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The general relativistic effects on the magnetic moment in Earth's gravity

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The magnetic moment of free fermions in the Earth's gravitational field has been studied on the basis of general relativity. Adopting the Schwarzschild metric for the background spacetime, the dipole coupling between the magnetic moment and the magnetic field has been found to be dependent on the gravity in the calculation up to the post-Newtonian order $O(1/c^2)$. The gravity dependence can be formulated by employing the effective value of the magnetic moment as a gravity-dependent quantity $\boldsymbol{\mu}_{\mathrm{m}}^{\mathrm{eff}} = (1 + 3\phi/c^2)\, \boldsymbol{\mu}_{\mathrm{m}}$ for the cases of minimal coupling, non-minimal coupling, and a mixture of the two. The gravitationally induced anomaly is found to be canceled in the experimental values of the anomalous magnetic moment measured in the Penning trap and storage ring methods.

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1. Introduction

The magnetic moment is regarded as one of the fundamental properties of elementary particles. The magnetic moment of fermions with a rest mass of m, a charge of e, and a spin of $\boldsymbol{S} = \boldsymbol{\sigma}/2$ is defined as

$$\boldsymbol{\mu}_{\mathrm{m}} \equiv \frac{\mathrm{g}}{2}\frac{e}{m}\boldsymbol{S}, \tag{1}$$

where g is the constant factor specific to each elementary particle, referred to as the gyromagnetic ratio or Landé g-factor[1]. Here we consider particles with a spin of 1/2 such as electrons, protons, and atoms placed in the Earth's gravitational field and consider if their magnetic moments, defined by Eq. (1), are independent of gravity in the same way as the mass or the electric charge. We focus on the influence of the spacetime distortion on the basis of general relativity, which is not a quantum mechanical effect mediated by the graviton at the Planck scale but a classical mechanical effect remaining even in low-energy regions.

Below, we consider observables measured using instruments fixed on the Earth's surface. General relativity requires that the translational motion of a charged particle with an electric charge of e obeys the geodesic equation

$$\frac{Du^\mu}{d\tau} \equiv \frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda = \frac{e}{m} F^{\mu\nu} u_\nu, \tag{2}$$

[1] In this paper we use units $\hbar = c = 1$.

where τ is the proper time, $u^\mu = (u^0, \boldsymbol{u})$ is the four velocity vector, $F^{\mu\nu}$ is the electromagnetic tensor, and $\Gamma^\mu_{\nu\lambda}$ is the Christoffel symbol. The Earth's gravitational field can be described by the Schwarzschild metric, approximated up to the post-Newtonian order $O(\epsilon^2)$ in the expansion series of $\epsilon = 1/c$, as

$$\begin{aligned} ds^2 &= g_{\mu\nu}\, dx^\mu\, dx^\nu \\ &= \epsilon^{-2}\big(1 + \epsilon^2 2\phi + \epsilon^4 2\phi^2\big)dt^2 - (1 - \epsilon^2 2\phi)\ (dx^2 + dy^2 + dz^2) + O(\epsilon^4), \end{aligned} \tag{3}$$

where $\phi = -GM/r$ is the Earth's gravitational potential at a distance of $r = \sqrt{x^2 + y^2 + z^2}$ from the center of a uniformly distributed spherical mass M with a radius of R (see Appendix B). Equation (2) becomes

$$\begin{aligned} \frac{d\boldsymbol{\beta}}{dt} = &\big(1 + (2\gamma^2 + 1)\epsilon^2\phi\big)\,\frac{e}{\gamma\, m}\,\big(\boldsymbol{E} + \boldsymbol{\beta} \times \boldsymbol{B} - (1 - 4\gamma^2\epsilon^2\phi)\,\boldsymbol{\beta}\,(\boldsymbol{\beta}\cdot\boldsymbol{E})\big) \\ &- \epsilon(1 + \beta^2)\nabla\phi + \epsilon(4\boldsymbol{\beta}\cdot\nabla\phi)\boldsymbol{\beta} \quad + O(\epsilon^4), \end{aligned} \tag{4}$$

where $\boldsymbol{\beta} = \boldsymbol{v}/c$, $\gamma = 1/\sqrt{1-\beta^2}$, and

$$\begin{aligned} u^0 = \frac{dt}{d\tau} &= 1/\sqrt{(1 + 2\epsilon^2\phi + 2\epsilon^4\phi^2) - (1 - 2\epsilon^2\phi)\beta^2} \\ &= \gamma\big(1 - (2\gamma^2 - 1)\epsilon^2\phi\big) \quad + O(\epsilon^4). \end{aligned} \tag{5}$$

Here we suggest that the **E** and **B** represent the electric and magnetic fields to be measured by the observer on the Earth's surface, and correspond to the field vectors in the flat spacetime.

At the limit of $\phi \to 0$, Eq. (2) results in

$$\frac{d\boldsymbol{\beta}}{dt} = \frac{e}{\gamma\, m}\big(\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B} - (\boldsymbol{E}\cdot\boldsymbol{\beta})\boldsymbol{\beta}\big), \tag{6}$$

which is the well-known equation of motion of charged particles in the flat spacetime (Minkowski spacetime) consistently with special relativity. Additionally, in the region of $\beta = v/c \ll 1$, it results in the Lorentz equation of motion

$$\frac{d(m\boldsymbol{v})}{dt} = e(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}). \tag{7}$$

For simplicity, we consider the case where the electric field is zero and the motion is limited on the horizontal plane and obtain

$$\frac{d\boldsymbol{\beta}}{dt} \simeq (1 + 3\epsilon^2\phi)\,\frac{e}{m}\boldsymbol{\beta}\times\boldsymbol{B}. \tag{8}$$

According to the correspondence in classical mechanics, the magnetic moment $\boldsymbol{\mu}_{\mathrm{m}}$ can be related to the angular momentum of a charged particle moving on a circle under a constant magnetic field, which leads to $|\boldsymbol{\mu}_{\mathrm{m}}| = e/(2m)$. Equation (8) can be written as

$$\frac{d\boldsymbol{\beta}}{dt} \simeq (1 + 3\epsilon^2\phi)\ 2\,|\boldsymbol{\mu}_{\mathrm{m}}|\,\boldsymbol{\beta}\times\boldsymbol{B}. \tag{9}$$

The factor $(1+3\epsilon^2\phi)$ represents the effect of the spacetime curvature due to the Earth's gravitational field. Here we note that $\boldsymbol{\mu}_{\mathrm{m}}$ is defined in the flat spacetime, while the magnetic field $\boldsymbol{B}$ is measured by the observer on the Earth's surface. Therefore, in this paper, we define the effective magnetic moment as

$$\boldsymbol{\mu}_{\rm m}^{\rm eff} \simeq (1 + 3\epsilon^2\phi)\ \boldsymbol{\mu}_{\rm m}, \tag{10}$$

to include the gravitational effect. The above discussion remains within the classical picture since the lepton is not treated as a fermion with quantized spin and it is not trivial to extend the influence of gravity for quantum mechanical particles. Therefore the effective magnetic moment should be examined based on the formalism compatible with both quantum mechanics and general relativity.

In quantum field theory, the magnetic moment of a fermion is related to the form factors F_1 and F_2 defined in the vertex function (see Refs. [1,2])

$$\mathcal{M}^\mu(p', p) = \gamma^\mu F_1(q^2) + \frac{i}{2m}\sigma^{\mu\nu} q_\nu F_2(q^2), \tag{11}$$

and the g-factor in the flat spacetime is defined as

$$\mathrm{g} = 2\left(F_1(0) + F_2(0)\right). \tag{12}$$

The origin of the magnetic moment varies according to the type of fermions as $F_1(0) = 1, F_2(0) = 0$ for Dirac particles, i.e., fermions with $\mathrm{g} = 2$, the internal structure and radiative corrections of which can be ignored; $F_1(0) = 0$, $F_2(0) \neq 0$ for neutrons and neutral atoms, i.e., fermions with $\mathrm{g} \neq 2$ and $e = 0$; and $F_1(0) = 1$, $F_2(0) \neq 0$ for protons and ions, i.e., fermions with $\mathrm{g} \neq 2$ and $e \neq 0$. We applied quantum mechanics in the curved spacetime in general relativity for the three types:

Type A: Dirac particles with $\mathrm{g} = 2$ and $e \neq 0$ $\quad (F_1(0) = 1,\ F_2(0) = 0)$,
Type B: fermions with $\mathrm{g} \neq 2$ and $e = 0$ $\quad (F_1(0) = 0,\ F_2(0) \neq 0)$,
Type C: fermions with $\mathrm{g} \neq 2$ and $e \neq 0$ $\quad (F_1(0) = 1,\ F_2(0) \neq 0)$.

The coupling between the fermions and the electromagnetic field corresponds to minimal coupling for Type A, non-minimal coupling for Type B, and a mixture of minimal and non-minimal coupling for Type C.

In this paper, we generalize the Dirac equation in curved spacetime by employing the covariant formulation. For simplicity, we employ the Schwarzschild metric as the background spacetime, derive the Hamiltonian of free fermions to be observed using instruments fixed on the Earth's surface up to the post-Newtonian order $O(1/c^2)$ for the above three types and evaluate the effective value of the magnetic moment. We introduce the *effective* magnetic moment of fermions in the Earth's gravitational field and discuss the influence on precision measurements of the anomalous magnetic moment of electrons and muons.

2. Effective magnetic moment in curved spacetime

The influence of general relativity in the Hamiltonian of fermions has been considered and reported in several papers [3–5]. Wajima et al. [5] employed the covariant Dirac equation

$$\left(\gamma^\mu(i\nabla_\mu) - \epsilon^{-1}m\right)\Psi = 0 \tag{13}$$

and regarded its time component

$$\begin{aligned}\mathcal{H} &\equiv i\partial_0 \\ &= -i\Gamma_0 + \frac{1}{g^{00}}\gamma^0\gamma^i(-i\partial_i - i\Gamma_i) + \epsilon^{-1}\frac{1}{g^{00}}\gamma^0 m\end{aligned} \tag{14}$$

as the Hamiltonian and calculated its expectation value up to the post-Newtonian order $O(1/c^2)$, where Ψ is the spinor function, $\nabla_\mu = \partial_\mu + \Gamma_\mu$ the covariant derivative, Γ_μ the spin connection, γ^μ the γ-matrices in the curved spacetime (Appendix A), and $\epsilon = 1/c$ a small quantity to be used as the variable for the power series expansion to identify the post-Newtonian approximation. Since the electric charge and the magnetic moment were not involved in their method, we develop and generalize their method to derive the Hamiltonian containing gravitational effects in general relativity on the basis of the covariant Dirac equation with the electric charge and the magnetic moment. Here we adopt the Heisenberg equation for the magnetic moment $\boldsymbol{\mu}_{\mathrm{m}}$

$$\frac{d\boldsymbol{S}}{dt} = \frac{1}{i}\Big[\boldsymbol{S}, \mathcal{H}\Big] = \boldsymbol{\mu}_{\mathrm{m}} \times \boldsymbol{B} \tag{15}$$

as the definition of the magnetic moment in the flat spacetime. We generalize Eq. (15) using the general relativistic Hamiltonian to evaluate the effective magnetic moment $\boldsymbol{\mu}_{\mathrm{m}}^{\mathrm{eff}}$.

2.1. *Dirac particles with* g = 2 *(Type A)*

In this section, we consider Type A, for which the form factors are $F_1(0) = 1$ and $F_2(0) = 0$, corresponding to point-like fermions, i.e., electrons and muons without radiative corrections. The covariant Dirac equation for this case is given with minimal coupling with the four potential A_μ as

$$\Big(\gamma^\mu\, i\,(\nabla_\mu + i\, eA_\mu) - \epsilon^{-1} m\Big)\ \Psi = 0, \tag{16}$$

which leads to the Hamiltonian

$$\mathcal{H}^{\mathrm{A}} = -i\,\Gamma_0 + eA_0 + \frac{1}{g^{00}}\gamma^0\gamma^i(-i\,\partial_i - i\,\Gamma_i + eA_i) + \epsilon^{-1}\frac{1}{g^{00}}\gamma^0 m, \tag{17}$$

which is the general form of the Hamiltonian of Dirac particles with g = 2 with the electric charge and the magnetic moment. It is uniquely determined for a given metric of the spacetime. Adopting the Schwarzschild metric (3), we can rewrite Eq. (17), after some algebra to rearrange it (see Appendix D), as

$$\begin{aligned}\mathcal{H}^{\mathrm{A}} &= \epsilon^{-2}\rho_3 m + \epsilon^{-1}\rho_1\boldsymbol{\sigma}\cdot(\boldsymbol{p} - e\boldsymbol{A}) + eA_0 + \rho_3 m\phi + \epsilon\,\rho_1\Big\{\boldsymbol{\sigma}\cdot(\boldsymbol{p} - e\boldsymbol{A}),\, \phi\Big\} + \epsilon^2\rho_3\frac{m\phi^2}{2}\\ &\quad + O(\epsilon^4).\end{aligned} \tag{18}$$

This Hamiltonian is a 4×4 matrix. We can obtain a 2×2 even matrix by applying the Foldy–Wouthuysen–Tani transformation [6–8]. Its left-up 2×2 matrix (the large component) $\mathcal{H}_+^{\mathrm{A}}$ given as

$$\begin{aligned}\mathcal{H}_+^{\mathrm{A}} &= \epsilon^{-2} m\\ &\quad + \frac{(\boldsymbol{\sigma}\cdot(\boldsymbol{p} - e\boldsymbol{A}))^2}{2m} + m\phi + eA_0\\ &\quad + \epsilon^2\Bigg(\frac{m\phi^2}{2} - \frac{(\boldsymbol{\sigma}\cdot(\boldsymbol{p} - e\boldsymbol{A}))^4}{8m^3} + \frac{3}{8m}\Big\{\boldsymbol{\sigma}\cdot(\boldsymbol{p} - e\boldsymbol{A}),\, \Big\{\boldsymbol{\sigma}\cdot(\boldsymbol{p} - e\boldsymbol{A}),\, \phi\Big\}\Big\}\\ &\qquad - \frac{1}{8m^2}\Big[\boldsymbol{\sigma}\cdot(\boldsymbol{p} - e\boldsymbol{A}), \Big[\boldsymbol{\sigma}\cdot(\boldsymbol{p} - e\boldsymbol{A}), eA_0\Big]\Big]\Bigg) + O(\epsilon^4)\end{aligned} \tag{19}$$

corresponds to the Hamiltonian of the Dirac particle in curved spacetime. Substituting $\boldsymbol{\mu}_\mathrm{m} = e\boldsymbol{S}/m = e\boldsymbol{\sigma}/2m$, Eq. (19) leads to

$$
\begin{aligned}
\mathcal{H}_+^\mathrm{A} = \epsilon^{-2}\, m \\
&+ \frac{(\boldsymbol{p} - e\boldsymbol{A})^2}{2m} - \boldsymbol{\mu}_\mathrm{m} \cdot \boldsymbol{B} + m\phi + eA_0 \\
&+ \epsilon^2 \left(\frac{m\phi^2}{2} - \frac{(\boldsymbol{p} - e\boldsymbol{A})^4}{8m^3} - \frac{(\boldsymbol{\mu}_\mathrm{m} \cdot \boldsymbol{B})^2}{2m} - \frac{1}{4m^2} \nabla^2 (\boldsymbol{\mu}_\mathrm{m} \cdot \boldsymbol{B}) - 3\phi\ \boldsymbol{\mu}_\mathrm{m} \cdot \boldsymbol{B} \right. \\
&\quad + \frac{1}{2m^2} (\boldsymbol{p} - e\boldsymbol{A}) \cdot \big(\boldsymbol{\mu}_\mathrm{m} \cdot \boldsymbol{B}\, (\boldsymbol{p} - e\boldsymbol{A})\big) + \frac{3}{2m} (\boldsymbol{p} - e\boldsymbol{A}) \cdot \big(\phi\, (\boldsymbol{p} - e\boldsymbol{A})\big) \\
&\quad - \frac{3\phi}{2mr^2} \boldsymbol{S} \cdot \big(\boldsymbol{r} \times (\boldsymbol{p} - e\boldsymbol{A})\big) \\
&\quad \left. - \frac{\mu_\mathrm{m}}{4m}\ \nabla \cdot (\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}) - \frac{1}{2m} \boldsymbol{\mu}_\mathrm{m} \cdot \Big((\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}) \times (\boldsymbol{p} - e\boldsymbol{A}) \Big) \right) + O(\epsilon^4). \qquad (20)
\end{aligned}
$$

Terms of the order of ϵ^{-2} and ϵ^0 are well known, while some of terms of the order of ϵ^2 are newly introduced coupling terms between the gravitational potential ϕ, the magnetic moment $\boldsymbol{\mu}_\mathrm{m}$, and the magnetic field $\boldsymbol{B}$. The effective magnetic moment is given as

$$
\boldsymbol{\mu}_\mathrm{m}^\mathrm{eff} \simeq\ (1 + 3\epsilon^2\phi)\ \boldsymbol{\mu}_\mathrm{m} \qquad \text{(for Type A)} \qquad (21)
$$

for sufficiently slow particles moving in a static spacetime. The result is consistent with the *gravitationally modified magnetic moment* obtained in the *TH*$\epsilon\mu$ formalism [9,10][2]. The above discussions are restricted to Dirac fermions minimally coupled to electromagnetic fields. Non-minimal coupling cases such as neutrons and spatially spread cases such as fermions dressed with renormalization and quantum radiative corrections are not involved.

2.2. *Fermions with* g $\neq 2, e = 0$ *(Type B)*

Neutrons and neutral atoms correspond to the case of form factors of $F_1(0) = 0$ and $F_2(0) \neq 0$ since $e = 0$. Since they cannot minimally couple to the electromagnetic field, their covariant Dirac equation contains the following non-minimal coupling between the magnetic moment μ_m and the electromagnetic tensor $F_{\mu\nu}$:

$$
\left(\gamma^\mu\, i\, \nabla_\mu - \epsilon^{-1} m - \epsilon\, \frac{1}{2} \mu_\mathrm{m}\, \sigma^{\mu\nu} F_{\mu\nu} \right) \Psi = 0, \qquad (22)
$$

where $\sigma^{\mu\nu}$ and $F_{\mu\nu}$ are second-rank covariant tensors (see Appendix A). The Hamiltonian is calculated as

$$
\mathcal{H}^\mathrm{B} = -i\,\Gamma_0 + \frac{1}{g^{00}} \gamma^0 \gamma^i (-i\,\partial_i - i\,\Gamma_i) + \epsilon^{-1} \frac{1}{g^{00}} \gamma^0 m \left(1 + \epsilon^2\, \frac{\mu_\mathrm{m}}{2m}\, \sigma^{\mu\nu} F_{\mu\nu} \right). \qquad (23)
$$

[2] The *TH*$\epsilon\mu$ formalism with the Schwarzschild metric corresponds to $T = 1 + \epsilon^2 2\phi + \epsilon^4 2\phi^2$ and $H = 1 - \epsilon^2 2\phi$, which leads to the gravitational dependence of $T^{1/2}/H = 1 + 3\epsilon^2\phi$ for the magnetic moment [10].

In the same manner as described in the previous section, the large component in the Schwarzschild spacetime $\mathcal{H}_+^{\mathrm{B}}$ is obtained as

$$\begin{aligned}
\mathcal{H}_+^{\mathrm{B}} &= \epsilon^{-2} m \\
&\quad + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2}{2m} + m\phi - \boldsymbol{\mu}_{\mathrm{m}}\cdot\boldsymbol{B} \\
&\quad + \epsilon^2\Bigg(\frac{m\phi^2}{2} - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})^4}{8m^3} - 3\,\phi\,\boldsymbol{\mu}_{\mathrm{m}}\cdot\boldsymbol{B} + \frac{3}{8m}\{\boldsymbol{\sigma}\cdot\boldsymbol{p},\,\{\boldsymbol{\sigma}\cdot\boldsymbol{p},\,\phi\}\} \\
&\qquad + \frac{1}{8m^2}\Big\{\boldsymbol{\sigma}\cdot\boldsymbol{p},\,\{\boldsymbol{\sigma}\cdot\boldsymbol{p},\,\boldsymbol{\mu}_{\mathrm{m}}\cdot\boldsymbol{B}\}\Big\}\Bigg) \\
&\quad - \epsilon^2\frac{i}{2m}\big[\boldsymbol{\sigma}\cdot\boldsymbol{p},\ \boldsymbol{\mu}_{\mathrm{m}}\cdot\boldsymbol{E}\big] + O(\epsilon^4) \\
&= \epsilon^{-2} m \\
&\quad + \frac{\boldsymbol{p}^2}{2m} + m\phi - \boldsymbol{\mu}_{\mathrm{m}}\cdot\boldsymbol{B} \\
&\quad + \epsilon^2\Bigg(\frac{m\phi^2}{2} - \frac{\boldsymbol{p}^4}{8m^3} - 3\phi\,\boldsymbol{\mu}_{\mathrm{m}}\cdot\boldsymbol{B} + \frac{3}{2m}\boldsymbol{p}\cdot(\phi\,\boldsymbol{p}) - \frac{3\phi}{2mr^2}\boldsymbol{S}\cdot(\boldsymbol{r}\times\boldsymbol{p}) \\
&\qquad - \frac{1}{8m^2}\nabla^2(\boldsymbol{\mu}_{\mathrm{m}}\cdot\boldsymbol{B}) + \frac{1}{4m^2}(\boldsymbol{\mu}_{\mathrm{m}}\cdot\boldsymbol{p})\,(\boldsymbol{B}\cdot\boldsymbol{p}) \\
&\qquad + \frac{1}{4m^2}(\boldsymbol{B}\cdot\boldsymbol{p})\,(\boldsymbol{\mu}_{\mathrm{m}}\cdot\boldsymbol{p}) + \frac{\mu_{\mathrm{m}}}{4m^2}(\nabla\times\boldsymbol{B})\times\boldsymbol{p} \\
&\qquad - \frac{1}{2m}\mu_{\mathrm{m}}\,\nabla\cdot\boldsymbol{E} - \frac{1}{m}\boldsymbol{\mu}_{\mathrm{m}}\cdot(\boldsymbol{E}\times\boldsymbol{p}) - \frac{i}{2m}\,\boldsymbol{\mu}_{\mathrm{m}}\cdot(\nabla\times\boldsymbol{E})\Bigg) + O(\epsilon^4).
\end{aligned} \tag{24}$$

Accordingly, the effective magnetic moment is obtained as

$$\boldsymbol{\mu}_{\mathrm{m}}^{\mathrm{eff}} \simeq (1 + 3\,\epsilon^2\,\phi)\ \boldsymbol{\mu}_{\mathrm{m}} \qquad \text{(for Type B).} \tag{25}$$

This result shows that the effective magnetic moment with the general relativistic influence up to the order of $O(\epsilon^2)$ has a common form for both Type A and Type B, in which the minimal and non-minimal coupling terms are employed in the Dirac equations (16) and (22), respectively.

2.3. *Fermions with $g \neq 2$, $e \neq 0$ (Type C)*

Spread charged fermions have the form factors of $F_1(0) = 1$ and $F_2(0) \neq 0$. This case corresponds to electrons and muons dressed with quantum radiative corrections and also composite particles such as protons and atoms containing internal structure. The corresponding covariant Dirac equation contains both minimal coupling and non-minimal coupling terms as

$$\left(\gamma^\mu\, i\,(\nabla_\mu + i\,eA_\mu) - \epsilon^{-1} m - \epsilon\,\frac{\mathrm{g}-2}{2}\frac{e}{4m}\sigma^{\mu\nu}F_{\mu\nu}\right)\Psi = 0. \tag{26}$$

The influence of radiative corrections and internal structure is contained in the last term in Eq. (26), which can be confirmed to observe that it returns to Eq. (16) to eliminate the last term corresponding to the spread of fermions. The Hamiltonian in 4 × 4-matrix format is calculated as

$$\mathcal{H}^{\mathrm{C}} = -i\,\Gamma_0 + eA_0 + \frac{1}{g^{00}}\gamma^0\gamma^i(-i\,\partial_i - i\,\Gamma_i + eA_i) + \epsilon^{-1}\frac{1}{g^{00}}\gamma^0 m\left(1 + \epsilon^2\,\frac{\mathrm{g}-2}{2}\frac{e}{4m^2}\sigma^{\mu\nu}F_{\mu\nu}\right), \tag{27}$$

and the large component in the 2 × 2-matrix format Hamiltonian $\mathcal{H}_+^{\mathrm{C}}$ as

$$\begin{aligned}
\mathcal{H}_+^{\mathrm{C}} = {}& \epsilon^{-2} m \\
&+ \frac{(\boldsymbol{\sigma}\cdot(\boldsymbol{p}-e\boldsymbol{A}))^2}{2m} + m\phi + eA_0 - \Delta\boldsymbol{\mu}_0\cdot\boldsymbol{B} \\
&+ \epsilon^2\Bigg(\frac{m\phi^2}{2} - \frac{(\boldsymbol{\sigma}\cdot(\boldsymbol{p}-e\boldsymbol{A}))^4}{8m^3} - 3\,\phi\,\Delta\boldsymbol{\mu}_0\cdot\boldsymbol{B} \\
&\qquad + \frac{3}{8m}\Big\{\boldsymbol{\sigma}\cdot(\boldsymbol{p}-e\boldsymbol{A}),\ \big\{\boldsymbol{\sigma}\cdot(\boldsymbol{p}-e\boldsymbol{A}),\ \phi\big\}\Big\} - \frac{1}{8m^2}\Big[\boldsymbol{\sigma}\cdot(\boldsymbol{p}-e\boldsymbol{A}), \big[\boldsymbol{\sigma}\cdot(\boldsymbol{p}-e\boldsymbol{A}), eA_0\big]\Big] \\
&\qquad + \frac{1}{8m^2}\Big\{\boldsymbol{\sigma}\cdot(\boldsymbol{p}-e\boldsymbol{A}),\ \{\boldsymbol{\sigma}\cdot(\boldsymbol{p}-e\boldsymbol{A}),\ \Delta\boldsymbol{\mu}_0\cdot\boldsymbol{B}\}\Big\}\Bigg) \\
&- \epsilon^2\frac{i}{2m}\big[\boldsymbol{\sigma}\cdot(\boldsymbol{p}-e\boldsymbol{A}), \Delta\boldsymbol{\mu}_0\cdot\boldsymbol{E}\big] + O(\epsilon^4) \\
= {}& \epsilon^{-2} m \\
&+ \frac{(\boldsymbol{p}-e\boldsymbol{A})^2}{2m} + m\phi + eA_0 - (\boldsymbol{\mu}_0 + \Delta\boldsymbol{\mu}_0)\cdot\boldsymbol{B} \\
&+ \epsilon^2\Bigg(\frac{m\phi^2}{2} - \frac{(\boldsymbol{p}-e\boldsymbol{A})^4}{8m^3} - 3\phi\ (\boldsymbol{\mu}_0 + \Delta\boldsymbol{\mu}_0)\cdot\boldsymbol{B} \\
&\qquad - \frac{(\boldsymbol{\mu}_0\cdot\boldsymbol{B})^2}{2m} - \frac{1}{4m^2}\nabla^2(\boldsymbol{\mu}_0\cdot\boldsymbol{B}) + \frac{1}{2m^2}(\boldsymbol{p}-e\boldsymbol{A})\cdot\big(\boldsymbol{\mu}_0\cdot\boldsymbol{B}\ (\boldsymbol{p}-e\boldsymbol{A})\big) \\
&\qquad + \frac{3}{2m}(\boldsymbol{p}-e\boldsymbol{A})\cdot\big(\phi(\boldsymbol{p}-e\boldsymbol{A})\big) - \frac{3\phi}{2mr^2}\boldsymbol{S}\cdot\big(\boldsymbol{r}\times(\boldsymbol{p}-e\boldsymbol{A})\big) \\
&\qquad - \frac{1}{8m^2}\nabla^2(\Delta\boldsymbol{\mu}_0\cdot\boldsymbol{B}) + \frac{1}{4m^2}\big(\Delta\boldsymbol{\mu}_0\cdot(\boldsymbol{p}-e\boldsymbol{A})\big)\ \big(\boldsymbol{B}\cdot(\boldsymbol{p}-e\boldsymbol{A})\big) \\
&\qquad + \frac{1}{4m^2}\big(\boldsymbol{B}\cdot(\boldsymbol{p}-e\boldsymbol{A})\big)\ \big(\Delta\boldsymbol{\mu}_0\cdot(\boldsymbol{p}-e\boldsymbol{A})\big) + \frac{\Delta\mu_0}{4m^2}\,(\nabla\times\boldsymbol{B})\times\boldsymbol{p} \\
&\qquad - \frac{\mu_0}{4m}\,\nabla\cdot(\boldsymbol{E} + \frac{\partial\boldsymbol{A}}{\partial t}) - \frac{1}{2m}\boldsymbol{\mu}_0\cdot\big((\boldsymbol{E} + \frac{\partial\boldsymbol{A}}{\partial t})\times(\boldsymbol{p}-e\boldsymbol{A})\big) \\
&\qquad - \frac{\Delta\mu_0}{2m}\nabla\cdot\boldsymbol{E} - \frac{1}{m}\Delta\boldsymbol{\mu}_0\cdot(\boldsymbol{E}\times\boldsymbol{p}) - \frac{i}{2m}\Delta\boldsymbol{\mu}_0\cdot(\nabla\times\boldsymbol{E})\Bigg) + O(\epsilon^4),
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
\boldsymbol{\mu}_0 &\equiv \frac{e}{m}\boldsymbol{S} = \frac{e}{2m}\boldsymbol{\sigma} \qquad \text{(Bohr magneton)}, \\
\Delta\boldsymbol{\mu}_0 &\equiv \boldsymbol{\mu}_m - \boldsymbol{\mu}_0 = \frac{\mathrm{g}-2}{2}\frac{e}{2m}\boldsymbol{\sigma}.
\end{aligned} \tag{29}$$

Accordingly, the effective magnetic moment is

$$\boldsymbol{\mu}_{\mathrm{m}}^{\mathrm{eff}} \simeq (1 + 3\epsilon^2\phi)\ \boldsymbol{\mu}_{\mathrm{m}} \qquad \text{(for Type C)}. \tag{30}$$

Equation (30) can be understood as the general expression of the effective magnetic moment of fermions including leptons and hadrons. Although the g-factor of proton or atoms largely deviates from $g = 2$, the effective magnetic moment is given in the same form as Type A: Dirac particles with minimal coupling (16) and Type B: neutral fermions with non-minimal coupling (22).

2.4. *Effective anomalous magnetic moment in curved spacetime*

The effective magnetic moment to be observed with the instrument fixed on the Earth's surface has been obtained as Eq. (30) up to the post-Newtonian order $O(1/c^2)$ adopting the Schwarzschild metric as the background spacetime, common to leptons and hadrons of Type A, Type B, and Type C. The effective values of the gyromagnetic ratio (g-factor) and the anomalous magnetic moment $a \equiv g/2 - 1 = \mu_{\mathrm{m}}/\mu_0 - 1$, to be measured in ground-based experiments, can be defined as

$$\begin{aligned} g^{\mathrm{eff}} &\simeq (1 + 3\,\epsilon^2\,\phi)\ g, \\ a^{\mathrm{eff}} &\simeq a\, + 3\,(1 + a)\epsilon^2\phi, \end{aligned} \tag{31}$$

which implies that the g-factor and the anomalous magnetic moment depend on the gravitational field as a result of the influence of the curved spacetime in general relativity. For an example, the anomalous magnetic moment of fermions with $g \simeq 2$ is evaluated as $|a^{\mathrm{eff}}| \simeq |3\epsilon^2\phi| \simeq 2.1 \times 10^{-9}$ on the Earth's surface.

This implies that the Earth's gravitational field induces an anomalous magnetic moment of 2.1×10^{-9} for electrons, muons, etc. regardless of the type of fermions, in addition to quantum radiative corrections.

2.5. *Parametric post-Newtonian (PPN) approximation*

Here we consider the case in which the parametric post-Newtonian (PPN) approximation is adopted as the background spacetime:

$$ds^2 = \epsilon^{-2}\big(1 + \epsilon^2 2\phi + \epsilon^4 2\beta_*\phi^2\big)dt^2 - (1 - \epsilon^2 2\gamma_*\phi)\ (dx^2 + dy^2 + dz^2). \tag{32}$$

This is a primitive version of one of the possible extensions of general relativity [11], in which the parameter choice of $\beta_* \neq 1$ and $\gamma_* \neq 1$ corresponds to the scalar–tensor gravity model and the choice of $\beta_* = 1$, $\gamma_* = 1$ corresponds to standard general relativity. In the same manner as in the case of general relativity, the large-component Hamiltonian can be obtained via the covariant Dirac equation (26) using the PPN metric as

$$\begin{aligned} \mathcal{H}_+^{\mathrm{PPN}} = \epsilon^{-2} m & \\ & + eA_0 + m\phi + \frac{(\boldsymbol{p} - e\boldsymbol{A})^2}{2m} - (\boldsymbol{\mu}_0 + \Delta\boldsymbol{\mu}_0)\cdot\boldsymbol{B} \\ & + \epsilon^2\bigg((\beta_* - \frac{1}{2})m\phi^2 - (1 + 2\gamma_*)\phi\ (\boldsymbol{\mu}_0 + \Delta\boldsymbol{\mu}_0)\cdot\boldsymbol{B} \\ & \qquad - \frac{(\boldsymbol{p} - e\boldsymbol{A})^4}{8m^3} - \frac{(\boldsymbol{\mu}_0\cdot\boldsymbol{B})^2}{2m} - \frac{1}{4m^2}\nabla^2(\boldsymbol{\mu}_0\cdot\boldsymbol{B}) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2m^2}(\boldsymbol{p}-e\boldsymbol{A})\cdot\big(\boldsymbol{\mu}_0\cdot\boldsymbol{B}\,(\boldsymbol{p}-e\boldsymbol{A})\big)\\
&+\frac{(1+2\gamma_*)}{2m}(\boldsymbol{p}-e\boldsymbol{A})\cdot\big(\phi(\boldsymbol{p}-e\boldsymbol{A})\big)-\frac{(1+2\gamma_*)\phi}{2mr^2}\boldsymbol{S}\cdot\big(\boldsymbol{r}\times(\boldsymbol{p}-e\boldsymbol{A})\big)\\
&-\frac{1}{8m^2}\nabla^2(\Delta\boldsymbol{\mu}_0\cdot\boldsymbol{B})+\frac{\Delta\mu_0}{4m^2}(\nabla\times\boldsymbol{B})\times\boldsymbol{p}\\
&+\frac{1}{4m^2}\big(\Delta\boldsymbol{\mu}_0\cdot(\boldsymbol{p}-e\boldsymbol{A})\big)\big(\boldsymbol{B}\cdot(\boldsymbol{p}-e\boldsymbol{A})\big)+\frac{1}{4m^2}\big(\boldsymbol{B}\cdot(\boldsymbol{p}-e\boldsymbol{A})\big)\big(\Delta\boldsymbol{\mu}_0\cdot(\boldsymbol{p}-e\boldsymbol{A})\big)\\
&-\frac{\mu_0}{4m}\nabla\cdot(\boldsymbol{E}+\frac{\partial\boldsymbol{A}}{\partial t})-\frac{1}{2m}\boldsymbol{\mu}_0\cdot\Big((\boldsymbol{E}+\frac{\partial\boldsymbol{A}}{\partial t})\times(\boldsymbol{p}-e\boldsymbol{A})\Big)\\
&-\frac{\Delta\mu_0}{2m}\nabla\cdot\boldsymbol{E}-\frac{1}{m}\Delta\boldsymbol{\mu}_0\cdot(\boldsymbol{E}\times\boldsymbol{p})-\frac{i}{2m}\Delta\boldsymbol{\mu}_0\cdot(\nabla\times\boldsymbol{E})\Big)+O(\epsilon^4). \qquad (33)
\end{aligned}
$$

Accordingly, the effective magnetic moment can be derived, in the same manner, as

$$
\boldsymbol{\mu}_{\mathrm{m}}^{\mathrm{eff}}\simeq\big(1+(1+2\gamma_*)\,\epsilon^2\phi\big)\;\boldsymbol{\mu}_{\mathrm{m}}\qquad\text{(for PPN)}. \qquad (34)
$$

We note that the parameter of the theoretical model of gravity γ_* explicitly appears in the gravitationally induced correction term. This implies that the precise determination of the magnetic moment in a specific local circumstance may determine the model parameter γ_*.

3. Penning trap experiment to measure electron $\mathrm{g}-2$ in the Earth's gravity

The g-factor of leptons exactly equals 2 according to the Dirac theory (g_l=2, for $l=e,\mu,\tau$), which is one of the important consequences of the Dirac equation for free fermions with an electric charge of e and a spin of $1/2$ minimally coupling to an electromagnetic field. However, the real value of g_l deviates from 2 by a fraction of about 0.002 as a result of the quantum radiative corrections. This deviation is referred to as the anomalous magnetic moment, defined as

$$
a_l\equiv\frac{\mathrm{g}_l}{2}-1=\frac{\mu_{\mathrm{m}}^l}{\mu_{\mathrm{B}}^l}-1\quad(=0.001\cdots), \qquad (35)
$$

where μ_{m}^l is the magnetic moment and μ_{B}^l is the Bohr magneton. The Bohr magneton is defined as μ_{B}^l=$(e/m_l)\,S$ for the lepton mass m_l and spin S. For electrons, a comparison between the theoretical value, which was obtained with higher orders of quantum radiative corrections based on the standard model of elementary particles (SM) [12,15], and the experimental value, which was obtained in the precision measurement [13–16], shows the extremely precise agreement up to the 12th digit:

$$
a_{e(\mathrm{EXP})}-a_{e(\mathrm{SM})}=-0.91\;(0.82)\times10^{-12}, \qquad (36)
$$

as shown in Table 1, which clearly validates the accuracy of the quantum electrodynamics (QED) employed in the standard model of elementary particles [12,15,16].

3.1. Interpretation of the experimental result of electron $\mathrm{g}-2$

Below, we examine the interpretation of the experimental result of the Penning trap experiment [13–15].

The cyclotron frequency Ω_{c} and the spin precession frequency Ω_{s} of an electron moving in a uniform magnetic field are measured in the Penning trap experiment [13–15]. The anomalous magnetic

Table 1. Comparison of the theoretical value and the experimental value of the anomalous magnetic moment of the electron.

	Anomalous magnetic moment ($a_e \equiv g_e/2 - 1$)	Ref.
Experiment	$a_{e(\mathrm{EXP})} = 1159\,652\,180.73\,(0.28) \times 10^{-12}$	[13–16]
Theory	$a_{e(\mathrm{SM})} = 1159\,652\,181.643(0.77) \times 10^{-12}$	[12,15]
Difference (Deviation)	$a_{e(\mathrm{EXP})} - a_{e(\mathrm{SM})} = -0.91(0.82) \times 10^{-12}$ ($1.1\,\sigma$)	

moment of electrons a_e is regarded as being equal to the ratio of the anomalous spin precession frequency $\Omega_\mathrm{a} \equiv \Omega_\mathrm{s} - \Omega_\mathrm{c}$ to the cyclotron frequency Ω_c as

$$a_e^{\mathrm{conv}}{}_{(\mathrm{EXP})} = \frac{\Omega_\mathrm{a}}{\Omega_\mathrm{c}} = \frac{\Omega_\mathrm{s}}{\Omega_\mathrm{c}} - 1. \tag{37}$$

The $a_e^{\mathrm{conv}}{}_{(\mathrm{EXP})}$ has been treated as the electron anomalous magnetic moment in the conventional analysis without consideration of the gravitational effect. However, this experiment was carried out in the Earth's gravitational field. Thus, the experimentally measured frequencies in Eq. (37) are the effective frequencies in a curved spacetime. Hereafter, we derive the effective values of the cyclotron frequency $\Omega_\mathrm{c}^{\mathrm{eff}}$ and the spin precession frequency $\Omega_\mathrm{s}^{\mathrm{eff}}$ in a curved spacetime.

General relativity requires that the translational motion of a free particle with an electric charge of e is described by the covariant equation (2). Assuming that the Earth's gravitational field can be described with the Schwarzschild metric shown in Eq. (3), the equation of translational motion of a free electron can be written as Eq. (4). In general, the solution of this equation gives a spiral motion that is a combination of translational acceleration and rotational motion. The cyclotron frequency is a measure of the rotational motion. In other words, the cyclotron frequency corresponds to the frequency of the rotational motion of the constant-norm vector parallel to the velocity. We define the unit vector parallel to the 3D velocity $\boldsymbol{\beta}$ as

$$\boldsymbol{\beta} \equiv \beta\ \hat{\boldsymbol{\beta}}, \qquad \hat{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{\beta}} = \frac{\boldsymbol{\beta} \cdot \boldsymbol{\beta}}{\beta^2} = 1. \tag{38}$$

Here we use $\boldsymbol{\beta} \cdot \nabla\phi = 0$ and assume that the effects of the gravitational gradient are negligibly small; the rotational component of the translational motion is given as

$$\begin{aligned} \frac{d\hat{\boldsymbol{\beta}}}{dt} &= -\big(1 + (2\gamma^2 + 1)\epsilon^2\phi\big)\ \frac{e}{m}\left(\frac{\boldsymbol{B}}{\gamma} - \frac{\gamma}{\gamma^2 - 1}(\boldsymbol{\beta} \times \boldsymbol{E})\right) \times \hat{\boldsymbol{\beta}} \\ &= \boldsymbol{\Omega}_\mathbf{c}{}^{\mathrm{eff}}\ \times\ \hat{\boldsymbol{\beta}}. \end{aligned} \tag{39}$$

Using the cyclotron frequency in the flat spacetime $\boldsymbol{\Omega}_\mathrm{c} = -e/m\big(\boldsymbol{B}/\gamma - \gamma/(\gamma^2 - 1)\ (\boldsymbol{\beta} \times \boldsymbol{E})\big)$, the effective cyclotron frequency can be written as

$$\boldsymbol{\Omega}_\mathrm{c}^{\mathrm{eff}} = \ \big(1 + (2\gamma^2 + 1)\epsilon^2\phi\big)\ \boldsymbol{\Omega}_\mathrm{c} + O(\epsilon^4), \tag{40}$$

which represents the cyclotron frequency in the Earth's gravitational field based on general relativity.

In the measurement of the electron $g_e - 2$ experiment (i.e., the Penning trap experiment), the electron velocity is sufficiently small as $\beta \ll 1$ and the effective value of the electron's cyclotron frequency is given as

$$\boldsymbol{\Omega}_{\mathrm{c}}^{\mathrm{eff}} = (1 + 3\epsilon^2\phi)\ \boldsymbol{\Omega}_{\mathrm{c}}. \tag{41}$$

We can interpret that the $\boldsymbol{\Omega}_{\mathrm{c}}^{\mathrm{eff}}$ is used as the experimentally measured cyclotron frequency to be observed with the ground-based instrument.

Here we analyze the behavior of an electron spin in the Earth's gravitational field. The time evolution of an electron spin in the flat spacetime is described by the equation

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{\mu}_{\mathrm{m}} \times \boldsymbol{B} = \boldsymbol{\Omega}_{\mathrm{s}} \times \boldsymbol{S}, \tag{42}$$

where $\boldsymbol{\Omega}_{\mathrm{s}} = -(\mathrm{g}_e/2)(e/m)\boldsymbol{B}$. Rewriting this equation with the four vector, we obtain

$$\frac{dS^\mu}{d\tau} = \frac{\mathrm{g}_e}{2}\frac{e}{m}\left(F^{\mu\nu}S_\nu + \frac{1}{c^2}u^\mu(S_\lambda F^{\lambda\nu}u_\nu)\right) - \frac{1}{c^2}u^\mu(S_\lambda\frac{du^\lambda}{d\tau}), \tag{43}$$

which is known as the Bargmann–Michel–Telegdi (BMT) equation [22]. In the same manner as in Eq. (2), employing the covariant derivative in the BMT equation, we obtain the covariant equation of spin motion in general relativity as

$$\frac{DS^\mu}{d\tau} = \frac{\mathrm{g}_e}{2}\frac{e}{m}\left(F^{\mu\nu}S_\nu + \frac{1}{c^2}u^\mu(S_\lambda F^{\lambda\nu}u_\nu)\right) - \frac{1}{c^2}u^\mu(S_\lambda\frac{Du^\lambda}{d\tau}). \tag{44}$$

Here we substitute the Schwarzschild metric defined by Eq. (3) into the covariant BMT equation. In the same way as we have derived the cyclotron frequency based on the equation of translational motion of electrons, we consider the case where the gradient of the gravitational potential $\nabla\phi$ is negligibly small, such as the case where the motion is limited on a horizontal plane. We define the four spin vector as $s^\mu = (S^0, \boldsymbol{S})$; the spatial component of the generalized BMT equation (44) up to the post-Newtonian order $O(\epsilon^2)$ can be written as

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{S} \times \frac{\left(1 + \epsilon^2\phi(2\gamma^2 + 1)\right)}{\gamma}\frac{\mathrm{g}_e}{2}\frac{e}{m}\boldsymbol{B}. \tag{45}$$

We can take $\beta \ll 1$ in the measurement of electron $\mathrm{g}_e - 2$ (i.e., the Penning trap experiment), which implies that the effective value of the electron spin precession frequency to be observed with the ground-based instrument is given as

$$\boldsymbol{\Omega}_{\mathrm{s}}^{\mathrm{eff}} = (1 + 3\epsilon^2\phi)\ \boldsymbol{\Omega}_{\mathrm{s}}. \tag{46}$$

The experimental value of the anomalous magnetic moment obtained in the Penning trap experiment, denoted as $a_{e(\mathrm{EXP})}^{\mathrm{eff}}$ below, should be compared with the ratio of the kinematically calculated effective values $\Omega_{\mathrm{c}}^{\mathrm{eff}}$ and $\Omega_{\mathrm{s}}^{\mathrm{eff}}$. Substituting the effective values $\Omega_{\mathrm{c}}^{\mathrm{eff}}$ and $\Omega_{\mathrm{s}}^{\mathrm{eff}}$ into Eq. (37), we obtain

$$\begin{aligned} a_e^{\mathrm{eff}}{}_{(\mathrm{EXP})} &= \frac{\Omega_{\mathrm{s}}^{\mathrm{eff}}}{\Omega_{\mathrm{c}}^{\mathrm{eff}}} - 1 = \frac{(1 + 3\epsilon^2\phi)\ \Omega_{\mathrm{s}}}{(1 + 3\epsilon^2\phi)\ \Omega_{\mathrm{c}}} - 1 \\ &= \frac{\Omega_{\mathrm{s}}}{\Omega_{\mathrm{c}}} - 1 \\ &= a_e^{\mathrm{conv}}{}_{(\mathrm{EXP})}. \end{aligned} \tag{47}$$

Although both effective values $\Omega_{\mathrm{c}}^{\mathrm{eff}}$ and $\Omega_{\mathrm{s}}^{\mathrm{eff}}$ are dependent on the Earth's gravitational field, the gravitational effects are canceled in their ratio and $a_{e(\mathrm{EXP})}^{\mathrm{eff}}$ coincides with $a_{e(\mathrm{EXP})}^{\mathrm{conv}}$. It is reasonable to regard the high-precision agreement between the experimental and theoretical values as confirming the cancellation of gravitational effects. It also suggests that, if we take the ratio of the conventionally defined a_e in Eq. (35) to the Bohr magneton, the "true" anomalous magnetic moment $a_e^{\mathrm{eff}}{}_{(\mathrm{EXP})}$ deviates from the "conventional" value $a_e^{\mathrm{conv}}{}_{(\mathrm{EXP})}$ as

$$\begin{aligned} a_e^{\mathrm{eff}}{}_{(\mathrm{EXP})} &\equiv \frac{g_e^{\mathrm{eff}}}{2} - 1 = \frac{\mu_{\mathrm{m}}^{\mathrm{eff}}}{\mu_{\mathrm{B}}} - 1 = \frac{\Omega_{\mathrm{s}}^{\mathrm{eff}}}{\Omega_{\mathrm{c}}} - 1 \\ &= (1 + 3\epsilon^2\phi)\,\frac{\Omega_{\mathrm{s}}}{\Omega_{\mathrm{c}}} - 1 \\ &\neq \frac{\Omega_{\mathrm{s}}}{\Omega_{\mathrm{c}}} - 1 = a_e^{\mathrm{conv}}{}_{(\mathrm{EXP})}. \end{aligned} \tag{48}$$

Since the Earth's gravity induces an anomaly even for $g_e = 2$ additionally to radiative corrections, the comparison between the experimental value measured in the curved spacetime and the theoretical value calculated in the flat spacetime results in a difference of

$$\left| a_e^{\mathrm{eff}}{}_{(\mathrm{EXP})} - a_{e(\mathrm{SM})} \right| \simeq 3\epsilon^2|\phi| \sim 2.1 \times 10^{-9}. \tag{49}$$

Here we introduce the redefinition of the effective Bohr magneton including the general relativistic effects of the Earth's gravitational field as

$$\mu_{\mathrm{B}}^{\mathrm{eff}} \equiv \mu_{\mathrm{B}}^{\mathrm{eff}}(\phi) \simeq (1 + 3\epsilon^2\phi)\,\mu_{\mathrm{B}}. \tag{50}$$

In this case, we can treat $a_{e(\mathrm{EXP})}^{\mathrm{eff}}$ as an invariant quantity independent of the gravitational field if we cancel the general relativistic effects by redefining the anomalous magnetic moment using the ratio of the effective magnetic moment to the effective Bohr magneton as

$$\begin{aligned} a_e^{\mathrm{eff}}{}_{(\mathrm{EXP})} &\equiv \frac{\mu_{\mathrm{m}}^{\mathrm{eff}}}{\mu_{\mathrm{B}}^{\mathrm{eff}}} - 1 = \frac{\Omega_{\mathrm{s}}^{\mathrm{eff}}}{\Omega_{\mathrm{c}}^{\mathrm{eff}}} - 1 \\ &= a_e^{\mathrm{conv}}{}_{(\mathrm{EXP})}, \end{aligned} \tag{51}$$

instead of its ratio to the Bohr magneton in the flat spacetime. This redefinition restores the validity of the test of the standard model of elementary particles via the comparison of the theoretical value $a_{e(\mathrm{SM})}$ and the experimental value $a_e^{\mathrm{eff}}{}_{(\mathrm{EXP})}$.

4. Storage ring experiment to measure muon $g - 2$ in the Earth's gravity

In the storage ring experiments for $g_\mu - 2$ such as CERN [17,18] and BNL E821 [19–21], the muon anomalous magnetic moment $a_{\mu(\mathrm{EXP})}$ was determined from the relation

$$a_{\mu(\mathrm{EXP})} \equiv \frac{R}{\lambda - R}, \tag{52}$$

where $R \equiv \Omega_{\mathrm{a}}/\Omega_p$ is the ratio of the anomalous spin precession frequency Ω_{a} of muons to the proton spin precession frequency Ω_p measured using the nuclear magnetic resonance, and $\lambda \equiv \mu_\mu/\mu_p$ is the ratio of the magnetic moment of muons to that of protons. This can be understood as $a_{\mu(\mathrm{EXP})}$ being determined from the relation

$$a_{\mu(\mathrm{EXP})} = \frac{\Omega_{\mathrm{a}}}{\Omega_{\mathrm{L}} - \Omega_{\mathrm{a}}}, \tag{53}$$

where Ω_{L} is the Larmor precession frequency of muons. This interpretation can be obtained using the classical kinematical framework and is the basis of the design of the experiment to measure the muon $g_\mu - 2$ [15–21]. We reconsider this interpretation including the spacetime curvature in the Earth's gravitational field on the basis of general relativity.

4.1. Spin motion in flat spacetime

Here we give an overview of the procedure to evaluate the experimental value of the anomalous magnetic moment using the equation of motion of spin in the flat spacetime (see, e.g., Chap. 11 in Ref. [23]). The BMT equation (43) described in the laboratory frame (inertial frame) can be decomposed into a set of equations for time and spatial components as

$$\begin{aligned}\frac{dS^0}{d\tau} &= F_0 + \gamma^2\left(\boldsymbol{S}\cdot\frac{d\boldsymbol{\beta}}{d\tau}\right)\\ \frac{d\boldsymbol{S}}{d\tau} &= \boldsymbol{F} + \gamma^2\boldsymbol{\beta}\left(\boldsymbol{S}\cdot\frac{d\boldsymbol{\beta}}{d\tau}\right).\end{aligned} \tag{54}$$

The 3D spin vector $\boldsymbol{S}$ in the inertial frame and the 3D spin vector $\boldsymbol{s}$ are related through the Lorentz transformation

$$\begin{aligned}\boldsymbol{s} &= \boldsymbol{S} - \frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\cdot\boldsymbol{S})\boldsymbol{\beta}\\ &= \boldsymbol{S} - \frac{\gamma}{\gamma+1}S^0\boldsymbol{\beta}.\end{aligned} \tag{55}$$

The equation of motion of the spin in the rest frame $\boldsymbol{s}$ can be derived as

$$\begin{aligned}\frac{d\boldsymbol{s}}{d\tau} &= \left(\boldsymbol{F} - \frac{\gamma}{\gamma+1}F_0\,\boldsymbol{\beta}\right) + \frac{\gamma^2}{\gamma+1}\,\boldsymbol{s}\times\left(\boldsymbol{\beta}\times\frac{d\boldsymbol{\beta}}{d\tau}\right)\\ &= \boldsymbol{F}' + \frac{\gamma^2}{\gamma+1}\,\boldsymbol{s}\times\left(\boldsymbol{\beta}\times\frac{d\boldsymbol{\beta}}{d\tau}\right).\end{aligned} \tag{56}$$

Rewriting the proper time τ with the time in the inertial frame t, we can reproduce the familiar equation

$$\begin{aligned}\frac{d\boldsymbol{s}}{dt} &= \frac{\boldsymbol{F'}}{\gamma} + \frac{\gamma^2}{\gamma+1}\,\boldsymbol{s}\times\left(\boldsymbol{\beta}\times\frac{d\boldsymbol{\beta}}{dt}\right)\\ &= \frac{g_\mu}{2}\frac{e}{\gamma\, m}\,\boldsymbol{s}\times\boldsymbol{B'} + \frac{\gamma^2}{\gamma+1}\,\boldsymbol{s}\times\left(\boldsymbol{\beta}\times\frac{d\boldsymbol{\beta}}{dt}\right).\end{aligned} \tag{57}$$

We note that the last term in the equation of motion of spin in the rest frame (57) represents the relativistic term known as Thomas precession, which does not appear in the non-relativistic equation of motion of spin (42). The electric and magnetic fields in the inertial frame $\boldsymbol{E}$ and $\boldsymbol{B}$ are related to those in the rest frame $\boldsymbol{E}'$ and $\boldsymbol{B}'$ through the Lorentz transformation as

$$\begin{aligned}\boldsymbol{E'} &= \gamma(\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B}) - \frac{\gamma^2}{\gamma+1}\,(\boldsymbol{\beta}\cdot\boldsymbol{E})\,\boldsymbol{\beta}\\ \boldsymbol{B'} &= \gamma(\boldsymbol{B} - \boldsymbol{\beta}\times\boldsymbol{E}) - \frac{\gamma^2}{\gamma+1}\,(\boldsymbol{\beta}\cdot\boldsymbol{B})\,\boldsymbol{\beta}.\end{aligned} \tag{58}$$

The equation of motion of spin in the rest frame can be rewritten with the quantities in the inertial frame by substituting Eqs. (6) and (58), which results in

$$\begin{aligned}
\frac{d\boldsymbol{s}}{dt} = \frac{g\mu}{2}\frac{e}{m}\,\boldsymbol{s}\times\left(\boldsymbol{B}-\boldsymbol{\beta}\times\boldsymbol{E}-\frac{\gamma}{\gamma+1}\,(\boldsymbol{\beta}\cdot\boldsymbol{B})\,\boldsymbol{\beta}\right)& \\
+\frac{\gamma^2}{\gamma+1}\,\boldsymbol{s}\times\left(\boldsymbol{\beta}\times\frac{e}{m\gamma}\,\big(\boldsymbol{E}+\boldsymbol{\beta}\times\boldsymbol{B}-(\boldsymbol{\beta}\cdot\boldsymbol{E})\,\boldsymbol{\beta}\big)\right)& \\
=\boldsymbol{s}\times\frac{e}{m}\left((\frac{g\mu}{2}-1+\frac{1}{\gamma})\boldsymbol{B}-(\frac{g\mu}{2}-\frac{\gamma}{\gamma+1})\boldsymbol{\beta}\times\boldsymbol{E}-(\frac{g\mu}{2}-1)\frac{\gamma}{\gamma+1}\,(\boldsymbol{\beta}\cdot\boldsymbol{B})\,\boldsymbol{\beta}\right)& \\
=\boldsymbol{\Omega}_{\rm s}\times\boldsymbol{s}.& \qquad (59)
\end{aligned}$$

By using the spin precession frequency $\boldsymbol{\Omega}_{\rm s}$, the anomalous spin precession frequency $\boldsymbol{\Omega}_{\rm a}$ can be written as

$$\begin{aligned}
\boldsymbol{\Omega}_{\rm a} &= \boldsymbol{\Omega}_{\rm s}-\boldsymbol{\Omega}_{\rm c} \\
&= -\frac{e}{m}\left(a_\mu\,\boldsymbol{B}-(a_\mu-\frac{1}{\gamma^2-1})\,\boldsymbol{\beta}\times\boldsymbol{E}-a_\mu\frac{\gamma}{\gamma+1}\,(\boldsymbol{\beta}\cdot\boldsymbol{B})\,\boldsymbol{\beta}\right), \qquad (60)
\end{aligned}$$

with $a_\mu \equiv g_\mu/2-1$. We assume that the motion is confined in the horizontal plane and the magnetic field is applied vertically. In addition, we assume that the electric term (the second term) can be canceled as $\big(a_\mu-1/(\gamma^2-1)\big)=0$ by selecting a muon momentum of $\gamma\simeq 29.3$, which is referred to as the magic momentum. The anomalous magnetic moment of the muon in the flat spacetime, which is defined as $a_{\mu(\rm EXP)}\equiv -\Omega_{\rm a}/(e\,B/m)$, can be written as

$$\begin{aligned}
a_{\mu(\rm EXP)} &\equiv -\frac{\Omega_{\rm a}}{(\frac{eB}{m})}=\frac{\Omega_{\rm a}}{\Omega_{\rm L}-\Omega_{\rm a}}=\frac{\Omega_{\rm a}/\Omega_p}{\Omega_{\rm L}/\Omega_p-\Omega_{\rm a}/\Omega_p} \\
&=\frac{R}{\lambda-R}, \qquad (61)
\end{aligned}$$

by rewriting the magnetic field $\boldsymbol{B}$ using the Larmor precession frequency $\Omega_{\rm L}=-(g_\mu/2)(e/m)\,B=-(1+a_\mu)(e/m)B$. Equation (61) represents the experimental value in the conventional interpretation. The experimental values of the anomalous magnetic moment $a_{\mu(\rm EXP)}$ using Eqs. (60) and (61) are evaluated within the classical kinematical framework. The relations were applied to achieve high precision by employing the magic momentum to suppress the contribution of electric fields.

4.2. Spin motion in curved spacetime

In this section, we apply the procedure of the evaluation of the experimental value of the anomalous magnetic moment in the flat spacetime to that in the curved spacetime.

The BMT equation in the curved spacetime in the inertial frame (44) can be decomposed into the time and spatial components as

$$\begin{aligned}
&\frac{dS^0}{d\tau}+\epsilon^2(S^0\boldsymbol{u}+u^0\boldsymbol{S})\cdot\nabla\phi \\
&\qquad = F_0+(1-4\gamma^2\epsilon^2\phi)\;\gamma^2\left(\boldsymbol{S}\cdot\frac{d\boldsymbol{\beta}}{d\tau}\right)+O(\epsilon^4)
\end{aligned}$$

$$\frac{d\boldsymbol{S}}{d\tau} + \epsilon\ u^0 S^0\,\nabla\phi + \epsilon^2\big((\boldsymbol{u}\cdot\boldsymbol{S})\nabla\phi - (\boldsymbol{S}\cdot\nabla\phi)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla\phi)\boldsymbol{S}\big)$$
$$= \boldsymbol{F} + (1-4\gamma^2\epsilon^2\phi)\ \gamma^2\boldsymbol{\beta}\left(\boldsymbol{S}\cdot\frac{d\boldsymbol{\beta}}{d\tau}\right) + O(\epsilon^4). \tag{62}$$

Considering that the contribution arising from the gradient of the gravitational potential $\nabla\phi$ is sufficiently small to be ignored in the post-Newtonian order (see Appendix E), Eq. (62) can be simplified as

$$\frac{dS^0}{d\tau} = F_0 + (1-4\gamma^2\epsilon^2\phi)\ \gamma^2\left(\boldsymbol{S}\cdot\frac{d\boldsymbol{\beta}}{d\tau}\right) + O(\epsilon^4)$$
$$\frac{d\boldsymbol{S}}{d\tau} = \boldsymbol{F} + (1-4\gamma^2\epsilon^2\phi)\ \gamma^2\boldsymbol{\beta}\left(\boldsymbol{S}\cdot\frac{d\boldsymbol{\beta}}{d\tau}\right) + O(\epsilon^4). \tag{63}$$

Following the procedure in the flat spacetime, we derive the relation between the spins in the inertial frame and the rest frame using the Lorentz transformation. The Lorentz transformation can be naturally defined in the flat spacetime since the Lorentz invariance is satisfied at any point. However, the Lorentz transformation cannot be defined in the general coordinate system (curved spacetime), which disturbs the applicability of the same procedure. We avoid the difficulty by employing the local inertial frame to satisfy the local Lorentz invariance (see Appendix C). The four vector in the general coordinate system $x^\mu = (x^0, \boldsymbol{x})$ can be related to the four vector in the local inertial frame $x^{(a)} = (\tilde{x}^0, \tilde{\boldsymbol{x}})$, using the tetrad $e^{(a)}_\mu$, as

$$x^{(a)} \equiv e^{(a)}_\mu\, x^\mu = (\tilde{x}^0, \tilde{\boldsymbol{x}}). \tag{64}$$

Using the explicit expression of the tetrad up to the post-Newtonian order $O(\epsilon^2)$ in the Schwarzschild metric (shown in Eq. (B.7)), the transformation from the general coordinate system into the local inertial frame can be simplified as

$$\begin{aligned}
\tilde{S}^0 &= (1+\epsilon^2\phi)\,S^0\\
\tilde{\boldsymbol{S}} &= (1-\epsilon^2\phi)\,\boldsymbol{S}\\
\tilde{\boldsymbol{\beta}} &= (1-2\epsilon^2\phi)\,\boldsymbol{\beta}\\
\tilde{t} &= (1+\epsilon^2\phi)\ t\\
\tilde{\gamma} &= \big(1-2\epsilon^2\phi\,(\gamma^2-1)\big)\,\gamma\\
\tilde{\boldsymbol{E}} &= \boldsymbol{E}\\
\tilde{\boldsymbol{B}} &= (1+2\epsilon^2\phi)\boldsymbol{B}\\
&\cdots\quad \text{etc.}
\end{aligned} \tag{65}$$

We rewrite the BMT equation in the curved spacetime (63) using the local inertial frame and obtain

$$\frac{d\tilde{S}^0}{d\tau} = \tilde{F}_0 + \tilde{\gamma}^2\left(\tilde{\boldsymbol{S}}\cdot\frac{d\tilde{\boldsymbol{\beta}}}{d\tau}\right) + O(\epsilon^4)$$
$$\frac{d\tilde{\boldsymbol{S}}}{d\tau} = \tilde{\boldsymbol{F}} + \tilde{\gamma}^2\tilde{\boldsymbol{\beta}}\left(\tilde{\boldsymbol{S}}\cdot\frac{d\tilde{\boldsymbol{\beta}}}{d\tau}\right) + O(\epsilon^4), \tag{66}$$

which is an analogue to the relation in the flat spacetime. The local Lorentz transformation is obtained as

$$\begin{aligned}\tilde{\boldsymbol{s}} &= \tilde{\boldsymbol{S}} - \frac{\tilde{\gamma}}{\tilde{\gamma}+1}(\tilde{\boldsymbol{\beta}} \cdot \tilde{\boldsymbol{S}})\tilde{\boldsymbol{\beta}} \\ &= \tilde{\boldsymbol{S}} - \frac{\tilde{\gamma}}{\tilde{\gamma}+1}\tilde{S}^0\tilde{\boldsymbol{\beta}}. \end{aligned} \tag{67}$$

In the same manner as the flat spacetime, using the BMT equation in the curved spacetime (66) and the local Lorentz transformation (67), the equation of motion of spin in the rest frame can be written as

$$\begin{aligned}\frac{d\tilde{\boldsymbol{s}}}{d\tau} &= \left(\tilde{\boldsymbol{F}} - \frac{\tilde{\gamma}}{\tilde{\gamma}+1}\tilde{F}_0\,\tilde{\boldsymbol{\beta}}\right) + \frac{\tilde{\gamma}^2}{\tilde{\gamma}+1}\,\tilde{\boldsymbol{s}} \times \left(\tilde{\boldsymbol{\beta}} \times \frac{d\tilde{\boldsymbol{\beta}}}{d\tau}\right) \\ &= \tilde{\boldsymbol{F}}' + \frac{\tilde{\gamma}^2}{\tilde{\gamma}+1}\,\tilde{\boldsymbol{s}} \times \left(\tilde{\boldsymbol{\beta}} \times \frac{d\tilde{\boldsymbol{\beta}}}{d\tau}\right), \end{aligned} \tag{68}$$

which is similar to the case of the flat spacetime. This relation gives a relation similar to the case of the flat spacetime:

$$\begin{aligned}\frac{d\tilde{\boldsymbol{s}}}{d\tilde{t}} &= \frac{\tilde{\boldsymbol{F}}'}{\tilde{\gamma}} + \frac{\tilde{\gamma}^2}{\tilde{\gamma}+1}\,\tilde{\boldsymbol{s}} \times \left(\tilde{\boldsymbol{\beta}} \times \frac{d\tilde{\boldsymbol{\beta}}}{d\tilde{t}}\right) \\ &= \frac{g_\mu}{2}\frac{e}{\tilde{\gamma}\, m}\,\tilde{\boldsymbol{s}} \times \tilde{\boldsymbol{B}}' + \frac{\tilde{\gamma}^2}{\tilde{\gamma}+1}\,\tilde{\boldsymbol{s}} \times \left(\tilde{\boldsymbol{\beta}} \times \frac{d\tilde{\boldsymbol{\beta}}}{d\tilde{t}}\right), \end{aligned} \tag{69}$$

by replacing the proper time τ with the time in the inertial frame $\tilde{t}$. Here we notice that the electric and magnetic fields in the rest frame $\tilde{\boldsymbol{E}}'$ and $\tilde{\boldsymbol{B}}'$ are given through a local Lorentz transformation, using the electric ands magnetic fields in the inertial frame $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{B}}$, as

$$\begin{aligned}\tilde{\boldsymbol{E}}' &= \tilde{\gamma}(\tilde{\boldsymbol{E}} + \tilde{\boldsymbol{\beta}} \times \tilde{\boldsymbol{B}}) - \frac{\tilde{\gamma}^2}{\tilde{\gamma}+1}\,(\tilde{\boldsymbol{\beta}} \cdot \tilde{\boldsymbol{E}})\,\tilde{\boldsymbol{\beta}} \\ \tilde{\boldsymbol{B}}' &= \tilde{\gamma}(\tilde{\boldsymbol{B}} - \tilde{\boldsymbol{\beta}} \times \tilde{\boldsymbol{E}}) - \frac{\tilde{\gamma}^2}{\tilde{\gamma}+1}\,(\tilde{\boldsymbol{\beta}} \cdot \tilde{\boldsymbol{B}})\,\tilde{\boldsymbol{\beta}}. \end{aligned} \tag{70}$$

Thus, the equation of motion of spin in the rest frame $\tilde{\boldsymbol{s}}$ can be written using the electric and magnetic fields in the inertial frame $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{B}}$ as

$$\begin{aligned}\frac{d\tilde{\boldsymbol{s}}}{d\tilde{t}} &= \frac{g_\mu}{2}\frac{e}{m}\,\tilde{\boldsymbol{s}} \times \left(\tilde{\boldsymbol{B}} - \tilde{\boldsymbol{\beta}} \times \tilde{\boldsymbol{E}} - \frac{\tilde{\gamma}}{\tilde{\gamma}+1}\,(\tilde{\boldsymbol{\beta}} \cdot \tilde{\boldsymbol{B}})\,\tilde{\boldsymbol{\beta}}\right) \\ &\quad + \frac{\tilde{\gamma}^2}{\tilde{\gamma}+1}\,\tilde{\boldsymbol{s}} \times \left(\tilde{\boldsymbol{\beta}} \times \frac{e}{m\tilde{\gamma}}\,\left(\tilde{\boldsymbol{E}} + \tilde{\boldsymbol{\beta}} \times \tilde{\boldsymbol{B}} - (\tilde{\boldsymbol{\beta}} \cdot \tilde{\boldsymbol{E}})\,\tilde{\boldsymbol{\beta}}\right)\right) \\ &= \tilde{\boldsymbol{s}} \times \frac{e}{m}\left((\frac{g_\mu}{2} - 1 + \frac{1}{\tilde{\gamma}})\tilde{\boldsymbol{B}} - (\frac{g_\mu}{2} - \frac{\tilde{\gamma}}{\tilde{\gamma}+1})\tilde{\boldsymbol{\beta}} \times \tilde{\boldsymbol{E}} - (\frac{g_\mu}{2} - 1)\frac{\tilde{\gamma}}{\tilde{\gamma}+1}(\tilde{\boldsymbol{\beta}} \cdot \tilde{\boldsymbol{B}})\,\tilde{\boldsymbol{\beta}}\right). \end{aligned} \tag{71}$$

Equation (71) has the same form as the equation of spin motion (59). We convert the electric and magnetic fields $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{B}}$ in the local inertial frame to those in the general coordinate system to consider the experimental values with the ground-based instrument.

The equation of motion of spin $\boldsymbol{s}$ in the rest frame represented in the general coordinate system is obtained as

$$
\begin{aligned}
\frac{d\boldsymbol{s}}{dt} &= \boldsymbol{s} \times \frac{e}{m}\Bigg((\frac{g_\mu}{2} - 1 + \frac{1}{\tilde{\gamma}})(1+3\epsilon^2\phi)\boldsymbol{B} \\
&\qquad - (\frac{g_\mu}{2} - \frac{\tilde{\gamma}}{\tilde{\gamma}+1})(1-\epsilon^2\phi)\boldsymbol{\beta}\times\boldsymbol{E} \\
&\qquad - (\frac{g_\mu}{2} - 1)\frac{\tilde{\gamma}}{\tilde{\gamma}+1}(1-\epsilon^2\phi)(\boldsymbol{\beta}\cdot\boldsymbol{B})\,\boldsymbol{\beta}\Bigg) \\
&= \boldsymbol{s} \times \frac{e}{m}\Bigg((\frac{g_\mu}{2} - 1 + \frac{1}{\gamma}[1+2\epsilon^2\phi(\gamma^2-1)])(1+3\epsilon^2\phi)\boldsymbol{B} \\
&\qquad - (\frac{g_\mu}{2} - \frac{\gamma}{\gamma+1}[1-2\epsilon^2\phi(\gamma-1)])(1-\epsilon^2\phi)\boldsymbol{\beta}\times\boldsymbol{E} \\
&\qquad - (\frac{g_\mu}{2} - 1)\frac{\gamma}{\gamma+1}[1-\epsilon^2\phi(2\gamma-1)](\boldsymbol{\beta}\cdot\boldsymbol{B})\,\boldsymbol{\beta}\Bigg) \\
&= \boldsymbol{\Omega}_{\rm s}^{\rm eff} \times \boldsymbol{s}.
\end{aligned}
\tag{72}
$$

Putting $a_\mu \equiv g_\mu/2 - 1$ and using the effective value of the spin precession frequency in the curved spacetime (72) and the effective value of the cyclotron frequency in the curved spacetime (40), the effective value of the anomalous spin precession frequency $\boldsymbol{\Omega}_{\rm a}^{\rm eff}$ is obtained as

$$
\begin{aligned}
\boldsymbol{\Omega}_{\rm a}^{\rm eff} &= \boldsymbol{\Omega}_{\rm s}^{\rm eff} - \boldsymbol{\Omega}_{\rm c}^{\rm eff} \\
&= -\frac{e}{m}\Bigg((1+3\epsilon^2\phi)\,a_\mu\mathbf{B} \\
&\qquad - \left[a_\mu - \frac{1}{\gamma^2-1} - \epsilon^2\phi\Big(4 + a_\mu + \frac{3}{\gamma^2-1}\Big)\right]\boldsymbol{\beta}\times\mathbf{E} \\
&\qquad - a_\mu\,\frac{\gamma}{\gamma+1}\big(1-\epsilon^2\phi(2\gamma-1)\big)(\boldsymbol{\beta}\cdot\mathbf{B})\,\boldsymbol{\beta}\Bigg).
\end{aligned}
\tag{73}
$$

When the motion is confined on the horizontal plane, a magnetic field is applied vertically, and the momentum is chosen to cancel the contribution of the electric field, the effective value of the anomalous magnetic moment of the muon is interpreted as

$$
a_{\mu({\rm EXP})}^{\rm eff} = \frac{\Omega_{\rm a}^{\rm eff}}{\Omega_{\rm L}^{\rm eff} - \Omega_{\rm a}^{\rm eff}}. \tag{74}
$$

The Larmor precession frequency in the curved spacetime can be interpreted as a special solution of the BMT equation with u^μ=$(u^0, \mathbf{0})$ and s^μ=$(0, \boldsymbol{s})$, which corresponds to the null translational motion, as

$$
\boldsymbol{\Omega}_{\rm L}^{\rm eff} = -(1+3\epsilon^2\phi)\,\frac{g_\mu}{2}\frac{e}{m}\boldsymbol{B} = (1+3\epsilon^2\phi)\,\boldsymbol{\Omega}_{\rm L}. \tag{75}
$$

Therefore, we obtain

$$
\begin{aligned}
a^{\text{eff}}_{\mu(\text{EXP})} &= \frac{\Omega^{\text{eff}}_{\text{a}}}{\Omega^{\text{eff}}_{\text{L}} - \Omega^{\text{eff}}_{\text{a}}} \\
&= \frac{(1+3\epsilon^2\phi)\ \Omega_{\text{a}}}{(1+3\epsilon^2\phi)\ \Omega_{\text{L}} - (1+3\epsilon^2\phi)\ \Omega_{\text{a}}} \\
&= \frac{\Omega_{\text{a}}}{\Omega_{\text{L}} - \Omega_{\text{a}}} \\
&= a_{\mu(\text{EXP})},
\end{aligned} \tag{76}
$$

which show that the effective value of the anomalous magnetic moment in the curved spacetime is equal to that in the flat spacetime up to the post-Newtonian order $O(\epsilon^2)$ of general relativity. Thus the effective values of cyclotron frequency, spin precession frequency, and Larmor precession frequency have been derived based on the primitive consideration based on a kinematical framework. However those effective values in curved spacetime are, respectively, different from those values in the flat spacetime, the gravitational contribution is canceled in the ratio (76) and the anomalous magnetic moment in the curved spacetime coincides with the case in the flat spacetime.

4.3. *Gravitational influence on the storage ring experiment to measure muon* g − 2

We have found that the gravitational influence on the magnetic moment of muons in the storage ring experiment is canceled in the ratio based on the kinematical consideration up to the post-Newtonian order $O(\epsilon^2)$ using the Schwarzschild metric if the muon momentum is chosen to satisfy

$$
a_\mu - \frac{1}{\gamma^2 - 1} = 0, \tag{77}
$$

where a_μ is the experimental value of the anomalous magnetic moment. Consequently, the comparison between the experimental and theoretical values remains valid as before. However, Eq. (73) implies that general relativity modifies the contribution of the $\boldsymbol{\beta} \times \boldsymbol{E}$ term. The perfect cancellation of the $\boldsymbol{\beta} \times \boldsymbol{E}$ term corresponds to

$$
a_\mu - \frac{1}{\gamma^2 - 1} - \epsilon^2\phi\left(4 + a_\mu + \frac{3}{\gamma^2 - 1}\right) \ \sim \ (a_\mu + 2.8 \times 10^{-9}) - \frac{1}{\gamma^2 - 1} = 0, \tag{78}
$$

which introduces an offset to the experimental value of the anomalous magnetic moment as large as 2.8×10^{-9}. Therefore, if the $\boldsymbol{\beta} \times \boldsymbol{E}$ term is perfectly canceled in the experiment, $a_\mu + 2.8 \times 10^{-9}$ is measured instead of a_μ and the offset may be misidentified as a discrepancy between the theory and the experiment. The general relativistic offset may be clarified in future experiments with improved accuracy.

5. Conclusion

The magnetic moment of free fermions to be observed using instruments fixed on the Earth's surface has been examined by evaluating the effective magnetic moment including the influence of the gravitational field up to the post-Newtonian order $O(1/c^2)$, adopting the Schwarzschild metric as the background spacetime. We found that the magnetic moments of fermions are influenced by the spacetime curvature as $\boldsymbol{\mu}^{\text{eff}}_{\text{m}} = (1 + 3\phi/c^2)\ \boldsymbol{\mu}_{\text{m}}$ for the cases of minimal coupling, non-minimal coupling, and a mixture of the two. In the same manner, the effective values of the gyromagnetic ratio and the anomalous magnetic moment depend on the Earth's gravitational field as $g^{\text{eff}} \simeq (1 + 3\phi/c^2)$ g, $a^{\text{eff}} \simeq a + 3(1+a)\ \phi/c^2$, which seems inconsistent with the precise agreement between the experimental values and the theoretical values calculated in the flat spacetime.

However, experimental values of the anomalous magnetic moment were obtained as a ratio of the spin precession frequency and the cyclotron frequency in the Penning trap and the storage ring experiments, and the gravitational influence is canceled to recover the consistency between the experiment and the theory. Proper treatment of the experimental offset of the anomalous magnetic moment measured in the storage ring method would be necessary to compare the theoretical and experimental values, taking into consideration the change of the cancellation condition of the electric field contribution.

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Appendix A. γ-matrix

The γ-matrices in a curved spacetime satisfy

$$\begin{aligned} \{\gamma^\mu, \gamma^\nu\} &= \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} I_4, \\ \{\gamma_\mu, \gamma_\nu\} &= \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu} I_4, \end{aligned} \tag{A.1}$$

where I_4 is the 4×4 unit matrix, $g_{\mu\nu}$ the metric tensor of the curved spacetime. In the flat spacetime, the above relation leads to

$$\begin{aligned} \{\gamma^{(\alpha)}, \gamma^{(\beta)}\} &= \gamma^{(\alpha)}\gamma^{(\beta)} + \gamma^{(\beta)}\gamma^{(\alpha)} = 2\eta^{\alpha\beta} I_4, \\ \{\gamma_{(\alpha)}, \gamma_{(\beta)}\} &= \gamma_{(\alpha)}\gamma_{(\beta)} + \gamma_{(\beta)}\gamma_{(\alpha)} = 2\eta_{\alpha\beta} I_4, \end{aligned} \tag{A.2}$$

where $\eta_{\alpha\beta}$ is the metric tensor of the flat spacetime (Minkowski metric). We define the tetrad $e^\mu_{(\alpha)}$ satisfying the relation

$$g_{\mu\nu}\ e^\mu_{(\alpha)} e^\nu_{(\beta)} = \eta_{\alpha\beta} = \begin{pmatrix} \epsilon^{-2} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.3}$$

The relation between the γ-matrices in the curved spacetime and the flat spacetime can be written as

$$\gamma^\mu = \gamma^{(\alpha)}\ e^\mu_{(\alpha)}, \quad \gamma_\mu = \gamma_{(\alpha)}\ e^{(\alpha)}_\mu. \tag{A.4}$$

Using the tetrad $e^\mu_{(\alpha)}$ and γ-matrices, the spin connection Γ_μ can be written as

$$\Gamma_\mu = \frac{1}{4} g^{\lambda\nu}\, e^{(\alpha)}_\lambda\, \nabla_\mu\, e^{(\beta)}_\nu\ \frac{1}{2}\left[\gamma_{(\alpha)}, \gamma_{(\beta)}\right]. \tag{A.5}$$

Here we show the explicit representation of γ-matrices in the flat spacetime using the standard Dirac representation as

$$\gamma^{(0)} = \epsilon\beta = \epsilon \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \ \gamma^{(i)} = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix},$$

$$\gamma_{(0)} = \frac{1}{\epsilon} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma_{(i)} = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \tag{A.6}$$

where σ_i is the Pauli matrix satisfying

$$\sigma_i \, \sigma_j = \delta_{ij} + i \, \epsilon_{ijk} \, \sigma_k \tag{A.7}$$

and their explicit representation is

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.8}$$

The 4×4 ρ_i matrices are Dirac matrices satisfying

$$\rho_i \, \rho_j = \delta_{ij} + i \, \epsilon_{ijk} \, \rho_k \tag{A.9}$$

and their explicit representation is

$$\rho_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \ \rho_2 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \ \rho_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{A.10}$$

which leads to

$$\alpha_i = \rho_1 \sigma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \rho_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{A.11}$$

Here we define the second-rank tensor $\sigma^{(\alpha)(\beta)}$ as

$$\sigma^{(\alpha)(\beta)} = \frac{i}{2} \left[\gamma^{(\alpha)}, \gamma^{(\beta)} \right], \quad \gamma^{(\alpha)} \gamma^{(\beta)} = \eta^{\alpha\beta} - i \, \sigma^{(\alpha)(\beta)} \tag{A.12}$$

$$\sigma^{(0)(k)} = \frac{i}{2} \left[\gamma^{(0)}, \gamma^{(k)} \right] = i\epsilon\rho_1\sigma_k = i\epsilon\alpha_k$$

$$\sigma^{(i)(j)} = \frac{i}{2} \left[\gamma^{(i)}, \gamma^{(j)} \right] = \epsilon^{ijk} \sigma_k \tag{A.13}$$

$$\sigma_{(0)(k)} = \frac{i}{2} \left[\gamma_{(0)}, \gamma_{(k)} \right] = -i\epsilon^{-1}\rho_1\sigma_k = -i\epsilon^{-1}\alpha_k$$

$$\sigma_{(i)(j)} = \frac{i}{2} \left[\gamma_{(i)}, \gamma_{(j)} \right] = \epsilon_{ijk}\sigma_k. \tag{A.14}$$

Using the tetrad $e_\mu^{(\alpha)}$, the second-rank tensors $\sigma^{\mu\nu}$ and $F_{\mu\nu}$ can be written as

$$\sigma^{\mu\nu} = e^\mu_{(\alpha)} \, e^\nu_{(\beta)} \sigma^{(\alpha)(\beta)}$$

$$F_{\mu\nu} = e_\mu^{(\alpha)} \, e_\nu^{(\beta)} F_{(\alpha)(\beta)} \tag{A.15}$$

and

$$\sigma^{\mu\nu} F_{\mu\nu} = \sigma^{(\alpha)(\beta)} \, F_{(\alpha)(\beta)}. \tag{A.16}$$

Appendix B. Geometrical values in the Schwarzschild metric

The Schwarzschild metric can be written as

$$ds^2 = \left(1 - \epsilon^2 \frac{2GM}{r}\right) c^2 dt^2 - \frac{dr^2}{(1 - \epsilon^2 \frac{2GM}{r})} - r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) \tag{B.1}$$

in spherical coordinates. This can be rewritten, using isotropic coordinates and expanding up to the post-Newtonian order $O(\epsilon^2)$, as

$$\begin{aligned} ds^2 = \epsilon^{-2}(1 + \epsilon^2 2\phi + \epsilon^4 2\phi^2)\, dt^2 - (1 - \epsilon^2 2\phi)(dx^2 + dy^2 + dz^2) \\ + O(\epsilon^4), \end{aligned} \tag{B.2}$$

where $\phi = -GM/r$ is the Earth's gravitational potential. When the metric tensor $g_{\mu\nu}$ is uniquely defined, the Christoffel symbol $\Gamma^{\lambda}_{\mu\nu}$ and the tetrad $e^{\mu}_{(\alpha)}$ are also uniquely determined in the following.

B.1. *Metric tensor $g_{\mu\nu}$*

Using the definition and Eq. (B.2), the explicit expression of each component of the four metric tensor $g_{\mu\nu}$ and $g^{\mu\nu}$ is given up to the post-Newtonian order $O(\epsilon^2)$ as

$$\begin{aligned} g_{00} &= \epsilon^{-2}\left(1 + \epsilon^2 2\phi + \epsilon^4 2\phi^2\right) \\ g_{01} &= g_{02} = g_{03} = 0 \\ g_{ij} &= -(1 - 2\epsilon^2\phi)\ \delta_{ij} \\ g^{00} &= \epsilon^2\left(1 - 2\epsilon^2\phi + \epsilon^4 2\phi^2\right) \\ g^{01} &= g^{02} = g^{03} = 0 \\ g^{ij} &= -(1 + 2\epsilon^2\phi)\ \delta^{ij}. \end{aligned} \tag{B.3}$$

B.2. *Christoffel symbol $\Gamma^{\lambda}_{\mu\nu}$*

The Christoffel symbols are defined as

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} (g_{\kappa\mu;\nu} + g_{\kappa\nu;\mu} - g_{\mu\nu;\kappa}). \tag{B.4}$$

Substituting Eq. (B.3), we obtain explicit expressions for the Christoffel symbols up to the post-Newtonian order $O(\epsilon^2)$ as

$$\begin{aligned} &\Gamma^0_{00} = 0 \\ &\Gamma^1_{00} = \phi_1 \qquad \Gamma^2_{00} = \phi_2 \qquad \Gamma^3_{00} = \phi_3 \\ &\Gamma^0_{01} = \epsilon^2\phi_1 \qquad \Gamma^0_{02} = \epsilon^2\phi_2 \qquad \Gamma^0_{03} = \epsilon^2\phi_3 \\ &\Gamma^i_{0j} = 0 \\ &\Gamma^0_{ij} = 0 \end{aligned}$$

$$
\begin{aligned}
&\Gamma^1_{11} = -\epsilon^2\phi_1 && \Gamma^2_{11} = \epsilon^2\phi_2 && \Gamma^3_{11} = \epsilon^2\phi_3 \\
&\Gamma^1_{12} = -\epsilon^2\phi_2 && \Gamma^2_{12} = -\epsilon^2\phi_1 && \Gamma^3_{12} = 0 \\
&\Gamma^1_{13} = -\epsilon^2\phi_3 && \Gamma^2_{13} = 0 && \Gamma^3_{13} = -\epsilon^2\phi_1 \\
&\Gamma^1_{22} = \epsilon^2\phi_1 && \Gamma^2_{22} = -\epsilon^2\phi_2 && \Gamma^3_{22} = \epsilon^2\phi_3 \\
&\Gamma^1_{23} = 0 && \Gamma^2_{23} = -\epsilon^2\phi_3 && \Gamma^3_{23} = -\epsilon^2\phi_2 \\
&\Gamma^1_{33} = \epsilon^2\phi_1 && \Gamma^2_{33} = \epsilon^2\phi_2 && \Gamma^3_{33} = -\epsilon^2\phi_3,
\end{aligned}
\tag{B.5}
$$

where $\phi_i = \partial_i\phi = -\frac{\phi}{r^2}x_i$.

B.3. Tetrad $e^\mu_{(\alpha)}$

Using Eqs. (B.3) and (B.5), the tetrad $e^\mu_{(a)}$ satisfying

$$
g_{\mu\nu}\, e^\mu_{(a)} e^\nu_{(b)} = \eta_{(a)(b)} = \begin{pmatrix} \epsilon^{-2} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\tag{B.6}
$$

can be obtained up to the post-Newtonian order $O(\epsilon^2)$ as

$$
\begin{aligned}
e^0_{(0)} &= 1 - \epsilon^2\phi + \epsilon^4\frac{1}{2}\phi^2 \\
e^i_{(0)} &= e^0_{(i)} = 0 \\
e^i_{(j)} &= (1+\epsilon^2\phi)\delta^i_j \\
e^{(0)}_0 &= 1 + \epsilon^2\phi + \epsilon^4\frac{1}{2}\phi^2 \\
e^{(i)}_0 &= e^{(0)}_i = 0 \\
e^{(i)}_j &= (1-\epsilon^2\phi)\delta^i_j \\
e^{(0)0} &= \epsilon^2\left(1 - \epsilon^2\phi + \epsilon^4\frac{1}{2}\phi^2\right) \\
e^{(0)i} &= e^{(i)0} = 0 \\
e^{(i)j} &= -(1+\epsilon^2\phi)\delta^i_j.
\end{aligned}
\tag{B.7}
$$

Appendix C. Local inertial frame

The tetrad enables us to define the local inertial frame in which the local Lorentz invariance (LLI) is satisfied. The four vectors in the local inertial frame $x^{(\mu)}$ and in the general coordinate system x^μ are related to each other using the tetrad as

$$
x^{(a)} \equiv e^{(a)}_\mu x^\mu = (\tilde{x}^0, \tilde{\boldsymbol{x}}).
\tag{C.1}
$$

Using the representation of the tetrad up to the post-Newton order $O(\epsilon^2)$ shown in Eq. (B.7), the physical quantities relevant in this paper can be expressed as

$$\begin{aligned}
\tilde{S}^0 &= (1+\epsilon^2\phi)\, S^0 \\
\tilde{\boldsymbol{S}} &= (1-\epsilon^2\phi)\, \boldsymbol{S} \\
\tilde{\boldsymbol{\beta}} &= \frac{\tilde{\boldsymbol{u}}}{\tilde{u}^0} = (1-2\epsilon^2\phi)\, \boldsymbol{\beta} \\
\tilde{t} &= (1+\epsilon^2\phi)\ t \\
\tilde{\boldsymbol{x}} &= (1-\epsilon^2\phi)\ \boldsymbol{x} \\
\tilde{\gamma} &= \left(1-2\epsilon^2\phi\,(\gamma^2-1)\right)\gamma.
\end{aligned} \tag{C.2}$$

The conversion of a tensor of rank 2 (electromagnetic tensor) is given as

$$F_{(a)(b)} = e^{\mu}_{(a)} e^{\nu}_{(a)} F_{\mu\nu}, \tag{C.3}$$

which results in

$$\begin{aligned}
\tilde{\boldsymbol{E}} &= F_{(0)(\alpha)} = e^{0}_{(0)} e^{\alpha}_{(\alpha)} F_{0\alpha} = \boldsymbol{E} \\
\tilde{\boldsymbol{B}} &= F_{(\alpha)(\beta)} = e^{\alpha}_{(\alpha)} e^{\beta}_{(\beta)} F_{\mu\nu} = (1+2\epsilon^2\phi)\boldsymbol{B}.
\end{aligned} \tag{C.4}$$

Appendix D. Normalization of the Hamiltonian

The scalar product of a spinor function Ψ is defined as

$$\langle \Psi \mid \Psi \rangle = \int \Psi^{\dagger} \Psi \sqrt{h}\, d^3x, \tag{D.1}$$

where h_{ij} are the spatial components of the 4D tensor $g_{\mu\nu} : h_{ij} = -g_{ij}$. The expectation value of the Hamiltonian is given as

$$\langle \mathcal{H} \rangle = \langle \Psi \mid \mathcal{H} \mid \Psi \rangle = \int \Psi^{\dagger}\, \mathcal{H}\, \Psi \sqrt{h}\, d^3x. \tag{D.2}$$

Here we put

$$\Psi' = h^{\frac{1}{4}}\, \Psi, \qquad \mathcal{H}' = h^{\frac{1}{4}}\, \mathcal{H}\, h^{-\frac{1}{4}} \tag{D.3}$$

to obtain

$$\begin{aligned}
\langle \Psi \mid \mathcal{H} \mid \Psi \rangle &= \int \Psi^{\dagger}\, \mathcal{H}\, \Psi \sqrt{h}\, d^3x \\
&= \int \Psi'^{\dagger}\, \mathcal{H}'\, \Psi'\, d^3x \\
&= \left\langle \Psi' \mid \mathcal{H}' \mid \Psi' \right\rangle.
\end{aligned} \tag{D.4}$$

This corresponds to the normalization of the expectation value of the Hamiltonian into the expectation value in the local flat spacetime as $\mathcal{H}'$, which justifies the description of ϵ-dependent terms such as the post-Newtonian terms as a perturbation to the terms in the flat spacetime [5].

Appendix E. Typical scale of the gradient terms of the gravitational field

In general the equation of translational motion of a charged particle in curved spacetime involving general relativistic effects can be written as in Eq. (2). Using the Schwarzschild metric and considering up to the post-Newtonian order $O(\epsilon^2)$, the equation can be written as in Eq. (4). Here we estimate the magnitude of the contribution of the terms containing the gradient of the gravitational potential $\nabla\phi$ (last two terms in Eq. (4)) in the precise measurement of $g_\mu - 2$ in the storage ring apparatus.

The BNL E821 experiment [19–21] has apparatus parameters of $\gamma = 29.3$, $B = 1.45$ [T], $r = 7112$ [mm] and $\Omega_c = 42$ [MHz], and the magnitudes of the electromagnetic interaction terms are estimated as

$$\begin{aligned} f_{\mathrm{EM}} &= \left| \left(1 + (2\gamma^2 + 1)\epsilon^2\phi\right) \frac{e}{\gamma\, m} \boldsymbol{\beta} \times \boldsymbol{B} \right| \\ &= \left| \left(1 + (2\gamma^2 + 1)\epsilon^2\phi\right) \Omega_c \times \beta \right| \\ &\simeq 4.2 \times 10^7 \;\; [\mathrm{s}^{-1}]. \end{aligned} \tag{E.1}$$

On the other hand, the magnitudes of terms containing the gradient of the gravitational potential are

$$\begin{aligned} f_{\nabla\phi} &= \left| \epsilon(1 + \beta^2)\nabla\phi \right| \\ &\simeq 2 \times \frac{GM/R^2}{c} \\ &\simeq 6.5 \times 10^{-8} \; [\mathrm{s}^{-1}]. \end{aligned} \tag{E.2}$$

Therefore, the relative magnitudes of the contributions of the gravity gradient and the electromagnetic interaction are

$$\frac{f_{\nabla\phi}}{f_{\mathrm{EM}}} \simeq 1.5 \times 10^{-15}, \tag{E.3}$$

which shows that the gravity gradient contribution is 10^{-15} times smaller than the electromagnetic interaction and is even 10^{-5} times smaller than the post-Newtonian effects. Consequently, the contribution of the gravity gradient is negligible as long as the magnitude of the relevant contribution is 10^{-10} times that of the main contribution.

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