilde{\tau}_2, \tag{2.48}$$

which is solved by

$$g = c\sqrt{\tau_2\widetilde{\tau}_2}, \tag{2.49}$$

where c is a positive real constant, which can be absorbed by the coordinate change $cx^m \to x^m$.

The solution is summarized as follows:

$$L^\alpha{}_i = K(\tau), \quad \tau = \tau_1 + i\tau_2, \tag{2.50}$$

$$\widetilde{L}^{\widetilde{\alpha}}{}_{\widetilde{i}} = \widetilde{L}_0\begin{pmatrix} K(\widetilde{\tau}) & 0 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{L}_0 \in SL(3,\mathbb{R}), \quad \widetilde{\tau} = \widetilde{\tau}_1 + i\widetilde{\tau}_2, \tag{2.51}$$

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + \tau_2\widetilde{\tau}_2 dx^m dx^m. \tag{2.52}$$

This is the general form of $1/4$ BPS solutions. A solution is specified by two holomorphic functions $\tau(z)$ and $\widetilde{\tau}(z)$ and constant $\widetilde{L}_0 \in SL(3,\mathbb{R})$.

$1/2$ BPS solutions are realized as special cases of this solution. Let us consider the case in which the supersymmetries associated with $\epsilon_{\uparrow-\tilde{+}}$ and its conjugate $\epsilon_{\downarrow+\tilde{-}}$ are also preserved in addition to $\epsilon_{\uparrow+\tilde{+}}$ and its conjugate $\epsilon_{\downarrow-\tilde{-}}$. Equation (2.42) shows that $\epsilon_{\uparrow-\tilde{+}}$ and $\epsilon_{\downarrow+\tilde{-}}$ have negative six-dimensional chirality and we have $\mathcal{N} = (1,1)$ supersymmetry in this case. The Killing spinor equations including $\epsilon_{\uparrow-\tilde{+}}$ are

$$0 = \delta\lambda^{\ominus}_{\downarrow+\tilde{+}} = P^{\ominus\ominus}_{z^*}\epsilon_{\uparrow-\tilde{+}}, \tag{2.53}$$

$$0 = \delta\widetilde{\lambda}^{\widetilde{i}}_{\downarrow-\tilde{+}} = \frac{i}{\sqrt{2}}\widetilde{P}^{\widetilde{i}\widetilde{3}}_{z^*}\epsilon_{\uparrow-\tilde{+}}, \tag{2.54}$$

$$0 = \delta\widetilde{\lambda}^{\widetilde{i}}_{\downarrow-\tilde{-}} = i\widetilde{P}^{\widetilde{i}\widetilde{\oplus}}_{z^*}\epsilon_{\uparrow-\tilde{+}}, \tag{2.55}$$

$$0 = \delta\psi_{m,\uparrow-\tilde{+}} = D_m\epsilon_{\uparrow-\tilde{+}}. \tag{2.56}$$

We have the additional condition $P^{--}_{z^*} = 0$ from Eq. (2.53), and this requires τ to be anti-holomorphic. This means τ must be a constant. Then the other equations hold.

There is another type of $1/2$ BPS solution with $\epsilon_{\uparrow+\tilde{-}}, \epsilon_{\downarrow-\tilde{+}} \neq 0$. Equation (2.42) shows that these components have positive six-dimensional chirality, and we obtain $\mathcal{N} = (2,0)$ supersymmetry in six dimensions. The Killing spinor equations including $\epsilon_{\uparrow+\tilde{-}}$ are

$$0 = \delta\lambda^{\oplus}_{\downarrow-\tilde{-}} = P^{\oplus\oplus}_{z^*}\epsilon_{\uparrow+\tilde{-}}, \tag{2.57}$$

$$0 = \delta\widetilde{\lambda}^{\widetilde{i}}_{\downarrow+\tilde{-}} = \frac{i}{\sqrt{2}}\widetilde{P}^{\widetilde{i}\widetilde{3}}_{z^*}\epsilon_{\uparrow+\tilde{-}}, \tag{2.58}$$

Table 5. The non-trivial 5-brane solutions in $D = 8$ supergravity and worldvolume supersymmetries.

	$\mathcal{N} = (1,0)$	$\mathcal{N} = (1,1)$	$\mathcal{N} = (2,0)$
τ	holomorphic	constant	holomorphic
$\widetilde{\tau}$	holomorphic	holomorphic	constant

Table 6. The field contents of seven-dimensional maximal supergravity. Anti-symmetric tensor fields are omitted.

	$SO(5)_R$	
$e_M^{\widehat{M}}$	1	vielbein
$L^{\alpha}{}_i$	5	scalars
ψ_M	4	gravitino
λ_i	16	dilatino, $\rho_i \lambda_i = 0$

$$0 = \delta \widetilde{\widetilde{\lambda}}^{\widetilde{i}}_{\downarrow + \widetilde{+}} = \widetilde{\widetilde{P}}^{\widetilde{i}\widetilde{\ominus}}_{z^*} \epsilon_{\uparrow + \widetilde{-}}, \tag{2.59}$$

$$0 = \delta \psi_{m, \uparrow + \widetilde{-}} = D_m \epsilon_{\uparrow + \widetilde{-}}. \tag{2.60}$$

Equation (2.59) gives the new condition $\widetilde{\widetilde{P}}^{\widetilde{i}\widetilde{\ominus}}_{z^*} = 0$, and this means $\widetilde{\tau}$ is a constant. Then the other conditions are satisfied.

Finally, let us consider the case with $\epsilon_{\downarrow + \widetilde{+}}$ and $\epsilon_{\uparrow - \widetilde{-}}$ non-vanishing. The Killing spinor equations including $\epsilon_{\downarrow + \widetilde{+}}$ are

$$0 = \delta \lambda^{\oplus}_{\uparrow - \widetilde{+}} = P_z^{\oplus\oplus} \epsilon_{\downarrow + \widetilde{+}}, \tag{2.61}$$

$$0 = \delta \widetilde{\widetilde{\lambda}}^{\widetilde{i}}_{\uparrow + \widetilde{+}} = \frac{i}{\sqrt{2}} \widetilde{\widetilde{P}}^{\widetilde{i}\widetilde{3}}_z \epsilon_{\downarrow + \widetilde{+}}, \tag{2.62}$$

$$0 = \delta \widetilde{\widetilde{\lambda}}^{\widetilde{i}}_{\uparrow + \widetilde{-}} = i \widetilde{\widetilde{P}}^{\widetilde{i}\widetilde{\oplus}}_z \epsilon_{\downarrow + \widetilde{+}}, \tag{2.63}$$

$$0 = \delta \psi_m = D_m \epsilon_{\downarrow + \widetilde{+}}. \tag{2.64}$$

The first gives the additional condition $P_z^{\oplus\oplus} = 0$, which requires τ to be a constant, and the third gives $\widetilde{\widetilde{P}}^{\widetilde{i}\widetilde{\oplus}}_z = 0$, and this means constant $\widetilde{\tau}$. Then, the solution becomes the trivial flat solution, and all supersymmetries are preserved.

We summarize the non-trivial BPS solutions in Table 5. As shown there, two holomorphic functions τ and $\widetilde{\tau}$ correspond to two types of branes. Namely, singularities of τ and $\widetilde{\tau}$ give 5-branes with $\mathcal{N} = (2,0)$ and $\mathcal{N} = (1,1)$ supersymmetry, respectively. 1/4 BPS solutions with $\mathcal{N} = (1,0)$ supersymmetry are regarded as simple superposition of two types of branes.

The most general 1/4 BPS solutions are embedded in $SL(2) \times SL(2) \subset SL(2) \times SL(3)$. These two $SL(2)$ factors are manifest in the type IIB frame. Namely, the $SL(2)$ factor that is a subgroup of $SL(3)$ can be associated with the axio-dilaton field in type IIB theory, and the other $SL(2)$ is associated with the internal space T^2. From the viewpoint of F-theory the 1/4 BPS solution can be regarded as a compactification of the F-theory in a Calabi–Yau realized as T^4 fibration over $\mathbb{C}$.

2.4. $D = 7$

The seven-dimensional maximal supergravity has the field contents in Table 6 [21,22]. The scalar manifold of seven-dimensional maximal supergravity is $SL(5)/SO(5)$. There is no duality frame which manifests the whole of the duality group $SL(5,\mathbb{Z})$ and the R-symmetry group $SO(5)_R$. When we regard the system as the T^4 compactification of M-theory $SL(4)/SO(4)$ becomes manifest, while T^3 compactification of type IIB theory manifests $SL(3)/SO(3) \times SL(2)/SO(2)$. Combining these we obtain the full symmetry.

The relevant part of the Lagrangian is

$$\begin{aligned}\mathcal{L} &= \frac{e}{2}R - \frac{e}{2}(\overline{\psi}_M \Gamma^{MNP} D_N \psi_P) \\ &\quad - \frac{e}{2}P_{Mij}P^{Mij} - \frac{e}{2}(\overline{\lambda}^i \Gamma^M D_M \lambda_i) + \frac{e}{2}(\overline{\psi}_M \Gamma^N \Gamma^M \rho^i \lambda^j) P_{Nij},\end{aligned} \tag{2.65}$$

and the supersymmetry transformation rules for fermions are

$$\begin{aligned}\delta\psi_M &= D_M \epsilon, \\ \delta\lambda_i &= \frac{1}{2}P_{Mij}\Gamma^M \rho^j \epsilon.\end{aligned} \tag{2.66}$$

The transformation parameter ϵ belongs to the spinor representation of $H = SO(5)$, and the covariant derivative $D_M\epsilon$ includes the connection Q_{Mij}.

We are interested in codimension-2 brane solutions and we use x^μ ($\mu = 0, 1, \ldots, 4$) and x^m ($m = 5, 6$) for longitudinal and transverse coordinates, respectively. By assuming the Poincaré invariance in the five dimensions parallel to the brane, we take the following ansatz:

$$L = L(x^m), \quad e^{\widehat{\mu}} = \delta^{\widehat{\mu}}_\mu dx^\mu, \quad e^a = g(x^m)\delta^a_m dx^m, \quad \epsilon = \epsilon(x^m). \tag{2.67}$$

Let us first consider the case with the minimum number of unbroken supersymmetries. The supersymmetry parameter ϵ belongs to the **4** of $SO(5)_R$ symmetry, and in the minimum case we have only one non-vanishing component. Then the R-symmetry is broken to $SU(2) \times U(1) \subset SO(5)_R$.

It is convenient to consider the intermediate subgroup $SU(2)_l \times SU(2)_r \sim SO(4) \subset SO(5)_R$. The parameter ϵ is decomposed into four irreducible representations $(2,1)_{\pm\frac{1}{2}}$ and $(1,2)_{\pm\frac{1}{2}}$ of $SU(2)_l \times SU(2)_r \times SO(2)_{56}$. ($SO(2)_{56}$ is the local Lorentz symmetry in the transverse space.) We denote them as

$$\epsilon \to \{\epsilon_{s,a}, \epsilon_{s,\dot{a}}\}, \tag{2.68}$$

where $s = \uparrow, \downarrow$ represent the $SO(2)_{56}$ charges and a and $\dot{a}$ are indices for $SU(2)_l$ and $SU(2)_r$, respectively. The fields P and Q are decomposed as

$$Q_{ij} \to \{Q_{a\dot{a}}, Q_{(ab)}, Q_{(\dot{a}\dot{b})}\}, \quad P_{ij} \to \{P, P_{a\dot{b}}, P_{(ab)(\dot{a}\dot{b})}\}, \tag{2.69}$$

and the dilatino λ as

$$\lambda \to \{\lambda_{sa}, \lambda_{s\dot{a}}, \lambda_{s(ab)\dot{a}}, \lambda_{sa(\dot{a}\dot{b})}\}, \tag{2.70}$$

where a pair of indices in parenthesis are symmetric. If we choose $\epsilon_{\dot{1}}$ as the component for the unbroken supersymmetry, $SU(2)_r$ is broken to $U(1)_r$. The connection Q should take its value in $SU(2)_l \times U(1)_r$. Namely, $Q_{a\dot{a}} = Q_{\dot{1}\dot{1}} = Q_{\dot{2}\dot{2}} = 0$ and the only non-vanishing components are

$$Q_{(ab)}, \quad Q_{\dot{1}\dot{2}}. \tag{2.71}$$

The $\epsilon_{\uparrow\dot{a}}$ appear in the supersymmetry transformation as

$$\delta\lambda_{\downarrow a} = P_{z^* a\dot{b}}\epsilon_{\uparrow}^{\dot{b}}, \tag{2.72}$$

$$\delta\lambda_{\downarrow\dot{a}} = P_{z^*}\epsilon_{\uparrow\dot{a}}, \tag{2.73}$$

$$\delta\lambda_{\downarrow(ab)\dot{a}} = P_{z^*(ab)(\dot{a}\dot{b})}\epsilon_{\uparrow}^{\dot{b}}, \tag{2.74}$$

$$\delta\lambda_{\downarrow a(\dot{a}\dot{b})} = P_{z^* a(\dot{a}}\epsilon_{\uparrow\dot{b})}. \tag{2.75}$$

(We omitted the numerical coefficients that are not important here.) If we require $\delta\lambda = 0$ for $\epsilon_{\uparrow\dot{1}} \neq 0$, we obtain $P_{z^*} = P_{z^* a\dot{a}} = P_{z^*(ab)(\dot{a}\dot{2})} = 0$ and the only non-vanishing components of P_{z^*} are

$$P_{z^*(ab)(\dot{1}\dot{1})}. \tag{2.76}$$

(We also have similar conditions for P_z from the equations containing $\epsilon_{\downarrow\dot{2}} \sim (\epsilon_{\uparrow\dot{1}})^*$.) Equations (2.71) and (2.76) show that non-vanishing components of P and Q are associated with a subgroup $SO(4) \subset SO(5)_R$. As we mentioned above, the $SO(4)$ subgroup of $SO(5)_R$ can be realized geometrically if we regard the theory as T^4 compactification of M-theory.

We want to give the scalar fields L such that P and Q have only the non-vanishing components of Eqs. (2.76) and (2.71). Unfortunately, we have not obtained the answer. To simplify the problem, let us consider a restricted case with $P_{(12)} = 0$. Then $F^{(Q)} = P \wedge P$ takes value in the Cartan part of $SU(2) \times U(1)$, and we can take the gauge such that $Q_{(11)} = Q_{(22)} = 0$; then the non-vanishing components are

$$Q_{(12)}, \quad Q_{(\dot{1}\dot{2})}, \quad P_{(11)(\dot{1}\dot{1})}, \quad P_{(22)(\dot{1}\dot{1})}. \tag{2.77}$$

In this case, with an appropriate real basis, the 5×5 matrices P and Q are block diagonal matrices in the following form:

$$Q = \begin{pmatrix} Q' & & \\ & Q'' & \\ & & 0 \end{pmatrix}, \quad P = \begin{pmatrix} P' & & \\ & P'' & \\ & & 0 \end{pmatrix}. \tag{2.78}$$

Therefore, the solution reduces to the superposition of two copies of solutions for the $SL(2,\mathbb{R})/SO(2)$ scalar manifold. In the same way as in higher dimensions, each $SL(2,\mathbb{R})$ part can be expressed in terms of a holomorphic function. Let the two holomorphic functions be τ' and τ''. The solution is given by

$$L^\alpha{}_i = \begin{pmatrix} K(\tau') & & \\ & K(\tau'') & \\ & & 0 \end{pmatrix}, \tag{2.79}$$

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + \tau_2'\tau_2'' dx^m dx^m. \tag{2.80}$$

As a special case of this 1/4 BPS solution we can realize the 1/2 BPS solution. Let us consider the cases where there is another Killing spinor in addition to $\epsilon_{\dot{1}}$. There are two cases.

First, let us consider the case that ϵ_1 is also a Killing spinor. In this case, from

$$0 = \delta\lambda_{\downarrow(\dot{a}\dot{b})a} = P_{z^*(ab)(\dot{a}\dot{b})}\epsilon_{\uparrow}^{b} \tag{2.81}$$

we obtain $P_{z^*(a2)(\dot{a}\dot{b})} = 0$. Then the only non-vanishing component of P_{z^*} is $P_{z^*(11)(\dot{1}\dot{1})}$. In this case, just like the case of the 1/2 BPS solution in eight dimensions, we can show that one of τ' and τ'' must be a z-independent constant.

If two Killing spinors have the same $SO(4)_R$ chirality, Eq. (2.74) requires $P_{z^*(ab)(\dot{a}\dot{b})} = 0$. Namely, all components of P vanish. Because $F^{(Q)} = P \wedge P = 0$ we can choose a gauge with $Q = 0$. Therefore, the solution is trivial.

3. Conclusions

In this paper we investigated codimension-2 BPS solutions in maximal supergravities in 9, 8, and 7 dimensions. We assumed the Poincaré invariance along branes and vanishing of various gauge fields.

The scalar manifold of nine-dimensional maximal supergravity is $(SL(2,\mathbb{R})/SO(2)) \times \mathbb{R}$. In a BPS solution the scalar field associated with the factor $\mathbb{R}$ must be constant and play no role. Therefore, the solutions are essentially the same as those of type IIB supergravity in ten dimensions, and simply interpreted as the double dimensional reduction of type IIB 7-branes.

In eight dimensions, the scalar manifold consists of the two factors $SL(2,\mathbb{R})/SO(2)$ and $SL(3,\mathbb{R})/SO(3)$. The Killing spinor equations associated with these factors decouple, and we can solve them one by one. From the $SL(2,\mathbb{R})/SO(2)$ part we obtain 1/2 BPS branes on which six-dimensional $\mathcal{N} = (2,0)$ supersymmetry is realized, while from the $SL(3,\mathbb{R})/SO(3)$ part we obtain 1/2 BPS solutions on which six-dimensional $\mathcal{N} = (1,1)$ supersymmetry is realized. The latter is always embedded in $SL(2,\mathbb{R})/SO(2) \subset SL(3,\mathbb{R})/SO(3)$. These two types of solutions have essentially the same structure as the 7-brane solution in ten dimensions. Namely, each type of classical solution is specified by a holomorphic function and singularities of the function give branes. We also found 1/4 BPS solutions, which are specified by two holomorphic functions and are regarded as simple superposition of two types of 1/2 BPS solutions. If we regard the eight-dimensional supergravity as the T^2 compactification of type IIB theory, the two copies of $SL(2,\mathbb{R})/SO(2)$ are associated with the complex moduli of the torus and the axio-dilaton field in type IIB theory, and both are geometrically realized in F-theory.

In seven dimensions, a generic BPS solution is 1/4 BPS. We showed that such solution can be embedded in $SL(4,\mathbb{R})/SO(4) \subset SL(5,\mathbb{R})/SO(5)$. This means the solution can be realized as a geometric compactification of M-theory. We could not solve the Killing spinor solutions in the general situation. We introduced one additional restriction to simplify the problem, and then the solution is factorized into two copies of solutions associated with $SL(2,\mathbb{R})/SO(2)$. Again, similarly to the eight-dimensional case, the solution can be regarded as a simple superposition of two 1/2 BPS solutions.

Although our original motivation for this work was to find essentially new BPS branes, all the solutions we found have simple geometric realization in M- or F-theory.

Acknowledgements

We would like to thank Tetsuji Kimura for valuable discussions. The work of Y.I. was partially supported by Grant-in-Aid for Scientific Research (C) (No. 15K05044), Ministry of Education, Science and Culture, Japan.

Funding

Open Access funding: SCOAP^3.

Appendix. Notes on our notation

We denote the Dirac matrices for the $SO(2)_R$ group by ρ_i ($i = 1, 2$). We use the representation

$$\rho_1 = \sigma_x, \quad \rho_2 = \sigma_y, \tag{A1}$$

where $\sigma_{x,y,z}$ are the Pauli matrices. We use lower indices $a, b, \ldots$ for two-component spinors, and thus the matrices acting on them have lower and upper indices like $(\rho_i)_a{}^b$. For components of spinors we define the $SO(2)_R$ charge as eigenvalues of the generator $(-i/2)\rho_{12}$. This means the upper and the lower components of spinors carry the charges $+1/2$ and $-1/2$, respectively. To specify these components of a spinor χ we use the notation χ_+ and χ_-, respectively.

For the analysis of the Killing spinor equations it is convenient to use a complex basis for vectors. For example, for a vector v^i ($i = 1, 2$) we define $v^\oplus = v_\ominus = \frac{1}{\sqrt{2}}(v^1 + iv^2)$ and $v^\ominus = v_\oplus = \frac{1}{\sqrt{2}}(v^1 - iv^2)$. For ρ^i we have

$$\rho^\oplus = \rho_\ominus = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \quad \rho^\ominus = \rho_\oplus = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}. \tag{A2}$$

With this representation the lower index $\oplus$ and upper index $\ominus$ carry $SO(2)_R$ charge $+1$, while the lower $\ominus$ and upper $\oplus$ carry $SO(2)_R$ charge -1. This is checked by looking at the non-vanishing components of ρ^i. For example, the non-vanishing component of $\rho_\oplus$ is $(\rho_\oplus)_-{}^+$, and the total charge of this component must be zero. The statement above about $SO(2)_R$ charge is consistent with this.

For the local rotation symmetry in the transverse space to branes we use up and down for the $SO(2)$ charge (spin) $\pm 1/2$. With the choice of the Dirac matrices, the lower indices z and z^* carry spin $+1$ and -1, respectively.

In eight dimensions we also deal with $SO(3)_R$ symmetry. The notation is basically the same as the $SO(2)_R$ case except that we put tildes on variables and indices to distinguish them from $SO(2)_R$ objects. We specify components of spinors by eigenvalues of the Cartan generator $(-i/2)\widetilde{\rho}_{\widetilde{12}}$. $\widetilde{\chi}_{\widetilde{\pm}}$ carry the charge $\pm 1/2$, and a vector $\widetilde{v}$ has three components, $\widetilde{v}^{\widetilde{\ominus}} = \widetilde{v}_{\widetilde{\oplus}}$, $\widetilde{v}^{\widetilde{\oplus}} = \widetilde{v}_{\widetilde{\ominus}}$, and $\widetilde{v}^3 = \widetilde{v}_3$, that carry the Cartan charges $+1$, -1, and 0, respectively.

In the following we give relations of the fields in this paper and those in the references. We will not give detailed explanations for normalization of fields, spinor conventions, etc., because they are not important in our analysis of the Killing spinor equations. We focus on giving a rough correspondence between the fields used in this paper and those in the references.

$D = 10$

Ten-dimensional supergravity is given in Ref. [16]. The global symmetry $SL(2, \mathbb{R})$ is isomorphic to $SU(1, 1)$. In Ref. [16] the scalar fields are expressed as the matrix $V^a_\pm$, which is defined with a complex basis natural for $SU(1, 1)$. The real matrix $L^\alpha{}_i$ used in this paper is related to V by

$$\begin{pmatrix} V^1{}_- & V^1{}_+ \\ V^2{}_- & V^2{}_+ \end{pmatrix} = U \begin{pmatrix} L^1{}_1 & L^1{}_2 \\ L^2{}_1 & L^2{}_2 \end{pmatrix} U^\dagger, \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}. \tag{A3}$$

The dilatino field λ in Ref. [16] is defined as the field with $U(1)_R$ charge $\pm 3/2$, while we denote this as a field with vector and spinor indices. They are related by

$$\lambda \sim \lambda^\ominus_+, \quad \overline{\lambda} \sim \lambda^\oplus_-. \tag{A4}$$

Due to the ρ-traceless condition, $\lambda^\oplus_+ = \lambda^\ominus_- = 0$.

$D = 9$

The nine-dimensional maximal supergravity is given in Ref. [17]. Two dilatino fields in Ref. [17] are renamed as λ_i and $\widetilde{\lambda}$ to match the fields in the other dimensions. $SO(2)$ Dirac matrices are denoted by τ_i in Ref. [17] while we use ρ_i for them.

$D = 8$

The eight-dimensional maximal supergravity is given in Ref. [18]. The dilatino field χ_i in Ref. [18] does not satisfy the ρ-traceless condition, and we decompose it into the traceless part $\widetilde{\lambda}_i$ and the trace part λ_I. To make the $SL(2,\mathbb{R})/SO(2)$ structure manifest we combine the scalar fields ϕ and B in Ref. [18] into the matrix $L^A{}_I$. With a gauge choice like Eq. (2.15) these are related by $L = K(\tau)$ with $\tau = -2B + ie^{2\phi}$.

$D = 7$

The seven-dimensional maximal supergravity is given in Ref. [22]. In the reference the scalar matrix is denoted by Π instead of L. The notation for other fields is similar to ours.

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