

Axial $U(1)$ anomaly in a gravitational field via the gradient flow

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A regularization-independent universal formula for the energy–momentum tensor in gauge theory in the flat spacetime can be written down by employing the so-called Yang–Mills gradient flow. We examine a possible use of the formula in the calculation of the axial $U(1)$ anomaly in a gravitational field, the anomaly first obtained by Toshiei Kimura [Prog. Theor. Phys. **42**, 1191 (1969)]. As a general argument indicates, the formula reproduces the correct non-local structure of the (axial $U(1)$ current)–(energy–momentum tensor)–(energy–momentum tensor) triangle diagram in a way that is consistent with the axial $U(1)$ anomaly. On the other hand, the formula does not automatically reproduce the general coordinate (or translation) Ward–Takahashi relation, requiring corrections by local counterterms. This analysis thus illustrates the fact that the universal formula as it stands can be used only in on-shell correlation functions, in which the energy–momentum tensor does not coincide with other composite operators in coordinate space.
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1. Introduction

Almost half a century ago, just nine months after the appearance of two seminal papers on the axial $U(1)$ anomaly in an electromagnetic field [1,2], Kimura noticed in a lesser-known but remarkable paper [3] that a similar anomalous non-conservation of the axial vector current also occurs in a gravitational field. His result was

$$D^\alpha \langle \bar{\psi}(x) \gamma_\alpha \gamma_5 \psi(x) \rangle - 2m_0 \langle \bar{\psi}(x) \gamma_5 \psi(x) \rangle = \frac{1}{384\pi^2} \epsilon_{\mu\nu\rho\sigma} R^{\mu\nu\lambda\tau}(x) R^{\rho\sigma}_{\lambda\tau}(x), \quad (1.1)$$

where m_0 is the bare mass of the fermion and $R^{\mu\nu\rho\sigma}(x)$ is the Riemann curvature. This axial $U(1)$ anomaly, also obtained in Refs. [4,5] (see also Refs. [6,7]), was the first example of the quantum anomaly related to the gravitational interaction, a subject that was to be extensively explored somewhat later in a wider context [8,9].

Recently, by employing the so-called Yang–Mills gradient flow [10–14] (see Refs. [15,16] for reviews) and the small flow time expansion [13], a regularization-independent universal formula for the energy–momentum tensor in gauge theory in the flat spacetime has been constructed [17,18] (see also Ref. [19] for a review); the formula is then applied to the computation of thermodynamic quantities in lattice QCD [20–26]. References [27–48] represent a partial list of developments relating to the gradient flow.

In this paper, we examine a possible use of the universal formula in the calculation of the axial $U(1)$ anomaly (1.1); we will obtain Eq. (1.1) by expansion around the flat spacetime. Precisely speaking, the

anomaly is a clash between the axial $U(1)$ Ward–Takahashi (WT) relation and the general coordinate (and the local Lorentz) WT relation. A general argument given in Ref. [3], which is analogous to the argument in Ref. [1], shows that the anomaly (1.1) is independent of the adopted regularization as long as the regularization is physically sensible and one imposes the general coordinate WT relation; the structure (1.1) is robust in this sense.

In what follows, we will observe that the universal formula does not automatically reproduce the correct WT relation associated with the general coordinate (or translation in the flat spacetime) WT relation. The resulting correlation functions, however, can be modified by adding appropriate *local terms* so that the translation WT relation holds. Then, as the general argument implies, we have Eq. (1.1). This shows that the universal formula reproduces the correct *non-local structure* of the (axial $U(1)$ current)–(energy–momentum tensor)–(energy–momentum tensor) triangle diagram in a way that is consistent with the axial $U(1)$ anomaly. This is expected without any calculation from the construction of the universal formula [17,18], but to check this point explicitly is certainly assuring. On the other hand, this analysis illustrates that the universal formula as it stands can be used only in on-shell correlation functions, in which the energy–momentum tensor does not coincide with other composite operators in coordinate space, because it does not automatically reproduce the translation WT relation when operators coincide. How to remedy this point in (a generalization of) the universal formula is a forthcoming challenge.

This paper is organized as follows. In Sect. 2, we summarize the naively expected form of the axial $U(1)$ and the general coordinate (or translation) WT relations in the flat spacetime limit. The breaking of these relations is regarded as the quantum anomaly. In Sect. 3, using the universal formulas for the energy–momentum tensor of the Dirac fermion [18] and the axial $U(1)$ current [35,42], we compute the total divergences of the triangle diagram and extract the parts potentially corresponding to the anomaly. Each of the axial $U(1)$ current and the energy–momentum tensors can possess a different flow time, t_1 , t_2 , and t_3 ; these are eventually taken to be zero. We adopt a particular ordering of the limits, which turns out to considerably simplify the calculation. We find that the translation WT relation does not hold. In Sect. 4, we seek an appropriate local term added to the triangle diagram, which restores the translation WT relation. Although our analysis here is quite analogous to that of Ref. [35] on the triangle anomaly in gauge theory, partially due to the fact that the translation WT relation also contains some two-point functions, the analysis is much more complicated. Finally, in Sect. 5, by adding appropriate local terms, we obtain the expansion of Eq. (1.1) around the flat spacetime. Section 6 is devoted to the conclusion.

2. Naively expected form of Ward–Takahashi relations

We consider the Dirac fermion in the curved spacetime with a Euclidean signature. The curved space indices are denoted by Greek letters while the local Lorentz indices are denoted by Latin letters. Letting $e_\mu^a(x)$ be the vierbein, the raising and lowering of the former indices are done by the metric $g_{\mu\nu}(x) \equiv \delta_{ab}e_\mu^a(x)e_\nu^b(x)$ and its inverse matrix $g^{\mu\nu}(x)$; those of the latter indices are, on the other hand, done by the Kronecker deltas, δ_{ab} and δ^{ab} .

The action of the Dirac fermion in the curved spacetime is given by

$$S = \int d^4x e(x) \bar{\psi}(x) \left(\frac{1}{2} \overleftrightarrow{D} + m_0 \right) \psi(x), \quad (2.1)$$

where $e(x) \equiv \det e_\mu^a(x)$, $\overleftrightarrow{D} \equiv D - \overleftarrow{D}$, and

$$D \equiv e_\mu^a(x) \gamma^a \left[\partial_\mu + \frac{1}{4} \omega_\mu^{bc}(x) \sigma_{bc} \right] \equiv \gamma^\mu(x) D_\mu, \quad (2.2)$$

$$\overleftarrow{D} \equiv \left[\overleftarrow{\partial}_\mu - \frac{1}{4} \omega_\mu^{bc}(x) \sigma_{bc} \right] e_\mu^a(x) \gamma^a \equiv \overleftarrow{D}_\mu \gamma^\mu(x). \quad (2.3)$$

γ^a is the Dirac matrix satisfying $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$ and $\sigma_{ab} \equiv \frac{1}{2}[\gamma_a, \gamma_b]$. $\omega_\mu^{ab}(x)$ is the spin connection, which is defined by

$$\omega_\mu^{ab}(x) \equiv \frac{1}{2} e^{a\rho}(x) e^{b\sigma}(x) [C_{\rho\sigma\mu}(x) - C_{\sigma\rho\mu}(x) - C_{\mu\rho\sigma}(x)], \quad (2.4)$$

$$C_{\mu\rho\sigma}(x) \equiv e_\mu^a(x) [\partial_\rho e_{a\sigma}(x) - \partial_\sigma e_{a\rho}(x)]. \quad (2.5)$$

The coupling of the fermion to a gravitational field is given by the energy–momentum tensor:

$$\begin{aligned} T_{\mu\nu}(x) &\equiv \frac{1}{e} e_\mu^a(x) \frac{\delta}{\delta e^{va}(x)} S \\ &= \frac{1}{2} \bar{\psi}(x) \gamma_\mu \overleftrightarrow{D}_\nu \psi(x) - g_{\mu\nu} \bar{\psi}(x) \left(\frac{1}{2} \overleftrightarrow{D} + m_0 \right) \psi(x) + \frac{1}{4} \epsilon_{\mu\nu\rho\sigma} D^\rho [\bar{\psi}(x) \gamma_5 \gamma^\sigma \psi(x)] \\ &= T_{\mu\nu}^{\text{sym.}}(x) + T_{\mu\nu}^{\text{anti-sym.}}(x), \end{aligned} \quad (2.6)$$

where we have defined

$$T_{\mu\nu}^{\text{sym.}}(x) \equiv \frac{1}{4} \bar{\psi}(x) \left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu \right) \psi(x) - g_{\mu\nu} \bar{\psi}(x) \left(\frac{1}{2} \overleftrightarrow{D} + m_0 \right) \psi(x) \quad (2.7)$$

$$T_{\mu\nu}^{\text{anti-sym.}}(x) \equiv \frac{1}{4} \bar{\psi}(x) \left[\sigma_{\mu\nu}(x) (D + m_0) + (\overleftarrow{D} - m_0) \sigma_{\mu\nu}(x) \right] \psi(x), \quad (2.8)$$

where $\sigma_{\mu\nu}(x) \equiv e_\mu^a(x) e_\nu^b(x) \sigma_{ab}$. We note that the anti-symmetric part of the energy–momentum tensor $T_{\mu\nu}^{\text{anti-sym.}}(x)$ is proportional to the equation of motion of the fermion.

Now, in order to determine the precise form of the quantum anomalies, it is crucial to clearly recognize the form of naively expected WT relations. For simplicity, we consider the massless fermion $m_0 = 0$ in what follows.

We start from the WT relation associated with the axial $U(1)$ symmetry. For this, we take the correlation function

$$\langle T_{\mu\nu}^{\text{sym.}}(y) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \equiv \int d\mu T_{\mu\nu}^{\text{sym.}}(y) T_{\rho\sigma}^{\text{sym.}}(z) e^{-S}, \quad (2.9)$$

where $d\mu$ denotes the functional integration measure for the fermion field, and make the change of integration variables in the form of a localized axial $U(1)$ transformation:¹

$$\delta\psi(x) = i\theta(x) \gamma_5 \psi(x), \quad \delta\bar{\psi}(x) = i\theta(x) \bar{\psi}(x) \gamma_5. \quad (2.10)$$

Noting that the action (2.1) changes under this change of variables as

$$\delta S = -i \int d^4x e(x) \theta(x) D^\alpha j_{5\alpha}(x), \quad j_{5\alpha}(x) \equiv \bar{\psi}(x) \gamma_\alpha \gamma_5 \psi(x), \quad (2.11)$$

¹ The chiral matrix γ_5 is defined by $\gamma_5 \equiv \frac{1}{4!} \epsilon_{abcd} \gamma^a \gamma^b \gamma^c \gamma^d$ by the totally anti-symmetric tensor ϵ_{abcd} being normalized as $\epsilon_{0123} = 1$.

and considering the flat spacetime limit $e_\mu^a(x) \rightarrow \delta_\mu^a$, neglecting a possible breaking of the symmetry associated with the regularization, we have the identity

$$\begin{aligned}
& \partial_\alpha^x \langle j_{5\alpha}(x) T_{\mu\nu}^{\text{sym.}}(y) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \\
& + \partial_\alpha^x \delta(y-x) \left\langle \frac{1}{2} \bar{\psi}(y) \gamma_5 (\gamma_\mu \delta_{\nu\alpha} + \gamma_\nu \delta_{\mu\alpha} - 2\delta_{\mu\nu} \gamma_\alpha) \psi(y) T_{\rho\sigma}^{\text{sym.}}(z) \right\rangle \\
& + \partial_\alpha^x \delta(z-x) \left\langle T_{\mu\nu}^{\text{sym.}}(y) \frac{1}{2} \bar{\psi}(z) \gamma_5 (\gamma_\rho \delta_{\sigma\alpha} + \gamma_\sigma \delta_{\rho\alpha} - 2\delta_{\rho\sigma} \gamma_\alpha) \psi(z) \right\rangle \\
& = \partial_\alpha^x \langle j_{5\alpha}(x) T_{\mu\nu}^{\text{sym.}}(y) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \\
& = 0,
\end{aligned} \tag{2.12}$$

where the first equality follows from the covariance under the Lorentz and parity transformations in the flat spacetime. The breaking of this naively expected relation is thus regarded as a quantum anomaly.

Next, we consider the WT relation associated with the general coordinate invariance and the local Lorentz symmetry. For this, we start with

$$\langle j_{5\alpha}(x) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \equiv \int d\mu j_{5\alpha}(x) T_{\rho\sigma}^{\text{sym.}}(z) e^{-S}, \tag{2.13}$$

and consider the following form of the change of integration variables:

$$\begin{aligned}
\delta\psi(x) &= \xi^\mu(x) \partial_\mu \psi(x) + \frac{1}{4} \xi^\mu(x) \omega_\mu^{ab}(x) \sigma_{ab} \psi(x) = \xi^\mu(x) D_\mu \psi(x), \\
\delta\bar{\psi}(x) &= \xi^\mu(x) \partial_\mu \bar{\psi}(x) - \frac{1}{4} \xi^\mu(x) \bar{\psi}(x) \sigma_{ab} \omega_\mu^{ab}(x) = \xi^\mu(x) D_\mu \bar{\psi}(x).
\end{aligned} \tag{2.14}$$

This is a particular combination of the general coordinate transformation and the local Lorentz transformation. Under this change of integration variables, from the fact that the action does not change if the vierbein is also changed by the same set of transformations, we have

$$\delta S = - \int d^4x e(x) \xi^\nu(x) D^\mu T_{\mu\nu}(x), \tag{2.15}$$

where the total energy–momentum tensor is given by Eq. (2.6). Considering the flat spacetime limit, we thus have the identity

$$\begin{aligned}
& \partial_\mu^y \langle j_{5\alpha}(x) T_{\mu\nu}^{\text{sym.}}(y) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \\
& + \partial_\mu^y \langle j_{5\alpha}(x) T_{\mu\nu}^{\text{anti-sym.}}(y) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \\
& + \delta(x-y) \partial_\nu^x \langle j_{5\alpha}(x) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \\
& + \delta(z-y) \partial_\nu^z \langle j_{5\alpha}(x) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \\
& + \partial_\beta^z \delta(z-y) \left\langle j_{5\alpha}(x) \frac{1}{4} \bar{\psi}(z) (\gamma_\rho \delta_{\sigma\beta} + \gamma_\sigma \delta_{\rho\beta} - 2\delta_{\rho\sigma} \gamma_\beta) \overleftrightarrow{\partial}_\nu \psi(z) \right\rangle \\
& = \partial_\mu^y \langle j_{5\alpha}(x) T_{\mu\nu}^{\text{sym.}}(y) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \\
& + \partial_\mu^y \langle j_{5\alpha}(x) T_{\mu\nu}^{\text{anti-sym.}}(y) T_{\rho\sigma}^{\text{sym.}}(z) \rangle
\end{aligned}$$

$$\begin{aligned}
& + \partial_\beta^y \delta(y-z) \langle j_{5\alpha}(x) \mathcal{O}_{1\beta, \nu, \rho\sigma}(z) \rangle \\
& = 0,
\end{aligned} \tag{2.16}$$

where again the first equality follows from the covariance under the Lorentz and parity transformations. We have introduced the combination

$$\mathcal{O}_{1\beta, \nu, \rho\sigma}(x) \equiv -\frac{1}{4} \bar{\psi}(x) (\gamma_\rho \delta_{\sigma\beta} + \gamma_\sigma \delta_{\rho\beta} - 2\delta_{\rho\sigma} \gamma_\beta) \overleftrightarrow{\partial}_\nu \psi(x). \tag{2.17}$$

On the other hand, by considering the change of integration variables of the form of the local Lorentz transformation,

$$\delta\psi(x) = -\frac{1}{4} \theta^{ab}(x) \sigma_{ab} \psi(x), \quad \delta\bar{\psi}(x) = \frac{1}{4} \bar{\psi}(x) \sigma_{ab} \theta^{ab}(x), \tag{2.18}$$

in Eq. (2.13), we have the following identity in the flat spacetime limit:

$$\begin{aligned}
& \langle j_{5\alpha}(x) T_{\mu\nu}^{\text{anti-sym.}}(y) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \\
& = -\delta(x-y) \left\langle \frac{1}{4} \bar{\psi}(x) [\gamma_\alpha \gamma_5, \sigma_{\mu\nu}] \psi(x) T_{\rho\sigma}^{\text{sym.}}(z) \right\rangle \\
& \quad - \delta(z-y) \left\langle j_{5\alpha}(x) \frac{1}{16} \bar{\psi}(z) [\gamma_\rho \overleftrightarrow{\partial}_\sigma + \gamma_\sigma \overleftrightarrow{\partial}_\rho - 2\delta_{\rho\sigma} \overleftrightarrow{\partial}, \sigma_{\mu\nu}] \psi(z) \right\rangle \\
& \quad - \partial_\beta^z \delta(z-y) \left\langle j_{5\alpha}(x) \frac{1}{16} \bar{\psi}(z) \{\gamma_\rho \delta_{\sigma\beta} + \gamma_\sigma \delta_{\rho\beta} - 2\delta_{\rho\sigma} \gamma_\beta, \sigma_{\mu\nu}\} \psi(z) \right\rangle \\
& = \delta(z-y) \langle j_{5\alpha}(x) \mathcal{O}_{2\mu\nu, \rho\sigma}(z) \rangle \\
& \quad + \partial_\beta^y \delta(z-y) \langle j_{5\alpha}(x) \mathcal{O}_{3\beta, \mu\nu, \rho\sigma}(z) \rangle,
\end{aligned} \tag{2.19}$$

where the last equality again follows from the covariance under the Lorentz and parity transformations and we have defined

$$\mathcal{O}_{2\mu\nu, \rho\sigma}(x) \equiv -\frac{1}{16} \bar{\psi}(x) [\gamma_\rho \overleftrightarrow{\partial}_\sigma + \gamma_\sigma \overleftrightarrow{\partial}_\rho - 2\delta_{\rho\sigma} \overleftrightarrow{\partial}, \sigma_{\mu\nu}] \psi(x), \tag{2.20}$$

$$\mathcal{O}_{3\beta, \mu\nu, \rho\sigma}(x) \equiv \frac{1}{16} \bar{\psi}(x) \{\gamma_\rho \delta_{\sigma\beta} + \gamma_\sigma \delta_{\rho\beta} - 2\delta_{\rho\sigma} \gamma_\beta, \sigma_{\mu\nu}\} \psi(x). \tag{2.21}$$

Thus, combining Eqs. (2.16) and (2.19), we have the relation in the flat spacetime limit:

$$\begin{aligned}
& \partial_\mu^y \langle j_{5\alpha}(x) T_{\mu\nu}^{\text{sym.}}(y) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \\
& \quad + \partial_\beta^y \delta(y-z) \langle j_{5\alpha}(x) \mathcal{O}_{1\beta, \nu, \rho\sigma}(z) \rangle \\
& \quad + \partial_\mu^y \delta(y-z) \langle j_{5\alpha}(x) \mathcal{O}_{2\mu\nu, \rho\sigma}(z) \rangle \\
& \quad + \partial_\mu^y \partial_\beta^y \delta(y-z) \langle j_{5\alpha}(x) \mathcal{O}_{3\beta, \mu\nu, \rho\sigma}(z) \rangle \\
& = 0.
\end{aligned} \tag{2.22}$$

Equation (2.22) is the naively expected form of the WT relation associated with the general coordinate invariance and the local Lorentz symmetry (in the flat spacetime limit). Thus, the breaking of this relation should be regarded as a quantum anomaly. It can be confirmed that one can directly derive the WT relation (2.22) only by using the translational invariance in the flat spacetime (see

Appendix A). The last three two-point functions including $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$ play a crucial role in the following analysis of the anomaly. The contribution of such “two-sided diagrams” in addition to the triangle diagram, which have no analogue in the axial $U(1)$ anomaly in gauge theory, has of course already been noted in Ref. [3] (through a somewhat different derivation from ours).

3. Computation of anomalies

3.1. Definition of the three-point function

Now, for the vector-like gauge theory in the flat spacetime, we know representations of the axial vector current [35,42] and the symmetric energy–momentum tensor [17,18,20] by the small flow time limit of flowed fields. In the zeroth order in the gauge coupling, the representations are rather trivial:

$$j_{5\alpha}(t, x) \equiv \bar{\chi}(t, x) \gamma_\alpha \gamma_5 \chi(t, x), \quad (3.1)$$

$$T_{\mu\nu}^{\text{sym.}}(t, x) \equiv \frac{1}{4} \bar{\chi}(t, x) \left(\gamma_\mu \overleftrightarrow{\partial}_\nu + \gamma_\nu \overleftrightarrow{\partial}_\mu - 2\delta_{\mu\nu} \overleftrightarrow{\partial} \right) \chi(t, x), \quad (3.2)$$

where $\chi(t, x)$ and $\bar{\chi}(t, x)$ are flowed fermion fields and eventually we have to take the small flow time limit $t \rightarrow 0$ in the correlation functions. Then, using the tree-level propagator of the flowed fermion field [14],²

$$\langle \chi(t, x) \bar{\chi}(s, y) \rangle_0 = \int_p e^{ip(x-y)} \frac{e^{-(t+s)p^2}}{i\cancel{p}}, \quad (3.4)$$

we have

$$\begin{aligned} & \langle j_{5\alpha}(x) T_{\mu\nu}^{\text{sym.}}(y) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \\ & \equiv \lim_{t_1 \equiv t \rightarrow 0} \lim_{t_2 = t_3 \rightarrow 0} \langle j_{5\alpha}(t_1, x) T_{\mu\nu}^{\text{sym.}}(t_2, y) T_{\rho\sigma}^{\text{sym.}}(t_3, z) \rangle \\ & = \lim_{t_1 \equiv t \rightarrow 0} \lim_{t_2 = t_3 \rightarrow 0} \int_{p, q, k} e^{ip(x-y)} e^{iq(y-z)} e^{ik(z-x)} e^{-(t_1+t_2)p^2} e^{-(t_2+t_3)q^2} e^{-(t_3+t_1)k^2} \\ & \quad \times \frac{i}{16} \left(\text{tr} \left\{ \gamma_\alpha \gamma_5 \frac{1}{\cancel{p}} \left[\gamma_\mu (p+q)_\nu + \gamma_\nu (p+q)_\mu - 2\delta_{\mu\nu} (\cancel{p} + \cancel{q}) \right] \frac{1}{\cancel{q}} \right. \right. \\ & \quad \times \left. \left[\gamma_\rho (q+k)_\sigma + \gamma_\sigma (q+k)_\rho - 2\delta_{\rho\sigma} (\cancel{q} + \cancel{k}) \right] \frac{1}{\cancel{k}} \right\} \\ & \quad - \text{tr} \left\{ \gamma_\alpha \gamma_5 \frac{1}{\cancel{k}} \left[\gamma_\rho (q+k)_\sigma + \gamma_\sigma (q+k)_\rho - 2\delta_{\rho\sigma} (\cancel{q} + \cancel{k}) \right] \frac{1}{\cancel{q}} \right. \\ & \quad \times \left. \left. \left[\gamma_\mu (p+q)_\nu + \gamma_\nu (p+q)_\mu - 2\delta_{\mu\nu} (\cancel{p} + \cancel{q}) \right] \frac{1}{\cancel{p}} \right\} \right) \\ & = \lim_{t \rightarrow 0} \int_{p, q, k} e^{ip(x-y)} e^{iq(y-z)} e^{ik(z-x)} e^{-tp^2} e^{-tk^2} \end{aligned}$$

² Throughout this paper, we use the abbreviation

$$\int_p \equiv \int \frac{d^4 p}{(2\pi)^4}. \quad (3.3)$$

$$\begin{aligned}
& \times \frac{i}{16} \left(\text{tr} \left\{ \gamma_\alpha \gamma_5 \frac{1}{\not{p}} \left[\gamma_\mu (p+q)_\nu + \gamma_\nu (p+q)_\mu - 2\delta_{\mu\nu} (\not{p} + \not{q}) \right] \frac{1}{\not{q}} \right. \right. \\
& \quad \times \left. \left[\gamma_\rho (q+k)_\sigma + \gamma_\sigma (q+k)_\rho - 2\delta_{\rho\sigma} (\not{q} + \not{k}) \right] \frac{1}{\not{k}} \right\} \\
& \quad - \text{tr} \left\{ \gamma_\alpha \gamma_5 \frac{1}{\not{k}} \left[\gamma_\rho (q+k)_\sigma + \gamma_\sigma (q+k)_\rho - 2\delta_{\rho\sigma} (\not{q} + \not{k}) \right] \frac{1}{\not{q}} \right. \\
& \quad \times \left. \left. \left[\gamma_\mu (p+q)_\nu + \gamma_\nu (p+q)_\mu - 2\delta_{\mu\nu} (\not{p} + \not{q}) \right] \frac{1}{\not{p}} \right\} \right). \quad (3.5)
\end{aligned}$$

In this definition, we have adopted a particular ordering of the small time limit; we first set $t_2 = t_3 \rightarrow 0$ and then $t_1 \equiv t \rightarrow 0$. Because of the Gaussian damping factors $e^{-(t_1+t_2)p^2}$, $e^{-(t_2+t_3)q^2}$, and $e^{-(t_3+t_1)k^2}$ in the first definition, the momentum integration is absolutely convergent as long as $t_1 \equiv t > 0$; we can thus trivially take the first limit $t_2 = t_3 \rightarrow 0$ inside the momentum integration. It turns out that this particular ordering considerably simplifies the actual calculation of the anomalies below.³ We also note that the expression (3.5) does not require further regularization; in other words, Eq. (3.5) is independent of the adopted regularization in the limit in which the regulator is sent to infinity. This shows the universality of the representations (3.1) and (3.2), although this finiteness is trivial in the present zeroth-order perturbation theory in the gauge coupling.

3.2. Anomaly in the axial WT relation

We are primarily interested in the anomalous divergence of the axial vector current, the breaking of the WT relation (2.12). From our definition (3.5), after careful rearrangements, we find the identity (omitting the symbol $\lim_{t \rightarrow 0}$)⁴

$$\begin{aligned}
& \partial_\alpha^x \langle j_{5\alpha}(x) T_{\mu\nu}^{\text{sym.}}(y) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \\
& \quad + \partial_\alpha^x \delta(y-x) \left\langle \frac{1}{2} \bar{\psi}(y) \gamma_5 (\gamma_\mu \delta_{\nu\alpha} + \gamma_\nu \delta_{\mu\alpha} - 2\delta_{\mu\nu} \gamma_\alpha) \psi(y) T_{\rho\sigma}^{\text{sym.}}(z) \right\rangle \\
& \quad + \partial_\alpha^x \delta(z-x) \left\langle T_{\mu\nu}^{\text{sym.}}(y) \frac{1}{2} \bar{\psi}(z) \gamma_5 (\gamma_\rho \delta_{\sigma\alpha} + \gamma_\sigma \delta_{\rho\alpha} - 2\delta_{\rho\sigma} \gamma_\alpha) \psi(z) \right\rangle \\
& = \partial_\alpha^x \text{F.T.} e^{-tp^2} e^{-tk^2} \frac{-i}{8} \text{tr} \left\{ \gamma_5 (\gamma_\mu \delta_{\nu\alpha} + \gamma_\nu \delta_{\mu\alpha} - 2\delta_{\mu\nu} \gamma_\alpha) \frac{1}{\not{q}} \left[\gamma_\rho (q+k)_\sigma + \gamma_\sigma (q+k)_\rho \right] \frac{1}{\not{k}} \right\} \\
& \quad + \partial_\alpha^x \text{F.T.} e^{-tp^2} e^{-tk^2} \frac{i}{8} \text{tr} \left\{ \gamma_5 \frac{1}{\not{p}} \left[\gamma_\mu (p+q)_\nu + \gamma_\nu (p+q)_\mu \right] \frac{1}{\not{q}} (\gamma_\rho \delta_{\sigma\alpha} + \gamma_\sigma \delta_{\rho\alpha} - 2\delta_{\rho\sigma} \gamma_\alpha) \right\} \\
& \quad + \text{F.T.} e^{-tp^2} e^{-tk^2} \frac{1}{8} \text{tr} \left\{ \gamma_5 \left[\gamma_\mu (q+k)_\nu + \gamma_\nu (q+k)_\mu \right] \frac{1}{\not{q}} \left[\gamma_\rho (q+k)_\sigma + \gamma_\sigma (q+k)_\rho \right] \frac{1}{\not{k}} \right\} \\
& \quad + \text{F.T.} e^{-tp^2} e^{-tk^2} \frac{1}{8} \text{tr} \left\{ \gamma_5 \frac{1}{\not{p}} \left[\gamma_\mu (p+q)_\nu + \gamma_\nu (p+q)_\mu \right] \frac{1}{\not{q}} \left[\gamma_\rho (p+q)_\sigma + \gamma_\sigma (p+q)_\rho \right] \right\}, \quad (3.6)
\end{aligned}$$

³ This simplification should be related to the fact that the energy-momentum tensor induces the correct translation on composite operators of the flowed fields with non-zero flow times (in our present case $j_{5\alpha}(t, x)$), a fact emphasized in this context in Refs. [27, 38, 45].

⁴ We have noted that the spinor trace with γ_5 requires at least other four Dirac matrices.

where F.T. denotes the Fourier transformation:

$$\text{F.T.} \equiv \int_{p,q,k} e^{ip(x-y)} e^{iq(y-z)} e^{ik(z-x)} \times. \quad (3.7)$$

On the left-hand side of Eq. (3.6), the two-point functions have been defined by

$$\begin{aligned} & \left\langle \frac{1}{2} \bar{\psi}(y) \gamma_5 (\gamma_\mu \delta_{\nu\alpha} + \gamma_\nu \delta_{\mu\alpha} - 2\delta_{\mu\nu} \gamma_\alpha) \psi(y) T_{\rho\sigma}^{\text{sym.}}(z) \right\rangle \\ & \equiv \int_{q,k} e^{i(q-k)(y-z)} e^{-tq^2} e^{-tk^2} \\ & \quad \times \frac{i}{8} \text{tr} \left\{ \gamma_5 (\gamma_\mu \delta_{\nu\alpha} + \gamma_\nu \delta_{\mu\alpha} - 2\delta_{\mu\nu} \gamma_\alpha) \frac{1}{q} [\gamma_\rho (q+k)_\sigma + \gamma_\sigma (q+k)_\rho] \frac{1}{k} \right\} = 0, \end{aligned} \quad (3.8)$$

$$\begin{aligned} & \left\langle T_{\mu\nu}^{\text{sym.}}(y) \frac{1}{2} \bar{\psi}(z) \gamma_5 (\gamma_\rho \delta_{\sigma\alpha} + \gamma_\sigma \delta_{\rho\alpha} - 2\delta_{\rho\sigma} \gamma_\alpha) \psi(z) \right\rangle \\ & \equiv \int_{p,q} e^{i(-p+q)(y-z)} e^{-tp^2} e^{-tq^2} \\ & \quad \times \frac{-i}{8} \text{tr} \left\{ \gamma_5 \frac{1}{p} [\gamma_\mu (p+q)_\nu + \gamma_\nu (p+q)_\mu] \frac{1}{q} (\gamma_\rho \delta_{\sigma\alpha} + \gamma_\sigma \delta_{\rho\alpha} - 2\delta_{\rho\sigma} \gamma_\alpha) \right\} = 0. \end{aligned} \quad (3.9)$$

These regularized two-point correlation functions identically vanish, as should be the case from the Lorentz and parity covariance.

In deriving Eq. (3.6), we first apply ∂_α^x to Eq. (3.5). In the integrand, this produces the factor $p - k$; each term of this is canceled by $1/p$ and $1/k$. We then use the identities $p + q = (p - k) + (q + k)$ and $q + k = (k - p) + (p + q)$ and express the momentum $(p - k)_\alpha$ by the derivative $-i\partial_\alpha^x$. These manipulations give rise to the right-hand side of Eq. (3.6). The last two terms on the left-hand side of Eq. (3.6) are simply zero, as noted above; the inclusion of these terms, however, clearly shows the correspondence to the naively expected axial WT relation (2.12).

Thus, comparing Eq. (3.6) and Eq. (2.12), we find that the anomalous breaking of the axial symmetry is given by the right-hand side of Eq. (3.6). We note that in Eq. (3.6) if Gaussian factors such as e^{-tp^2} and e^{-tk^2} are simply unity (i.e., if we could naively set $t \rightarrow 0$ before the momentum integration), then the right-hand side identically vanishes. The fact is that there are Gaussian factors and they give rise to a non-vanishing result. After a straightforward calculation in the $t \rightarrow 0$ limit, we find

$$\begin{aligned} & \partial_\alpha^x \langle j_{5\alpha}(x) T_{\mu\nu}^{\text{sym.}}(y) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \\ & = \int_{p,q} e^{ip(x-y)} e^{iq(x-z)} \frac{1}{(4\pi)^2} \\ & \quad \times \left\{ \frac{1}{12} \epsilon_{\mu\rho\beta\gamma} p_\beta q_\gamma \left[q_\nu p_\sigma + \delta_{\nu\sigma} \left(-\frac{1}{t} + p^2 + pq + q^2 \right) \right] + (\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma) \right\}. \end{aligned} \quad (3.10)$$

3.3. Anomaly in the translation WT relation

Next, we investigate the anomalous breaking of the translation WT relation (2.22). From Eq. (3.5), after careful rearrangements by using the relation

$$\begin{aligned}
\gamma_\mu(p+q)_\nu + \gamma_\nu(p+q)_\mu &= 2\gamma_\mu(p+q)_\nu + \not{p}\sigma_{\mu\nu} - \sigma_{\mu\nu}\not{q} - \frac{1}{2}\gamma_\mu\gamma_\nu(\not{p}-\not{q}) + \frac{1}{2}(\not{p}-\not{q})\gamma_\nu\gamma_\mu \\
&= 2\gamma_\mu(p+q)_\nu - \sigma_{\mu\nu}\not{p} + \not{q}\sigma_{\mu\nu} + \frac{1}{2}\gamma_\mu\gamma_\nu(\not{p}-\not{q}) - \frac{1}{2}(\not{p}-\not{q})\gamma_\nu\gamma_\mu,
\end{aligned} \tag{3.11}$$

we have the identity⁵

$$\begin{aligned}
&\partial_\mu^y \langle j_{5\alpha}(x) T_{\mu\nu}^{\text{sym.}}(y) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \\
&\quad + \delta(x-y) \partial_\nu^x \langle j_{5\alpha}(x) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \\
&\quad + \delta(y-z) \partial_\nu^z \langle j_{5\alpha}(x) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \\
&\quad - \partial_\mu^y \delta(x-y) \left\langle \frac{1}{4} \bar{\psi}(x) [\gamma_\alpha \gamma_5, \sigma_{\mu\nu}] \psi(x) T_{\rho\sigma}^{\text{sym.}}(z) \right\rangle \\
&\quad + \partial_\beta^y \delta(y-z) \langle j_{5\alpha}(x) \mathcal{O}_{1\beta, \nu, \rho\sigma}(z) \rangle \\
&\quad + \partial_\mu^y \delta(y-z) \langle j_{5\alpha}(x) \mathcal{O}_{2\mu\nu, \rho\sigma}(z) \rangle \\
&\quad + \partial_\mu^y \partial_\beta^y \delta(y-z) \langle j_{5\alpha}(x) \mathcal{O}_{3\beta, \mu\nu, \rho\sigma}(z) \rangle \\
&= (\partial_\nu^x + \partial_\nu^y) \text{F.T.} e^{-tp^2} e^{-tk^2} \frac{-i}{4} \text{tr} \left\{ \gamma_\alpha \gamma_5 \frac{1}{\not{q}} [\gamma_\rho(q+k)_\sigma + \gamma_\sigma(q+k)_\rho] \frac{1}{\not{k}} \right\} \\
&\quad - \partial_\mu^y \text{F.T.} e^{-tp^2} e^{-tk^2} \\
&\quad \times \left(\frac{-i}{16} \text{tr} \left\{ \gamma_\alpha \gamma_5 \sigma_{\mu\nu} \frac{1}{\not{q}} [\gamma_\rho(q+k)_\sigma + \gamma_\sigma(q+k)_\rho - 2\delta_{\rho\sigma}(\not{q} + \not{k})] \frac{1}{\not{k}} \right\} \right. \\
&\quad \left. + \frac{-i}{16} \text{tr} \left\{ \gamma_\alpha \gamma_5 \frac{1}{\not{k}} [\gamma_\rho(q+k)_\sigma + \gamma_\sigma(q+k)_\rho - 2\delta_{\rho\sigma}(\not{q} + \not{k})] \frac{1}{\not{q}} \sigma_{\mu\nu} \right\} \right) \\
&\quad + \text{F.T.} e^{-tp^2} e^{-tk^2} \frac{1}{4} \text{tr} \left\{ \gamma_\alpha \gamma_5 (q+k)_\nu \frac{1}{\not{q}} [\gamma_\rho(q+k)_\sigma + \gamma_\sigma(q+k)_\rho] \frac{1}{\not{k}} \right\} \\
&\quad + \text{F.T.} e^{-tp^2} e^{-tk^2} \frac{-1}{4} \text{tr} \left\{ \gamma_\alpha \gamma_5 (p+k)_\nu \frac{1}{\not{p}} [\gamma_\rho(p+k)_\sigma + \gamma_\sigma(p+k)_\rho] \frac{1}{\not{k}} \right\} \\
&\quad + \partial_\beta^y \partial_\nu^x \text{F.T.} e^{-tp^2} e^{-tk^2} \frac{1}{4} \text{tr} \left[\gamma_\alpha \gamma_5 \frac{1}{\not{p}} (\gamma_\rho \delta_{\sigma\beta} + \gamma_\sigma \delta_{\rho\beta} - 2\delta_{\rho\sigma} \gamma_\beta) \frac{1}{\not{k}} \right].
\end{aligned} \tag{3.12}$$

In this expression, the two-point functions on the left-hand side have been defined by

$$\begin{aligned}
&\partial_\nu^x \langle j_{5\alpha}(x) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \\
&\equiv \partial_\nu^x \int_{q,k} e^{i(q-k)(x-z)} e^{-tq^2} e^{-tk^2} \frac{-i}{4} \text{tr} \left\{ \gamma_\alpha \gamma_5 \frac{1}{\not{q}} [\gamma_\rho(q+k)_\sigma + \gamma_\sigma(q+k)_\rho] \frac{1}{\not{k}} \right\} = 0,
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
&\partial_\nu^z \langle j_{5\alpha}(x) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \\
&\equiv \partial_\nu^z \int_{p,k} e^{i(p-k)(x-z)} e^{-tp^2} e^{-tk^2} \frac{-i}{4} \text{tr} \left\{ \gamma_\alpha \gamma_5 \frac{1}{\not{p}} [\gamma_\rho(p+k)_\sigma + \gamma_\sigma(p+k)_\rho] \frac{1}{\not{k}} \right\} = 0,
\end{aligned} \tag{3.14}$$

$$\left\langle \frac{1}{4} \bar{\psi}(x) [\gamma_\alpha \gamma_5, \sigma_{\mu\nu}] \psi(x) T_{\rho\sigma}^{\text{sym.}}(z) \right\rangle$$

⁵ We have again noted that the spinor trace with γ_5 requires at least other four Dirac matrices.

$$\begin{aligned}
&\equiv \int_{q,k} e^{i(q-k)(x-z)} e^{-tq^2} e^{-tk^2} \frac{-i}{16} \text{tr} \left\{ [\gamma_\alpha \gamma_5, \sigma_{\mu\nu}] \frac{1}{\not{q}} \left[\gamma_\rho (q+k)_\sigma + \gamma_\sigma (q+k)_\rho - 2\delta_{\rho\sigma} (\not{q} + \not{k}) \right] \frac{1}{\not{k}} \right\} \\
&= 0,
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
&\langle j_{5\alpha}(x) \mathcal{O}_{1\beta, \nu, \rho\sigma}(z) \rangle \\
&\equiv \int_{p,k} e^{i(p-k)(x-z)} e^{-tp^2} e^{-tk^2} \frac{1}{4} \text{tr} \left[\gamma_\alpha \gamma_5 \frac{1}{\not{p}} (\gamma_\rho \delta_{\sigma\beta} + \gamma_\sigma \delta_{\rho\beta} - 2\delta_{\rho\sigma} \gamma_\beta) i(p+k)_\nu \frac{1}{\not{k}} \right],
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&\langle j_{5\alpha}(x) \mathcal{O}_{2\mu\nu, \rho\sigma}(z) \rangle \\
&\equiv \int_{p,k} e^{i(p-k)(x-z)} e^{-tp^2} e^{-tk^2} \frac{-i}{16} \text{tr} \left[\gamma_\alpha \gamma_5 \frac{1}{\not{k}} \left[\gamma_\rho (p+k)_\sigma + \gamma_\sigma (p+k)_\rho - 2\delta_{\rho\sigma} (\not{p} + \not{k}), \sigma_{\mu\nu} \right] \frac{1}{\not{p}} \right],
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
&\langle j_{5\alpha}(x) \mathcal{O}_{3\beta, \mu\nu, \rho\sigma}(z) \rangle \\
&\equiv \int_{p,k} e^{i(p-k)(x-z)} e^{-tp^2} e^{-tk^2} \frac{1}{16} \text{tr} \left(\gamma_\alpha \gamma_5 \frac{1}{\not{p}} \{ \gamma_\rho \delta_{\sigma\beta} + \gamma_\sigma \delta_{\rho\beta} - 2\delta_{\rho\sigma} \gamma_\beta, \sigma_{\mu\nu} \} \frac{1}{\not{k}} \right).
\end{aligned} \tag{3.18}$$

The two-point functions in Eqs. (3.13)–(3.15) identically vanish as should be the case from the Lorentz and parity covariance.

On the right-hand side of Eq. (3.12), the last two lines change sign under the change of integration variables, $p \rightarrow -k$ and $k \rightarrow -p$. Thus, those two lines identically vanish. The other three terms do not vanish and after a tedious calculation in the limit $t \rightarrow 0$, we have

$$\begin{aligned}
&\partial_\mu^y \langle j_{5\alpha}(x) T_{\mu\nu}^{\text{sym.}}(y) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \\
&\quad + \partial_\beta^y \delta(y-z) \langle j_{5\alpha}(x) \mathcal{O}_{1\beta, \nu, \rho\sigma}(z) \rangle \\
&\quad + \partial_\mu^y \delta(y-z) \langle j_{5\alpha}(x) \mathcal{O}_{2\mu\nu, \rho\sigma}(z) \rangle \\
&\quad + \partial_\mu^y \partial_\beta^y \delta(y-z) \langle j_{5\alpha}(x) \mathcal{O}_{3\beta, \mu\nu, \rho\sigma}(z) \rangle \\
&= \int_{p,q} e^{ip(x-y)} e^{iq(x-z)} \frac{1}{(4\pi)^2} \\
&\quad \times \left\{ p_\beta q_\gamma \epsilon_{\alpha\beta\gamma\nu} \left(\frac{1}{12} p_\rho p_\sigma + \frac{1}{12} p_\rho q_\sigma + \frac{1}{12} q_\rho p_\sigma - \frac{1}{8} p q \delta_{\rho\sigma} \right) \right. \\
&\quad + p_\beta q_\gamma \epsilon_{\alpha\beta\gamma\rho} \left[-\frac{1}{12} q_\nu p_\sigma + \frac{1}{24} q_\nu q_\sigma + \delta_{\nu\sigma} \left(\frac{1}{12t} - \frac{1}{16} p^2 - \frac{1}{12} p q - \frac{5}{48} q^2 \right) \right] + (\rho \leftrightarrow \sigma) \\
&\quad + p_\beta \epsilon_{\alpha\beta\nu\rho} \left[p_\sigma \left(-\frac{1}{24t} + \frac{1}{48} p^2 + \frac{1}{12} p q + \frac{5}{48} q^2 \right) + q_\sigma \left(-\frac{1}{24} p q \right) \right] + (\rho \leftrightarrow \sigma) \\
&\quad \left. + q_\beta \epsilon_{\alpha\beta\nu\rho} \left[p_\sigma \left(-\frac{1}{12t} + \frac{1}{48} p^2 + \frac{1}{12} p q + \frac{5}{48} q^2 \right) + q_\sigma \left(-\frac{1}{24} p q \right) \right] + (\rho \leftrightarrow \sigma) \right\}.
\end{aligned} \tag{3.19}$$

3.4. Axial anomaly in the two-point functions

For the subsequent analysis, we still need to know possible anomalous breakings of the axial WT relations for the two-point functions (3.16)–(3.18). From the structure of the fermion bi-linear operators, $\mathcal{O}_1$ (Eq. (2.17)), $\mathcal{O}_2$ (Eq. (2.20)), and $\mathcal{O}_3$ (Eq. (2.21)), we would expect, as the axial $U(1)$ WT relations, $\partial_\alpha^x \langle j_{5\alpha}(x) \mathcal{O}_{1\beta, \nu, \rho\sigma}(z) \rangle = \partial_\alpha^x \langle j_{5\alpha}(x) \mathcal{O}_{2\mu\nu, \rho\sigma}(z) \rangle = \partial_\alpha^x \langle j_{5\alpha}(x) \mathcal{O}_{3\beta, \mu\nu, \rho\sigma}(z) \rangle = 0$. It turns out that our definition (3.18) is not compatible with this expectation and

$$\partial_\alpha^x \langle j_{5\alpha}(x) \mathcal{O}_{1\beta, \nu, \rho\sigma}(z) \rangle = 0, \quad (3.20)$$

$$\partial_\alpha^x \langle j_{5\alpha}(x) \mathcal{O}_{2\mu\nu, \rho\sigma}(z) \rangle = 0, \quad (3.21)$$

$$\begin{aligned} & \partial_\alpha^x \langle j_{5\alpha}(x) \mathcal{O}_{3\beta, \mu\nu, \rho\sigma}(z) \rangle \\ &= i \int_q e^{iq(x-z)} \frac{1}{(4\pi)^2} (\epsilon_{\rho\mu\nu\gamma} \delta_{\sigma\beta} + \epsilon_{\sigma\mu\nu\gamma} \delta_{\rho\beta} - 2\delta_{\rho\sigma} \epsilon_{\beta\mu\nu\gamma}) q_\gamma \left(\frac{1}{8t} - \frac{1}{12} q^2 \right). \end{aligned} \quad (3.22)$$

As we will see, however, the axial anomaly in the two-point function (3.22) can be removed by adding an appropriate local term to the two-point function (3.18).

4. Local counterterms

Now, the anomalous breaking of the axial WT relation in Eq. (3.10) would have intrinsic meaning only when we require the validity of the translation WT relation (2.22). That is, we still have the freedom to modify the local part of the three-point correlation function (3.5) by adding a “local counterterm” $\mathcal{C}_{\alpha, \mu\nu, \rho\sigma}(x, y, z)$. In the momentum space, it must be a cubic polynomial of external momenta. We see that the general form of the counterterm that is consistent with the symmetric structure of the three-point function (3.5) is given by

$$\begin{aligned} & \mathcal{C}_{\alpha, \mu\nu, \rho\sigma}(x, y, z) \\ &= \int_{p, q} e^{ip(x-y)} e^{iq(x-z)} \frac{i}{(4\pi)^2} \\ & \quad \times [\epsilon_{\alpha\mu\rho\beta} p_\beta \delta_{\nu\sigma} (c_0 + c_1 p^2 + c_2 p q + c_3 q^2) - \epsilon_{\alpha\mu\rho\beta} q_\beta \delta_{\nu\sigma} (c_0 + c_3 p^2 + c_2 p q + c_1 q^2) \\ & \quad + \epsilon_{\alpha\mu\rho\beta} p_\beta (d_1 p_\nu p_\sigma + d_2 p_\nu q_\sigma + d_3 q_\nu p_\sigma + d_4 q_\nu q_\sigma) \\ & \quad - \epsilon_{\alpha\mu\rho\beta} q_\beta (d_4 p_\nu p_\sigma + d_2 p_\nu q_\sigma + d_3 q_\nu p_\sigma + d_1 q_\nu q_\sigma) \\ & \quad + \epsilon_{\alpha\mu\beta\gamma} p_\beta q_\gamma (e_1 p_\nu \delta_{\rho\sigma} + e_2 p_\rho \delta_{\nu\sigma} + e_3 q_\nu \delta_{\rho\sigma} + e_4 q_\rho \delta_{\nu\sigma}) \\ & \quad - \epsilon_{\alpha\rho\beta\gamma} p_\beta q_\gamma (e_3 p_\sigma \delta_{\mu\nu} + e_4 p_\mu \delta_{\nu\sigma} + e_1 q_\sigma \delta_{\mu\nu} + e_2 q_\mu \delta_{\nu\sigma}) \\ & \quad + \epsilon_{\mu\rho\beta\gamma} p_\beta q_\gamma (f_1 p_\alpha \delta_{\nu\sigma} + f_2 p_\nu \delta_{\alpha\sigma} + f_3 p_\sigma \delta_{\alpha\nu} + f_1 q_\alpha \delta_{\nu\sigma} + f_3 q_\nu \delta_{\alpha\sigma} + f_2 q_\sigma \delta_{\alpha\nu}) \\ & \quad + (\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma)], \end{aligned} \quad (4.1)$$

where c_i , d_i , e_i , and f_i are constants. The basic idea is to choose the coefficients c_i , d_i , e_i , and f_i so that the right-hand side of Eq. (3.19) vanishes after the addition $\langle j_{5\alpha}(x) T_{\mu\nu}^{\text{sym.}}(y) T_{\rho\sigma}^{\text{sym.}}(z) \rangle + \mathcal{C}_{\alpha, \mu\nu, \rho\sigma}(x, y, z)$. Then, to the axial anomaly (3.10), the counterterm contributes by

$$\begin{aligned} & \partial_\alpha^x \mathcal{C}_{\alpha, \mu\nu, \rho\sigma}(x, y, z) \\ &= \int_{p, q} e^{ip(x-y)} e^{iq(x-z)} \frac{1}{(4\pi)^2} \epsilon_{\mu\rho\beta\gamma} p_\beta q_\gamma \end{aligned}$$

$$\begin{aligned}
& \times \{ (d_1 + d_4 - f_2 - f_3)p_\nu p_\sigma + (2d_2 - 2f_2)p_\nu q_\sigma \\
& + (2d_3 - 2f_3)q_\nu p_\sigma + (d_1 + d_4 - f_2 - f_3)q_\nu q_\sigma \\
& + \delta_{\nu\sigma} [2c_0 + (c_1 + c_3 - f_1)p^2 + (2c_2 - 2f_1)pq + (c_1 + c_3 - f_1)q^2] \} \\
& + (\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma).
\end{aligned} \tag{4.2}$$

A complication arises, however, since we may also modify the two-point functions (3.16)–(3.18) appearing in the relation (3.19) by adding local terms. We choose the counterterms for the two-point functions such that the axial $U(1)$ WT relations hold for the two-point functions.

For the two-point function (3.16), we thus require the validity of the axial WT relation

$$\partial_\alpha^x [\langle j5_\alpha(x) \mathcal{O}_{1\beta,\nu,\rho\sigma}(z) \rangle + \mathcal{S}_{1\alpha,\beta,\nu,\rho\sigma}(x, z)] = 0, \tag{4.3}$$

where $\mathcal{S}_{1\alpha,\beta,\nu,\rho\sigma}(x, z)$ is a local term. Equation (3.20), however, shows that there is no axial anomaly in this two-point function and thus we should require $\partial_\alpha^x \mathcal{S}_{1\alpha,\beta,\nu,\rho\sigma}(x, z) = 0$. It turns out that the most general form of such a local term is

$$\begin{aligned}
& \mathcal{S}_{1\alpha,\beta,\nu,\rho\sigma}(x, z) \\
& = \int_q e^{iq(x-z)} \frac{i}{(4\pi)^2} \\
& \quad \times \left\{ \epsilon_{\alpha\beta\nu\rho} q_\sigma [\tilde{c}_0 + (\tilde{c}_1 + \tilde{c}_2)q^2] + (\rho \leftrightarrow \sigma) \right. \\
& \quad + \epsilon_{\beta\nu\rho\gamma} q_\gamma [\delta_{\alpha\sigma}(\tilde{c}_0 + \tilde{c}_1 q^2) + \tilde{c}_2 q_\alpha q_\sigma] + (\rho \leftrightarrow \sigma) \\
& \quad + \epsilon_{\alpha\beta\nu\gamma} q_\gamma [\delta_{\rho\sigma}(\tilde{d}_0 + \tilde{d}_1 q^2) + \tilde{d}_2 q_\rho q_\sigma] \\
& \quad + \epsilon_{\alpha\beta\rho\gamma} q_\gamma [\delta_{\nu\sigma}(\tilde{e}_0 + \tilde{e}_1 q^2) + \tilde{e}_2 q_\nu q_\sigma] + (\rho \leftrightarrow \sigma) \\
& \quad \left. + \epsilon_{\alpha\nu\rho\gamma} q_\gamma [\delta_{\beta\sigma}(\tilde{f}_0 + \tilde{f}_1 q^2) + \tilde{f}_2 q_\beta q_\sigma] + (\rho \leftrightarrow \sigma) \right\},
\end{aligned} \tag{4.4}$$

where $\tilde{c}_i$, $\tilde{d}_i$, $\tilde{e}_i$, and $\tilde{f}_i$ are constants.

Similarly, for the two-point function (3.17), requiring

$$\partial_\alpha^x [\langle j5_\alpha(x) \mathcal{O}_{2\mu\nu,\rho\sigma}(z) \rangle + \mathcal{S}_{2\alpha,\mu\nu,\rho\sigma}(x, z)] = 0 \tag{4.5}$$

implies $\partial_\alpha^x \mathcal{S}_{2\alpha,\mu\nu,\rho\sigma}(x, z) = 0$ because of Eq. (3.21) and the possible form of the counterterm is given by

$$\begin{aligned}
& \mathcal{S}_{2\alpha,\mu\nu,\rho\sigma}(x, z) \\
& = \int_q e^{iq(x-z)} \frac{i}{(4\pi)^2} \\
& \quad \times \left\{ \epsilon_{\alpha\mu\nu\rho} q_\sigma [\tilde{c}'_0 + (\tilde{c}'_1 + \tilde{c}'_2)q^2] + (\rho \leftrightarrow \sigma) \right. \\
& \quad + \epsilon_{\mu\nu\rho\beta} q_\beta [\delta_{\alpha\sigma}(\tilde{c}'_0 + \tilde{c}'_1 q^2) + \tilde{c}'_2 q_\alpha q_\sigma] + (\rho \leftrightarrow \sigma) \\
& \quad + \epsilon_{\alpha\mu\nu\beta} q_\beta [\delta_{\rho\sigma}(\tilde{d}'_0 + \tilde{d}'_1 q^2) + \tilde{d}'_2 q_\rho q_\sigma] \\
& \quad + \epsilon_{\alpha\mu\rho\beta} q_\beta [\delta_{\nu\sigma}(\tilde{e}'_0 + \tilde{e}'_1 q^2) + \tilde{e}'_2 q_\nu q_\sigma] - \epsilon_{\alpha\nu\rho\beta} q_\beta [\delta_{\mu\sigma}(\tilde{e}'_0 + \tilde{e}'_1 q^2) + \tilde{e}'_2 q_\mu q_\sigma] \\
& \quad \left. + (\rho \leftrightarrow \sigma) \right\},
\end{aligned} \tag{4.6}$$

where $\tilde{c}'_i$, $\tilde{d}'_i$, and $\tilde{e}'_i$ are constants.

Finally, after some examination, we find that the most general form of the counterterm for the function (3.18) is given by

$$\begin{aligned}
& \mathcal{S}_{3\alpha,\beta,\mu\nu,\rho\sigma}(x,z) \\
&= \int_q e^{iq(x-z)} \frac{1}{(4\pi)^2} \\
&\quad \times \left\{ \epsilon_{\alpha\rho\mu\nu} (c'_0 \delta_{\sigma\beta} + c'_1 q^2 \delta_{\sigma\beta} + c'_2 q_\sigma q_\beta) + (\rho \leftrightarrow \sigma) \right. \\
&\quad + \epsilon_{\alpha\beta\mu\nu} (d'_0 \delta_{\rho\sigma} + d'_1 q^2 \delta_{\rho\sigma} + d'_2 q_\rho q_\sigma) \\
&\quad + [\epsilon_{\alpha\beta\mu\rho} (e'_0 \delta_{\nu\sigma} + e'_1 q^2 \delta_{\nu\sigma} + e'_2 q_\nu q_\sigma) - \epsilon_{\alpha\beta\nu\rho} (e'_0 \delta_{\mu\sigma} + e'_1 q^2 \delta_{\mu\sigma} + e'_2 q_\mu q_\sigma) + (\rho \leftrightarrow \sigma)] \\
&\quad + \epsilon_{\rho\mu\nu\gamma} q_\gamma (f'_1 q_\beta \delta_{\alpha\sigma} + f'_2 q_\sigma \delta_{\alpha\beta} + f'_3 q_\alpha \delta_{\beta\sigma}) + (\rho \leftrightarrow \sigma) \\
&\quad + \epsilon_{\beta\mu\nu\gamma} q_\gamma (g'_1 q_\rho \delta_{\alpha\sigma} + g'_1 q_\sigma \delta_{\alpha\rho} + g'_2 q_\alpha \delta_{\rho\sigma}) \\
&\quad + [\epsilon_{\beta\mu\rho\gamma} q_\gamma (h'_1 q_\nu \delta_{\alpha\sigma} + h'_2 q_\sigma \delta_{\alpha\nu} + h'_3 q_\alpha \delta_{\nu\sigma}) \\
&\quad \quad - \epsilon_{\beta\nu\rho\gamma} q_\gamma (h'_1 q_\mu \delta_{\alpha\sigma} + h'_2 q_\sigma \delta_{\alpha\mu} + h'_3 q_\alpha \delta_{\mu\sigma}) + (\rho \leftrightarrow \sigma)] \\
&\quad + \epsilon_{\alpha\mu\nu\gamma} q_\gamma (c''_1 q_\rho \delta_{\beta\sigma} + c''_1 q_\sigma \delta_{\beta\rho} + c''_2 q_\beta \delta_{\rho\sigma}) \\
&\quad + [\epsilon_{\alpha\beta\mu\gamma} q_\gamma (d''_1 q_\rho \delta_{\nu\sigma} + d''_1 q_\sigma \delta_{\nu\rho} + d''_2 q_\nu \delta_{\rho\sigma}) - \epsilon_{\alpha\beta\nu\gamma} q_\gamma (d''_1 q_\rho \delta_{\mu\sigma} + d''_1 q_\sigma \delta_{\mu\rho} + d''_2 q_\mu \delta_{\rho\sigma})] \\
&\quad + \epsilon_{\alpha\beta\rho\gamma} q_\gamma (e''_1 q_\mu \delta_{\nu\sigma} - e''_1 q_\nu \delta_{\mu\sigma}) + (\rho \leftrightarrow \sigma) \\
&\quad + [\epsilon_{\alpha\mu\rho\gamma} q_\gamma (f''_1 q_\beta \delta_{\nu\sigma} + f''_2 q_\nu \delta_{\beta\sigma} + f''_3 q_\sigma \delta_{\beta\nu}) \\
&\quad \quad - \epsilon_{\alpha\nu\rho\gamma} q_\gamma (f''_1 q_\beta \delta_{\mu\sigma} + f''_2 q_\mu \delta_{\beta\sigma} + f''_3 q_\sigma \delta_{\beta\mu}) + (\rho \leftrightarrow \sigma)] \left. \right\}. \tag{4.7}
\end{aligned}$$

We choose the coefficients c'_0 etc. so that the addition of $\mathcal{S}_{1\alpha,\beta,\mu\nu,\rho\sigma}(x,z)$ to the two-point function cancels the anomalous breaking (3.22). That is, we require

$$\partial_\alpha^x [\langle j_{5\alpha}(x) \mathcal{O}_{3\beta,\mu\nu,\rho\sigma}(z) \rangle + \mathcal{S}_{3\alpha,\beta,\mu\nu,\rho\sigma}(x,z)] = 0. \tag{4.8}$$

This yields

$$\begin{aligned}
c'_0 &= \frac{1}{8t}, & c'_1 &= f'_3 - \frac{1}{12}, & c'_2 &= f'_1 + f'_2, \\
d'_0 &= -\frac{1}{4t}, & d'_1 &= g'_2 + \frac{1}{6}, & d'_2 &= 2g'_1, \\
e'_0 &= 0, & e'_1 &= h'_3, & e'_2 &= h'_1 + h'_2.
\end{aligned} \tag{4.9}$$

Now, we require that the translation WT relation (2.22) holds by adding the above local terms to the correlation functions. That is, our requirement is

$$\begin{aligned}
& \partial_\mu^y [\langle j_{5\alpha}(x) T_{\mu\nu}^{\text{sym.}}(y) T_{\rho\sigma}^{\text{sym.}}(z) \rangle + \mathcal{C}_{\alpha,\mu\nu,\rho\sigma}(x,y,z)] \\
& \quad + \partial_\mu^y \partial_\beta^y \delta(y-z) [\langle j_{5\alpha}(x) \mathcal{O}_{3\beta,\mu\nu,\rho\sigma}(z) \rangle + \mathcal{S}_{1\alpha,\beta,\mu\nu,\rho\sigma}(x,z)] \\
& \quad + \partial_\beta^y \delta(y-z) [\langle j_{5\alpha}(x) \mathcal{O}_{1\beta,\nu,\rho\sigma}(z) \rangle + \mathcal{S}_{2\alpha,\beta,\nu,\rho\sigma}(x,z)] \\
& \quad + \partial_\mu^y \delta(y-z) [\langle j_{5\alpha}(x) \mathcal{O}_{2\mu\nu,\rho\sigma}(z) \rangle + \mathcal{S}_{3\alpha,\mu\nu,\rho\sigma}(x,z)] \\
&= 0.
\end{aligned} \tag{4.10}$$

The resulting relations among the coefficients in the counterterms are summarized in Appendix B. From those relations, we see that some coefficients are still left unfixed, but the coefficients in the expression (4.2) are completely determined as

$$d_1 + d_4 - f_2 - f_3 = 0, \quad (4.11)$$

$$2d_2 - 2f_2 = 0, \quad (4.12)$$

$$2d_3 - 2f_3 = \frac{1}{12}, \quad (4.13)$$

$$2c_0 = \frac{1}{12t}, \quad (4.14)$$

$$c_1 + c_3 - f_1 = -f_1 = -\frac{1}{12}, \quad (4.15)$$

$$2c_2 - 2f_1 = -\frac{1}{4}. \quad (4.16)$$

This gives

$$\begin{aligned} & \partial_\alpha^x \mathcal{C}_{\alpha,\mu\nu,\rho\sigma}(x,y,z) \\ &= \int_{p,q} e^{ip(x-y)} e^{iq(x-z)} \frac{1}{(4\pi)^2} \epsilon_{\mu\rho\beta\gamma} p_\beta q_\gamma \left\{ \frac{1}{12} q_\nu p_\sigma + \delta_{\nu\sigma} \left[\frac{1}{12t} - \frac{1}{12} p^2 - \frac{1}{4} pq - \frac{1}{12} q^2 \right] \right\} \\ & \quad + (\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma). \end{aligned} \quad (4.17)$$

5. Final steps

We are now able to write down the axial $U(1)$ anomaly in the three-point function (3.5) under the requirement of the translation WT relation (4.10); the latter requirement is accomplished by the counterterm (4.1). Then the axial $U(1)$ anomaly is given by the sum of Eqs. (3.10) and (4.17), i.e.,

$$\begin{aligned} & \partial_\alpha^x \left[\langle j_{5\alpha}(x) T_{\mu\nu}^{\text{sym.}}(y) T_{\rho\sigma}^{\text{sym.}}(z) \rangle + \mathcal{C}_{\alpha,\mu\nu,\rho\sigma}(x,y,z) \right] \\ &= \int_{p,q} e^{ip(x-y)} e^{iq(x-z)} \frac{1}{(4\pi)^2} \left[\frac{1}{6} \epsilon_{\mu\rho\beta\gamma} p_\beta q_\gamma (q_\nu p_\sigma - \delta_{\nu\sigma} pq) + (\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma) \right]. \end{aligned} \quad (5.1)$$

This is the most non-trivial result of this paper.

Going back to Eq. (2.6), the energy–momentum tensor also has a part that is anti-symmetric under the exchange of indices, $T_{\mu\nu}^{\text{anti-sym.}}(x)$ (Eq. (2.8)). We may redefine the energy–momentum tensor $T_{\mu\nu}(x)$ by simply removing this anti-symmetric part because $T_{\mu\nu}^{\text{anti-sym.}}(x)$ in Eq. (2.8) is proportional to the equation of motion; its effect on the correlation functions must be at most local contact terms, as the Schwinger–Dyson equation implies. We can in fact corroborate this argument by explicit calculations by using some regularizing prescription for $T_{\mu\nu}^{\text{anti-sym.}}(x)$. Here, however, we are content with the above argument and set $T_{\mu\nu}(x) \rightarrow T_{\mu\nu}^{\text{sym.}}(x)$ in what follows.

We now re-express Eq. (5.1) as the anomalous divergence of the axial $U(1)$ current in the curved spacetime. We expand $D^\alpha \langle j_{5\alpha}(x) \rangle_g$, the divergence of the axial vector current in the curved spacetime, as the power series of the vierbein around the flat spacetime:

$$\begin{aligned} D^\alpha \langle j_{5\alpha}(x) \rangle_g &= D^\alpha \langle j_{5\alpha}(x) \rangle + \int d^4y \delta e^{\mu a}(y) \frac{\delta}{\delta e^{\mu a}(x)} D^\alpha \langle j_{5\alpha}(x) \rangle \\ & \quad + \frac{1}{2!} \int d^4y \delta e^{\mu a}(y) \int d^4z \delta e^{\nu b}(z) \frac{\delta}{\delta e^{\mu a}(y)} \frac{\delta}{\delta e^{\nu b}(z)} D^\alpha \langle j_{5\alpha}(x) \rangle + O(\delta e^3), \end{aligned} \quad (5.2)$$

where $\delta e^{\mu a}(x) \equiv e^{\mu a}(x) - \delta^{\mu a}$ and it is understood that the right-hand side is evaluated in the flat spacetime with appropriate local counterterms specified as above. Noting

$$\langle j_{5\alpha}(x) \rangle = 0, \quad \langle j_{5\alpha}(x) T_{\mu\nu}^{\text{sym.}}(y) \rangle = 0, \quad \left\langle \partial_\alpha j_{5\alpha}(x) \frac{\delta}{\delta e^{\rho b}(z)} T_{\mu\nu}^{\text{sym.}}(y) \right\rangle = 0, \quad (5.3)$$

as far as the regularization preserves the Lorentz and parity covariance, we have

$$\begin{aligned} & D^\alpha \langle j_{5\alpha}(x) \rangle_g \\ &= \frac{1}{2!} \int d^4y \delta e^{\mu a}(y) e_a^\rho(y) \int d^4z \delta e^{\nu b}(z) e_b^\sigma(z) \langle \partial_\alpha j_{5\alpha}(x) T_{\rho\mu}^{\text{sym.}}(y) T_{\sigma\nu}^{\text{sym.}}(z) \rangle + O(\delta e^3) \\ &= \frac{1}{2!} \int d^4y \frac{1}{2} \delta g^{\mu\nu}(y) \int d^4z \frac{1}{2} \delta g^{\rho\sigma}(z) \partial_\alpha^x \langle j_{5\alpha}(x) T_{\mu\nu}^{\text{sym.}}(y) T_{\rho\sigma}^{\text{sym.}}(z) \rangle + O(\delta g^3), \\ &= \frac{1}{(4\pi)^2} \frac{1}{12} \epsilon_{\mu\nu\rho\sigma} \partial_\mu \partial_\lambda \delta g_{\nu\tau}(x) [\partial_\rho \partial_\lambda \delta g_{\sigma\tau}(x) - \partial_\rho \partial_\tau \delta g_{\sigma\lambda}(x)] + O(\delta g^3), \end{aligned} \quad (5.4)$$

where we have used $[\delta/\delta e^{\nu a}(x)]S = e_a^\mu(x) T_{\mu\nu}^{\text{sym.}}(x)$, $\delta g^{\mu\nu}(x) = \delta e^{\mu a}(x) e_a^\nu(x) + e^{\mu a}(x) \delta e_a^\nu(x)$, and Eq. (5.1) in the last equality. Comparing this with the expansion of the curvature,⁶

$$\epsilon_{\mu\nu\rho\sigma} R^{\mu\nu\lambda\tau}(x) R^{\rho\sigma}_{\lambda\tau}(x) = 2\epsilon_{\mu\nu\rho\sigma} \partial_\mu \partial_\lambda \delta g_{\nu\tau}(x) [\partial_\rho \partial_\lambda \delta g_{\sigma\tau}(x) - \partial_\rho \partial_\tau \delta g_{\sigma\lambda}(x)] + O(\delta g^3), \quad (5.5)$$

we finally observe Eq. (1.1) for $m_0 = 0$.

6. Conclusion

In this paper, we have examined a possible use of the universal formula for the energy–momentum tensor in gauge theory in the flat spacetime through the Yang–Mills gradient flow [17,18]. As a general argument indicates, after choosing local counterterms appropriately so as to restore the translation WT relation, we obtain the correct axial $U(1)$ anomaly in Eq. (1.1) (in the flat space limit).

From the present analysis, we can learn the following feature of the universal formula of the energy–momentum tensor. The universal formula is based on the gradient flow and its small flow time expansion of Ref. [13]. The latter asserts that any composite operator of flowed fields as $t \rightarrow 0$ can be expressed as an asymptotic series of renormalized composite operators of unflowed fields with increasing mass dimensions. When two composite operators of flowed fields collide in coordinate space to form another composite operator, we have to consider the expansion in terms of another set of renormalized composite operators of unflowed fields. Consequently, it is not obvious what happens when the universal formula of the energy–momentum tensor collides with other composite operators, such as the axial $U(1)$ current or the energy–momentum tensor, in coordinate space. Our present analysis illustrates that the formula in fact does not automatically fulfill the translation WT relation precisely when the formula coincides with other composite operators in coordinate space. On the other hand, our finding that *local* counterterms are sufficient to restore the translation WT relation ensures the expectation that the formula fulfills the translation WT relation when the energy–momentum tensor is in isolation in coordinate space; for this case, the translation WT relation is simply the conservation law of the energy–momentum tensor.

⁶ Our definition of the Riemann curvature is $R^\alpha_{\rho\mu\nu} \equiv \partial_\mu \Gamma^\alpha_{\rho\nu} - \partial_\nu \Gamma^\alpha_{\rho\mu} + \Gamma^\alpha_{\lambda\mu} \Gamma^\lambda_{\rho\nu} - \Gamma^\alpha_{\lambda\nu} \Gamma^\lambda_{\rho\mu}$, where $\Gamma^\lambda_{\mu\nu} \equiv \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$ is the Christoffel symbol.

Thus, our analysis has revealed that the universal formula as it stands can be used only in on-shell correlation functions (i.e., correlation functions in which the energy–momentum tensor does not coincide with other composite operators in coordinate space). The incorporation of this point into (a generalization of) the universal formula is a forthcoming challenge.⁷ A related issue is the possible generalization of the gradient flow to the curved spacetime. A possible generalization is

$$\partial_t B_\mu(t, x) = g^{\nu\rho}(x) D_\nu G_{\rho, \mu}(t, x), \quad B_\mu(t = 0, x) = A_\mu(x), \quad (6.1)$$

$$\partial_t \chi(t, x) = g^{\mu\nu}(x) D_\mu D_\nu \chi(t, x), \quad \chi(t = 0, x) = \psi(x), \quad (6.2)$$

$$\partial_t \bar{\chi}(t, x) = \bar{\chi}(t, x) g^{\mu\nu}(x) \overleftarrow{D}_\mu \overleftarrow{D}_\nu, \quad \bar{\chi}(t = 0, x) = \bar{\psi}(x). \quad (6.3)$$

It then appears interesting to see whether this setup improves the covariance under the general coordinate transformation and the restoration of the associated WT relations for the energy–momentum tensor.

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A. Derivation of the WT relation (2.22) in the flat spacetime

In the functional integral corresponding to the correlation function $\langle j_{5\alpha}(x) T_{\rho\sigma}^{\text{sym.}}(z) \rangle$ in the flat spacetime, where

$$T_{\mu\nu}^{\text{sym.}}(x) \equiv \frac{1}{4} \bar{\psi}(x) \left(\gamma_\mu \overleftrightarrow{\partial}_\nu + \gamma_\nu \overleftrightarrow{\partial}_\mu \right) \psi(x) - \delta_{\mu\nu} \bar{\psi}(x) \left(\frac{1}{2} \overleftrightarrow{\not{\partial}} + m_0 \right) \psi(x), \quad (A1)$$

we consider the change of integration variables of the form of the localized translation:

$$\delta\psi(x) = \xi_\mu(x) \partial_\mu \psi(x), \quad \delta\bar{\psi}(x) = \xi_\mu(x) \partial_\mu \bar{\psi}(x). \quad (A2)$$

Since the action in the flat spacetime

$$S = \int d^4x \bar{\psi}(x) \left(\frac{1}{2} \overleftrightarrow{\not{\partial}} + m_0 \right) \psi(x) \quad (A3)$$

changes under Eq. (A2) as

$$\delta S = - \int d^4x \xi_\nu(x) \partial_\mu T_{\mu\nu}^{\text{can.}}(x), \quad (A4)$$

where $T_{\mu\nu}^{\text{can.}}(x)$ is the *canonical* energy–momentum tensor, defined by

$$T_{\mu\nu}^{\text{can.}}(x) \equiv \frac{1}{2} \bar{\psi}(x) \gamma_\mu \overleftrightarrow{\partial}_\nu \psi(x) - \delta_{\mu\nu} \bar{\psi}(x) \left(\frac{1}{2} \overleftrightarrow{\not{\partial}} + m_0 \right) \psi(x), \quad (A5)$$

⁷ We would like to thank Shinya Aoki for an interesting suggestion on this issue.

we have

$$\begin{aligned}
& \partial_\mu^y \langle j_{5\alpha}(x) T_{\mu\nu}^{\text{can.}}(y) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \\
& + \delta(x-y) \partial_\nu^x \langle j_{5\alpha}(x) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \\
& + \delta(z-y) \partial_\nu^z \langle j_{5\alpha}(x) T_{\rho\sigma}^{\text{sym.}}(z) \rangle \\
& + \partial_\beta^y \delta(y-z) \langle j_{5\alpha}(x) \mathcal{O}_{1\beta,\nu,\rho\sigma}(z) \rangle \\
& = 0,
\end{aligned} \tag{A6}$$

where $\mathcal{O}_{1\beta,\nu,\rho\sigma}(x)$ is the combination defined in Eq. (2.17).

We next note that the symmetric energy–momentum tensor (A1) and the canonical energy–momentum tensor (A5) are related as (this is the relation attributed to Belinfante and Rosenfeld (see Ref. [49]))

$$\begin{aligned}
T_{\mu\nu}^{\text{sym.}}(x) &= T_{\mu\nu}^{\text{can.}}(x) - \frac{1}{4} \bar{\psi}(x) \left[\sigma_{\mu\nu} (\not{\partial} + m_0) + (\overleftarrow{\not{\partial}} - m_0) \sigma_{\mu\nu} \right] \psi(x) \\
&+ \frac{1}{8} \partial_\rho \left[\bar{\psi}(x) (\gamma_\mu \gamma_\nu \gamma_\rho - \gamma_\rho \gamma_\nu \gamma_\mu) \psi(x) \right].
\end{aligned} \tag{A7}$$

On the right-hand side, since the last term has no total divergence with respect to the index μ , it can be neglected in the following discussion. The second term is proportional to the equations of motion and its insertion in $\langle j_{5\alpha}(x) T_{\rho\sigma}^{\text{sym.}}(z) \rangle$ can be determined by the Schwinger–Dyson equation as

$$\begin{aligned}
& \left\langle j_{5\alpha}(x) \left(-\frac{1}{4} \right) \bar{\psi}(y) \left[\sigma_{\mu\nu} (\not{\partial} + m_0) + (\overleftarrow{\not{\partial}} - m_0) \sigma_{\mu\nu} \right] \psi(y) T_{\rho\sigma}^{\text{sym.}}(z) \right\rangle \\
&= \delta(x-y) \left\langle \frac{1}{4} \bar{\psi}(x) [\gamma_\alpha \gamma_5, \sigma_{\mu\nu}] \psi(x) T_{\rho\sigma}^{\text{sym.}}(z) \right\rangle \\
&\quad - \delta(z-y) \langle j_{5\alpha}(x) \mathcal{O}_{2\mu\nu,\rho\sigma}(z) \rangle \\
&\quad - \partial_\beta^y \delta(z-y) \langle j_{5\alpha}(x) \mathcal{O}_{3\beta,\mu\nu,\rho\sigma}(z) \rangle,
\end{aligned} \tag{A8}$$

where the combinations $\mathcal{O}_{2\mu\nu,\rho\sigma}(x)$ and $\mathcal{O}_{3\beta,\mu\nu,\rho\sigma}(x)$ are given in Eqs. (2.20) and (2.21), respectively. Finally, combining Eqs. (A6), (A7), and (A8), we have Eq. (2.22).

B. Relations among counterterm coefficients

For the coefficients in Eqs. (4.1), (4.4), (4.6), and (4.7), the requirements of the validity of the WT relations (4.8) (i.e., Eq. (4.9)) and (4.10) yield the following relations:

$$c_0 = \frac{\tilde{f}_0 - \tilde{e}'_0}{3} = \frac{1}{24t}, \tag{B1}$$

$$c_1 = -f_1'' - f_2'' + f_3'' + \frac{1}{48}, \tag{B2}$$

$$c_2 = -\frac{1}{24}, \tag{B3}$$

$$c_3 = f_1'' + f_2'' - f_3'' - \frac{1}{48} = -c_1, \tag{B4}$$

$$d_1 = \tilde{f}_2 - \tilde{e}'_2 - \frac{1}{24}, \tag{B5}$$

$$d_2 = \tilde{f}_2 - \tilde{e}'_2 + f_3'', \quad (\text{B6})$$

$$d_3 = \tilde{f}_2 - \tilde{e}'_2 + f_3'' + \frac{1}{24}, \quad (\text{B7})$$

$$d_4 = \tilde{f}_2 - \tilde{e}'_2 + 2f_3'' + \frac{1}{24}, \quad (\text{B8})$$

$$e_1 = f_3'', \quad (\text{B9})$$

$$e_2 = f_1'' + f_2'' - \frac{1}{24}, \quad (\text{B10})$$

$$e_3 = f_3'' + \frac{1}{16}, \quad (\text{B11})$$

$$e_4 = f_1'' + f_2'' - \frac{1}{12}, \quad (\text{B12})$$

$$f_1 = \frac{1}{12}, \quad (\text{B13})$$

$$f_2 = \tilde{f}_2 - \tilde{e}'_2 + f_3'', \quad (\text{B14})$$

$$f_3 = \tilde{f}_2 - \tilde{e}'_2 + f_3'', \quad (\text{B15})$$

and

$$c'_0 = \tilde{f}_0 - \tilde{e}'_0 = \frac{1}{8t}, \quad (\text{B16})$$

$$c'_1 = h'_3, \quad (\text{B17})$$

$$c'_2 = h'_1 + h'_2 + 2\tilde{f}_2 - 2\tilde{e}'_2 + 2f_3'', \quad (\text{B18})$$

$$d'_0 = -2(\tilde{f}_0 - \tilde{e}'_0) = -\frac{1}{4t}, \quad (\text{B19})$$

$$d'_1 = g'_2 + \frac{1}{6}, \quad (\text{B20})$$

$$d'_2 = 2g'_1, \quad (\text{B21})$$

$$e'_0 = 0, \quad (\text{B22})$$

$$e'_1 = h'_3, \quad (\text{B23})$$

$$e'_2 = h'_1 + h'_2, \quad (\text{B24})$$

$$f'_1 = h'_1 + \tilde{f}_2 - \tilde{e}'_2 + f_3'', \quad (\text{B25})$$

$$f'_2 = h'_2 + \tilde{f}_2 - \tilde{e}'_2 + f_3'', \quad (\text{B26})$$

$$f'_3 = h'_3 + \frac{1}{12}, \quad (\text{B27})$$

$$c''_1 = d''_1 - f''_1 - f''_2, \quad (\text{B28})$$

$$c''_2 = d''_2 - 2f''_3, \quad (\text{B29})$$

$$e''_1 = f''_3 - f''_1, \quad (\text{B30})$$

$$\tilde{c}_0 = -\tilde{c}'_0, \quad (\text{B31})$$

$$\tilde{c}_1 = -\tilde{c}'_1, \quad (\text{B32})$$

$$\tilde{c}_2 = -\tilde{c}'_2, \quad (\text{B33})$$

$$\tilde{d}_0 = -\tilde{d}'_0, \quad (\text{B34})$$

$$\tilde{d}_1 = -\tilde{d}'_1, \quad (\text{B35})$$

$$\tilde{d}_2 = -\tilde{d}'_2, \quad (\text{B36})$$

$$\tilde{e}_0 = \tilde{f}_0 - 2\tilde{e}'_0 = -\tilde{e}'_0 + \frac{1}{8t}, \quad (\text{B37})$$

$$\tilde{e}_1 = -\tilde{e}'_1 - f''_1 - f''_2 + f''_3 - \frac{1}{12}, \quad (\text{B38})$$

$$\tilde{e}_2 = \tilde{f}_2 - 2\tilde{e}'_2, \quad (\text{B39})$$

$$\tilde{f}_0 = \tilde{e}'_0 + \frac{1}{8t}, \quad (\text{B40})$$

$$\tilde{f}_1 = \tilde{e}'_1 - f''_1 - f''_2 + f''_3 - \frac{1}{12}. \quad (\text{B41})$$

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