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Unequal binary configurations of interacting Kerr black holes

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ABSTRACT

Stationary axisymmetric binary configurations of unequal Kerr sources with a massless strut among them are developed in a physical representation. In order to describe interacting black holes, the axis conditions in the most general case are solved analytically deriving the corresponding 5-parametric asymptotically flat exact solution. In addition, we obtain concise formulas for the black hole horizons, the interaction force, as well as the thermodynamical characteristics of each source in terms of physical Komar parameters: mass M_i, angular momentum J_i, and coordinate distance R, where such parameters are part of a cubic equation which can be interpreted as a dynamical law for interacting black holes with struts. Some limits are obtained and discussed.

1. Introduction

Nowadays, the coalescence process among two interacting black hole (BH) sources has been considered an outstanding candidate to study and detect gravitational waves (GW) by the LIGO and Virgo Collaborations [1]. Due mainly to the fact that the numerical simulations are the main tool at the moment of considering the merger process (MP) between two BHs, it motivates scientists around the world to match this huge discovery with models within the framework of exact solutions. Nevertheless, it seems quite complicated to construct exact results describing physical models that can take into account all the possible interactions between the components of the binary system (BS) during the MP. In this respect, the double-Kerr-NUT (DKN) solution [2] developed by Kramer and Neugebauer almost four decades ago permits us to describe dynamical scenarios between two massive rotating sources in stationary axisymmetric spacetime, but in the absence of a supporting strut (conical singularity (CS) [3,4]) the notion of dealing with BHs is spoiled due to the appearance of ring singularities off the axis. It is worth mentioning that rotating BS develop ring singularities off the axis if at least one of the masses turns out to be negative [5–7], regardless if the positive mass theorem [8,9] is fulfilled in the binary configuration. The last point suggests us to focus our attention in configurations of unequal binary BHs with a massless strut in between, with the main purpose of describing their dynamical and physical properties before the coalescence process may occur. However, until this day solving analytically the axis conditions (AC) in the most general case has been one of the main technical (highly complicated) problems to treat binary configurations of interacting BHs.

The present paper aims at solving for the first time the AC in order to derive a 5-parametric subclass of the DKN solution [2]. It represents the most general case with regard to the description of the dynamical interaction of binary configurations of unequal Kerr BHs with struts, with the main distinctive of being characterized by five arbitrary physical Komar parameters [10]: the masses M_i and angular momenta J_i, as well as the coordinate distance R. These parameters constitute a cubic algebraic equation, which in the absence of the strut is reduced to the equilibrium law for two nonequal Kerr particles [11]. We obtain simple formulas for both event horizons as a function of physical Komar parameters, with the main objective to determine some dynamical and thermodynamical characteristics of the BS. An interesting feature in our numerical analysis of the interaction force related to the CS reveals the existence of equilibrium states (without strut) during the MP, where both BHs are endowed with positive masses. As a consequence, we conclude that the CS cannot be eliminated when the MP is taking place, otherwise closed timelike curves (CTC) will appear outside the ergoregion of the new source that has been formed during such a process, therefore, it cannot be considered a BH.

2. The DKN solution in a physical representation

The well-known DKN solution constructed by Kramer and Neugebauer long time ago [2] represents a superposition of two massive rotating sources in stationary spacetimes. It was devel-

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oped by employing Bäcklund transformations [13] as a modern generation technique of exact solutions in Einstein's vacuum equations. Moreover, the DKN solution can also be derived through the Sibgatullin's method (SM) [14] which is also very fit to describe electrovacuum spacetimes [15]. Both approaches start with a particular form of the Ernst potential [16] on the symmetry axis (the axis data), which is then extended in the whole spacetime. According to Ernst formalism [16], the vacuum Einstein field equations are reduced into a new complex equation for solving

$$(\mathcal{E}+\bar{\mathcal{E}})(\mathcal{E}_{\rho\rho}+\rho^{-1}\mathcal{E}_{\rho}+\mathcal{E}_{zz})=2(\mathcal{E}_{\rho}^{2}+\mathcal{E}_{z}^{2}), \tag{1}$$

being $\mathcal{E}=f+i\Psi$ defined in Weyl–Papapetrou cylindrical coordinates (ρ, z), where the subscript ρ or z denotes partial differentiation. In this regard, the line element for stationary axisymmetric spacetimes is given by [17]

$$ds^{2}=f^{-1}\left[e^{2\gamma}(d\rho^{2}+dz^{2})+\rho^{2}d\varphi^{2}\right]-f(dt-\omega d\varphi)^{2}, \tag{2}$$

where the metric functions $\omega(\rho, z)$ and $\gamma(\rho, z)$ can be derived from the following system of differential equations:

$$\begin{aligned}&\omega_{\rho}=-\rho f^{-2}\Psi_{z}, \quad \omega_{z}=\rho f^{-2}\Psi_{\rho},\\ &4\gamma_{\rho}=\rho f^{-2}\left(|\mathcal{E}_{\rho}|^{2}-|\mathcal{E}_{z}|^{2}\right), \quad 2\gamma_{z}=\rho f^{-2}\mathrm{Re}(\mathcal{E}_{\rho}\bar{\mathcal{E}}_{z}),\end{aligned} \tag{3}$$

once we know a peculiar form of the Ernst potential $\mathcal{E}$. For solving the nonlinear Eq. (1) by using the SM, the axis data for the Ernst potential in vacuum systems adopts the most general representation as follows [15]:

$$\mathcal{E}(\rho=0, z):=e(z)=1+\sum_{i=1}^{2}\frac{e_{i}}{z-\beta_{i}}, \tag{4}$$

where $\{e_i, \beta_i\}$, $i=1,2$, are arbitrary complex constants related to the Geroch–Hansen (GH) multipole moments [18,19]. At the same time, the SM begins with the characteristic equation

$$e(z)+\bar{e}(z)=0, \tag{5}$$

being α_n, for $n=\overline{1,4}$, the roots of Eq. (5) that locate the sources on the symmetry axis. In order to change the old parameters $\{e_i, \beta_i\}$ by the new ones $\{\alpha_n, \beta_i\}$, Eq. (4) is placed into Eq. (5)

$$2+\sum_{i=1}^{2}\left(\frac{e_{i}}{z-\beta_{i}}+\frac{\bar{e}_{i}}{z-\bar{\beta}_{i}}\right)=\frac{2\prod_{n=1}^{4}(z-\alpha_{n})}{\prod_{i=1}^{2}(z-\beta_{i})(z-\bar{\beta}_{i})}, \tag{6}$$

to obtain

$$e_{1}=\frac{2\prod_{n=1}^{4}(\beta_{1}-\alpha_{n})}{(\beta_{1}-\beta_{2})(\beta_{1}-\bar{\beta}_{1})(\beta_{1}-\bar{\beta}_{2})}, \qquad e_{2}=e_{1(1\leftrightarrow 2)}. \tag{7}$$

The DKN solution can be performed directly from the last formulas of [15], with $N=2$ and taking into account an absence of the electromagnetic field ($\Phi=0$), where such a metric contains eight arbitrary real parameters. However, although the SM allows us to build the DKN solution in all the spacetime, the solution itself lacks of physical meaning at the moment one wishes to study the dynamical interaction between two rotating sources, therefore it is mandatory to solve the AC. At this point, it is worthwhile to stress the fact that solving analytically these conditions cannot be assumed as a trivial problem, and for that reason only identical cases taking the advantage of their symmetry property on the equatorial plane have been considered until this work [20–24]. Before solving the AC, we are going to depict the DKN problem with

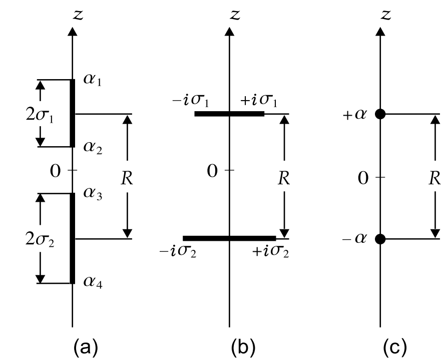

Fig. 1. Schematic representation of different types of unequal Kerr sources: (a) BH configuration $\sigma_i^2>0$; (b) hyperextreme sources if $\sigma_i \to i\sigma_i$ (or $\sigma_i^2<0$); (c) the extreme limit case $\sigma_i=0$.

an aspect which contains a more physical representation. To do this, we notice that the Fodor–Hoenselaers–Perjés procedure [25] permits us to calculate from Eq. (4) the first GH multipolar terms like the total mass of the system M, NUT charge J_0 [12], and total angular momentum J of the system, which are given by

$$-\frac{e_{1}+e_{2}}{2}=M+iJ_{0}, \qquad -\frac{\mathrm{Im}[e_{1}\beta_{1}+e_{2}\beta_{2}]}{2}=J. \tag{8}$$

Replacing Eq. (7) into Eq. (8) yields the relation for the total mass

$$\beta_{1}+\beta_{2}+\bar{\beta}_{1}+\bar{\beta}_{2}+\sum_{n=1}^{4}\alpha_{n}=-2M \tag{9}$$

where the parameters α_n can be rewritten in terms of the relative distance R and the half-length rod σ_i as follows:

$$\alpha_{1,2}=\frac{R}{2}\pm\sigma_{1}, \qquad \alpha_{3,4}=-\frac{R}{2}\pm\sigma_{2}. \tag{10}$$

Thereby we have reduced only one parameter of the DKN solution. It is important to mention that σ_i can take real positive or pure imaginary values representing BHs (subextreme sources) or relativistic disks (hyperextreme sources), respectively, as shown in Fig. 1. In what follows in this research we will consider only and exclusively BHs. In addition, to solve Eq. (9) one might choose the ansatz for β_i

$$\beta_{1,2}=\frac{-M+i\mathrm{q}\pm\sqrt{p+i\delta}}{2}, \tag{11}$$

where after using the following parametrization:

$$\begin{aligned}&p=R^{2}-\Delta+2\left(\epsilon_{1}-\frac{\epsilon_{2}R}{M}\right)+\frac{2\mathrm{q}(P_{1}+P_{2})}{M},\\ &\delta=-2(2P_{2}+M\mathrm{q}),\\ &\epsilon_{1,2}:=\sigma_{1}^{2}\pm\sigma_{2}^{2}, \qquad \Delta:=M^{2}-\mathrm{q}^{2},\end{aligned} \tag{12}$$

it guides us to simple expressions for the NUT charge and total angular momentum

$$\begin{aligned}&J_{0}=\frac{\mathrm{q}}{2M}\left(\frac{N}{D}\right), \qquad J=M\mathrm{q}-\frac{P_{1}-P_{2}}{2}+\frac{J_{0}P_{2}}{\mathrm{q}},\\ &N:=[\mathrm{q}(P_{1}+P_{2})-\epsilon_{2}R]^{2}\\ &\quad -M^{2}\left[4P_{1}P_{2}+(R^{2}-\Delta)(2\epsilon_{1}-\Delta)+\epsilon_{2}^{2}\right],\\ &D:=\mathrm{q}^{2}\left[M(R^{2}+M^{2}-2\Delta)+2\left(\mathrm{q}(P_{1}+P_{2})+M\epsilon_{1}-R\epsilon_{2}\right)\right]\\ &\quad -M(2P_{2}+M\mathrm{q})^{2},\end{aligned} \tag{13}$$

and now the axis data of the Ernst potential given by Eq. (4) and satisfying Eq. (5) results to be

$$\begin{aligned} \mathcal{E}(0,z) &= \frac{e_1}{e_2}, \qquad e_1 = z^2 - [M + i(\mathfrak{q} + 2J_0)]z + P_+ + iP_1 \\ &\quad - 2iJ_0[M - i\mathfrak{q} + P_2/\mathfrak{q}], \\ e_2 &= z^2 + (M - i\mathfrak{q})z + P_- + iP_2 \\ P_\pm &:= \frac{M(2\Delta - R^2) - 2[M\epsilon_1 \pm R\epsilon_2 \mp \mathfrak{q}(P_1 + P_2)]}{4M}. \end{aligned} \tag{14}$$

The Ernst potential and full metric in the entire spacetime can be reduced eventually until get the following concise form [26]:

$$\begin{aligned} \mathcal{E} &= \frac{\Lambda + \Gamma}{\Lambda - \Gamma}, \quad f = \frac{\Lambda\bar{\Lambda} - \Gamma\bar{\Gamma}}{(\Lambda - \Gamma)(\bar{\Lambda} - \bar{\Gamma})}, \\ \omega &= 2\mathfrak{q} - \frac{2\mathrm{Im}\left[(\Lambda - \Gamma)\bar{\mathcal{G}}\right]}{\Lambda\bar{\Lambda} - \Gamma\bar{\Gamma}}, \quad e^{2\gamma} = \frac{\Lambda\bar{\Lambda} - \Gamma\bar{\Gamma}}{\kappa_o\bar{\kappa}_o r_1 r_2 r_3 r_4}, \\ \Lambda &= 4\sigma_1\sigma_2(\mathfrak{r}_1 - \mathfrak{r}_3)(\mathfrak{r}_2 - \mathfrak{r}_4) - [R^2 - (\sigma_1 - \sigma_2)^2] \\ &\quad \times (\mathfrak{r}_1 - \mathfrak{r}_2)(\mathfrak{r}_3 - \mathfrak{r}_4), \\ \Gamma &= 2\sigma_2(R^2 + \epsilon_2)(\mathfrak{r}_1 - \mathfrak{r}_2) + 2\sigma_1(R^2 - \epsilon_2)(\mathfrak{r}_3 - \mathfrak{r}_4) \\ &\quad - 4\sigma_1\sigma_2 R(\mathfrak{r}_1 - \mathfrak{r}_4 + \mathfrak{r}_2 - \mathfrak{r}_3), \\ \mathcal{G} &= -z\Gamma + \sigma_1(R^2 - \epsilon_2)(\mathfrak{r}_3 - \mathfrak{r}_4)(\mathfrak{r}_1 + \mathfrak{r}_2 + R) \\ &\quad + \sigma_2(R^2 + \epsilon_2)(\mathfrak{r}_1 - \mathfrak{r}_2)(\mathfrak{r}_3 + \mathfrak{r}_4 - R) - 2\sigma_1\sigma_2 \\ &\quad \{2R[\mathfrak{r}_1\mathfrak{r}_2 - \mathfrak{r}_3\mathfrak{r}_4 - \sigma_1(\mathfrak{r}_1 - \mathfrak{r}_2) + \sigma_2(\mathfrak{r}_3 - \mathfrak{r}_4)] \\ &\quad + \epsilon_2(\mathfrak{r}_1 - \mathfrak{r}_4 + \mathfrak{r}_2 - \mathfrak{r}_3)\}, \qquad \mathfrak{r}_i := \mathfrak{a}_i r_i \\ \mathfrak{a}_1 &= \frac{s_{1+}}{\bar{s}_{1+}}, \quad \mathfrak{a}_2 = \frac{s_{1-}}{\bar{s}_{1-}}, \quad \mathfrak{a}_3 = \frac{s_{2-}}{\bar{s}_{2-}}, \quad \mathfrak{a}_4 = \frac{s_{2+}}{\bar{s}_{2+}}, \\ \kappa_o &= \frac{64M^3\sigma_1\sigma_2(R^4 - 2\epsilon_1 R^2 + \epsilon_2^2)D}{\bar{s}_{1+}\bar{s}_{1-}\bar{s}_{2+}\bar{s}_{2-}}, \\ s_{1\pm} &= \mathfrak{q}(P_1 + P_2) - M(\Delta + MR) \\ &\quad - (R + M)(\epsilon_2 \pm 2M\sigma_1) + iM[2P_2 - \mathfrak{q}(R \pm 2\sigma_1)], \\ s_{2\pm} &= \mathfrak{q}(P_1 + P_2) - M(\Delta - MR) \\ &\quad - (R - M)(\epsilon_2 \pm 2M\sigma_2) + iM[2P_2 + \mathfrak{q}(R \pm 2\sigma_2)], \\ r_{1,2} &= \sqrt{\rho^2 + (z - R/2 \mp \sigma_1)^2}, \\ r_{3,4} &= \sqrt{\rho^2 + (z + R/2 \mp \sigma_2)^2}, \end{aligned} \tag{15}$$

where $|\mathfrak{a}_j| \equiv 1$, for $\sigma_i^2 > 0$. The above metric Eq. (15) represents the DKN solution, which contains seven parameters into the set $\{M, R, \mathfrak{q}, \sigma_1, \sigma_2, P_1, P_2\}$, and in the absence of J_0 is reduced to the one considered earlier in Ref. [26]. As a matter of fact, the advantage of this particular choice of the axis data Eq. (14) is that it gives us more information about several options to eliminate the NUT charge J_0 with the main purpose to obtain an asymptotically flat exact solution, and later on, to solve the axis condition in between sources.

2.1. Two unequal Kerr BH sources apart by a strut: solving the axis conditions

By construction the metric Eq. (15) contains an elementary flatness in the upper part of the symmetry axis; it means that the conditions: $\omega(\rho = 0, \alpha_1 < z < \infty) = 0$, and $\gamma(\rho = 0, \alpha_1 < z < \infty) = \gamma(\rho = 0, -\infty < z < \alpha_4) = 0$ are automatically satisfied. Therefore, the remaining AC on the symmetry axis are

$$\omega(\rho = 0, \alpha_2 < z < \alpha_3) = 0, \quad \omega(\rho = 0, -\infty < z < \alpha_4) = 0, \tag{16}$$

which in terms of the canonical parameters $\{\alpha_n, \beta_i\}$ are given by [27]

$$\mathrm{Im}\left[\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & \pm\gamma_{11} & \pm\gamma_{12} & \gamma_{13} & \gamma_{14} \\ 1 & \pm\gamma_{21} & \pm\gamma_{22} & \gamma_{23} & \gamma_{24} \\ 0 & \kappa_{11} & \kappa_{12} & \kappa_{13} & \kappa_{14} \\ 0 & \kappa_{21} & \kappa_{22} & \kappa_{23} & \kappa_{24} \end{vmatrix}\right] = 0, \tag{17}$$

$$\gamma_{jn} = (\alpha_n - \beta_j)^{-1}, \qquad \kappa_{jn} = (\alpha_n - \bar{\beta}_j)^{-1}.$$

The condition with + sign is equivalent to kill the NUT charge (gravitomagnetic monopole), that is given explicitly above by Eq. (13), while the other one containing a − sign disconnects the region in between sources; it means that after solving such a condition, the mass in the middle region does not contribute to the total ADM mass [28], thus both sources will be apart by a massless strut. Due to the fact that we have at hand several possibilities among the seven parameters that could eliminate J_0, without taking into account the trivial case $\mathfrak{q} = 0$, we are going to select the option

$$\epsilon_1 = \frac{\Delta}{2} + \frac{[\mathfrak{q}(P_1 + P_2) - \epsilon_2 R]^2 - M^2(4P_1P_2 + \epsilon_2^2)}{2M^2(R^2 - \Delta)}. \tag{18}$$

Surprisingly, the extremely complicated axis condition (with − sign) from Eq. (17) eventually is reduced to a quadratic equation for ϵ_2, P_1 and P_2, namely

$$\begin{aligned} &\mathfrak{q}(R + M)\left[(R + M)(R^2 - \Delta) - M\mathfrak{q}^2\right]\epsilon_2^2 \\ &+ \left[M(R^2 - \Delta)^2 + 2(R + M)\left(\Delta(R^2 - \Delta) - M\mathfrak{q}^2 R\right)\right] \\ &\times (P_1 + P_2)\epsilon_2 + \mathfrak{q}[M^2(R^2 + MR + \mathfrak{q}^2)(P_1 - P_2)^2 \\ &+ (\Delta + MR)(\Delta - MR - R^2)(P_1 + P_2)^2] - M^2(R^2 - \Delta) \\ &\times \{[M\mathfrak{q}^2 + (R + M)(R^2 + MR + \mathfrak{q}^2)](P_1 - P_2) \\ &- M\mathfrak{q}(R + M)(R^2 - \Delta)\} = 0. \end{aligned} \tag{19}$$

This quadratic equation is solved by adopting the following parametrization:

$$\begin{aligned} \epsilon_2 &= -\frac{(\Delta + MR)(P_1 + P_2) + Mr(R^2 - \Delta)}{\mathfrak{q}(R + M)}, \\ P_{1,2} &= \frac{M\mathfrak{q}^2 - (R + M)(R^2 - \Delta)}{2P_0} r - \frac{\mathfrak{q}^2(R^2 + MR + \mathfrak{q}^2)}{2(R + M)P_0}\frac{s^2}{r} \\ &\quad \pm \frac{R^2 - \Delta}{2(R + M)} s \mp \frac{\mathfrak{q}(R^2 - \Delta)}{2(R + M)} + \frac{\mathfrak{q}(R^2 + MR + 2\mathfrak{q}^2)}{2(R + M)}\frac{s}{r} \\ &\quad - \frac{\mathfrak{q}^2 P_0}{2(R + M)}\frac{1}{r}, \qquad P_0 := (R + M)^2 + \mathfrak{q}^2, \end{aligned} \tag{20}$$

where it is observed that there exists a symmetry property in our ansatz that solves the AC since $P_{1,2} \to -P_{2,1}$, $\epsilon_2 \to -\epsilon_2$, and $\epsilon_1 \to \epsilon_1$, under the transformations $s \to s$, $r \to -r$. This special characteristic means that we are interchanging the location of the components of the BS as well as their physical properties.

3. Physical representation for the BH horizons

In order to obtain a real physical representation of the double-Kerr solution we must calculate the Komar parameters of the BS. To perform such a task we will use the well-known Tomimatsu formulas [29] for stationary axisymmetric spacetimes in vacuum

$$\begin{aligned} M_i &= -\frac{1}{8\pi}\int_{H_i} \omega\,\mathrm{Im}(\mathcal{E}_z)\,d\varphi dz, \\ J_i &= -\frac{1}{8\pi}\int_{H_i} \omega\left(1 + \frac{1}{2}\omega\,\mathrm{Im}(\mathcal{E}_z)\right)d\varphi dz, \end{aligned} \tag{21}$$

where the integrals are evaluated over the BH horizons, which are defined as null hypersurfaces $H_i = \{\alpha_{2i} \le z \le \alpha_{2i-1}, \varphi \le 2\pi, \ \rho \to 0\}$, $i = 1, 2$. Substituting Eqs. (15), (18), and (20) inside Eq. (21), one obtains the individual mass and angular momentum for each BH

$$M_{1,2} = \frac{M}{2} \pm \frac{\mathfrak{q}[(R+M)^2 + \mathfrak{q}^2] - (R^2 + MR + \mathfrak{q}^2)s}{2r(R+M)}, \qquad J_{1,2} = M_{1,2}\left(\frac{s \pm r}{2}\right). \tag{22}$$

It follows that the total mass $M = M_1 + M_2$ and total angular momentum $J = J_1 + J_2$, where $s = a_1 + a_2$ and $r = a_1 - a_2$, being $a_i \equiv J_i/M_i$ the individual angular momentum per unit mass. On the other hand, from Eq. (13) the following relation arises

$$J - M\mathfrak{q} = \frac{(R^2 - \Delta)(\mathfrak{q} - a_1 - a_2)}{2(R+M)}, \tag{23}$$

and it is reduced to a dynamical law for interacting Kerr sources with struts via the expressions contained in Eq. (22), namely

$$\mathfrak{q}^3 - (a_1 + a_2)\mathfrak{q}^2 + (R + M_1 + M_2)^2\mathfrak{q} - (R + M_1 + M_2) \times [a_1(R + M_1 - M_2) + a_2(R - M_1 + M_2)] = 0. \tag{24}$$

Finally, combining Eqs. (18), (20), and (24), the explicit formulas for both unequal σ_i in terms of Komar physical parameters are given by

$$\sigma_1 = \sqrt{M_1^2 - a_1^2 + \gamma_{12}}, \qquad \sigma_2 = \sqrt{M_2^2 - a_2^2 + \gamma_{21}},$$
$$\gamma_{12} := 4a_1 M_2 \frac{a_1 M_2 \mathfrak{q}^2 + [M_1(\mathfrak{q} + a_1 - a_2) + a_1 R]P_0}{P_0^2},$$
$$\gamma_{21} := 4a_2 M_1 \frac{a_2 M_1 \mathfrak{q}^2 + [M_2(\mathfrak{q} - a_1 + a_2) + a_2 R]P_0}{P_0^2}, \tag{25}$$

where $\sigma_2 = \sigma_{1(1\leftrightarrow 2)}$. The solving of the AC Eq. (17) and the physical functional form of each half-length horizon σ_i are two of the principal results of this paper. It is worth mentioning that both horizons can be entirely depicted in terms of the five parameters $\{M_1, M_2, a_1, a_2, R\}$ after solving analytically the above cubic Eq. (24), whose roots explicitly are

$$\mathfrak{q}_{(k)} = -\mathfrak{b}_1 + e^{i2\pi k/3}\left[\mathfrak{b}_o + \sqrt{\mathfrak{b}_o^2 - \mathfrak{a}_o^3}\right]^{1/3} + e^{-i2\pi k/3}\mathfrak{a}_o\left[\mathfrak{b}_o + \sqrt{\mathfrak{b}_o^2 - \mathfrak{a}_o^3}\right]^{-1/3}, \quad k = 0, 1, 2,$$
$$\mathfrak{a}_o := \mathfrak{b}_1^2 - \mathfrak{b}_2, \qquad \mathfrak{b}_o := (1/2)\left[3\mathfrak{b}_1\mathfrak{b}_2 - \mathfrak{b}_3 - 2\mathfrak{b}_1^3\right],$$
$$\mathfrak{b}_1 := -(a_1 + a_2)/3, \qquad \mathfrak{b}_2 := (R+M)^2/3,$$
$$\mathfrak{b}_3 := -(R+M)[a_1(R + M_1 - M_2) + a_2(R - M_1 + M_2)]. \tag{26}$$

The parameter $\mathfrak{q}$ is the key for a better understanding of the dynamical interaction between two Kerr sources, and since this dynamical law is represented by a cubic equation, there exists at least one real root which in this case is given by the phase $k = 0$. So, we have that $\mathfrak{q}$ can take positive or negative values depending on whether the configuration is co or counter-rotating as shown in Fig. 2. The constant line $\mathfrak{q} = 0$ gives us the following condition among two nonequal counter-rotating Kerr BHs [27]:

$$J_2 = -\frac{J_1 M_2}{M_1}\left(\frac{R + M_1 - M_2}{R - M_1 + M_2}\right), \tag{27}$$

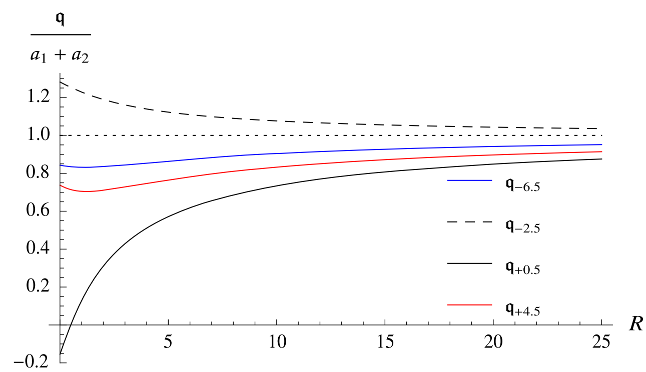

Fig. 2. Several figures of $\mathfrak{q}$ in the unequal case, for $M_1 = 1$ $M_2 = 2$, $a_1 = 1.5$, and different angular momentum values per unit mass a_2 indicated by the subindex.

whose identical case $M_1 = M_2 = m$ and $a_1 = -a_2 = a$, represents one of the most simple exact solutions in which there is no need to solve the AC. It was derived first in Refs. [20,22,23]. Moreover, after the redefinition $M_1 = M_2 = m$, $a_1 = a_2 = a \equiv j/m$, and $\mathfrak{q} = 2q$, from Eq. (24) one recovers the cubic equation for identical corotating Kerr sources [24]

$$q^3 - \frac{j}{m}q^2 + \frac{(R+2m)^2}{4}q - \frac{R(R+2m)j}{4m} = 0. \tag{28}$$

3.1. Physical and thermodynamical properties

The first physical property we will consider in this two-body configuration is the interaction force associated with the strut; a line source of pressure deforming the BH horizons. It can be computed by means of the formula [4,30]

$$\mathcal{F} = \frac{1}{4}(e^{-\gamma_s} - 1), \tag{29}$$

being γ_s the metric function γ evaluated on the middle region corresponding to the CS among sources. It is really amazing how simple turns out to be the formula for the interaction force between two Kerr sources, which takes the final form

$$\mathcal{F} = \frac{M_1 M_2[(R+M)^2 - \mathfrak{q}^2]}{(R^2 - M^2 + \mathfrak{q}^2)[(R+M)^2 + \mathfrak{q}^2]}. \tag{30}$$

If $\mathfrak{q} = 0$ and there is no rotation ($a_i = 0$) we recover the first known expression of the force for two Schwarzschild BHs [3]

$$\mathcal{F} = \frac{M_1 M_2}{R^2 - (M_1 + M_2)^2}. \tag{31}$$

The force tends to zero as the sources move further and further away from each other. In this case if $R \to \infty$, the parameter $\mathfrak{q} \to a_1 + a_2$, and the force contains the following aspect:

$$\mathcal{F} \simeq \frac{M_1 M_2}{R^2}[1 + \frac{M^2 - 3(a_1 + a_2)^2}{R^2} + \frac{4(a_1 + a_2)[J + 4(M_2 a_1 + M_1 a_2)]}{R^3} + O\left(\frac{1}{R^4}\right)], \tag{32}$$

which is the formula already given by Dietz and Hoenselaers [31] with an extra term containing more information about the spin–spin interaction at large distances. It should be remarked that in the limit $R \to \infty$ is also recovered from Eq. (25) the expression $\sigma_i = \sqrt{M_i^2 - J_i^2/M_i^2}$ for one isolated Kerr BH. In addition, we observe from Eq. (30) that the strut disappears with the condition

$\mathfrak{q} = -\varepsilon(R + M)$, $\varepsilon = \pm 1$, leading us to the non-trivial expressions for σ_i obtainable from Eq. (25) in this equilibrium situation [11]

$$\begin{aligned} &\sigma_1 = \sqrt{M_1^2 - a_1^2 + \delta_{12}}, \qquad \sigma_2 = \sqrt{M_2^2 - a_2^2 + \delta_{21}}, \\ &\delta_{12} := M_2 a_1 \frac{a_1(M + M_1 + 2R) - 2M_1[a_2 + \varepsilon(M + R)]}{(M + R)^2}, \\ &\delta_{21} := M_1 a_2 \frac{a_2(M + M_2 + 2R) - 2M_2[a_1 + \varepsilon(M + R)]}{(M + R)^2}, \end{aligned} \tag{33}$$

whereas the equilibrium law is derived from Eq. (24) and given by [11]

$$J_1 + J_2 + R\,(J_1/M_1 + J_2/M_2) + \varepsilon(R + M_1 + M_2)^2 = 0. \tag{34}$$

Turning now our attention to the thermodynamical features of the BS, where is well-known that each BH fulfills the Smarr formula [32]

$$M_i = \frac{\kappa_i S_i}{4\pi} + 2\Omega_i J_i = \sigma_i + 2\Omega_i J_i, \qquad i = 1, 2, \tag{35}$$

being κ_i the so-called surface gravity of the ith BH, which is related to the corresponding angular velocity Ω_i by means of [31, 33]

$$\kappa_i = \sqrt{-\Omega_i^2 e^{-2\gamma^{H_i}}}, \qquad \Omega_i := \omega_i^{-1}, \tag{36}$$

where ω_i and γ^{H_i} are the constant values of the metric functions ω and γ on the axis part associated to each horizon H_i. Additionally, S_i is the area of the horizon. Taking into account Eqs. (24) and (25) it follows that the angular velocities, surface gravities and the area of the horizons acquire the final compact expressions

$$\begin{aligned} &\Omega_1 = \frac{M_1 - \sigma_1}{2J_1} = \frac{J_1 F_1}{2M_1^2(M_1 + \sigma_1)}, \\ &\kappa_1 = \frac{\sigma_1 P_0[(R + \sigma_1)^2 - \sigma_2^2]}{[P_0(M_1 + \sigma_1) - 2M_1 a_1 \mathfrak{q}]^2 + a_1^2(R^2 - \Delta)^2}, \\ &S_1 = 4\pi \frac{[P_0(M_1 + \sigma_1) - 2M_1 a_1 \mathfrak{q}]^2 + a_1^2(R^2 - \Delta)^2}{P_0[(R + \sigma_1)^2 - \sigma_2^2]}, \\ &F_1 := 1 - \frac{4M_2}{a_1}\left[\frac{a_1 M_2 \mathfrak{q}^2 + [M_1(\mathfrak{q} + a_1 - a_2) + a_1 R]P_0}{P_0^2}\right], \\ &\Omega_2 = \Omega_{1(1\leftrightarrow 2)}, \quad \kappa_2 = \kappa_{1(1\leftrightarrow 2)}, \quad S_2 = S_{1(1\leftrightarrow 2)}. \end{aligned} \tag{37}$$

On the other hand, in order to interpret the interaction between BHs, by looking once more the denominator of the above formula of the force, at first sight it seems that the merger limit occurs whether R tends to a minimal value given by $R_0 = \sqrt{M^2 - \mathfrak{q}^2} = \sigma_1 + \sigma_2$, from which the interaction force $\mathcal{F} \to \infty$. At this particular value of the distance we notice from Eq. (24) [or Eq. (23)] that $\mathfrak{q} = J/M$, and therefore the values of the half-length horizons are

$$\begin{aligned} &\sigma_1 = \sqrt{M_1^2 - \frac{a_1 M_1^2[a_1(R_0 + M_1 - M_2) + 2M_2 a_2]}{(M_1 + M_2)^2(R_0 + M_1 + M_2)}}, \\ &\sigma_2 = \sqrt{M_2^2 - \frac{a_2 M_2^2[a_2(R_0 - M_1 + M_2) + 2M_1 a_1]}{(M_1 + M_2)^2(R_0 + M_1 + M_2)}}. \end{aligned} \tag{38}$$

Let us now consider that both BHs become extremal; i.e., $\sigma_1 = \sigma_2 = 0$, thus the minimal value $R_0 = 0$ befalls when $\mathfrak{q} = M$, and because $\mathfrak{q} = J/M$, the MP will produce a single extreme BH of mass $M = M_1 + M_2$ and total angular momentum $J = J_1 + J_2$, satisfying a well-known relation given by

$$J_1 + J_2 = (M_1 + M_2)^2. \tag{39}$$

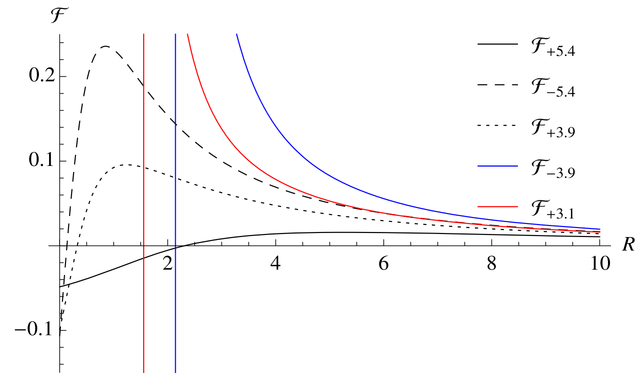

Fig. 3. The behavior of the interaction force for $M_1 = 1$, $M_2 = 2$, $a_1 = 1.5$, and several angular momentum values per unit mass a_2 labeled with a subindex. The merger limit is indicated by a vertical asymptote.

Then, a natural question arises: What values the angular momenta of the BHs are taking during the MP in this extreme case? The answer of this question comes immediately from Eq. (38), by settling $\sigma_1 = \sigma_2 = 0$; having that

$$J_1 = M_1(M_1 + M_2), \qquad J_2 = M_2(M_1 + M_2), \tag{40}$$

whose sum $J_1 + J_2$ clearly recovers Eq. (39). Apparently, this description of the MP corresponds to a BS composed by corotating sources; in agreement with Ref. [23] in the identical case. Moreover, if $\mathfrak{q} = 0$, and therefore $R = M_1 + M_2$, the BH horizons become statics during the merger limit since $\sigma_1 = M_1$ and $\sigma_2 = M_2$ [27], as well as identical when both sources result to be extreme [23,27]. In both static cases the merger limit begins to form a single Schwarzschild BH containing a total mass equal to the sum $M_1 + M_2$ whose area of its horizon satisfies the relation $S = S_1 + S_2 = 16\pi(M_1 + M_2)^2$.

It is well-known that the concept of two rotating BHs in equilibrium without a supporting strut before the system merges is not possible since at least one source develops a ring singularity off the axis. This pathology has been associated mainly to the presence of negative masses in the DKN solution [5–7], being also observed when the total ADM mass [28] of the BS is positive, and the solution does not violate the positive mass theorem [8,9]. Fig. 3 supports such a statement, where it is observed that the force is never crossing the horizontal axis if both masses are positive, but apparently it does during the MP. So, this bring us to consider the following scenario: The BHs are interacting with a CS among them before they reach the merger limit; this is due to the fact that the spin–spin interaction is not enough to counterbalance the gravitational attraction. Later on, the MP is taking place and the BS begins to form a single source of mass $M = M_1 + M_2$ and angular momentum $J = J_1 + J_2$. If the MP is occurring, we are wondering the following questions:

i) the strut can be removed during the MP?

ii) the new source that is being generated is a BH?

To answer the first question, we find very illustrative to plot the stationary limit surfaces (SLS) for both BHs until get the point in which the strut is eliminated. In Figs. 4(a), 4(b), and 4(c) we have assigned the initial values $M_1 = 1$, $M_2 = 2$, $a_1 = 0.7$, and $a_2 = 1.5$, where Figs. 4(a) and 4(b) show two disconnected ergoregions at the coordinate distances $R = 3.27$ and $R = 3.252$, respectively. The merger limit is given at $R = 2.7348$ and the MP has begun, as it is shown in Fig. 4(c). Finally, Fig. 4(d) depicts the case when the MP is taking place, where the second BH must increase rapidly its spin to the value $a_2 = 4.2$ in order to repel the first BH and establishes an equilibrium situation at $R = 0.08443$; therefore, the CS has been removed.

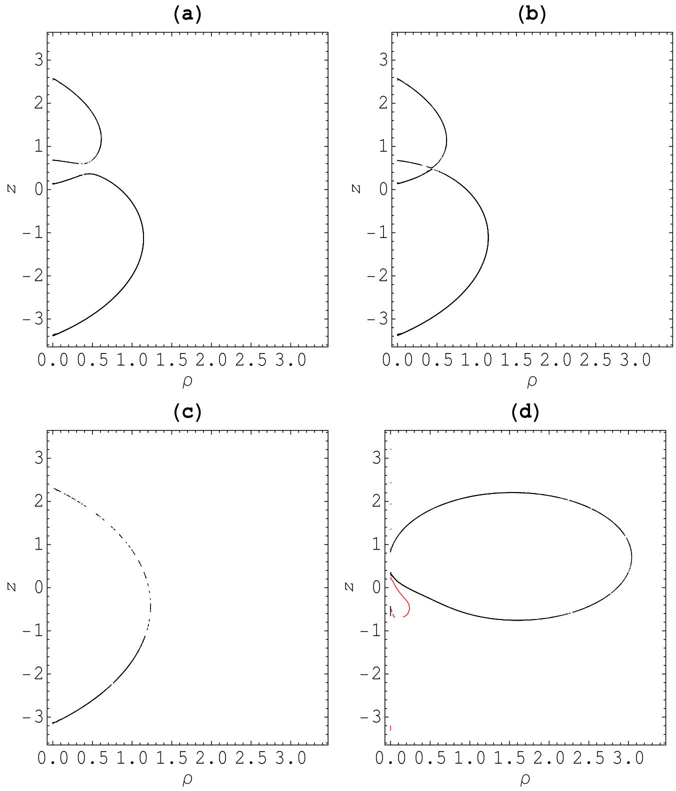

Fig. 4. The SLS ($f = 0$) of the BS with a strut among sources for the values $M = 1$, $M = 2$, $a_1 = 0.7$, and $a_2 = 1.5$, as well as different values for R; (a) $R = 3.27$, (b) $R = 3.252$, (c) $R = 2.7348$ (the merger limit). (d) During the MP at $R = 0.08443$, the second BH tends to increase its rotation to compensate the repulsion of the first one and remove the strut, where CTC appear outside the ergoregion.

Table 1
Numerical values for equilibrium states during the MP, fixing the values in the masses $M_1 = 1$ and $M_2 = 2$.

σ_1	σ_2	a_1	a_2	R	q
1.36892	0.96607	1.2	−5.2	0.09545	−3.09545
0.78173	0.39310	0.7	4.2	0.08443	3.08443
1.22305	1.01419	−0.7	4.9	0.05394	3.05394
1.40981	0.29261	−1.44	5.42	0.18168	3.18168
2.36235	1.27626	5.0	−7.4	0.20998	−3.20998

With regard to the second question, on the nature of the new source. Table 1 shows a set of numerical values satisfying equilibrium states during the MP. In all cases we have evaluated the condition $M^2 - J^2/M^2$, which results to be always negative and the new source develops CTC outside the ergoregion. This interesting aspect is noticed in Fig. 4(d) after plotting the region where the metric coefficient $g_{\varphi\varphi}$ of the line element Eq. (2), satisfies $g_{\varphi\varphi} = -\rho^2 f^{-1} + f\omega^2 > 0$. On the other hand, for a wide range of numerical values we have not found CTC outside the ergoregion, when the strut is not eliminated during the MP and the condition $M^2 - J^2/M^2 \geq 0$ is fulfilled. Therefore, we conclude that the CS should not be removable even when the MP is occurring, in order to not break the notion of being treating with binary BHs.

4. Final remarks

This paper is devoted to conclude one of the main problems during almost the last four decades that might help to study in the most general case dynamical and thermodynamical aspects of two interacting BHs in stationary axisymmetric vacuum systems: *the solving of the axis conditions*. Our suitable parametrization of the double Kerr solution [2] including the NUT charge led us to consider the desirable parametrization which eventually simplified and helped us to solve the axis condition in between sources. After that, we have been capable to obtain nontrivial expressions for the BH horizons σ_i, $i = 1, 2$, in terms of the five arbitrary physical Komar parameters $\{M_1, M_2, a_1, a_2, R\}$ as well as the thermodynamical features contained within the Smarr formula [32]. These five physical parameters satisfy a dynamical law for interacting BHs with struts; a cubic algebraic equation, which is reduced to the equilibrium law for two arbitrary Kerr sources [11] in the absence of the strut. The numerical analysis reveals that the presence of the CS cannot be avoided if the final state of the MP should be a BH. It is worthwhile to mention that the physical representation considered in this work is more transparent at the moment of considering astrophysical phenomena related to GW, like the collision of two BHs, since the quasinormal modes of GW will be in terms of these physical parameters. We are convinced that our results will help not only to derive further exact models regarding geodesics around binary BHs with the main objective to research GW during the MP, but also they can be useful at the moment of studying lensing/shadow models to probe the horizon geometry of binary BHs that could be observed in experiments such as the Event Horizon Telescope [34].

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