

Holographic subregion complexity of a (1+1)-dimensional p -wave superconductor

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Received January 25, 2019; Revised April 19, 2019; Accepted May 7, 2019; Published June 29, 2019

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 We analyze the holographic subregion complexity in a three-dimensional black hole with vector hair. This three-dimensional black hole is dual to a (1+1)-dimensional p -wave superconductor. We probe the black hole by changing the size of the interval and by fixing q or T . We show that the universal part is finite across the superconductor phase transition and has competitive behaviors different from the finite part of the entanglement entropy. The behavior of the subregion complexity depends on the gravitational coupling constant divided by the gauge coupling constant. When this ratio is less than the critical value, the subregion complexity increases as temperature becomes low. This behavior is similar to that of the holographic (1+1)-dimensional s -wave superconductor [M. K. Zangeneh, Y. C. Ong, and B. Wang, Phys. Lett. B **771**, 130 (2014)]. When the ratio is larger than the critical value, the subregion complexity has a non-monotonic behavior as a function of q or T . We also find a discontinuous jump of the subregion complexity as a function of the size of the interval. The subregion complexity has a maximum when it wraps almost the entire spatial circle. Due to competitive behaviors between the normal and condensed phases, the universal term in the condensed phase becomes even smaller than that of the normal phase by probing the black hole horizon at a large interval. This implies that the condensate formed decreases the subregion complexity as in the case of the entanglement entropy.

Subject Index B02, B21, B22, E00

1. Introduction

Entanglement entropy is an important non-local quantity in quantum information, capturing geometric aspects of field theories (e.g. an area law [1,2] and the strong subadditivity). The entanglement entropy counts the number of degrees of freedom in the quantum entangled state [3–6], while it turns out to be an order parameter of the phase transition as in the Wilson loop operator in gauge theories (see quantum critical phase transitions [7]). Duality between strongly coupled gauge theories and the weakly coupled gravity, called the gauge/gravity correspondence [8], has been a powerful tool for analyzing entanglement entropy.¹ The gravity dual to entanglement entropy is given by the minimal surface called the Ryu–Takayanagi surface [12–14], which is a useful way of analyzing the entanglement entropy of strongly coupled systems. The holographic entanglement entropy has been an order parameter of the confinement/deconfinement phase transition [15–19] and a probe of superconductor phase transitions [20–27].

Also, the complexity in quantum information describes the minimal number of gates of any quantum circuit to obtain a desired target state from a reference state. The holographic dual of the

¹ Non-local quantities such as the Wilson loop operator were analyzed by using the minimal surface of the string worldsheet in the gauge/gravity correspondence [9–11].

complexity has recently been noted. First, the holographic complexity was conjectured by Susskind in dual black holes [28,29]. The holographic complexity in a black hole is given by the surface of the Einstein–Rosen bridge. The holographic complexity grows linearly in time as the length of the surface grows. The complexity of a state is in proportion to the volume of the codimension-one maximal bulk surface V in general (complexity=volume conjecture $C \sim V/\kappa^2 l$). However, the length scale l in the complexity=volume conjecture is unclear for separate backgrounds. On the other hand, the complexity=action conjecture improves the ambiguity of the length. Following this conjecture, the Einstein–Hilbert action in the Wheeler–DeWitt patch turns out to be the holographic dual of complexity, the coefficient of which will have a physical meaning [30,31].²

In this paper, we compute the holographic complexity of subregions. Namely, we evaluate the holographic complexity of the mixed state by tracing out states of a separate region. The holographic subregion complexity is proportional to the volume surrounded by the minimal surface (Ryu–Takayanagi surface) [34] as follows (see also a generalization in Ref. [35]):

$$C = \frac{\text{Volume}(\gamma_A)}{\kappa^2 R}, \quad (1)$$

where γ_A is an area of the extremal surface and R is the radius of curvatures in the background.³ The subregion complexity leads to a discontinuous jump at the transition, which is confirmed by computing the integral of the volume form [36] and using the Gauss–Bonnet theorem [38,39]. In the context of the tensor network, numerical results in the Ising model in a squared lattice reproduce an expected linear law behavior of the holographic subregion complexity.⁴ The holographic complexity has also been computed for probing string backgrounds [40] and anisotropic black branes [41].

The motivation for this paper is to further analyze the behavior of the holographic subregion complexity across a holographic superconductor phase transition. Since the holographic subregion complexity is surrounded by a Ryu–Takayanagi surface, the black hole horizon can be probed by changing the size of the interval, similar to holographic entanglement entropy. We are interested in the finiteness of the universal term, unlike the entanglement entropy which is divergent at a critical point of a two-dimensional quantum critical phase transition of infinite length [4,7]. For a finite length, on the other hand, finite size effects should be taken into account. In Ref. [25], the holographic subregion complexity was used as the probe through the holographic (1+1)-dimensional s -wave superconductor phase transition. By improving the result of Ref. [42], the universal terms of the holographic complexity are numerically shown to become finite through the s -wave superconductor phase transition. In the holographic s -wave superconductor, the universal term does not behave as in the entanglement entropy through the superconductor phase transition. On the other hand, this is not the case for the Schwarzschild and Reissner–Nordstrom anti-de Sitter (AdS) black holes [26], where the behavior of the holographic complexity mimics that of the holographic entanglement entropy in some regions. Thus, it is interesting to analyze the holographic complexity in other holographic superconductor models to show the finiteness of the universal terms and to investigate the difference from the holographic entanglement entropy.

² Moreover, an optimized way of doing path integrals has been proposed to understand complexity in quantum field theory [32,33].

³ More precisely, this volume is the codimension-one maximal volume attached to the extremal surface as well as the entangling region [36,37].

⁴ Numerical results do not reproduce the finite part due to the lack of rotational symmetry in a square lattice [39].

For the computation of the subregion complexity, we focus on a specific holographic model dual to the (1+1)-dimensional p -wave superconductor phase transition. The three-dimensional $SU(2)$ Yang–Mills term and the Einstein–Hilbert action are dual to a (1+1)-dimensional p -wave superconductor, as proven in the probe limit [43–45]. In the large- N limit, one can evade the Coleman–Mermin–Wagner theorem in this lower-dimensional system [46]: quantum fluctuations preventing the formation of condensates are suppressed in the large- N limit. The holographic entanglement entropy is computed in a fully backreacted metric of a (1+1)-dimensional p -wave superconductor across the phase transition [27]. The backreacted metric turns out to be a black hole with vector hair in the condensed phase, while it turns out to be the AdS_3 charged black hole in the normal phase. It is shown that the order of the p -wave superconductor phase transition varies depending on the strength of the coupling constant (see Appendix A).

In this paper we compute the holographic subregion complexity in a fully backreacted metric of a (1+1)-dimensional p -wave superconductor. We make use of the divergent form of the holographic complexity analyzed in Refs. [37,47]. After analyzing the coefficient of the divergent term by varying the size of the subregion, we specify the size dependence of this coefficient. Subtracting the divergent term, we analyze the finite part of the subregion complexity. The finite part of the subregion complexity should also depend on the strength of the coupling constant. In the main section, we show that the subregion complexity as a function of T or q suddenly jumps near the phase transition for a large ratio of the gravitational coupling constant to the gauge coupling constant.

In Sect. 2 we review the three-dimensional Einstein–Hilbert and $SU(2)$ Yang–Mills action, which are dual to the (1+1)-dimensional p -wave superconductor. To analyze the holographic subregion complexity, we compute the backreaction of the Yang–Mills term into the metric. In Sect. 3 we compute both the holographic entanglement entropy and the holographic subregion complexity in the holographic (1+1)-dimensional p -wave superconductor phase transition. We compute the holographic subregion complexity by fixing q or T (or both quantities). In Sect. 4, we analyze the renormalized entanglement entropy as a universal term of the entanglement entropy. We compare it with the finite term of both the holographic entanglement entropy and the subregion complexity.

2. Backreactions of the Yang–Mills term

The $SU(2)$ Yang–Mills theory for the AdS_3 black hole has been a holographic model of the p -wave superconductor [43,44]. In this section we review the holographic p -wave superconductor in the three-dimensional Einstein–Hilbert action with the $SU(2)$ Yang–Mills term. We consider the Einstein–Hilbert action and the $SU(2)$ Yang–Mills term as

$$I_G = \frac{1}{2\kappa^2} \int d^3x \sqrt{-g} \left(R + \frac{2}{L^2} \right) - \frac{1}{2g_{YM}^2} \int d^3x \sqrt{-g} \text{tr}(F_{\mu\nu} F^{\mu\nu}), \quad (2)$$

where the field strength is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$. Note that this normalization of the Yang–Mills term is convenient when it is compared with that of the Maxwell theory. That is, using $\text{tr}(T^a T^b) = \delta^{ab}/2$, the kinetic term is as in $F_{\mu\nu}^a F_{\mu\nu}^a / 4g_{YM}^2$.

By performing the coordinate transformation, a general ansatz for the metric is given by

$$ds^2 = \frac{L^2}{z^2} \left(-f(z) dt^2 + dy^2 + \frac{dz^2}{h(z)f(z)} \right), \quad (3)$$

where y is compactified with the periodicity $y \sim y + 2\pi L$. The function $f(z)$ is the blackening factor which gives the position of the black hole horizon at $z = z_h$. The ansatz for the background

non-Abelian gauge field becomes in the radial gauge (see also Refs. [48,49])

$$A = \frac{1}{2}(\phi(z)\sigma^3 dt + w(z)\sigma^1 dy), \quad A_z^b = 0, \quad (4)$$

where σ^a ($a = 1, 2, 3$) are Pauli matrices.

The Einstein equations derived from Eq. (2) turn out to be following three equations:

$$\begin{aligned} \frac{f(zhf' + zf'h - 2fh + 2)}{z^2} - \frac{\tilde{\kappa}^2 z^2 (\phi^2 w^2 + fh(\phi'^2 + fw'^2))}{L^2} &= 0, \\ \frac{2z^2 hf'' + z^2 f'h' - 4zhf' - 2zf'h + 4fh - 4}{2z^2} + \frac{\tilde{\kappa}^2 z^2 (\phi^2 w^2 - fh(fw'^2 + \phi'^2))}{L^2 f} &= 0, \\ -\frac{zhf' - 2fh + 2}{z^2 fh} - \frac{\tilde{\kappa}^2 z^2 (fh(fw'^2 - \phi'^2) + \phi^2 w^2)}{L^2 f^2 h} &= 0, \end{aligned} \quad (5)$$

where the last equation is the z -component corresponding to the constraint equation. Here, we have introduced a parameter $\tilde{\kappa} = \kappa/g_{YM}$, which has dimension -1 . In addition, the equations of motion in terms of Yang–Mills fields are written as

$$\begin{aligned} -\sqrt{hf}(z\sqrt{h}\phi')' + zw^2\phi &= 0, \\ \sqrt{hf}(z\sqrt{h}fw')' + z\phi^2 w &= 0. \end{aligned} \quad (6)$$

Due to the dependence of these equations of motion only on the dimensionless combination $\tilde{\kappa}/L$, L is set to be 1 in the remainder of this section.

2.1. The normal phase

We then solve the Einstein equation of motion derived from the action in Eq. (2). In the normal phase the y -component of the gauge field is zero, while non-zero $A_t^3 = \phi$ produces the charge density and breaks $SU(2)$ gauge symmetry into $U(1)_3$. The energy–momentum tensor turns out to be those without non-linear terms. We then know the charged AdS_3 black hole solution [50,51] with the unit AdS radius as the solution to the Einstein equation of motion as follows:

$$ds_{\text{normal}}^2 = \frac{1}{z^2} \left(-f(z)dt^2 + dy^2 + \frac{dz^2}{f(z)} \right), \quad \phi(z) = q \log \left(\frac{z}{z_0} \right), \quad (7)$$

where $f(z) = 1 - (z/z_0)^2 + \tilde{\kappa}^2 q^2 z^2 \log(z/z_0)$ and the black hole horizon is located at $z = z_0$. Here, $\phi(z)$ is required to be regular at the position of the horizon $z = z_0$. The squared horizon position is inversely proportional to the regularized mass $M_0 = (L/z_0)^2$ which satisfies the Bogomol'nyi–Prasad–Sommerfield-like (BPS) bound $M_0 \geq \tilde{\kappa}^2 q^2 / 2$ [50]. The BPS-like bound is saturated at zero temperature.

Due to the non-normalizable log term, the gauge field obeys the alternative boundary condition, for which the charge density q is considered as the source. The chemical potential turns out to be $\mu = -q \log(z_0)$.

2.2. The condensed phase

In this section we consider the condensed phase, where both $\phi(z)$ and $w(z)$ are non-zero. The charged AdS_3 black hole is unstable when q is large. The black hole acquires the vector hair to go to the stable configuration called the condensed phase. A vector operator dual to $w(z)$ condenses in the

condensed phase, while it breaks parity symmetry as well as remaining $U(1)_3$ spontaneously. The critical point for the p -wave superconductor phase transition is determined from the scaling analysis. In the probe limit, the critical charge density is $q_c = 21.7T_H$ [43]. We analyze the order of the phase transition between the normal phase and the condensed phase by varying the coupling constant $\tilde{\kappa}$.

At the AdS boundary $z \rightarrow 0$, the fields are expanded as

$$\begin{aligned}\phi(z) &\sim q \log(z) + \mu_0, \\ w(z) &\sim W_0 + v_w \log(z), \\ f(z) &\sim \frac{1}{n_0} \left(1 - \frac{z^2}{z_0^2}\right) + \tilde{\kappa}^2 q^2 z^2 \log(z), \\ h(z) &\sim n_0,\end{aligned}\tag{8}$$

where μ_0 is the chemical potential, the parameter W_0 is the vacuum expectation value of a vector order parameter, and v_w is the source conjugate to W_0 . z_0 and n_0 are constant parameters.

The black hole horizon is expected at $z = z_h$ and satisfies $f(z_h) = 0$. The regular boundary condition $\phi(z_h) = 0$ is imposed at the black hole horizon. The analytic expansion near the black hole horizon $z = z_h$ is given by

$$\begin{aligned}\phi(z) &= \alpha_1(z_h - z) + \cdots, \\ w(z) &= \beta_1 + \beta_2(z_h - z) + \cdots, \\ f(z) &= \delta_1(z - z_h) + \cdots, \\ h(z) &= \gamma_1 + \gamma_2(z_h - z) + \cdots,\end{aligned}\tag{9}$$

where $\alpha_1, \beta_1, \beta_2, \delta_1, \gamma_1$, and γ_2 are constants. The Hawking temperature of this black hole solution turns out to be

$$T_H = \frac{1}{4\pi} |f'(z_h)| \sqrt{h(z_h)} = \frac{|\delta_2| \sqrt{\gamma_1}}{4\pi}.\tag{10}$$

Substituting the expansions in Eq. (9) into the equations of motion of Eqs. (5) and (6), we obtain the four independent parameters $\alpha_1, \beta_1, \gamma_1$, and z_h . The other parameters are fixed by these four and $\tilde{\kappa}$.

We then solve the equations of motion starting from the black hole horizon. The constraint equation of Eq. (5) is solved at the black hole horizon. We numerically solve the first two equations of Eqs. (5) and (6), specifying regularity conditions at the horizon $z = z_h$.

At the AdS boundary we specify the boundary conditions $W_0, v_w = 0, n_0 = 1$. The vanishing source v_w shows the superconductor boundary condition, which describes the spontaneous symmetry breaking of residual $U(1)$ symmetry generated by A_μ^3 . The superconductor boundary condition is similar to the one imposed on the charged scalar [52,53] in the holographic s -wave superconductor.

Note that there are scaling symmetries in the equations of motion as follows:

$$(t, y, z) \rightarrow \Lambda_0^{-1}(t, y, z), \quad \phi \rightarrow \Lambda_0 \phi, \quad w \rightarrow \Lambda_0 w,\tag{11}$$

$$f \rightarrow \Lambda_0^2 f, \quad h \rightarrow \Lambda_0^{-2} h, \quad \phi \rightarrow \Lambda_0 \phi.\tag{12}$$

One can use the first symmetry to fix $z_h = 1$. The second symmetry can be used to fix the parameter $n_0 = 1$, which yields the standard asymptotic AdS₃ metric.

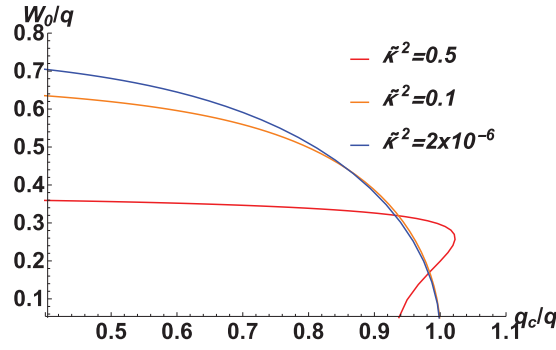


Fig. 1. W_0 normalized by q plotted as a function of q_c/q at fixed temperature $T_H = 0.15$. W_0 increases from zero at $q = q_c$ when $\tilde{\kappa}^2 < 0.31$. In contrast, W_0 jumps to be non-zero at $q = q_c$ when $\tilde{\kappa}^2 > 0.31$.

The behavior of W_0/q is plotted as a function of q_c/q in Fig. 1 at fixed temperature $T_H = 0.15$. In the figure, the critical charge density q_c is determined from the thermodynamic stability between the normal and condensed phases; see Appendix A. The critical value q_c depends on the coupling constant $\tilde{\kappa}$: $q_c = 189T_H$, $33.5T_H$, and $21.7T_H$ for $\tilde{\kappa}^2 = 0.5$, 0.1 , and 2×10^{-6} , respectively.⁵ For all coupling constants $\tilde{\kappa}$, W_0 is zero at small charge density. When $\tilde{\kappa}^2 < 0.31$, W_0 suddenly increases from zero at the critical density $q = q_c$. The condensate W_0 has the scaling behavior $\sim 1.18\sqrt{1 - q_c/q}$. This implies a second-order phase transition. When $\tilde{\kappa}^2 > 0.31$, W_0 jumps to be non-zero at the critical density $q = q_c$, where W_0 does not follow a scaling behavior.

3. Holographic complexity of the subregion

In this section we compute the holographic complexity of the subregion in the holographic $d = 1 + 1$ p -wave superconductor phase transition. We analyze the time-independent subregion complexity via holography [34]. We start with the metric of the three-dimensional black hole, Eq. (3).

Recall that the holographic entanglement entropy is proportional to the area of a minimal surface γ_A . The one-dimensional strip subregion of size l is considered. Using the metric of Eq. (3), the embedding scalar of the surface γ_A satisfies the equation of motion

$$z' = \sqrt{h(z)f(z) \left(\frac{z_*^2}{z^2} - 1 \right)}, \quad (13)$$

where $z = z_*$ is the turning point for the surface. Integrating the equation of motion, the embedding scalar turns out to be

$$x(z) = \int_z^{z_*} dz \frac{1}{\sqrt{h(z)f(z) \left(\frac{z_*^2}{z^2} - 1 \right)}}; \quad (14)$$

$x(z)$ satisfies $x(\epsilon) = l/2$ as well as $x(z_*) = 0$. Due to the symmetry of the curve at the turning point $z = z_*$, the factor of $1/2$ appears in front of l . The minimal surface ends on the particular end points. The holographic entanglement entropy is the minimal surface divided by the gravitational constant:

$$S^{\text{EE}} = \frac{2\pi}{\kappa^2}(\gamma_A) = \frac{4\pi}{\kappa^2} \int_\epsilon^{z_*} dz \frac{1}{z \sqrt{f(z)h(z) \left(\frac{z_*^2}{z^2} - 1 \right)}}. \quad (15)$$

⁵ When $\tilde{\kappa}^2 > 0.31$, the log behavior of fields of the gravity dual would affect the scaling behavior of q_c .

The divergent part of S^{EE} is of the form $S^{\text{EE}} \sim \frac{4\pi}{\kappa^2} \log(\epsilon)$. Apart from the divergence, the finite part of the entanglement entropy $S_{\text{fin}}^{\text{EE}}$ is interesting to analyze. Note that the divergent log term depends on the regularization $\log \epsilon \rightarrow \log \epsilon - \log \Lambda_0$. To have a finite term independent of the regularization, instead, one needs to have the dimensionless combination inside the logarithm $\log \epsilon/T$. The finite part is also shifted by $\log T$, which should be just a constant and will not affect the analysis.

The holographic entanglement entropy $S_{\text{fin}}^{\text{EE}}$ in the normal phase was analyzed by using charged black holes with hyperbolic horizons [54] and second-order excitations [55,56]. The charge q dependence of the finite part $S_{\text{fin}}^{\text{EE}}$ was analyzed in Ref. [27], with $\tilde{\kappa}^2 < 0.31$. The finite part $S_{\text{fin}}^{\text{EE}}$ has a cusp at the intersecting critical point between the normal and condensed phases. The finite part behaves non-monotonically in the regime where the amount of the entanglement in the charge sector competes with the effect of the condensate. When $\tilde{\kappa}^2 > 0.31$, $S_{\text{fin}}^{\text{EE}}$ is analyzed in Appendix B. While the holographic entanglement entropy turns out to be multivalued in a region of $q < q_i$, the holographic entanglement entropy behaves similarly as for $\tilde{\kappa}^2 < 0.31$ at large $q > q_i$.

Unlike the charge density q dependence, we do not find any critical sizes of phase transition varying T/T_i . In particular, the finite part always decreases with a decrease of T/T_i at a low enough temperature. The amount of quantum entanglement decreases due to both the decrease of temperature and the formed condensate. Thus, we do not have competition of two effects, namely, the formation of the condensate and the decrease of temperature. To confirm the behavior of the entanglement entropy, we perform an alternative computation of the renormalized entanglement entropy in Sect. 4. It is finite entropy independent of the cutoff.

By contrast, the holographic complexity of the subregion is proposed to be proportional to the volume surrounded by the minimal surface γ_A . This subregion has the size l . Following Ref. [34], the holographic complexity is defined as

$$C = \frac{\text{volume}(\gamma_A)}{\kappa^2}. \quad (16)$$

Substituting the metric of Eq. (3) into the formula in Eq. (16), the holographic complexity of the subregion turns out to be

$$C(\epsilon) = \frac{c}{6\pi} \int_{\epsilon}^{z_*} \int_0^{x(z)} \frac{dz dx}{z^2 \sqrt{f(z)h(z)}} = \frac{c}{6\pi} \int_{\epsilon}^{z_*} \frac{x(z) dz}{z^2 \sqrt{f(z)h(z)}}, \quad (17)$$

where the central charge is defined as $c = 12\pi/\kappa^2$.

By using the scaling symmetry of Eq. (11), the physics parameters, the entanglement entropy, and the subregion complexity are transformed into

$$T \rightarrow \Lambda_0 T, \quad l \rightarrow \Lambda_0^{-1} l, \quad q \rightarrow \Lambda_0 q, \quad S^{\text{EE}} \rightarrow S^{\text{EE}}, \quad C \rightarrow C. \quad (18)$$

Due to the presence of scaling symmetry, it is convenient to use dimensionless parameters such as l and ql .

One needs to use the numerics to compute the holographic subregion complexity. First, we find z_* by following the argument around Eq. (14) and fixing the parameter l . Secondly, we obtain $x(z)$ from Eq. (14) to perform the double integration in Eq. (17). The subregion complexity is divergent itself. It can be shown that the divergent part of $C(\epsilon)$ is proportional to only $1/\epsilon$. The coefficient of the divergent part is given by

$$l_d = -\frac{\kappa^2 \epsilon_1 \epsilon_2 (C(\epsilon_1) - C(\epsilon_2))}{\epsilon_1 - \epsilon_2}. \quad (19)$$

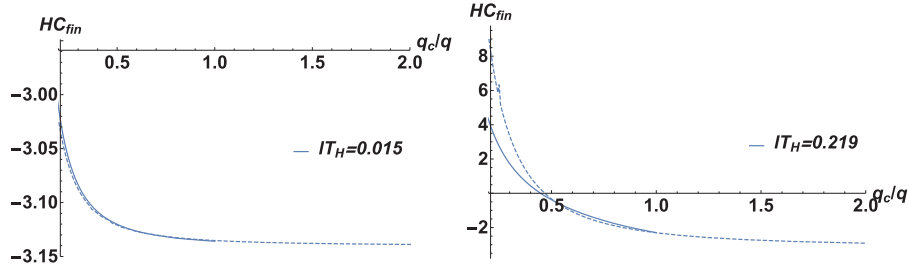


Fig. 2. The normalized finite part HC_{fin} as a function of q_c/q with fixed T_H , l , and fixed $\tilde{\kappa}^2 = 0.1$ ($q_c = 33.5T_H = 5.02$). Left: When $lT_H \ll 1$, the holographic complexity coincides between the normal and condensed phases. Right: Due to the formed condensate, the holographic complexity of the condensed phase turns out to be smaller than that of the normal phase at high charge density.

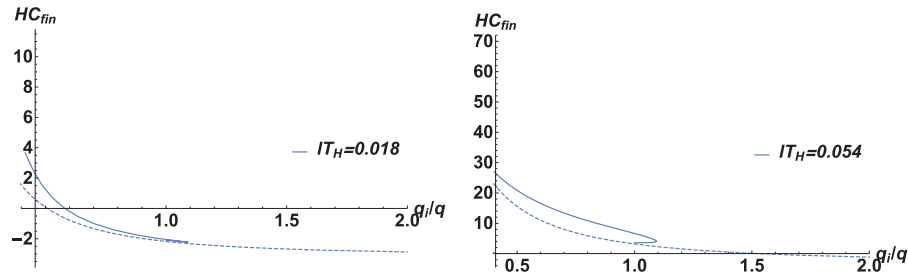


Fig. 3. The normalized finite part HC_{fin} as a function of q_i/q with fixed T_H , l , and fixed $\tilde{\kappa}^2 = 0.5$ ($q_c = 189T_H = 28.3$). In the left figure, HC_{fin} of the condensed phase starts from $q = q_i$. It turns out to be multi-valued when the charge density is small $q < q_i$. The open angle θ_o between two phases is almost π . In the right figure, the open angle θ_o is smaller at fixed $lT_H = 0.054$.

The parameter l_d should be a function of the size of the interval l .⁶ One needs to subtract this singular part to pick up the finite contribution $\kappa^2 HC_{\text{fin}}$.

3.1. The holographic complexity as a function of q or T

We consider two separate coupling constants, $\tilde{\kappa}^2 = 0.1$ and 0.5 , in our numerical computation. We specify the coefficient l_d and the finite part $\kappa^2 HC_{\text{fin}}$ for each coupling constant. We find that l_d is linearly equal to l in the numerics. This linear behavior of the subregion complexity is also observed in the Ising model on the squared lattice in the context of the tensor network [39].

Subtracting the singular part with a coefficient in Eq. (19), we compute the finite part of the subregion complexity $\kappa^2 HC_{\text{fin}}$. We plot the finite part $\kappa^2 HC_{\text{fin}}$, fixing the size of the interval l , temperature T_H , and $\tilde{\kappa}^2$, in Figs. 2 and 3. We plot the finite part $\kappa^2 HC_{\text{fin}}$ fixing l , q , and $\tilde{\kappa}^2$ in Figs. 4 and 5. In both cases, when the size l is smaller than $1/T_H$ (the extremal limit) or $1/q$, the finite part $\kappa^2 HC_{\text{fin}}$ in the condensed phase (solid curves) is almost equal to that of the normal phase (dashed curves). The two behave differently when $lT_H \gg 1$ or $lq \gg 1$. Note that the intersecting point arises from $q = q_i$ ($T = T_i$). While the intersecting point coincides with the critical point $q_i = q_c$ for

⁶ In Ref. [39], the two-dimensional holographic subregion complexity including the Ricci scalar has the divergent structure $\frac{l}{\epsilon} + a_1$ in an AdS_3 black hole by applying the two-dimensional Gauss–Bonnet theorem. Now, l is the size of the interval and a_1 is an Euler number plus a constant. Due to the same theorem, the subregion complexity of the AdS black hole with vector hair also has the same form in the presence of the Ricci scalar in Eq. (3). Without the Ricci scalar, however, a_1 is not topological without the Gauss–Bonnet term.

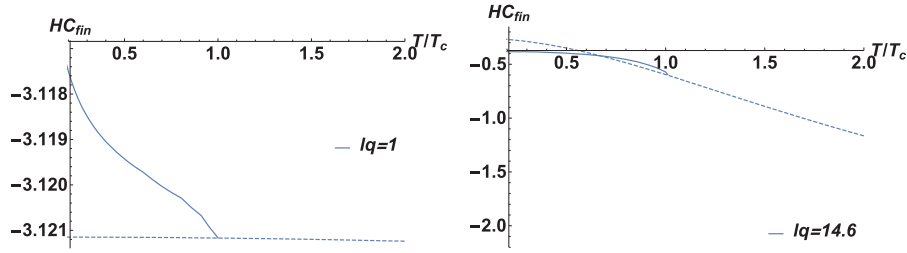


Fig. 4. The finite part HC_{fin} as a function of T/T_c with fixed q, l , and fixed $\tilde{\kappa}^2 = 0.1$ ($T_c = 0.03q = 0.3$). In the normal phase, HC_{fin} is always a decreasing function of T/T_c . Due to the formed condensate, HC_{fin} in the normal phase becomes smaller than that of the normal phase.

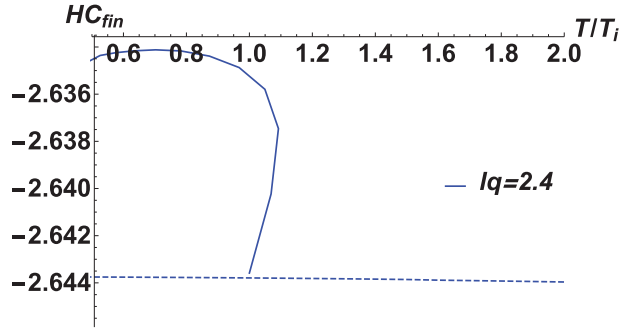


Fig. 5. The finite part HC_{fin} as a function of T/T_c with fixed q, l , and fixed $\tilde{\kappa}^2 = 0.5$ ($T_c \sim 0.0053q = 0.053$). In the normal phase, HC_{fin} decreases with an increase of T/T_c . By contrast, the holographic complexity in the condensed phase turns out to be multi-valued for a specific range of parameters. It increases with a decrease of T/T_c after the phase transition point, having a peak at low temperature.

$\tilde{\kappa}^2 = 0.1$ (< 0.31), $q_i \neq q_c$ for $\tilde{\kappa}^2 = 0.5$ (> 0.31). When $\tilde{\kappa}^2 = 0.5$ (> 0.31), the intersecting point does not seem to have a physical meaning.

In the normal phase of both coupling constants, the finite part $\kappa^2 HC_{\text{fin}}$ decreases with an increase of T/T_c or q_c/q . When $\tilde{\kappa}^2 = 0.1$, the finite part $\kappa^2 HC_{\text{fin}}$ in the condensed phase behaves similarly. This implies that the ordered phase at high density is a more complicated system. When $\tilde{\kappa}^2 = 0.5$, the finite part $\kappa^2 HC_{\text{fin}}$ turns out to be multi-valued for a specific range of the parameter q_c/q or T/T_c . As opposed to small $\tilde{\kappa} < 0.31$, the finite part HC_{fin} in the condensed phase has a peak at the intermediate regime after increasing at low temperature in Fig. 5. These behaviors are separate from the holographic entanglement entropy, which decreases at low temperature or high density.

One can define opening angles θ_0 around the intersecting point q_i or T_i between the two curves of the two phases. The opening angles θ_0 increase when $\tilde{\kappa}^2$ increases, as observed in the holographic s -wave superconductor [25]. When $\tilde{\kappa}^2 < 0.31$, θ_0 is small, being similar to the probe limit. When $\tilde{\kappa}^2 > 0.31$, θ_0 can turn out to be larger than $\pi/2$.

The extremal limit $T_{\text{Hl}} \ll 1$ of the opening angles is interesting. In the extremal limit and for $\tilde{\kappa}^2 < 0.31$, the holographic complexity of the condensed phase coincides with that of the normal phase. The opening angles between the normal and condensed phases are small enough in the extremal limit. By contrast, while the holographic complexity coincides between two phases for $\tilde{\kappa}^2 > 0.31$ in the extremal limit, the opening angles between two phases are large enough; see the right-hand side of Fig. 3. The opening angles decrease with an increase of temperature.

In summary, we analyzed the T or q dependence of the subregion complexity. The universal part HC_u is always finite in both phases. When lq or lT is much smaller than 1 (the pure AdS_3 limit),

the two curves almost agree (cf. the case of the holographic entanglement entropy). By increasing lq or lT , the volume surface probes the region near the black hole horizon. As seen in Figs. 2 and 4 for $\tilde{\kappa}^2 = 0.1$, HC_u in the condensed phase becomes even smaller than that of the normal phase at low temperature or high density. As seen from the analysis of the holographic entanglement entropy, the condensate dominates the charged degrees of freedom in a large subregion. This implies that the condensate decreases HC_u with the small backreaction. When $\tilde{\kappa}$ is large ($\tilde{\kappa}^2 = 0.5$), however, a large l was not be able to be chosen due to a numerical problem.

3.2. The holographic complexity as a function of lq

In this section we compute $\kappa^2 HC_{\text{fin}}$ as well as $S_{\text{fin}}^{\text{EE}}$ as a function of lq . After increasing the size l , the minimal surface is attached to the black hole horizon. Part of the surface attached to the black hole horizon explains the thermal entropy. Following Ref. [57], moreover, the minimal surface wraps the black hole horizon of a Bañados–Teitelboim–Zanelli (BTZ) black hole for a large enough size of the interval, which gives the contribution of the thermal entropy. Thus, we introduce the surface

$$S^{\text{EE}} = S_{\text{ent}} + S^{\text{EE}}(2\pi - l), \quad (20)$$

where S_{ent} is thermal entropy. That is, the difference $S^{\text{EE}}(2\pi - \delta) - S^{\text{EE}}(\delta)$ ($\delta \ll 1$) is equal to the thermal entropy S_{ent} . This equality is also satisfied by the entanglement entropy of two-dimensional free massless Dirac fermions on the two-torus [57].

Actually, the surface wrapping the black hole horizon minimizes the holographic entanglement entropy when the size l is larger than a critical size l_c . There is a phase transition at the critical size l_c as a function of l . In Fig. 6, the finite part of the entanglement entropy is plotted as a function of lq , where $q = 5$. After varying the size of the interval, there is a phase transition of the entanglement entropy. The critical size of the phase transition turns out to be $l_c q = 29.1, 25.7, 20.5$ for $T/T_c = 1, 0.48, 0.17$ ($T_c = 0.03q = 0.149$), respectively. The dashed green curve means the finite part in the normal phase. The critical size becomes $l_c q = 27.5$. The critical size decreases with a decrease of temperature. The blue curve is almost the same as in the charged AdS_3 black hole with the same parameters. The finite part of the entanglement entropy becomes small with a decrease of temperature.

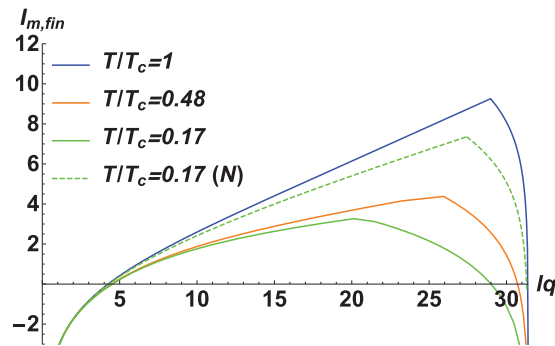


Fig. 6. The finite part of the entanglement entropy $\kappa^2 S_{\text{fin}}^{\text{EE}}/(2\pi) = I_{m,\text{fin}}$ as a function of lq , where $q = 5$. The critical size of the phase transition turns out to be $l_c q = 29.1, 25.7, 20.5$ for $T/T_c = 1, 0.48, 0.17$, respectively. The dashed green curve is the finite part in the normal phase with the same charge density. The critical size is $l_c q = 27.5$. The finite part of the entanglement entropy decreases with a decrease of temperature.

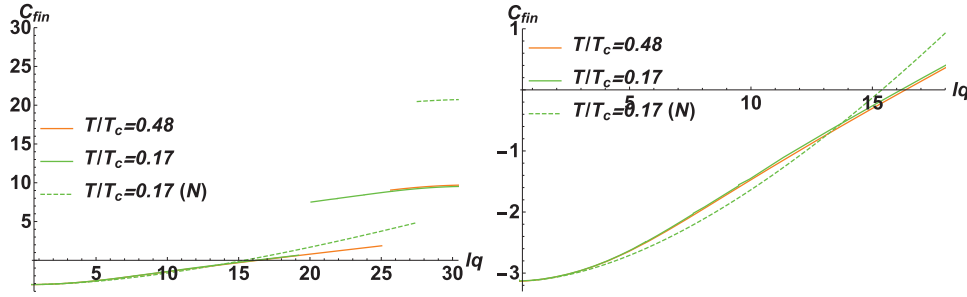


Fig. 7. Left: The normalized finite part HC_{fin} suddenly jumps at a critical size $l_c q = 25.7, 20.5$ ($q = 5$) for fixed $T/T_c = 0.48, 0.17$, respectively. The critical values decrease with a decrease of T/T_c . The dashed green curve is HC_{fin} in the normal phase with the same charge density. The critical size is $l_c q = 27.5$. Right: Close-up figure of the left-hand side. The finite part HC_{fin} in the normal phase becomes even larger than that in the condensed phase when lq becomes large.

Due to the minimal surface wrapping the black hole horizon at a large size l , the complexity also has the following form:

$$C = C_{\text{entire}} - C(2\pi - l), \quad (21)$$

where C_{entire} is the subregion complexity of the entire spatial boundary,

$$C_{\text{entire}} = \frac{c}{3} \int_{\epsilon}^{z_*} dz \frac{1}{z^2 \sqrt{f(z)h(z)}}. \quad (22)$$

C_{entire} does not give any finite part in an AdS_3 black hole. In Eq. (21), the singular part is proportional to the size l due to the cancellation between two terms. The finite part of the holographic complexity $\kappa^2 HC_{\text{fin}}$ is plotted as a function of a dimensionless size lq at fixed temperature in Fig. 7. The finite part $\kappa^2 HC_{\text{fin}}$ increases with an increase of the size of the interval. Due to the topological phase transition of the minimal surface surrounding the volume of the subregion complexity at critical sizes, the finite part $\kappa^2 HC_{\text{fin}}$ suddenly jumps at a critical size $l_c q = 25.7, 20.5$ ($q = 5$) for $T/T_c = 0.48, 0.17$, respectively. The dashed green curve is HC_{fin} in the normal phase. The critical size is $l_c q = 27.5$. The critical size l_c is not dependent on the magnitude of the subregion complexity but the magnitude of the holographic entanglement entropy. The critical size decreases with a decrease of temperature. $lq = 10\pi$ corresponds to the entire spatial boundary. In the figure, the finite part turns out to be maximum at $lq = 10\pi$. The finite part HC_{fin} in the normal phase becomes even larger than that in the condensed phase when lq becomes large. Recall that the formed condensate dominates the charged degrees of freedom in the large separation. Because the volume surface at a large size probes the black hole horizon as observed in the holographic entanglement entropy, it implies that the formed condensate decreases the subregion complexity.

The discontinuous jump of the subregion complexity was originally found in Refs. [36,39] by computing topologically different configurations of two-dimensional volume surfaces (i.e. the disc as well as the annulus). In an AdS_3 black hole, this finite part at zero charge density approaches π after the jump from $-\pi$ to the topologically different configuration. That is, the opposite sign of the finite part. The magnitude of the jump is $\Delta C = 2\pi$, being independent of temperature. After switching on the charge density, the difference ΔHC_{fin} ($l = l_c$) is not independent of the charge density but almost 2π in the condensed phase, while the jump is larger than 2π in the normal phase in Fig. 7.

In summary, the singular part of the subregion complexity gives an expected linear behavior, which is divergent like l/ϵ for those in either eq. (17) or eq. (21). The subregion complexity jumps at the critical length. Depending on charge density, the difference $\Delta HC_{\text{fin}}(l = l_c)$ changes more in the normal phase. Interestingly, the maximum of the subregion complexity is not a constant depending on charge density q or temperature T . As shown in Fig. 7, HC_{fin} in the condensed phase can become smaller than the one in the normal phase. Except for the discontinuous phase transition, the formation of the condensate dominates the charged degrees of freedom and decreases HC_{fin} at the large length.

4. The holographic renormalized entanglement entropy

Motivated by the analysis of the universal term in the entanglement entropy, we compute the renormalized entanglement entropy (renormalized EE). The renormalized entanglement entropy is defined as $S^{\text{ren}} = l \partial S_1 / \partial l$ in two dimensions [58,59], where S_1 is the entanglement entropy. It becomes UV finite and is independent of the cutoff. Due to the UV finiteness, the renormalized EE is considered as a universal term in the entanglement entropy. This entropy is related to the degrees of freedom at the scale of the length l . Moreover, the renormalized EE satisfies the C-theorem only in special cases.

We consider the two-dimensional thermal conformal field theory (CFT) at finite temperature. Substituting the entanglement entropy $S_1 = \frac{c}{3} \log \left(\frac{\beta}{\epsilon \pi} \sinh \left(\frac{\pi l}{\beta} \right) \right)$, the renormalized entanglement entropy becomes

$$S^{\text{ren}} = \frac{c\pi l}{3\beta} \coth \left(\frac{\pi l}{\beta} \right) \sim \begin{cases} \frac{c}{3} & (lT \ll 1), \\ \frac{c\pi l}{3\beta} & (lT \gg 1). \end{cases} \quad (23)$$

In the small- lT limit, moreover, it approaches the vacuum behavior, which does not depend on l . In the large- lT limit, it approaches the thermal entropy with a positive coefficient times l [60]. This behavior is called a crossover.

We apply the renormalized entanglement entropy (renormalized EE) to a holographic (1+1)-dimensional p -wave superconductor. The renormalized EE in a three-dimensional gravity is defined as

$$S^{\text{ren}} = l \frac{\partial S^{\text{EE}}}{\partial l}, \quad (24)$$

where S^{EE} is the Ryu–Takayanagi formula $S^{\text{EE}} = \frac{2\pi}{\kappa^2} (\gamma_A)$. The renormalized EE of an AdS_3 black hole is given by Eq. (23) when the minimal surface does not wrap the black hole horizon.

In Fig. 8, the renormalized EE is plotted as a function of l with a fixed q . When l approaches zero, the renormalized EE goes to a constant, while it is proportional to the thermal entropy density at large l . So, it obeys the crossover as for an AdS_3 black hole. The renormalized EE in the condensed phase becomes smaller than the critical behavior $q = q_c$ at large l . This implies that the formation of the condensate dominates the charged degrees of freedom by probing the black hole horizon.

In Fig. 9, the renormalized EE is plotted as a function of q/q_c when l is fixed (left: $\tilde{\kappa}^2 = 0.1$; right: $\tilde{\kappa}^2 = 0.5$). The renormalized EE behaves qualitatively similarly to the finite part of the holographic entanglement entropy. When $q > q_c$ and the backreaction is small $\tilde{\kappa}^2 = 0.1$, it increases at small lT_H and decreases at large lT_H . The renormalized EE has a non-monotonic behavior and extremal values among a range of size l . This implies that there are competing contributions between

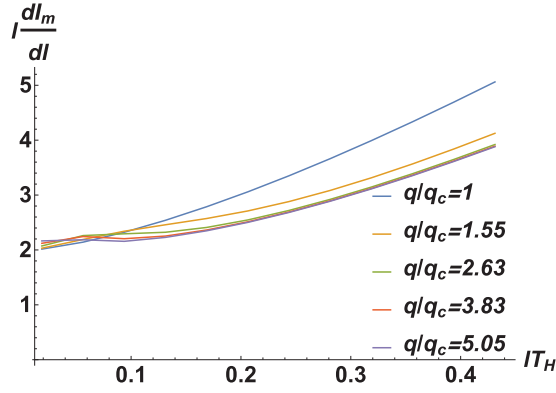


Fig. 8. The renormalized EE $\kappa^2 S^{\text{ren}}/2\pi (= l \partial I_m / \partial l)$ as a function of l at fixed temperature $T_H = 0.15$ ($\tilde{\kappa}^2 = 0.1$). Each curve describes the renormalized EE at fixed charges. It obeys the crossover as for an AdS_3 black hole. The renormalized EE becomes small at large q and l , where the formed condensate dominates the charged degrees of freedom.

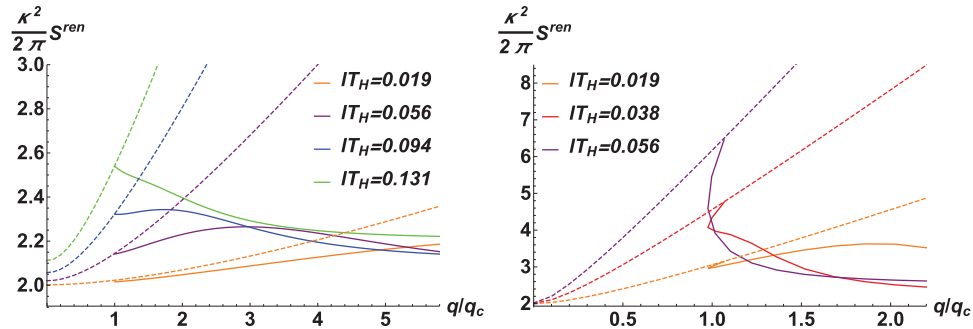


Fig. 9. The renormalized entanglement entropy as a function of q/q_c . Left: $\tilde{\kappa}^2 = 0.1$. Right: $\tilde{\kappa}^2 = 0.5$. In both cases, the renormalized entanglement entropy decreases at a large size l and charge density q .

the charged degrees of freedom and the formed condensate. By probing the black hole horizon, both quantities capture the physics of the formed condensate.

5. Discussion

We have computed the holographic subregion complexity in a fully backreacted metric of the (1+1)-dimensional p -wave superconductor phase transition. We computed the subregion complexity by fixing q or T (or both quantities). We confirm that the universal part HC_u is finite across the phase transition and has competitive behaviors different from the finite part of the entanglement entropy, as seen in Figs. 6 and 7. We probed the black hole with a vector hair by changing the size of the subregion complexity. By increasing the size lq or lT , the volume surface of the subregion complexity approaches the black hole horizon. As observed in Figs. 2 and 4 for $\tilde{\kappa}^2 = 0.1$, HC_u in the condensed phase is even smaller than in the normal phase at low temperature. As seen from the analysis of the holographic entanglement entropy, the formed condensate dominates the charged degrees of freedom in the large size. This implies that the formed condensate decreases HC_u .

The leading divergence of the subregion complexity was shown to be linear to the size of the interval $C \propto l/\epsilon$ in either Eq. (17) or Eq. (21). Even if the Ryu–Takayanagi surface wraps the black hole horizon at a large size of the interval, cancellations occur between two terms in Eq. (21). The same linear behavior was observed in the Ising model on the square lattice.

The finite part of the subregion complexity was plotted by fixing the size of the interval l . In the extremal limit $lT_H \ll 1$ or $lq \ll 1$, $\kappa^2 HC_{\text{fin}}$ almost agreed between the normal and condensed phases except for the region around the intersecting point, where the curve in the condensed phase ended. The curve in the condensed phase behaved differently when $lT_H \gg 1$ (or $lq \gg 1$). Moreover, it depended on the coupling constant. When $\tilde{\kappa}^2 = 0.1 < 0.31$, $\kappa^2 HC_{\text{fin}}$ decreased with an increase of T/T_c or q_c/q . This implies that the system at high charge density and low temperature is complicated. This result of the subregion complexity in addition to the finiteness of the universal part agrees with those of the holographic (1+1)-dimensional s -wave superconductor [25].

The order of phase transition is varied in the holographic (1+1)-dimensional p -wave superconductor with a large amount of backreaction $\tilde{\kappa}^2 = 0.5 > 0.31$, while it does not vary in the holographic (1+1)-dimensional s -wave superconductor [42,61,62] with the backreaction. Due to the large amount of backreaction, the condensate does not behave as in mean field theories (second-order phase transition) but suddenly jumps to a finite value at the critical point. This large amount of backreaction also causes non-monotonic behavior of the finite part $\kappa^2 HC_{\text{fin}}$ in the condensed phase, while $\kappa^2 HC_{\text{fin}}$ in the normal phase behaves monotonically. Moreover, the finite part turns out to be multi-valued as a function of T/T_c or q_c/q .

We plotted the finite part of the subregion complexity $\kappa^2 HC_{\text{fin}}$ as a function of lq , fixing q and T . The formation of the condensate almost did not vary the finite part $\kappa^2 HC_{\text{fin}}$, while the charge density varies it. Wrapping almost the entire space circle maximized the subregion complexity. We found a discontinuous jump of the finite part $\kappa^2 HC_{\text{fin}}$. The magnitude of the discontinuous jump depended on the charge density, unlike that for the AdS_3 black hole. Note that the magnitude of the jump is larger than the jump in the Ising model on the squared lattice $\Delta HC_{\text{fin}} \sim 4 \pm 0.3 < 2\pi$ [39]. This discrepancy will come from the broken rotational symmetry of the squared lattice as well as the difference between the two models.

Finally, we computed the renormalized EE, which was considered as a universal term of the entanglement entropy. We showed that the renormalized EE had a behavior similar to the finite part of the holographic entanglement entropy: when the backreaction was small, it had a monotonic behavior at both small- and large- l limits. On the other hand, the renormalized EE behaved non-monotonically for an intermediate region of size l . These can be understood as competition between charged degrees of freedom and the formed condensate. We found that the renormalized EE obeyed a crossover as seen for the AdS_3 black hole: it approaches a constant for very small size and is linearly proportional to l for a large interval. We noticed that the renormalized EE and $\kappa^2 HC_{\text{fin}}$ had some common properties such as a decrease due to the formed condensate, approaching a constant in the very small lq limit. We did not compare both quantities in the large size limit due to the presence of the discontinuous phase transition. To compare the renormalized EE with the subregion complexity, it will be interesting to explore the large size limit in higher-dimensional holographic models.

Acknowledgements

Special thanks to J. Sun for collaboration during early stages of this work. We would like to thank A. Gadde, B. S. Kim, J. H. Lee, S. Lin, and S. Pujari for helpful discussions and comments. Discussions during the workshop in Fudan University, “String Theory and Quantum Field Theory,” were useful in completing this work. M.F. is supported by the NSFC under the grant No. 11850410431.

Funding

Open Access funding: SCOAP³.

Appendix A. The free energy

To compute the free energy using the AdS/CFT correspondence, we analyze a finite on-shell action in the presence of the Gibbons–Hawking term and a term of the Legendre transformation. These terms reflect a well-defined variation principle. Due to the divergent on-shell action, we must also use counter-terms to cancel divergence [63–65]:

$$\begin{aligned} I_K &= \frac{1}{\kappa^2} \int d^2x \sqrt{-\gamma} (K + \alpha_1), \\ I_A &= \frac{1}{g_{YM}^2} \int d^2x \operatorname{tr} \left(-2\sqrt{-g} A_i F^{zi} + \alpha_2 \log \left(\frac{\epsilon}{L} \right) \sqrt{-\gamma} F_{iz} F^{iz} \right), \end{aligned} \quad (\text{A.1})$$

where $\alpha_1 = -1/L$ and $\alpha_2 = L$. The log term in the second line adds a scale L in the Lagrangian. Summing the three contributions in Eqs. (2) and (A.1), one can obtain the finite renormalized action and the free energy as follows:

$$\begin{aligned} I_{\text{tot}} &= I_G + I_K + I_A, \\ F &= -\frac{I_{\text{tot}}}{\beta}, \end{aligned} \quad (\text{A.2})$$

where a dictionary of the AdS/CFT correspondence has been used in the last line.

By using the analytic solution of the charged AdS₃ black hole in Eq. (7), we can integrate the free energy in the normal phase to give

$$F = \frac{V_1}{\beta} \left(-\frac{L}{2\kappa^2 z_h^2} + \frac{q^2(1 + \log(z_h/L))}{2g_{YM}^2 L} \right), \quad (\text{A.3})$$

where V_1 is the volume of the entire spatial circle. In the above free energy, the presence of the log term varies the scaling transformation: the scaling transformation gives an additional term in the free energy [51]. The variation of the free energy in terms of T, q gives thermodynamic quantities such as the entropy $S = -dF/dT$ and the chemical potential $\mu = dF/dq$, respectively.

In contrast, numerical computation is required to compute the free energy in the condensed phase. The difference of the normalized free energy $2\kappa^2 \Delta I_G \equiv 2\kappa^2 (F_{\text{SF}} - F_{\text{n}})$ is plotted as a function of the normalized temperature T/T_c in Fig. A.1. For small $\tilde{\kappa}$ ($\tilde{\kappa}^2 < 0.31$) and fixed charge density, the solution of the condensed phase always has lower free energy. When $\tilde{\kappa}$ is large ($\tilde{\kappa}^2 > 0.31$), ΔI_G is multi-valued at temperatures larger than T_c ; there occurs the swallow tail of the first-order phase transition. In contrast, it is plotted as a function of the normalized charge density in Fig. A.2. While

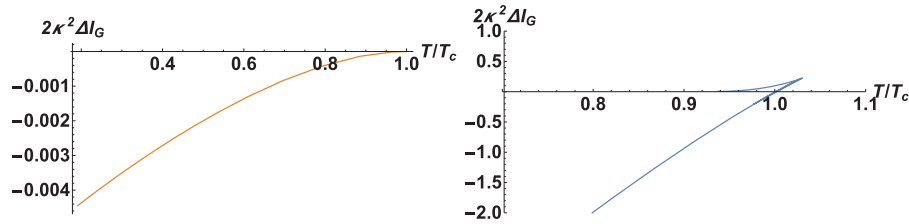


Fig.A.1. The difference in the normalized free energy $2\kappa^2 \Delta I_G$ is plotted at fixed charge density. Left: Fixed $\tilde{\kappa}^2 = 0.1$ and $q = 0.5$. The negative value denotes that the condensed phase is favored at temperatures lower than T_c ($= 0.03q = 0.0149$). Right: Fixed $\tilde{\kappa}^2 = 0.5$ and $q = 30.2$. ΔI_G is multi-valued at some regime of $T > T_c$ ($\sim 0.0053q = 0.16$). There is the swallow tail of the first-order phase transition.

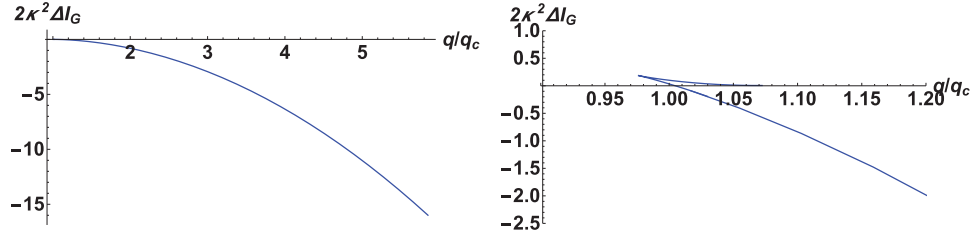


Fig. A.2. The difference in the normalized free energy $2\kappa^2\Delta I_G$ is plotted at fixed temperature $T_H = 0.15$. Left: Fixed $\tilde{\kappa}^2 = 0.1$. It shows that, at large charge density $q > q_c (= 33.5T_H = 5.02)$, the condensed phase is favored. Right: Fixed $\tilde{\kappa}^2 = 0.5$. There is the swallow tail of the first-order phase transition at the critical charge density $q = q_c (= 189T_H = 28.3)$.

the phase transition to the condensed phase occurs at low temperature $T < T_c$, it occurs at large charge density $q > q_c$.

Appendix B. The holographic entanglement entropy in the (1+1)-dimensional p -wave superconductor

In this section we compute the finite part of the holographic entanglement entropy. Due to the subtraction of the divergent part from S^{EE} , we define $I_{m,\text{fin}} \equiv \frac{\kappa^2}{2\pi} S^{\text{EE}} - 2 \log(\epsilon)$. When $\tilde{\kappa}^2 = 0.5$, the finite part of the holographic entanglement entropy is plotted as a function of q/q_i in Fig. B.1. It has a cusp at the intersecting point between the two curves of the normal and condensed phases at the critical point. The finite part $I_{m,\text{fin}}$ behaves non-monotonically. Compared with $\tilde{\kappa}^2 < 0.31$, the holographic entanglement entropy turns out to be multi-valued in the region of small $q < q_i$. For large $q > q_i$, by contrast, the behavior of the holographic entanglement entropy is qualitatively similar that for $\tilde{\kappa}^2 < 0.31$. By increasing l , the increasing behavior of $I_{m,\text{fin}}(l, q)$ with an increase of q is varied into decreasing behavior at large q . This phase transition occurs because S^{EE} probes the formation of the condensate at large intervals l . The decreased degrees of freedom due to the formation of the condensate overcomes the increasing entanglement of the charged states.

In Figs. B.2 and B.3, the finite part of the holographic entanglement entropy is plotted as a function of T/T_i . There is a cusp between the two curves of the normal and condensed phases. At low temperature, the finite part of the condensed phase is always lower than that of the normal phase. When $\tilde{\kappa}^2 = 0.5 > 0.31$, the finite part turns out to be multi-valued around the critical point T_c .

Unlike the entanglement entropy as a function of q , we do not find any critical sizes where the finite part of the entanglement entropy has the phase transition. The finite part $I_{m,\text{fin}}$ decreases with a

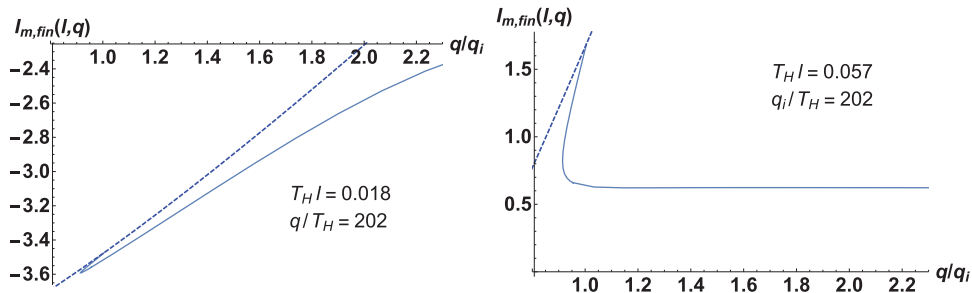


Fig. B.1. The finite part of the holographic entanglement entropy $\kappa^2 S^{\text{EE}}/(2\pi) \equiv I_m$ as a function of q/q_i , with fixed $\tilde{\kappa}^2 = 0.5$. The finite part in the condensed phase is always smaller than that in the normal phase. The holographic entanglement entropy behaves non-monotonically.

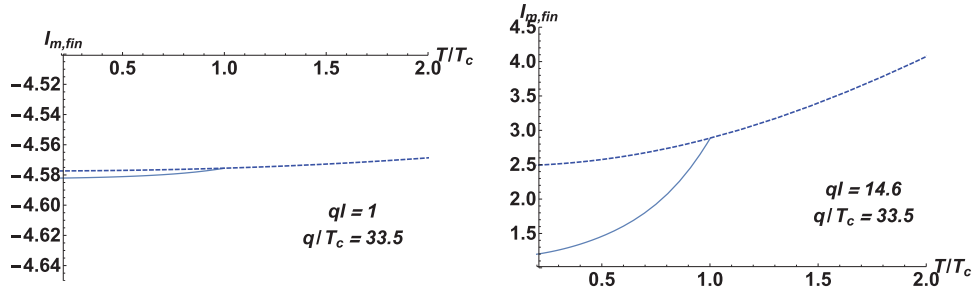


Fig. B.2. The finite part of the holographic entanglement entropy $\kappa^2 S^{\text{EE}}/(2\pi) \equiv I_m$ as a function of T/T_i , for fixed $\tilde{\kappa}^2 = 0.1$ and a fixed size of the interval. By decreasing T/T_c , the superconductor phase appears below the critical temperature. The finite part in the condensed phase is always smaller than that in the normal phase.

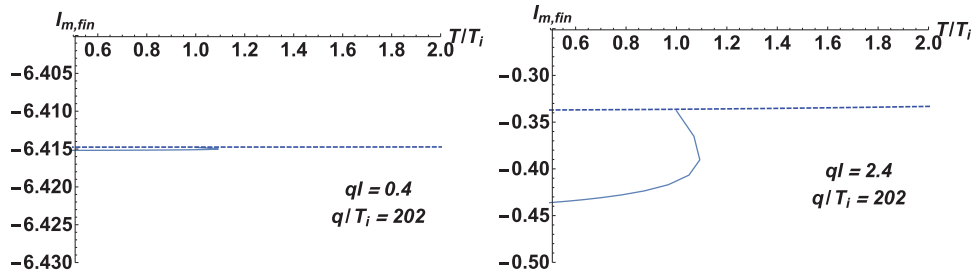


Fig. B.3. The same quantity for fixed $\tilde{\kappa}^2 = 0.5$ and a fixed size of the interval. The finite part in the condensed phase is always smaller than that in the normal phase. The finite part in the condensed phase turns out to be multi-valued around the critical point T_c .

decrease of T/T_i at a low enough temperature, while the finite part increases with an increase of q/q_i at a high charge density for a small size. This implies that by decreasing temperature, the amount of the quantum entanglement decreases. Simultaneously, the condensate is formed and the degrees of freedom decrease [27].

Appendix C. Holographic subregion complexity in asymptotically AdS backgrounds

In this section we analytically compute the holographic subregion complexity in asymptotically AdS backgrounds for comparison. The subregion complexity has a divergent part like $1/\epsilon$ and a finite part.

C.1. The subregion complexity of the pure AdS

We consider a Ryu–Takayanagi surface in the pure AdS₃ background (see Sect. 3 for the Ryu–Takayanagi surface). Solving the equation of motion in terms of the embedding scalar $x(z)$ in the AdS₃ with the unit radius, it is evaluated as

$$x(z) = \int_z^{z_*} \frac{dz}{\sqrt{\frac{z_*^2}{z^2} - 1}} = \sqrt{z_*^2 - z^2}, \quad (\text{C.1})$$

where z_* is the position of the turning point. The boundary condition $x(\epsilon) = l/2$ is imposed and then $l = 2z_*$.

Substituting Eq. (C.1) into the holographic subregion complexity in Eq. (17), we can analytically integrate it to give⁷

$$C = \frac{c}{12\pi} \left(\frac{l}{\epsilon} - \pi \right), \quad (\text{C.2})$$

where $c = 12\pi/\kappa^2$ is the central charge. The subregion complexity is divergent like $1/\epsilon$.

By considering a rectangular shape of the surface γ_A , the constant term of Eq. (C.2) vanishes as follows:

$$C = \frac{c}{12\pi} \int_{-l/2}^{l/2} dx \int_{\infty}^{\epsilon} dz \frac{1}{z^2} = \frac{cl}{12\pi\epsilon}. \quad (\text{C.3})$$

C.2. The subregion complexity of the BTZ black hole

We consider a Ryu–Takayanagi surface in a BTZ black hole in the unit AdS radius ($f(z) = 1 - z^2/z_h^2$ and $h(z) = 1$ in Eq. (3)). Solving the equation of motion in terms of the embedding scalar $x(z)$, it can be integrated analytically as follows:

$$x(z) = \int_z^{z_*} \frac{dz}{\sqrt{f(z) \left(\frac{z_*^2}{z^2} - 1 \right)}} = z_h \coth^{-1} \left(\sqrt{\frac{z_h^2 - z^2}{z_*^2 - z^2}} \right), \quad (\text{C.4})$$

where z_* is the position of the turning point. The boundary condition $x(\epsilon) = l/2$ shows that $l = 2z_h \tanh^{-1}(z_*/z_h)$.

Substituting Eq. (C.4) into the subregion complexity of Eq. (17), the complexity can be integrated out as follows:

$$C = \frac{c}{12\pi} \left(\frac{l}{\epsilon} - \pi \right). \quad (\text{C.5})$$

The subregion complexity of the whole spatial circle is easily evaluated. It does not have a finite term:

$$C = \frac{cl}{12\pi\epsilon}. \quad (\text{C.6})$$

At a large size l , the Ryu–Takayanagi surface wraps the black hole horizon. Accordingly, the subregion complexity at a large interval in Eq. (21) has a finite term of the opposite sign, as follows:

$$C = C_{\text{entire}} - C(2\pi - l) = \frac{c}{12\pi} \left(\frac{l}{\epsilon} + \pi \right). \quad (\text{C.7})$$

Thus, the subregion complexity has its maximum, similarly to the holographic superconductor in the main section, when it almost wraps the whole spatial circle.

⁷ Following Ref. [39], another definition of the subregion complexity turns out to be

$$C_2 = -\frac{1}{2} \int \sqrt{-g} R d^2x.$$

One can apply the Gauss–Bonnet theorem for the above formula. In asymptotically AdS₃ geometries, the divergent term is proportional to l/ϵ . The finite term consists of the Euler number and the contribution coming from the extrinsic geodesic. The above formula gives the same result as Eq. (17) when the Ricci scalar R is a constant.

References

- [1] L. Bombelli, R. K. Koul, J. Lee, and R. D. Sorkin, *Phys. Rev. D* **34**, 373 (1986).
- [2] M. Srednicki, *Phys. Rev. Lett.* **71**, 666 (1993) [arXiv:hep-th/9303048] [Search INSPIRE].
- [3] C. Holzhey, F. Larsen, and F. Wilczek, *Nucl. Phys. B* **424**, 443 (1994) [arXiv:hep-th/9403108] [Search INSPIRE].
- [4] P. Calabrese and J. Cardy, *J. Stat. Mech.* **0406**, P06002 (2004) [arXiv:hep-th/0405152] [Search INSPIRE].
- [5] P. Calabrese and J. Cardy, *J. Phys. A: Math. Theor.* **42**, 504005 (2009) [arXiv:0905.4013 [cond-mat.stat-mech]] [Search INSPIRE].
- [6] H. Casini and M. Huerta, *J. Phys. A: Math. Theor.* **42** (2009) 504007 [arXiv:0905.2562 [hep-th]] [Search INSPIRE].
- [7] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, *Phys. Rev. Lett.* **90**, 227902 (2003) [arXiv:quant-ph/0211074] [Search INSPIRE].
- [8] J. Maldacena, *Int. J. Theor. Phys.* **38**, 1113 (1999) [*Adv. Theor. Math. Phys.* **2**, 231 (1998)] [arXiv:hep-th/9711200] [Search INSPIRE].
- [9] J. Maldacena, *Phys. Rev. Lett.* **80**, 4859 (1998) [arXiv:hep-th/9803002] [Search INSPIRE].
- [10] S. J. Rey and J. T. Yee, *Eur. Phys. J. C* **22**, 379 (2001) [arXiv:hep-th/9803001] [Search INSPIRE].
- [11] N. Drukker, D. J. Gross, and H. Ooguri, *Phys. Rev. D* **60**, 125006 (1999) [arXiv:hep-th/9904191] [Search INSPIRE].
- [12] S. Ryu and T. Takayanagi, *Phys. Rev. Lett.* **96**, 181602 (2006) [arXiv:hep-th/0603001] [Search INSPIRE].
- [13] S. Ryu and T. Takayanagi, *J. High Energy Phys.* **0608**, 045 (2006) [arXiv:hep-th/0605073] [Search INSPIRE].
- [14] T. Nishioka, S. Ryu, and T. Takayanagi, *J. Phys. A: Math. Theor.* **42**, 504008 (2009) [arXiv:0905.0932 [hep-th]] [Search INSPIRE].
- [15] T. Nishioka and T. Takayanagi, *J. High Energy Phys.* **0701**, 090 (2007) [arXiv:hep-th/0611035] [Search INSPIRE].
- [16] I. R. Klebanov, D. Kutasov, and A. Murugan, *Nucl. Phys. B* **796**, 274 (2008) [arXiv:0709.2140 [hep-th]] [Search INSPIRE].
- [17] P. V. Buividovich and M. I. Polikarpov, *Phys. Lett. B* **670**, 141 (2008) [arXiv:0806.3376 [hep-th]] [Search INSPIRE].
- [18] M. Fujita, T. Nishioka, and T. Takayanagi, *J. High Energy Phys.* **0809**, 016 (2008) [arXiv:0806.3118 [hep-th]] [Search INSPIRE].
- [19] I. Bah, L. A. Pando Zayas, and C. A. Terrero-Escalante, *Int. J. Mod. Phys. A* **27**, 1250048 (2012) [arXiv:0809.2912 [hep-th]] [Search INSPIRE].
- [20] T. Albash and C. V. Johnson, *J. High Energy Phys.* **1205**, 079 (2012) [arXiv:1202.2605 [hep-th]] [Search INSPIRE].
- [21] R.-G. Cai, S. He, L. Li, and Y.-L. Zhang, *J. High Energy Phys.* **1207**, 088 (2012) [arXiv:1203.6620 [hep-th]] [Search INSPIRE].
- [22] R.-G. Cai, S. He, L. Li, and Y.-L. Zhang, *J. High Energy Phys.* **1207**, 027 (2012) [arXiv:1204.5962 [hep-th]] [Search INSPIRE].
- [23] R. E. Arias and I. S. Landea, *J. High Energy Phys.* **1301**, 157 (2013) [arXiv:1210.6823 [hep-th]] [Search INSPIRE].
- [24] X.-M. Kuang, E. Papantonopoulos, and B. Wang, *J. High Energy Phys.* **1405**, 130 (2014) [arXiv:1401.5720 [hep-th]] [Search INSPIRE].
- [25] M. K. Zangeneh, Y. C. Ong, and B. Wang, *Phys. Lett. B* **771**, 235 (2017) [arXiv:1704.00557 [hep-th]] [Search INSPIRE].
- [26] P. Roy and T. Sarkar, *Phys. Rev. D* **96**, 026022 (2017) [arXiv:1701.05489 [hep-th]] [Search INSPIRE].
- [27] S. R. Das, M. Fujita, and B. S. Kim, *J. High Energy Phys.* **1709**, 016 (2017) [arXiv:1705.10392 [hep-th]] [Search INSPIRE].
- [28] L. Susskind, *Fortsch. Phys.* **64**, 24 (2016) [arXiv:1403.5695 [hep-th]] [Search INSPIRE].
- [29] L. Susskind, *Fortsch. Phys.* **64**, 44 (2016) [arXiv:1402.5674 [hep-th]] [Search INSPIRE].
- [30] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle, and Y. Zhao, *Phys. Rev. Lett.* **116**, 191301 (2016) [arXiv:1509.07876 [hep-th]] [Search INSPIRE].

- [31] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle, and Y. Zhao, *Phys. Rev. D* **93**, 086006 (2016) [arXiv:1512.04993 [hep-th]] [Search INSPIRE].
- [32] P. Caputa, N. Kundu, M. Miyaji, T. Takayanagi, and K. Watanabe, *J. High Energy Phys.* **1711**, 097 (2017) [arXiv:1706.07056 [hep-th]] [Search INSPIRE].
- [33] A. Bhattacharyya, P. Caputa, S. R. Das, N. Kundu, M. Miyaji, and T. Takayanagi, *J. High Energy Phys.* **1807**, 086 (2018).
- [34] M. Alishahiha, *Phys. Rev. D* **92**, 126009 (2015) [arXiv:1509.06614 [hep-th]] [Search INSPIRE].
- [35] M. Alishahiha, K. Babaei Velni, and M. R. Mohammadi Mozaffar, arXiv:1809.06031 [hep-th] [Search INSPIRE].
- [36] O. Ben-Ami and D. Carmi, *J. High Energy Phys.* **1611**, 129 (2016) [arXiv:1609.02514 [hep-th]] [Search INSPIRE].
- [37] D. Carmi, R. C. Myers, and P. Rath, *J. High Energy Phys.* **1703**, 118 (2017) [arXiv:1612.00433 [hep-th]] [Search INSPIRE].
- [38] W.-C. Gan and F.-W. Shu, *Phys. Rev. D* **96**, 026008 (2017) [arXiv:1702.07471 [hep-th]] [Search INSPIRE].
- [39] R. Abt, J. Erdmenger, H. Hinrichsen, C. M. Melby-Thompson, R. Meyer, C. Northe, and I. A. Reyes, *Fortsch. Phys.* **66**, 1800034 (2018) [arXiv:1710.01327 [hep-th]] [Search INSPIRE].
- [40] A. Bhattacharya and S. Roy, arXiv:1807.06361 [hep-th] [Search INSPIRE].
- [41] S. A. Hosseini Mansoori, V. Jahnke, M. M. Qaemmaqami, and Y. D. Olivas, arXiv:1808.00067 [hep-th] [Search INSPIRE].
- [42] D. Momeni, S. A. H. Mansoori, and R. Myrzakulov, *Phys. Lett. B* **756**, 354 (2016) [arXiv:1601.03011 [hep-th]] [Search INSPIRE].
- [43] X. Gao, M. Kaminski, H.-B. Zeng, and H.-Q. Zhang, *J. High Energy Phys.* **1211**, 112 (2012) [arXiv:1204.3103 [hep-th]] [Search INSPIRE].
- [44] Y. Bu, *Phys. Rev. D* **86**, 106005 (2012) [arXiv:1205.1614 [hep-th]] [Search INSPIRE].
- [45] Y. Peng and G. Liu, *Int. J. Mod. Phys. A* **32**, 1750160 (2017).
- [46] D. Anninos, S. A. Hartnoll, and N. Iqbal, *Phys. Rev. D* **82**, 066008 (2010) [arXiv:1005.1973 [hep-th]] [Search INSPIRE].
- [47] E. Bakhshaei, A. Mollabashi, and A. Shirzad, *Eur. Phys. J. C* **77**, 665 (2017) [arXiv:1703.03469 [hep-th]] [Search INSPIRE].
- [48] S. S. Gubser and S. S. Pufu, *J. High Energy Phys.* **0811**, 033 (2008) [arXiv:0805.2960 [hep-th]] [Search INSPIRE].
- [49] M. Ammon, J. Erdmenger, V. Grass, P. Kerner, and A. O'Bannon, *Phys. Lett. B* **686**, 192 (2010) [arXiv:0912.3515 [hep-th]] [Search INSPIRE].
- [50] M. Cadoni and C. Monni, *Phys. Rev. D* **80**, 024034 (2009) [arXiv:0905.3517 [gr-qc]] [Search INSPIRE].
- [51] K. Jensen, *J. High Energy Phys.* **1101**, 109 (2011) [arXiv:1012.4831 [hep-th]] [Search INSPIRE].
- [52] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, *Phys. Rev. Lett.* **101**, 031601 (2008) [arXiv:0803.3295 [hep-th]] [Search INSPIRE].
- [53] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, *J. High Energy Phys.* **0812**, 015 (2008) [arXiv:0810.1563 [hep-th]] [Search INSPIRE].
- [54] A. Belin, L.-Y. Hung, A. Maloney, S. Matsuura, R. C. Myers, and T. Sierens, *J. High Energy Phys.* **1312**, 059 (2013) [arXiv:1310.4180 [hep-th]] [Search INSPIRE].
- [55] S. He, J.-R. Sun, and H.-Q. Zhang, *Nucl. Phys. B* **928**, 160 (2018) [arXiv:1411.6213 [hep-th]] [Search INSPIRE].
- [56] D. Momeni, H. Gholizade, M. Raza, and R. Myrzakulov, *Phys. Lett. B* **747**, 417 (2015) [arXiv:1503.02896 [hep-th]] [Search INSPIRE].
- [57] T. Azeyanagi, T. Nishioka, and T. Takayanagi, *Phys. Rev. D* **77**, 064005 (2008) [arXiv:0710.2956 [hep-th]] [Search INSPIRE].
- [58] H. Casini and M. Huerta, *Phys. Lett. B* **600**, 142 (2004) [arXiv:hep-th/0405111] [Search INSPIRE].
- [59] H. Casini and M. Huerta, *J. Phys. A: Math. Theor.* **40**, 7031 (2007) [arXiv:cond-mat/0610375] [cond-mat.stat-mech] [Search INSPIRE].
- [60] H. Liu and M. Mezei, *J. High Energy Phys.* **1304**, 162 (2013) [arXiv:1202.2070 [hep-th]] [Search INSPIRE].
- [61] J. Ren, *J. High Energy Phys.* **1011**, 055 (2010) [arXiv:1008.3904 [hep-th]] [Search INSPIRE].
- [62] Y. Liu, Q. Pan, and B. Wang, *Phys. Lett. B* **702**, 94 (2011) [arXiv:1106.4353 [hep-th]] [Search INSPIRE].

- [63] M. Henningson and K. Skenderis, J. High Energy Phys. **9807**, 023 (1998) [arXiv:hep-th/9806087] [Search INSPIRE].
- [64] S. de Haro, K. Skenderis, and S. N. Solodukhin, Commun. Math. Phys. **217**, 595 (2001) [arXiv:hep-th/0002230] [Search INSPIRE].
- [65] A. Karch, A. O’Bannon, and K. Skenderis, J. High Energy Phys. **0604**, 015 (2006) [arXiv:hep-th/0512125] [Search INSPIRE].