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Spinning vortex braneworld

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A spinning vortex is considered in the context of the braneworld. We numerically analyze the profiles of a stationary solution in a 6D U(1) gauge theory, and clarify their dependence on the angular velocity in the field space ω. We find that there is an upper limit on ω, and the vortex configuration should be parametrized by the angular momentum rather than ω. We also discuss matter modes localized on the vortex. We show that the vortex spin mixes the Kaluza–Klein (KK) masses and induces nonvanishing masses to the zero-modes. It also resolves the degeneracy in the KK spectrum that the static vortex had.

Subject Index B02, B06, B33, B43

1. Introduction

The braneworld scenario is interesting both from the phenomenological and the string-theoretical points of view, and has been extensively investigated in a vast amount of papers. However, many of them only discuss static brane configurations. This is mainly because such configurations are much easier to analyze, and moving branes generically violate the Lorentz symmetry on the branes. If we focus on a local region near the earth in the present universe, it may be a good approximation. However, when we discuss the cosmological evolution of the universe, we should take into account the brane motions in the past.

In the braneworld scenario, it is natural to imagine that branes were actively moving and colliding with each other in the early stage of the universe. Such motions become slower as the universe expands, and eventually the branes approach static configurations. However, brane motions in the past may affect the cosmological history since they generally lead to various symmetry breakings including the Lorentz violation. In addition, the brane collision process can induce inflation [1,2]. Hence it is quite important to understand how such brane motions affect the 4D effective theory.

There are various kinds of brane motions, such as translation, rotation, collision and merger of branes, and so on. In particular, when a brane is a field-theoretical soliton rather than the D-branes in string theory, it has a finite width. In such a case, deformations and spin of the brane are also possible. It is a nontrivial and intriguing subject to investigate how such brane motions affect the evolution of the 4D spacetime on the brane. This is the motivation for our work.

The simplest setup for the braneworld scenario is a 5D theory. The moving branes with codimension-one in 5D are discussed in Refs. [3–7], and it has been shown that their motions affect the evolution of our 4D spacetime significantly. Here we will consider the next simplest case, i.e., the codimension-two case. In this case, a rotation of the branes in the extra-dimensional space

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becomes possible.[1] Specifically, we focus on a vortex soliton in 6D theories. In this paper, as a first step for our purpose, we study a spinning vortex[2] in a 6D non-gravitational theory. Namely, we assume that the typical energy scale of the vortex is much smaller than the 6D Planck mass, and all the gravitational effects are negligible. The spin of the soliton may be understood as a trail of a grazing collision that the brane has experienced in the past. To simplify the discussion, we focus on a stationary field configuration.[3] We also discuss the localized modes of the matter fields on the vortex, and the violation of the 4D Lorentz symmetry that they feel.

The paper is organized as follows. In the next section, we briefly review the ANO vortex in the 6D Abelian–Higgs model, and discuss the possibility of its rotation. In Sect. 3, we extend the model to obtain a stationary solution for a spinning vortex. The profiles of the vortex background are numerically calculated, and their dependence on the angular velocity in the field space is clarified. In Sect. 4, we introduce a scalar matter field and discuss its Kaluza–Klein (KK) mass spectrum. In Sect. 5, we introduce the matter fermions and discuss how they are expanded into the KK modes in the presence of the spinning vortex background. We also comment on the violation of the 4D Lorentz symmetry in the 4D effective theory. Section 6 is devoted to the summary.

2. Case of the ANO vortex

First we consider the Abrikosov–Nielsen–Olesen (ANO) vortex [13,14]. The theory is the 6D Abelian–Higgs model whose Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F_{MN}F^{MN} - \mathcal{D}_M\phi^*\mathcal{D}^M\phi - \frac{\lambda}{2}\left(|\phi|^2 - v^2\right), \tag{2.1}$$

where $M, N = 0, 1, \ldots, 5$, and

$$\begin{aligned} F_{MN} &\equiv \partial_M A_N - \partial_N A_M, \\ \mathcal{D}_M\phi &\equiv (\partial_M - igA_M)\,\phi. \end{aligned} \tag{2.2}$$

The parameters λ and v are chosen to be positive. Since the scalar ϕ and the gauge field A_M have mass dimension 2 in 6D, the dimensions of the parameters are given by

$$[g] = -1, \qquad [\lambda] = -2, \qquad [v] = 2. \tag{2.3}$$

The equations of motion are

$$\begin{aligned} \partial_M F^{MN} - 2g\,\mathrm{Im}\left(\mathcal{D}^N\phi^*\phi\right) &= 0, \\ \mathcal{D}_M\mathcal{D}^M\phi - \lambda\phi\left(|\phi|^2 - v^2\right) &= 0. \end{aligned} \tag{2.4}$$

2.1. Static background

The ANO vortex is obtained by imposing the background ansatz,

$$\phi = vf(v^{1/2}r)e^{in\theta}, \qquad A_\theta = \frac{n\alpha(v^{1/2}r)}{g}, \qquad A_{M\neq\theta} = 0, \tag{2.5}$$

---

[1] Rotation of the D-branes in a compact space is discussed in Refs. [8–11].

[2] As another example of spinning codimension-two objects, a rotating hollow cylinder constructed by a domain wall is discussed in Ref. [12] in a 6D gravitational theory. In this case, the spin is necessary to stabilize the configuration against collapse due to the tension of the domain wall.

[3] In this paper, we only consider a classical motion. Our solution might no longer be stationary when quantum effects are taken into account.

where (r, θ) are the polar coordinates for the extra dimensions,

$$x^4 = r\cos\theta, \qquad x^5 = r\sin\theta, \tag{2.6}$$

and the integer n is the vortex number. The Hamiltonian density for this background is given by

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}\sum_{M=1}^{5} F_{0M}^2 + \frac{1}{4}\sum_{M,N=1}^{5} F_{MN}^2 + \sum_{M=0}^{5} |\mathcal{D}_M\phi|^2 + \frac{\lambda}{2}\left(|\phi|^2 - v^2\right)^2 \\ &= \frac{n^2 v\alpha'^2}{2g^2r^2} + v^3 f'^2 + \frac{n^2v^2}{r^2}(1-\alpha)^2 f^2 + \frac{\lambda v^4}{2}\left(f^2-1\right)^2.\end{aligned} \tag{2.7}$$

Thus, in order to have a finite vortex tension (i.e., 4D vacuum energy density) $\tau_3 = 2\pi\int_0^\infty dr\, r\mathcal{H}$, the dimensionless functions f and α should satisfy

$$\lim_{r\to\infty} f(v^{1/2}r) = \lim_{r\to\infty}\alpha(v^{1/2}r) = 1. \tag{2.8}$$

Besides, the regularity of the fields at the origin requires

$$f(0) = \alpha(0) = 0. \tag{2.9}$$

With these boundary conditions, we obtain the (static) vortex solution by solving the equations of motion (2.4).

2.2. *Background ansatz for a spinning vortex*

In order to search for a spinning vortex solution, we extend the background ansatz (2.5) as

$$\begin{aligned}\phi &= vf(v^{1/2}r)e^{in\theta + i\omega t}, \qquad A_0 = \frac{\omega\beta(v^{1/2}r)}{g}, \\ A_\theta &= \frac{n\alpha(v^{1/2}r)}{g}, \qquad A_{M\neq 0,\theta} = 0.\end{aligned} \tag{2.10}$$

In this case, the Hamiltonian density (2.7) becomes

$$\begin{aligned}\mathcal{H} = &\frac{1}{2}\left(\frac{\omega^2 v\beta'^2}{g^2} + \frac{n^2v\alpha'^2}{g^2r^2}\right) + \omega^2v^2(1-\beta)^2f^2 + v^3f'^2 + \frac{n^2v^2}{r^2}(1-\alpha)^2f^2 \\ &+ \frac{\lambda v^4}{2}\left(f^2-1\right)^2.\end{aligned} \tag{2.11}$$

Thus, from the condition that the vortex tension τ_3 should be finite and the regularity at the origin, the dimensionless functions f, α, and β must satisfy the following boundary conditions:

$$\begin{aligned}&\lim_{r\to\infty} f(v^{1/2}r) = \lim_{r\to\infty}\alpha(v^{1/2}r) = \lim_{r\to\infty}\beta(v^{1/2}r) = 1, \\ &f(0) = \alpha(0) = \beta'(0) = 0.\end{aligned} \tag{2.12}$$

Under the ansatz (2.10), the equations of motion (2.4) become

$$
\begin{aligned}
&f''(\rho)+\frac{f'(\rho)}{\rho}+\left\{\tilde{\omega}^2\left(1-\beta(\rho)\right)-\frac{n^2}{\rho^2}\left(1-\alpha(\rho)\right)^2-\tilde{\lambda}\left(f^2(\rho)-1\right)\right\}f(\rho)=0,\\
&\alpha''(\rho)-\frac{\alpha'(\rho)}{\rho}+2\tilde{g}^2\left(1-\alpha(\rho)\right)f^2(\rho)=0,\\
&\beta''(\rho)+\frac{\beta'(\rho)}{\rho}+2\tilde{g}^2\left(1-\beta(\rho)\right)f^2(\rho)=0,
\end{aligned}
\tag{2.13}
$$

where $\rho \equiv v^{1/2}r$ is a dimensionless radial coordinate and

$$
\tilde{\omega}\equiv\frac{\omega}{v^{1/2}},\qquad \tilde{\lambda}\equiv\lambda v,\qquad \tilde{g}\equiv g v^{1/2}
\tag{2.14}
$$

are dimensionless parameters.

In fact, Eq. (2.13) does not have a solution that satisfies the boundary conditions in Eq. (2.12). Let us focus on a region $\rho \gg 1$. There, the second term of the equation for β is neglected and $f(\rho) \simeq 1$. Thus the solution behaves as

$$
\beta(\rho)\simeq 1+Ce^{-\sqrt{2}\tilde{g}\rho},
\tag{2.15}
$$

where C is a real constant. When $C > 0$, we find that $\beta''(\rho) > 0$ and $\beta'(\rho) < 0$ for $\rho \gg 1$. Therefore,

$$
\beta''(\rho)=2\tilde{g}^2\left(1-\beta(\rho)\right)f^2(\rho)-\frac{\beta'(\rho)}{\rho}
\tag{2.16}
$$

is positive for all regions because the second term on the right-hand side, which is positive, gets bigger and bigger as we approach the origin while the contribution of the first term, which is negative, decreases. This indicates that $\beta'(\rho)$ is always negative for any value of ρ. For a similar reason, $\beta'(\rho)$ is always positive when $C < 0$. In both cases, we cannot satisfy the boundary conditions at the origin.[4] The only possible case is $C = 0$. In this case, $\beta(\rho) = 1$ is a solution that satisfies the boundary conditions in Eq. (2.12). However, this solution is gauge-equivalent to the static solution in Sect. 2.1. Namely, a stationary spinning vortex solution does not exist in this model.

This fact is expected from the following reason.[5] The vacuum of this model is

$$
\phi = ve^{i\delta},\qquad A_M=0,
\tag{2.17}
$$

where δ is a real constant. The fluctuation modes around this vacuum are as follows. The gauge boson gets a nonzero mass via the Higgs mechanism for the breaking of the U(1) gauge symmetry. The scalar field ϕ is decomposed as $\phi = (\varphi + v)\, e^{i(\delta+\chi)}$. The phase part χ is the would-be Nambu–Goldstone (NG) boson and is absorbed by the gauge boson, and the radial part φ gets a mass from the potential. Namely, no massless modes exist in the vacuum. This indicates that nonzero energy is necessary when we move the vortex in any direction. So we cannot rotate the vortex without an energy cost. This is the reason why there is no stationary spinning vortex solution in this model.

[4] In fact, $\lim_{\rho\to 0}\beta'(\rho) = -\mathrm{sign}\,(C)\infty$ in these cases.

[5] The author thanks Keisuke Ohashi for suggesting this perspective.

According to the above perspective, we need a massless mode corresponding to the fluctuation along the phase direction in order to have a stationary spinning vortex. In the next section, we will extend the model in such a way.

3. Stationary spinning vortex

3.1. Setup

We extend the previous model by adding an extra charged scalar field. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F_{MN}F^{MN} - \sum_{i=1,2}\mathcal{D}_M\phi_i^*\mathcal{D}^M\phi_i - U,$$
$$U = \sum_{i=1,2}\frac{\lambda_i}{2}\left(|\phi_i|^2 - v_i^2\right)^2 + \gamma\,|\phi_1|^2\,|\phi_2|^2 + U_0, \tag{3.1}$$

where λ_1, λ_2, v_1, v_2, and γ are positive constants, and

$$\mathcal{D}_M\phi_i = (\partial_M - igA_M)\,\phi_i. \tag{3.2}$$

The constant U_0 is irrelevant to the physics if we neglect the gravity.

The mass dimensions of the parameters are

$$[g] = -1, \qquad [\lambda_1] = [\lambda_2] = [\gamma] = -2, \qquad [v_1] = [v_2] = 2. \tag{3.3}$$

In addition to the U(1) gauge symmetry, the model has U(1) global symmetry, which is denoted as $\mathrm{U(1)_{gl}}$, under the transformation that changes the relative phase of ϕ_1 and ϕ_2.

The vacuum structure of this model is summarized in Appendix A. In the following, we will focus on the case that

$$\gamma v_1^2 > \lambda_2 v_2^2, \qquad \lambda_1 v_1^4 > \lambda_2 v_2^4. \tag{3.4}$$

Then the vacuum (i.e., the global minimum of U) is

$$|\phi_1| = v_1, \qquad \phi_2 = 0. \tag{3.5}$$

We will set the constant U_0 so that the vacuum energy is zero in the following. Namely,

$$U_0 = -\frac{\lambda_2 v_2^4}{2}. \tag{3.6}$$

The equations of motion are

$$\partial_M F^{MN} - 2g\mathrm{Im}\left(\mathcal{D}^N\phi_1^*\phi_1 + \mathcal{D}^N\phi_2^*\phi_2\right) = 0,$$
$$\mathcal{D}_M\mathcal{D}^M\phi_1 - \lambda_1\phi_1\left(|\phi_1|^2 - v_1^2\right) - \gamma\phi_1\,|\phi_2|^2 = 0,$$
$$\mathcal{D}_M\mathcal{D}^M\phi_2 - \lambda_2\phi_2\left(|\phi_2|^2 - v_2^2\right) - \gamma\,|\phi_1|^2\,\phi_2 = 0. \tag{3.7}$$

This theory has the (static) ANO vortex solution,

$$\phi_1 = v_1 f(v_1^{1/2}r), \qquad \phi_2 = 0,$$
$$A_\theta = \frac{n\alpha(v_1^{1/2}r)}{g}, \qquad A_{M\neq\theta} = 0. \tag{3.8}$$

3.2. Background ansatz for the spinning vortex

For the purpose of finding an axially symmetric stationary spinning vortex solution, we make the following ansatz for the background:[6]

$$\begin{aligned} \phi_1 &= v_1 f_1(v_1^{1/2} r) e^{in\theta}, \qquad \phi_2 = v_2 f_2(v_1^{1/2} r) e^{i\omega t}, \\ A_0 &= \frac{\omega\beta(v_1^{1/2} r)}{g}, \qquad A_\theta = \frac{n\alpha(v_1^{1/2} r)}{g}, \qquad A_{M\neq 0,\theta} = 0, \end{aligned} \tag{3.9}$$

where $f_{1,2}$, α, and β are dimensionless real functions, the integer n is the vortex number, and the real constant ω is the angular velocity.

Then the Hamiltonian density is

$$\begin{aligned} \mathcal{H} = {} & \frac{v_1}{2g^2}\left(\omega^2\beta'^2 + \frac{n^2\alpha'^2}{r^2}\right) + v_1^2\left\{\omega^2\beta^2 f_1^2 + v_1 f_1'^2 + \frac{n^2(1-\alpha)^2}{r^2} f_1^2\right\} \\ & + v_2^2\left\{\omega^2(1-\beta)^2 f_2^2 + v_1 f_2'^2 + \frac{n^2\alpha^2}{r^2} f_2^2\right\} \\ & + \frac{\lambda_1 v_1^4}{2}\left(f_1^2 - 1\right)^2 + \frac{\lambda_2 v_2^4}{2} f_2^2\left(f_2^2 - 2\right) + \gamma v_1^2 v_2^2 f_1^2 f_2^2. \end{aligned} \tag{3.10}$$

In order to have a finite vortex tension, we should require the boundary conditions at infinity:

$$\begin{aligned} \lim_{r\to\infty} f_1(v_1^{1/2} r) &= \lim_{r\to\infty} \alpha(v_1^{1/2} r) = 1, \\ \lim_{r\to\infty} f_2(v_1^{1/2} r) &= \lim_{r\to\infty} \beta(v_1^{1/2} r) = 0. \end{aligned} \tag{3.11}$$

From the regularity at the vortex core, we obtain the boundary conditions at the origin:

$$f_1(0) = \alpha(0) = 0, \qquad f_2'(0) = \beta'(0) = 0. \tag{3.12}$$

With our ansatz, the equations of motion in Eq. (3.7) are translated into the equations for the dimensionless functions as

$$\begin{aligned} f_1'' + \frac{f_1'}{\rho} + \left\{\tilde\omega^2\beta^2 - \frac{n^2}{\rho^2}(1-\alpha)^2 - \tilde\lambda_1\left(f_1^2 - 1\right) - \tilde\gamma\xi f_2^2\right\} f_1 = 0, \\ f_2'' + \frac{f_2'}{\rho} + \left\{\tilde\omega^2(1-\beta)^2 - \frac{n^2}{\rho^2}\alpha^2 - \tilde\lambda_2\xi\left(f_2^2 - 1\right) - \tilde\gamma f_1^2\right\} f_2 = 0, \\ \alpha'' - \frac{\alpha'}{\rho} + 2\tilde g^2\left\{(1-\alpha) f_1^2 - \xi\alpha f_2^2\right\} = 0, \\ \beta'' + \frac{\beta'}{\rho} - 2\tilde g^2\left\{\beta f_1^2 - \xi(1-\beta) f_2^2\right\} = 0, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} &\rho \equiv v_1^{1/2} r, \qquad \tilde\lambda_1 \equiv \lambda_1 v_1, \qquad \tilde\lambda_2 \equiv \lambda_2 v_1, \qquad \tilde\gamma \equiv \gamma v_1, \\ &\tilde g \equiv g v_1^{1/2}, \qquad \tilde\omega \equiv \frac{\omega}{v_1^{1/2}}, \qquad \xi \equiv \frac{v_2^2}{v_1^2} \end{aligned} \tag{3.14}$$

are dimensionless coordinate and parameters.

[6] In the following, we will normalize the dimensionful quantities except for f_2 by v_1.

Due to the $\mathrm{U}(1)_{\mathrm{gl}}$, this model has a massless mode corresponding to the fluctuation changing the relative phase between ϕ_1 and ϕ_2 at every spacetime point. Thus it is expected for the above equations to have a solution that satisfies the boundary conditions (3.11) and (3.12), in contrast to the previous model.

3.3. *Asymptotic behaviors of the solution*

From the equations in Eq. (3.13) with the boundary conditions (3.11) and (3.12), we can read off the asymptotic behaviors of the dimensionless functions. In a region $\rho \ll 1$, they behave as

$$\begin{aligned} f_1(\rho) &= C^0_{f_1}\rho^{|n|}\left\{1+\mathcal{O}(\rho^2)\right\}, \\ f_2(\rho) &= C^0_{f_2}+\mathcal{O}(\rho^2), \\ \alpha(\rho) &= C^0_\alpha\rho^2+\mathcal{O}(\rho^4), \\ \beta(\rho) &= C^0_\beta+\mathcal{O}(\rho^2), \end{aligned} \tag{3.15}$$

where $C^0_{f_1}$, $C^0_{f_2}$, C^0_α, and C^0_β are real constants.

Next we consider a region $\rho \gg 1$. Then, using Eq. (3.11), Eq. (3.13) is reduced to

$$\begin{aligned} \hat{f}_1''+\frac{\hat{f}_1'}{\rho}-\left(\tilde{\omega}^2\beta^2-\frac{n^2}{\rho^2}\hat{\alpha}^2+2\tilde{\lambda}_1\hat{f}_1-\tilde{\gamma}\xi f_2^2\right) &\simeq 0, \\ f_2''+\frac{f_2'}{\rho}+\left(\tilde{\omega}^2-\frac{n^2}{\rho^2}+\tilde{\lambda}_2\xi-\tilde{\gamma}\right)f_2 &\simeq 0, \\ \hat{\alpha}''-\frac{\hat{\alpha}'}{\rho}-2\tilde{g}^2\left(\hat{\alpha}-\xi f_2^2\right) &\simeq 0, \\ \beta''+\frac{\beta'}{\rho}-2\tilde{g}^2\left(\beta-\xi f_2^2\right) &\simeq 0, \end{aligned} \tag{3.16}$$

where

$$\hat{f}_1(\rho) \equiv 1-f_1(\rho), \qquad \hat{\alpha}(\rho) \equiv 1-\alpha(\rho). \tag{3.17}$$

The solution of the second equation is expressed by the (modified) Bessel function as

$$f_2(\rho) \simeq \begin{cases} K_n(\sqrt{a_2}\rho) & (\tilde{\omega}^2<\tilde{\gamma}-\tilde{\lambda}_2\xi) \\ J_n(\sqrt{|a_2|}\rho),\ Y_n(\sqrt{|a_2|}\rho) & (\tilde{\omega}^2>\tilde{\gamma}-\tilde{\lambda}_2\xi) \end{cases}, \tag{3.18}$$

up to the normalization factor, where

$$a_2 \equiv \tilde{\gamma}-\tilde{\lambda}_2\xi-\tilde{\omega}^2. \tag{3.19}$$

Namely, when $a_2>0$, it behaves as

$$f_2(\rho) \simeq \frac{C^\infty_{f_2}}{\sqrt{\rho}}e^{-\sqrt{a_2}\rho}, \tag{3.20}$$

where $C_{f_2}^{\infty}$ is a positive constant. Using this and the last two equations in Eq. (3.16), we find that

$$\alpha(\rho) \simeq \begin{cases} 1 - C_{\alpha}^{\infty}\sqrt{\rho}e^{-\sqrt{2}\tilde{g}\rho} & (\tilde{g}^2 < 2a_2) \\ 1 - \xi\left(C_{f_2}^{\infty}\right)^2 \dfrac{e^{-2\sqrt{a_2}\rho}}{\rho} & (\tilde{g}^2 > 2a_2) \end{cases},$$

$$\beta(\rho) \simeq \begin{cases} \dfrac{C_{\beta}^{\infty}}{\sqrt{\rho}}e^{-\sqrt{2}\tilde{g}\rho} & (\tilde{g}^2 < 2a_2) \\ \xi\left(C_{f_2}^{\infty}\right)^2 \dfrac{e^{-2\sqrt{a_2}\rho}}{\rho} & (\tilde{g}^2 > 2a_2) \end{cases}, \tag{3.21}$$

where C_{α}^{∞} and C_{β}^{∞} are real constants. Then, from the first equation in Eq. (3.16) with the above asymptotic forms, we obtain

$$f_1(\rho) \simeq \begin{cases} 1 - \dfrac{C_{f_1}^{\infty}}{\sqrt{\rho}}e^{-\sqrt{2\tilde{\lambda}_1}\rho} & \left(\tilde{\lambda}_1 < \min(4\tilde{g}^2, 2a_2)\right) \\ 1 - \dfrac{n^2(C_{\alpha}^{\infty})^2 - \tilde{\omega}^2(C_{\beta}^{\infty})^2}{2\tilde{\lambda}_1}\dfrac{e^{-2\sqrt{2}\tilde{g}\rho}}{\rho} & \left(4\tilde{g}^2 < \min(\tilde{\lambda}_1, 2a_2)\right) \\ 1 - \dfrac{\tilde{\gamma}\xi}{2\tilde{\lambda}_1}\dfrac{\left(C_{f_2}^{\infty}\right)^2}{\rho}e^{-2\sqrt{a_2}\rho} & \left(2a_2 < \min(\tilde{\lambda}_1, 4\tilde{g}^2)\right) \end{cases}, \tag{3.22}$$

where $C_{f_1}^{\infty}$ is a real constant.

Thus, when a_2 is small enough, the Hamiltonian density (3.10) is approximated as

$$\begin{aligned} \mathcal{H} = v_1^3 \Bigg[& \frac{1}{2\tilde{g}^2}\left(\tilde{\omega}^2\beta'^2 + \frac{n^2\alpha'^2}{\rho^2}\right) + \left\{\tilde{\omega}^2\beta^2 f_1^2 + f_1'^2 + \frac{n^2(1-\alpha)^2}{\rho^2}f_1^2\right\} \\ & + \xi\left\{\tilde{\omega}^2(1-\beta)^2 f_2^2 + f_2'^2 + \frac{n^2\alpha^2}{\rho^2}f_2^2\right\} \\ & + \frac{\tilde{\lambda}_1}{2}\left(f_1^2 - 1\right)^2 + \frac{\tilde{\lambda}_2\xi^2}{2}f_2^2\left(f_2^2 - 2\right) + \tilde{\gamma}\xi f_1^2 f_2^2 \Bigg] \\ \simeq v_1^3 & \left(\xi\tilde{\omega}^2 f_2^2 - \tilde{\lambda}_2\xi^2 f_2^2 + \tilde{\gamma}\xi f_2^2\right) \\ \simeq v_1^3 & \xi\left(C_{f_2}^{\infty}\right)^2\left(\tilde{\omega}^2 - \tilde{\lambda}_2\xi + \tilde{\gamma}\right)\frac{e^{-2\sqrt{a_2}\rho}}{\rho}, \end{aligned} \tag{3.23}$$

for $\rho \gg 1$. Therefore, the vortex tension $\tau_3 \equiv 2\pi\int_0^{\infty} d\rho\, \rho\mathcal{H}$ diverges when $a_2 \simeq 0$. This indicates that there is a maximum value of the (normalized) angular velocity $\tilde{\omega}$:[7]

$$\tilde{\omega}_{\max} \simeq \sqrt{\tilde{\gamma} - \tilde{\lambda}_2\xi}. \tag{3.24}$$

3.4. *Profiles of the solution*

The equations in Eq. (3.13) with the boundary conditions (3.11) and (3.12) can be solved numerically. Figure 1 shows the profiles of the solution. The solid, the dot-dashed, the dashed, and the dotted lines

[7] Notice that the constant $C_{f_2}^{\infty}$ depends on $\tilde{\omega}$. Thus, the actual divergent value of $\tilde{\omega}$ slightly deviates from Eq. (3.24). (See the next subsection.)

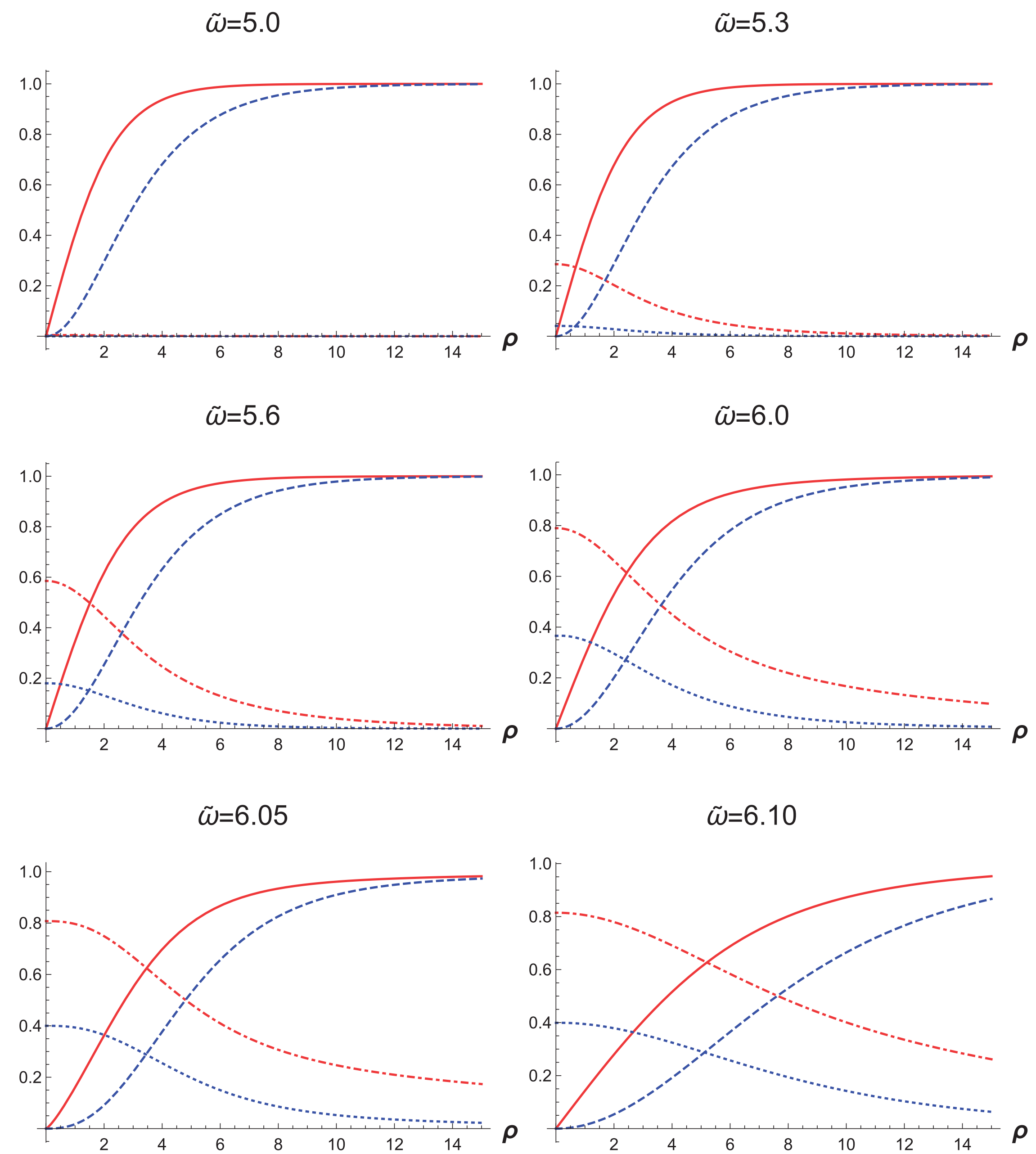

Fig. 1. The profiles of the dimensionless functions. The solid, the dot-dashed, the dashed and the dotted lines represent $f_1(\rho), f_2(\rho), \alpha(\rho)$, and $\beta(\rho)$, respectively. The parameters are chosen as $n = 1$, $\tilde{g} = 0.4$, $\tilde{\lambda}_1 = 0.3$, $\tilde{\lambda}_2 = 0.2$, $\tilde{\gamma} = 0.5$, and $\xi = 0.7$.

represent $f_1(\rho), f_2(\rho), \alpha(\rho)$ and $\beta(\rho)$, respectively. The parameters are chosen as $n = 1$, $\tilde{g} = 0.4$, $\tilde{\lambda}_1 = 0.3$, $\tilde{\lambda}_2 = 0.2$, $\tilde{\gamma} = 0.5$, and $\xi = 0.7$. For $\tilde{\omega} \lesssim 0.5$, $f_2(\rho)$ and $\beta(\rho)$ are exponentially small, and the background is almost that of the ANO vortex (3.8). For $0.5 \lesssim \tilde{\omega} \lesssim 0.61$, $f_2(\rho)$ and $\beta(\rho)$ grow as $\tilde{\omega}$ increases, and the profiles of $f_1(\rho)$ and $\alpha(\rho)$ are deformed due to the centrifugal force induced by the spin of the vortex. For $\tilde{\omega} > 0.61$, the functions do not decay enough in the region of $\rho \gg 1$, and cannot satisfy the boundary condition (3.11).

These behaviors of the functions can be understood by noticing that the centrifugal force is proportional to the angular momentum P_θ of the vortex, rather than the angular velocity in the field

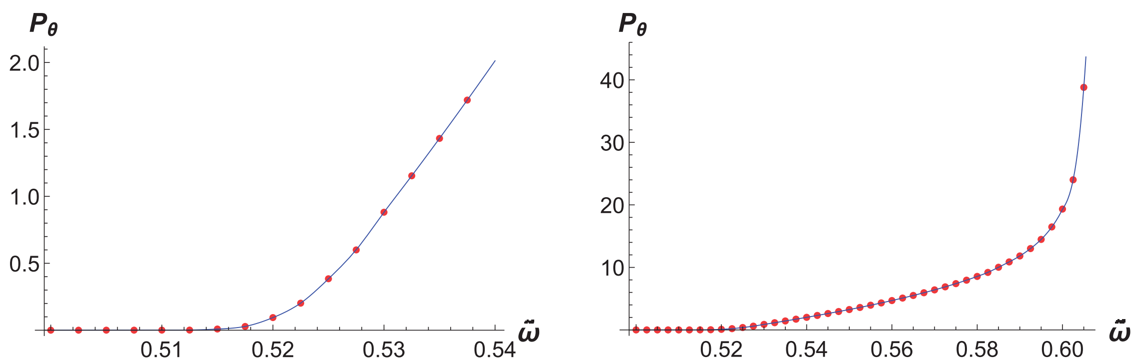

Fig. 2. The angular momentum P_θ as a function of $\tilde{\omega}$. The parameter choice is the same as that of Fig. 1. P_θ is normalized by $v_1^{5/2}$.

space ω. The angular momentum P_θ is given by

$$\begin{aligned} P_\theta &\equiv \int dx^4 dx^5\, \mathcal{P}_\theta = 2\pi \int_0^\infty d\rho\, \rho \mathcal{P}_\theta(\rho), \\ \mathcal{P}_\theta &\equiv -\left\{\mathcal{D}_\theta \phi_1^* \mathcal{D}^0 \phi_1 + \mathcal{D}_\theta \phi_2^* \mathcal{D}^0 \phi_2 + \text{h.c.}\right\} - F_{\theta r} F^{0r} \\ &= -n\tilde{\omega} v_1^{5/2} \left[\{1-\alpha(\rho)\}\, \beta(\rho) f_1^2(\rho) + \alpha(\rho)\, \{1-\beta(\rho)\} f_2^2(\rho) - \frac{\alpha'(\rho)\beta'(\rho)}{\tilde{g}^2}\right], \end{aligned} \tag{3.25}$$

where $\mathcal{P}_\theta(\rho)$ is the Noether current for the rotation in the x^4–x^5 plane. Figure 2 shows the relation between P_θ and ω. The overlap integrals in Eq. (3.25) are exponentially small for $\tilde{\omega} \lesssim 0.5$; P_θ increases linearly with $\tilde{\omega}$ in the region $0.52 \lesssim \tilde{\omega} \lesssim 0.6$, and it diverges at some value around $\tilde{\omega} = 0.61$. In fact, with our parameter choice, Eq. (3.24) is

$$\tilde{\omega}_{\text{max}} \simeq 0.6. \tag{3.26}$$

These behaviors indicate that we should parametrize the vortex configuration by P_θ rather than by ω.

4. Localized scalar modes

In this section, we introduce an additional scalar field Φ whose U(1) charge is q_Φ as a matter field. Its Lagrangian is given by

$$\mathcal{L}_\text{s} = -\mathcal{D}^M \Phi^* \mathcal{D}_M \Phi - M_\Phi^2 |\Phi|^2 - \left(\kappa_1 |\phi_1|^2 + \kappa_2 |\phi_2|^2\right) |\Phi|^2, \tag{4.1}$$

where $\kappa_{1,2} > 0$, and

$$\mathcal{D}_M \Phi \equiv (\partial_M - i q_\Phi g A_M)\, \Phi. \tag{4.2}$$

The equation of motion for Φ is

$$\mathcal{D}^M \mathcal{D}_M \Phi - M_\Phi^2 \Phi - \left(\kappa_1 |\phi_1|^2 + \kappa_2 |\phi_2|^2\right) \Phi = 0. \tag{4.3}$$

Substituting the background (3.9), the linearized equation of motion is given by

$$
\begin{aligned}
&\partial^\mu \partial_\mu \Phi + 2iq_\Phi \omega \beta(v_1^{1/2} r)\partial_0 \Phi \\
&+ \left\{ \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 - \frac{2iq_\Phi n\alpha(v_1^{1/2} r)}{r^2}\partial_\theta + q_\Phi^2 \omega^2 \beta^2(v_1^{1/2} r) - \frac{q_\Phi^2 n^2 \alpha^2(v_1^{1/2} r)}{r^2} \right\} \Phi \\
&- \left\{ M_\Phi^2 + \kappa_1 v_1^2 f_1^2(v_1^{1/2} r) + \kappa_2 v_2^2 f_2^2(v_1^{1/2} r) \right\} \Phi = 0.
\end{aligned} \tag{4.4}
$$

4.1. Mode expansion

The 6D scalar field Φ is decomposed into the KK modes as

$$
\Phi(x^\mu, r, \theta) = \sum_K h_\Phi^{(K)}(\rho, \theta)\varphi^{(K)}(x^\mu), \tag{4.5}
$$

where $\rho = v_1^{1/2} r$. We choose the mode functions as solutions of the following mode equation:

$$
\begin{aligned}
&\left\{ \partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\theta^2 - \frac{2iq_\Phi n\alpha(\rho)}{\rho^2}\partial_\theta^2 + q_\Phi^2 \tilde{\omega}^2 \beta^2(\rho) - \frac{q_\Phi^2 n^2 \alpha^2(\rho)}{\rho^2} \right. \\
&\left. - \tilde{M}_\Phi^2 - \tilde{\kappa}_1 f_1^2(\rho) - \tilde{\kappa}_2 \xi f_2^2(\rho) \right\} h_\Phi^{(K)} = -\tilde{m}_K^2 h_\Phi^{(K)},
\end{aligned} \tag{4.6}
$$

where

$$
\tilde{M}_\Phi^2 \equiv \frac{M_\Phi^2}{v_1}, \qquad \tilde{m}_K^2 \equiv \frac{m_K^2}{v_1}, \qquad \tilde{\kappa}_1 \equiv \kappa_1 v_1, \qquad \tilde{\kappa}_2 \equiv \kappa_2 v_1 \tag{4.7}
$$

are dimensionless (m_K is the KK mass).

Since Eq. (4.6) has rotational symmetry in the extra dimensions, the eigenvalues $\tilde{m}_K^2$ in Eq. (4.6) have degeneracy and the mode functions $h_\Phi^{(K)}(\rho, \theta)$ can be expressed as

$$
h_\Phi^{(k)[m]}(\rho, \theta) = b_\Phi^{(k)[m]}(\rho) e^{im\theta}, \tag{4.8}
$$

where m is an integer and labels the degenerate modes. Then, the mode equation becomes

$$
\left\{ -\partial_\rho^2 - \frac{1}{\rho}\partial_\rho + V(\rho) \right\} b_\Phi^{(k)[m]}(\rho) = \tilde{m}_k^2 b_\Phi^{(k)[m]}(\rho), \tag{4.9}
$$

where

$$
V(\rho) \equiv \frac{\{q_\Phi n\alpha(\rho) - m\}^2}{\rho^2} - q_\Phi^2 \tilde{\omega}^2 \beta^2(\rho) + \tilde{M}_\Phi^2 + \tilde{\kappa}_1 f_1^2(\rho) + \tilde{\kappa}_2 \xi f_2^2(\rho). \tag{4.10}
$$

A more explicit derivation of Eqs. (4.8) and (4.9) is given in Appendix B. This has the form of a 1D Schrödinger equation with the potential $V(\rho)$.

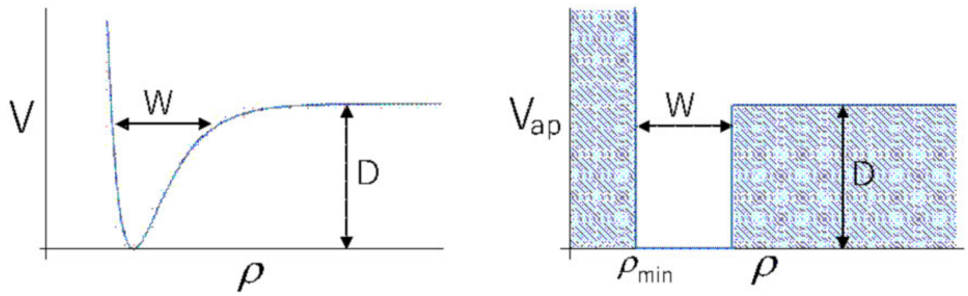

Fig. 3. The typical form of the potential $V(\rho)$ in the static vortex case (left figure), which is approximated by a simple function (right figure).

4.2. KK spectrum

We can obtain the KK mass spectrum by solving Eq. (4.9). However, it depends on many parameters and it is hard to express it analytically. Thus, we illustrate its property by analyzing the Schrödinger equation with the potential approximated by a simple function.

Let us first consider the static vortex case (i.e., $\omega = 0$). The typical form of the potential $V(\rho)$ in this case is shown by the left figure in Fig. 3. In order to see the properties of the spectrum in this system, we approximate $V(\rho)$ by the following simple function $V_{\rm ap}(\rho)$ up to a constant:

$$V_{\rm ap}(\rho) = \begin{cases} \infty & (\rho \le \rho_{\rm min}) \\ -D & (\rho_{\rm min} \le \rho \le \rho_{\rm min} + W) \\ 0 & (\rho > \rho_{\rm min} + W) \end{cases}, \tag{4.11}$$

where the constants D and W denote the depth and width of the potential (see the right figure in Fig. 3). Thus, Eq. (4.9) is approximated by

$$\left\{ -\partial_\rho^2 - \frac{1}{\rho}\partial_\rho + V_{\rm ap}(\rho) \right\} b_\Phi^{(k)[m]}(\rho) = E_{\rm eff} b_\Phi^{(k)[m]}(\rho), \tag{4.12}$$

where $E_{\rm eff} \equiv \tilde{m}_k^2 + C_E$ (C_E: constant). We will concentrate on the bound-state solutions whose eigenvalues $E_{\rm eff}$ satisfy $-D < E_{\rm eff} < 0$. Then, the solution of Eq. (4.12) is

$$b_\Phi^{(k)[m]}(\rho) = N_< \left\{ J_0\left(\sqrt{E_{\rm eff}+D}\rho\right) - \frac{J_0\left(\sqrt{E_{\rm eff}+D}\rho_{\rm min}\right)}{Y_0\left(\sqrt{E_{\rm eff}+D}\rho_{\rm min}\right)} Y_0\left(\sqrt{E_{\rm eff}+D}\rho\right) \right\}, \tag{4.13}$$

for $\rho_{\rm min} \le \rho \le \rho_{\rm min} + W$, and

$$b_\Phi^{(k)[m]}(\rho) = N_> K_0\left(\sqrt{-E_{\rm eff}}\rho\right), \tag{4.14}$$

for $\rho > \rho_{\rm min} + W$. Here, $J_0(z)$ and $Y_0(z)$ are the Bessel functions of the first and second kinds, and $K_0(z)$ is the modified Bessel function of the second kind. The mode function $b_\Phi^{(k)[m]}(\rho)$ and its derivative should be continuous at $\rho = \rho_{\rm min} + W$. In order for these conditions to satisfy with nonvanishing $N_<$ and $N_>$, it must be satisfied that

$$0 = \sqrt{-E_{\rm eff}} F_1(E_{\rm eff}) K_1\left(\sqrt{-E_{\rm eff}}\rho_{\rm b}\right) - \sqrt{E_{\rm eff}+D} F_2(E_{\rm eff}) K_0\left(\sqrt{-E_{\rm eff}}\rho_{\rm b}\right), \tag{4.15}$$

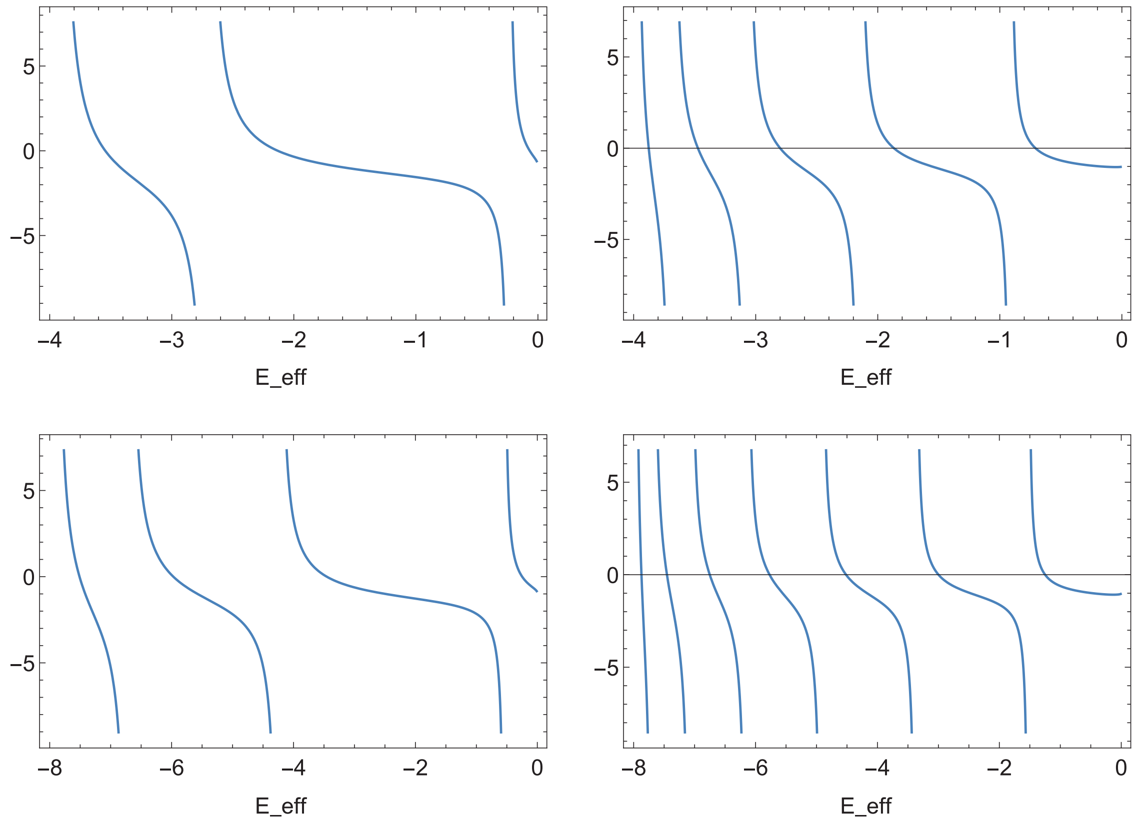

Fig. 4. The plots of $\mathcal{F}(E_{\text{eff}})$ in Eq. (4.17) for $(W, D) = (4, 4)$ (top left), $(W, D) = (8, 4)$ (top right), $(W, D) = (4, 8)$ (bottom left), and $(W, D) = (8, 8)$ (bottom right). We chose ρ_{min} to 1.0 for all plots.

where $\rho_{\text{b}} \equiv \rho_{\text{min}} + W$, and

$$F_1(E_{\text{eff}}) \equiv J_0\left(\sqrt{E_{\text{eff}}+D}\rho_{\text{b}}\right) Y_0\left(\sqrt{E_{\text{eff}}+D}\rho_{\text{min}}\right) - J_0\left(\sqrt{E_{\text{eff}}+D}\rho_{\text{min}}\right) Y_0\left(\sqrt{E_{\text{eff}}+D}\rho_{\text{b}}\right),$$
$$F_2(E_{\text{eff}}) \equiv J_1\left(\sqrt{E_{\text{eff}}+D}\rho_{\text{b}}\right) Y_0\left(\sqrt{E_{\text{eff}}+D}\rho_{\text{min}}\right) - J_0\left(\sqrt{E_{\text{eff}}+D}\rho_{\text{min}}\right) Y_1\left(\sqrt{E_{\text{eff}}+D}\rho_{\text{b}}\right). \quad (4.16)$$

Figure 4 shows the plots of

$$\mathcal{F}(E_{\text{eff}}) \equiv \frac{\sqrt{-E_{\text{eff}}}F_1(E_{\text{eff}})K_1\left(\sqrt{-E_{\text{eff}}}\rho_{\text{b}}\right)}{\sqrt{E_{\text{eff}}+D}F_2(E_{\text{eff}})K_0\left(\sqrt{-E_{\text{eff}}}\rho_{\text{b}}\right)} - 1. \quad (4.17)$$

The points of $\mathcal{F}(E_{\text{eff}}) = 0$ denote the eigenvalues. The plots show the following properties:

- The deeper the potential well of $V_{\text{ap}}(\rho)$ is, the more modes are bounded to the potential.
- The wider the well is, the more densely the eigenvalues are distributed.

Since the eigenvalue of Eq. (4.9) is the square of the (normalized) KK mass, it must be non-negative. In particular, we consider a case that the lowest-mass eigenvalue is zero $\tilde{m}_0^2 = 0$, which can be achieved by tuning the bulk squared mass M_Φ^2 appropriately.[8] This corresponds to the choice that the constant C_E is chosen as the lowest eigenvalue E_0.

We can numerically read off the depth D and the width W of the potential $V(\rho)$ (see Fig. 3). Qualitatively, D is a decreasing function of the penetration length of the vortex, which is read off from Eq. (3.21), and an increasing function of the correlation length, which is read off from Eq. (3.22). The width W is a decreasing function of the penetration length while having a nontrivial dependence on the correlation length.

The spin of the vortex induces nonvanishing $\beta(\rho)$ and $f_2(\rho)$. Both of them have support near the core of the vortex, and affect the shape of the potential $V(\rho)$. However, since the signs of their contributions to the potential in Eq. (4.10) are opposite, their effects on $\rho_{\rm min}$, D, and W depend on the parameters q_Φ, $\tilde{\omega}$, $\tilde{\kappa}_2$, and ξ. In particular, the negative contribution $q_\Phi^2\tilde{\omega}^2\beta^2(\rho)$ in Eq. (4.10) indicates that there can be a negative $\tilde{m}_k^2$ solution, which indicates that the background configuration is unstable. This reflects the fact that the vortex configuration becomes unstable when $\tilde{\omega}$ exceeds the critical value $\tilde{\omega}_{\rm max}$ mentioned in the previous section.

For $E_{\rm eff} > 0$, Eq. (4.12) has a continuous spectrum, which corresponds to unbounded states. Thus, Eq. (4.5) should be understood as

$$\Phi(x^\mu, r, \theta) = \sum_m \left\{ \sum_{k=0}^{k_{\rm max}-1} h_\Phi^{(k)[m]}(\rho,\theta)\varphi^{(k)[m]}(x^\mu) + \int_{\lambda_{\rm min}}^\infty d\lambda\, h_\Phi^{(\lambda)[m]}(\rho)\varphi^{(\lambda)[m]}(x^\mu) \right\}, \tag{4.18}$$

where $k_{\rm max}$ denotes the number of localized modes, and $\lambda_{\rm min} \equiv D - E_0$ is the lowest value of the continuous KK spectrum.

4.3. *Dispersion relations*

Making use of the orthonormal relations of the mode functions, we can rewrite the linearized equation of motion (4.4) as

$$\partial^\mu\partial_\mu\varphi^{(k)[m]} + 2iC_{k,l}^{[m]}\partial_0\varphi^{(l)[m]} - m_k^2\varphi^{(k)[m]} = 0, \tag{4.19}$$

where

$$\begin{aligned} C_{k,l}^{[m]} &\equiv \int d\rho d\theta\ \rho q_\Phi\omega\beta(\rho)h_\Phi^{(k)[m]*}(\rho,\theta)h_\Phi^{(l)[m]}(\rho,\theta) \\ &= 2\pi\int_0^\infty d\rho\ \rho q_\Phi\omega\beta(\rho)b_\Phi^{(k)[m]}(\rho)b_\Phi^{(l)[m]}(\rho). \end{aligned} \tag{4.20}$$

If we move to the momentum basis by the Fourier transformation, this is rewritten as

$$\left\{ \left(E^2 - \vec{p}^2 - m_k^2\right)\delta_{k,l} - 2C_{k,l}^{[m]}E \right\} \tilde{\varphi}^{(l)[m]}(p^\mu) = 0. \tag{4.21}$$

By diagonalizing the matrix on the left-hand side, we obtain the dispersion relation for each KK mode. When the angular momentum of the vortex is small, each element of $C_{k,l}^{[m]}$ is small. Then, the contribution from the off-diagonal elements of $C_{k,l}^{[m]}$ to the eigenvalues of the matrix in Eq. (4.21) is negligible, and the dispersion relation for $\varphi^{(k)[m]}$ is read off as

$$E^2 - \vec{p}^2 - m_k^2 - 2C_{k,k}^{[m]}E \simeq 0. \tag{4.22}$$

[8] Some symmetry, such as supersymmetry, can ensure this parameter tuning.

Thus, the energy is expressed as[9]

$$
\begin{aligned}
E &\simeq C_{k,k}^{[m]} + \sqrt{\vec{p}^2 + m_k^2 + C_{k,k}^{[m]2}} \\
&= C_{k,k}^{[m]} + \sqrt{m_k^2 + C_{k,k}^{[m]2}} + \frac{\vec{p}^2}{2\sqrt{m_k^2 + C_{k,k}^{[m]2}}} + \mathcal{O}(\vec{p}^4).
\end{aligned}
\tag{4.23}
$$

Therefore, we can identify the effective KK masses M_k as

$$
M_k \simeq \sqrt{m_k^2 + C_{k,k}^{[m]2}}. \tag{4.24}
$$

Even if a massless localized mode exists in the static vortex case, it will obtain a nonvanishing mass by the spin of the vortex. Furthermore, we should note that this effective mass depends on the KK label m. This means that the degeneracy in the KK spectrum, which Eq. (4.6) has, is resolved by the spin.

5. Localized fermion modes

In this section, we introduce matter fermions in the bulk, and consider the localized modes on the vortex brane. We introduce 6D Weyl fermions $\Psi_\pm$ whose Lagrangian is given by

$$
\mathcal{L}_{\mathrm{f}} = \sum_{\chi_6=\pm} i\bar{\Psi}_{\chi_6}\Gamma^M \mathcal{D}_M \Psi_{\chi_6} + \left\{ (y_1\phi_1 + y_2\phi_2)\, \bar{\Psi}_- \Psi_+ + \text{h.c.} \right\}, \tag{5.1}
$$

where χ_6 denotes the 6D chirality, and

$$
\mathcal{D}_M \Psi_\pm = (\partial_M - iq_\pm g A_M)\, \Psi_\pm. \tag{5.2}
$$

The constants $q_\pm$ are the U(1) charges of $\Psi_\pm$, respectively. Due to the charge conservation, they are related as

$$
q_+ - q_- + 1 = 0. \tag{5.3}
$$

The coupling constants y_1 and y_2 are chosen to be real and have the mass dimension

$$
[y_1] = [y_2] = -1. \tag{5.4}
$$

The notations for the gamma matrices and the fermions are collected in Appendix C.

5.1. Mode equations

The equations of motion for the fermions are

$$
\begin{aligned}
&i\Gamma^M \mathcal{D}_M \Psi_+ + \left(y_1\bar{\phi}_1 + y_2\bar{\phi}_2\right) \Psi_- = 0, \\
&i\Gamma^M \mathcal{D}_M \Psi_- + (y_1\phi_1 + y_2\phi_2)\, \Psi_+ = 0.
\end{aligned}
\tag{5.5}
$$

[9] The other solution $E \simeq C_{k,k}^{[m]} - \sqrt{\vec{p}^2 + m_k^2 + C_{k,k}^{[m]2}}$ corresponds to the annihilation mode of the anti-particle. We should also note that the energies of the particle and the anti-particle are not degenerate since the Lorentz symmetry is violated in our case.

In the two-component spinor notation, these are rewritten as

$$
\begin{aligned}
i\sigma^\mu\left(\partial_\mu - iq_+gA_\mu\right)\bar{\zeta}_+ - (\partial_4 + i\partial_5)\chi_+ + iq_+g(A_4 + iA_5)\chi_+ \left(y_1\bar{\phi}_1 + y_2\bar{\phi}_2\right)\chi_- = 0,\\
i\bar{\sigma}^\mu\left(\partial_\mu - iq_+gA_\mu\right)\chi_+ + (\partial_4 - i\partial_5)\bar{\zeta}_+ - iq_+g(A_4 - iA_5)\bar{\zeta}_+ + \left(y_1\bar{\phi}_1 + y_2\bar{\phi}_2\right)\bar{\zeta}_- = 0,\\
i\sigma^\mu\left(\partial_\mu - igq_-A_\mu\right)\bar{\zeta}_- - (\partial_4 - i\partial_5)\chi_- + iq_-g(A_4 - iA_5)\chi_- + (y_1\phi_1 + y_2\phi_2)\chi_+ = 0,\\
i\bar{\sigma}^\mu\left(\partial_\mu - iq_-gA_\mu\right)\chi_- + (\partial_4 + i\partial_5)\bar{\zeta}_- - iq_-g(A_4 + iA_5)\bar{\zeta}_- + (y_1\phi_1 + y_2\phi_2)\bar{\zeta}_+ = 0,
\end{aligned}
\tag{5.6}
$$

where the two-component spinors $\chi_\pm$ and $\bar{\zeta}_\pm$ are defined in Appendix C.

Since the background (3.9) breaks the 4D Lorentz symmetry SO(1,3) to SO(3), we need not discriminate the dotted and undotted indices. Thus, the linearized equations of motion for the fermions are expressed as

$$
\begin{aligned}
&-(i\partial_0 + q_+\omega\beta)\bar{\zeta}_+ + i\sigma^i\partial_i\bar{\zeta}_+\\
&-e^{i\theta}\left\{\partial_r + \frac{i}{r}(\partial_\theta - iq_+n\alpha)\right\}\chi_+ + \left(y_1v_1f_1e^{-in\theta} + y_2v_2f_2e^{-i\omega t}\right)\chi_- = 0,\\
&-(i\partial_0 + q_+\omega\beta)\chi_+ - i\sigma^i\partial_i\chi_+\\
&+e^{-i\theta}\left\{\partial_r - \frac{i}{r}(\partial_\theta - iq_+n\alpha)\right\}\bar{\zeta}_+ + \left(y_1v_1f_1e^{-in\theta} + y_2v_2f_2e^{-i\omega t}\right)\bar{\zeta}_- = 0,\\
&-(i\partial_0 + q_-\omega\beta)\bar{\zeta}_- + i\sigma^i\partial_i\bar{\zeta}_-\\
&-e^{-i\theta}\left\{\partial_r - \frac{i}{r}(\partial_\theta - iq_-n\alpha)\right\}\chi_- + \left(y_1v_1f_1e^{in\theta} + y_2v_2f_2e^{i\omega t}\right)\chi_+ = 0,\\
&-(i\partial_0 + q_-\omega\beta)\chi_- - i\sigma^i\partial_i\chi_-\\
&+e^{i\theta}\left\{\partial_r + \frac{i}{r}(\partial_\theta - iq_-n\alpha)\right\}\bar{\zeta}_- + \left(y_1v_1f_1e^{in\theta} + y_2v_2f_2e^{i\omega t}\right)\bar{\zeta}_+ = 0.
\end{aligned}
\tag{5.7}
$$

We have used the polar coordinates for the extra dimensions.

Each component of the fermions is decomposed into the KK modes as

$$
\begin{aligned}
\chi_\pm(x^\mu, r, \theta) &= \sum_K h^{(K)}_{\mathrm{R}\pm}(\rho,\theta)\eta^{(K)}(x^\mu),\\
\bar{\zeta}_\pm(x^\mu, r, \theta) &= \sum_K h^{(K)}_{\mathrm{L}\pm}(\rho,\theta)\eta^{(K)}(x^\mu),
\end{aligned}
\tag{5.8}
$$

where $\rho = v_1^{1/2}r$ is the dimensionless coordinate.[10] The sums in the above expansion contain integrals over the continuous spectrum, just like in Eq. (4.18). We choose the mode functions as solutions of the following mode equations:

$$
\begin{aligned}
-e^{i\theta}\left\{\partial_\rho + \frac{q_+n\alpha(\rho)}{\rho} + \frac{i}{\rho}\partial_\theta\right\}h^{(K)}_{\mathrm{R}+} + \left\{\tilde{y}_1f_1(\rho)e^{-in\theta} + \tilde{y}_2\xi^{1/2}f_2(\rho)e^{-i\omega t}\right\}h^{(K)}_{\mathrm{R}-} = \tilde{m}_K h^{(K)}_{\mathrm{L}+},\\
e^{-i\theta}\left\{\partial_\rho - \frac{q_+n\alpha(\rho)}{\rho} - \frac{i}{\rho}\partial_\theta\right\}h^{(K)}_{\mathrm{L}+} + \left\{\tilde{y}_1f_1(\rho)e^{-in\theta} + \tilde{y}_2\xi^{1/2}f_2(\rho)e^{-i\omega t}\right\}h^{(K)}_{\mathrm{L}-} = \tilde{m}_K h^{(K)}_{\mathrm{R}+},
\end{aligned}
$$

[10] Since we do not discriminate the undotted and dotted indices, each KK mode can reside in all fermions.

$$-e^{-i\theta}\left\{\partial_\rho-\frac{q_-n\alpha(\rho)}{\rho}-\frac{i}{\rho}\partial_\theta\right\}h_{\mathrm{R}-}^{(K)}+\left\{\tilde{y}_1f_1(\rho)e^{in\theta}+\tilde{y}_2\xi^{1/2}f_2(\rho)e^{i\omega t}\right\}h_{\mathrm{R}+}^{(K)}=\tilde{m}_Kh_{\mathrm{L}-}^{(K)},$$
$$e^{i\theta}\left\{\partial_\rho+\frac{q_-n\alpha(\rho)}{\rho}+\frac{i}{\rho}\partial_\theta\right\}h_{\mathrm{L}-}^{(K)}+\left\{\tilde{y}_1f_1(\rho)e^{in\theta}+\tilde{y}_2\xi^{1/2}f_2(\rho)e^{i\omega t}\right\}h_{\mathrm{L}+}^{(K)}=\tilde{m}_Kh_{\mathrm{R}-}^{(K)},\tag{5.9}$$

where

$$\tilde{m}_K\equiv\frac{m_K}{v_1^{1/2}},\qquad\tilde{y}_1\equiv y_1v_1^{1/2},\qquad\tilde{y}_2\equiv y_2v_1^{1/2}\tag{5.10}$$

are dimensionless (m_K is the KK mass).

Note that when $(h_{\mathrm{R}\pm}^{(K)},h_{\mathrm{L}\pm}^{(K)})$ are solutions with the eigenvalue $\tilde{m}_K$, the functions $(-h_{\mathrm{R}\pm}^{(K)},h_{\mathrm{L}\pm}^{(K)})$ or $(h_{\mathrm{R}\pm}^{(K)},-h_{\mathrm{L}\pm}^{(K)})$ become solutions with the eigenvalue $-\tilde{m}_K$. Thus, we label the KK modes in such a way that

$$\tilde{m}_{-K}=-\tilde{m}_K,\qquad h_{\mathrm{R}\pm}^{(-K)}(\rho,\theta)=-h_{\mathrm{R}\pm}^{(K)}(\rho,\theta),\qquad h_{\mathrm{L}\pm}^{(-K)}(\rho,\theta)=h_{\mathrm{L}\pm}^{(K)}(\rho,\theta),\tag{5.11}$$

for $n>0$ (n is the integer in Eq. (3.9)), and

$$\tilde{m}_{-K}=-\tilde{m}_K,\qquad h_{\mathrm{R}\pm}^{(-K)}(\rho,\theta)=h_{\mathrm{R}\pm}^{(K)}(\rho,\theta),\qquad h_{\mathrm{L}\pm}^{(-K)}(\rho,\theta)=-h_{\mathrm{L}\pm}^{(K)}(\rho,\theta),\tag{5.12}$$

for $n<0$. These are consistent with the fact that only one chiral component has zero-modes, which is explicitly shown in Appendix D.1.[11] Making use of Eq. (5.9) and performing the partial integrals, we can show that

$$\tilde{m}_K\int d\rho d\theta\;\rho\left(h_{\mathrm{R}+}^{(K)*}h_{\mathrm{R}+}^{(L)}+h_{\mathrm{R}-}^{(K)*}h_{\mathrm{R}-}^{(L)}\right)=\tilde{m}_L\int d\rho d\theta\;\rho\left(h_{\mathrm{L}+}^{(K)*}h_{\mathrm{L}+}^{(L)}+h_{\mathrm{L}-}^{(K)*}h_{\mathrm{L}-}^{(L)}\right),\tag{5.13}$$

which leads to

$$(\tilde{m}_K-\tilde{m}_L)\int d\rho d\theta\;\rho\left(h_{\mathrm{R}+}^{(K)*}h_{\mathrm{R}+}^{(L)}+h_{\mathrm{R}-}^{(K)*}h_{\mathrm{R}-}^{(L)}+h_{\mathrm{L}+}^{(K)*}h_{\mathrm{L}+}^{(L)}+h_{\mathrm{L}-}^{(K)*}h_{\mathrm{L}-}^{(L)}\right)=0.\tag{5.14}$$

Thus, we normalize the mode functions so that

$$\int d\rho d\theta\;\rho\left(h_{\mathrm{R}+}^{(K)*}h_{\mathrm{R}+}^{(L)}+h_{\mathrm{R}-}^{(K)*}h_{\mathrm{R}-}^{(L)}+h_{\mathrm{L}+}^{(K)*}h_{\mathrm{L}+}^{(L)}+h_{\mathrm{L}-}^{(K)*}h_{\mathrm{L}-}^{(L)}\right)=\delta_{K,L}.\tag{5.15}$$

5.2. $\boldsymbol{y_2=0}$ *case*

To make the discussion more specific, we focus on the case of $y_2=0$ in the following. Similar to the scalar case in the previous section, the eigenvalues $\tilde{m}_K$ in Eq. (5.9) have degeneracy, and we can separate the variables as

$$\begin{aligned}h_{\mathrm{R}+}^{(k)[m]}(\rho,\theta)&=b_{\mathrm{R}+}^{(k)[m]}(\rho)e^{im\theta},\\h_{\mathrm{R}-}^{(k)[m]}(\rho,\theta)&=b_{\mathrm{R}-}^{(k)[m]}(\rho)e^{i(m+n+1)\theta},\\h_{\mathrm{L}+}^{(k)[m]}(\rho,\theta)&=b_{\mathrm{L}+}^{(k)[m]}(\rho)e^{i(m+1)\theta},\\h_{\mathrm{L}-}^{(k)[m]}(\rho,\theta)&=b_{\mathrm{L}-}^{(k)[m]}(\rho)e^{i(m+n)\theta}.\end{aligned}\tag{5.16}$$

[11] We have assumed that $\tilde{y}_1\neq0$ and $\tilde{y}_2=0$ to specify the situation.

Then the mode equations in Eq. (5.9) are expressed as

$$
\begin{aligned}
-\left\{\partial_\rho+\frac{q_+n\alpha(\rho)-m}{\rho}\right\}b_{\mathrm{R}+}^{(k)[m]}(\rho)+\tilde{y}_1f_1(\rho)b_{\mathrm{R}-}^{(k)[m]}&=\tilde{m}_kb_{\mathrm{L}+}^{(k)[m]}(\rho),\\
-\left\{\partial_\rho-\frac{q_-n\alpha(\rho)-m-n-1}{\rho}\right\}b_{\mathrm{R}-}^{(k)[m]}(\rho)+\tilde{y}_1f_1(\rho)b_{\mathrm{R}+}^{(k)[m]}&=\tilde{m}_kb_{\mathrm{L}-}^{(k)[m]}(\rho),\\
\left\{\partial_\rho-\frac{q_+n\alpha(\rho)-m-1}{\rho}\right\}b_{\mathrm{L}+}^{(k)[m]}(\rho)+\tilde{y}_1f_1(\rho)b_{\mathrm{L}-}^{(k)[m]}(\rho)&=\tilde{m}_kb_{\mathrm{R}+}^{(k)[m]}(\rho),\\
\left\{\partial_\rho+\frac{q_-n\alpha(\rho)-m-n}{\rho}\right\}b_{\mathrm{L}-}^{(k)[m]}(\rho)+\tilde{y}_1f_1(\rho)b_{\mathrm{L}+}^{(k)[m]}(\rho)&=\tilde{m}_kb_{\mathrm{R}-}^{(k)[m]}(\rho).
\end{aligned}
\tag{5.17}
$$

The relations in Eq. (5.11) are translated into

$$
\tilde{m}_{-k}=-\tilde{m}_k,\qquad b_{\mathrm{R}\pm}^{(-k)[m]}(\rho)=-b_{\mathrm{R}\pm}^{(k)[m]}(\rho),\qquad b_{\mathrm{L}\pm}^{(-k)[m]}(\rho)=b_{\mathrm{L}\pm}^{(k)[m]}(\rho).
\tag{5.18}
$$

Since $h_{\mathrm{R}\pm}^{(k)[m]}(\rho,\theta)$ and $h_{\mathrm{L}\pm}^{(k)[m]}(\rho,\theta)$ are normalized by Eq. (5.15), $b_{\mathrm{R}\pm}^{(k)[m]}(\rho)$ and $b_{\mathrm{L}\pm}^{(k)[m]}(\rho)$ satisfy

$$
\int_0^\infty d\rho\;\rho\left(b_{\mathrm{R}+}^{(k)[m]}b_{\mathrm{R}+}^{(l)[m]}+b_{\mathrm{R}-}^{(k)[m]}b_{\mathrm{R}-}^{(l)[m]}+b_{\mathrm{L}+}^{(k)[m]}b_{\mathrm{L}+}^{(l)[m]}+b_{\mathrm{L}-}^{(k)[m]}b_{\mathrm{L}-}^{(l)[m]}\right)=\frac{\delta_{k,l}}{2\pi}.
\tag{5.19}
$$

Making use of the orthonormal relations, we can rewrite Eq. (5.7) as the linearized equations for the KK modes:

$$
-i\partial_0\eta^{(k)[m]}+i\sum_l B_{k,l}^{[m]}\sigma^i\partial_i\eta^{(l)[m]}-\sum_l C_{k,l}^{[m]}\eta^{(l)[m]}+m_k\eta^{(k)[m]}=0,
\tag{5.20}
$$

where

$$
\begin{aligned}
B_{k,l}^{[m]}&\equiv\int d\rho d\theta\;\rho\left(-h_{\mathrm{R}+}^{(k)[m]*}h_{\mathrm{R}+}^{(l)[m]}-h_{\mathrm{R}-}^{(k)[m]*}h_{\mathrm{R}-}^{(l)[m]}+h_{\mathrm{L}+}^{(k)[m]*}h_{\mathrm{L}+}^{(l)[m]}+h_{\mathrm{L}-}^{(k)[m]*}h_{\mathrm{L}-}^{(l)[m]}\right)\\
&=2\pi\int_0^\infty d\rho\;\rho\left(-b_{\mathrm{R}+}^{(k)[m]}b_{\mathrm{R}+}^{(l)[m]}-b_{\mathrm{R}-}^{(k)[m]}b_{\mathrm{R}-}^{(l)[m]}+b_{\mathrm{L}+}^{(k)[m]}b_{\mathrm{L}+}^{(l)[m]}+b_{\mathrm{L}-}^{(k)[m]}b_{\mathrm{L}-}^{(l)[m]}\right),\\
C_{k,l}^{[m]}&\equiv\int d\rho d\theta\;\rho\omega\beta\left(q_+h_{\mathrm{R}+}^{(k)[m]*}h_{\mathrm{R}+}^{(l)[m]}+q_-h_{\mathrm{R}-}^{(k)[m]*}h_{\mathrm{R}-}^{(l)[m]}+q_+h_{\mathrm{L}+}^{(k)[m]*}h_{\mathrm{L}+}^{(l)[m]}+q_-h_{\mathrm{L}-}^{(k)[m]*}h_{\mathrm{L}-}^{(l)[m]}\right)\\
&=2\pi\omega\int_0^\infty d\rho\;\rho\beta\left(q_+b_{\mathrm{R}+}^{(k)[m]}b_{\mathrm{R}+}^{(l)[m]}+q_-b_{\mathrm{R}-}^{(k)[m]}b_{\mathrm{R}-}^{(l)[m]}+q_+b_{\mathrm{L}+}^{(k)[m]}b_{\mathrm{L}+}^{(l)[m]}+q_-b_{\mathrm{L}-}^{(k)[m]}b_{\mathrm{L}-}^{(l)[m]}\right).
\end{aligned}
\tag{5.21}
$$

From Eqs. (5.18) and (5.19), we can see that

$$
B_{k,l}^{[m]}=\delta_{k,-l}.
\tag{5.22}
$$

Note also that the KK modes with different m are decoupled from each other in the linearized equations of motion (5.20). Thus, Eq. (5.20) is rewritten as

$$i\partial_0 \begin{pmatrix} \boldsymbol{\eta}_+^{[m]} \\ \eta^{(0)[m]} \\ \boldsymbol{\eta}_-^{[m]} \end{pmatrix} - i \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & 1 & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \end{pmatrix} \sigma^i \partial_i \begin{pmatrix} \boldsymbol{\eta}_+^{[m]} \\ \eta^{(0)[m]} \\ \boldsymbol{\eta}_-^{[m]} \end{pmatrix}$$
$$+ \begin{pmatrix} \mathbb{C}_+^{[m]} & \boldsymbol{C}_{0+}^{[m]} & \mathbb{C}_-^{[m]} \\ \boldsymbol{C}_{0+}^{[m]t} & C_{0,0}^{[m]} & \boldsymbol{C}_{0-}^{[m]t} \\ \mathbb{C}_-^{[m]} & \boldsymbol{C}_{0-}^{[m]} & \mathbb{C}_+^{[m]} \end{pmatrix} \begin{pmatrix} \boldsymbol{\eta}_+^{[m]} \\ \eta^{(0)[m]} \\ \boldsymbol{\eta}_-^{[m]} \end{pmatrix} - \begin{pmatrix} \mathbb{M} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbb{M} \end{pmatrix} \begin{pmatrix} \boldsymbol{\eta}_+^{[m]} \\ \eta^{(0)[m]} \\ \boldsymbol{\eta}_-^{[m]} \end{pmatrix} = \mathbf{0}, \tag{5.23}$$

where

$$\boldsymbol{\eta}_\pm^{[m]} \equiv \left(\eta^{(\pm 1)[m]}, \eta^{(\pm 2)[m]}, \eta^{(\pm 3)[m]}, \dots\right)^t,$$
$$\mathbb{M} \equiv \mathrm{diag}\,(m_1, m_2, m_3, \dots), \tag{5.24}$$

and the matrices $\mathbb{C}_\pm^{[m]}$ are defined by

$$(\mathbb{C}_+^{[m]})_{k,l} \equiv C_{k,l}^{[m]} = C_{-k,-l}^{[m]},$$
$$(\mathbb{C}_-^{[m]})_{k,l} \equiv C_{k,-l}^{[m]} = C_{-k,l}^{[m]}, \tag{5.25}$$

and the column vectors $\boldsymbol{C}_{0\pm}^{[m]}$ are defined by

$$(\boldsymbol{C}_{0+}^{[m]})_k \equiv C_{0,k}^{[m]} = C_{k,0}^{[m]},$$
$$(\boldsymbol{C}_{0-}^{[m]})_k \equiv C_{0,-k}^{[m]} = C_{-k,0}^{[m]}, \tag{5.26}$$

for $k, l > 0$. Note that $\mathbb{C}_\pm^{[m]}$ are Hermitian. Since Eq. (5.23) is rewritten as

$$i\sigma^i\partial_i \begin{pmatrix} \boldsymbol{\eta}_+^{[m]} \\ \eta^{(0)[m]} \\ \boldsymbol{\eta}_-^{[m]} \end{pmatrix} = \begin{pmatrix} \mathbb{C}_-^{[m]} & \boldsymbol{C}_{0-}^{[m]} & i\partial_0 + \mathbb{C}_+^{[m]} + \mathbb{M} \\ \boldsymbol{C}_{0+}^{[m]t} & i\partial_0 + C_{0,0}^{[m]} & \boldsymbol{C}_{0-}^{[m]t} \\ i\partial_0 + \mathbb{C}_+^{[m]} - \mathbb{M} & \boldsymbol{C}_{0+}^{[m]} & \mathbb{C}_-^{[m]} \end{pmatrix} \begin{pmatrix} \boldsymbol{\eta}_+^{[m]} \\ \eta^{(0)[m]} \\ \boldsymbol{\eta}_-^{[m]} \end{pmatrix}, \tag{5.27}$$

we obtain

$$\begin{pmatrix} \mathbb{C}_-^{[m]} & \boldsymbol{C}_{0-}^{[m]} & i\partial_0 + \mathbb{C}_+^{[m]} + \mathbb{M} \\ \boldsymbol{C}_{0+}^{[m]t} & i\partial_0 + C_{0,0}^{[m]} & \boldsymbol{C}_{0-}^{[m]t} \\ i\partial_0 + \mathbb{C}_+^{[m]} - \mathbb{M} & \boldsymbol{C}_{0+}^{[m]} & \mathbb{C}_-^{[m]} \end{pmatrix}^2 \begin{pmatrix} \boldsymbol{\eta}_+^{[m]} \\ \eta^{(0)[m]} \\ \boldsymbol{\eta}_-^{[m]} \end{pmatrix}$$
$$= \begin{pmatrix} \mathbb{C}_-^{[m]} & \boldsymbol{C}_{0-}^{[m]} & i\partial_0 + \mathbb{C}_+^{[m]} + \mathbb{M} \\ \boldsymbol{C}_{0+}^{[m]t} & i\partial_0 + C_{0,0}^{[m]} & \boldsymbol{C}_{0-}^{[m]t} \\ i\partial_0 + \mathbb{C}_+^{[m]} - \mathbb{M} & \boldsymbol{C}_{0+}^{[m]} & \mathbb{C}_-^{[m]} \end{pmatrix} i\sigma^i\partial_i \begin{pmatrix} \boldsymbol{\eta}_+^{[m]} \\ \eta^{(0)[m]} \\ \boldsymbol{\eta}_-^{[m]} \end{pmatrix}$$
$$= i\sigma^i\partial_i \begin{pmatrix} \mathbb{C}_-^{[m]} & \boldsymbol{C}_{0-}^{[m]} & i\partial_0 + \mathbb{C}_+^{[m]} + \mathbb{M} \\ \boldsymbol{C}_{0+}^{[m]t} & i\partial_0 + C_{0,0}^{[m]} & \boldsymbol{C}_{0-}^{[m]t} \\ i\partial_0 + \mathbb{C}_+^{[m]} - \mathbb{M} & \boldsymbol{C}_{0+}^{[m]} & \mathbb{C}_-^{[m]} \end{pmatrix} \begin{pmatrix} \boldsymbol{\eta}_+^{[m]} \\ \eta^{(0)[m]} \\ \boldsymbol{\eta}_-^{[m]} \end{pmatrix}$$
$$= i\sigma^i\partial_i \left(i\sigma^j\partial_j\right) \begin{pmatrix} \boldsymbol{\eta}_+^{[m]} \\ \eta^{(0)[m]} \\ \boldsymbol{\eta}_-^{[m]} \end{pmatrix} = -\vec{\nabla}^2 \begin{pmatrix} \boldsymbol{\eta}_+^{[m]} \\ \eta^{(0)[m]} \\ \boldsymbol{\eta}_-^{[m]} \end{pmatrix}. \tag{5.28}$$

In the momentum basis, this becomes

$$\begin{pmatrix} \mathbb{C}_{-}^{[m]} & \boldsymbol{C}_{0-}^{[m]} & -E+\mathbb{C}_{+}^{[m]}+\mathbb{M} \\ \boldsymbol{C}_{0+}^{[m]t} & -E+C_{0,0}^{[m]} & \boldsymbol{C}_{0-}^{[m]t} \\ -E+\mathbb{C}_{+}^{[m]}-\mathbb{M} & \boldsymbol{C}_{0+}^{[m]} & \mathbb{C}_{-}^{[m]} \end{pmatrix}^2 \begin{pmatrix} \tilde{\boldsymbol{\eta}}_{+}^{[m]} \\ \tilde{\eta}^{(0)[m]} \\ \tilde{\boldsymbol{\eta}}_{-}^{[m]} \end{pmatrix} = \vec{p}^2 \begin{pmatrix} \tilde{\boldsymbol{\eta}}_{+}^{[m]} \\ \tilde{\eta}^{(0)[m]} \\ \tilde{\boldsymbol{\eta}}_{-}^{[m]} \end{pmatrix}, \tag{5.29}$$

or

$$\begin{pmatrix} \mathbb{M}_{++}^{[m]} & \boldsymbol{M}_{+0}^{[m]} & \mathbb{M}_{+-}^{[m]} \\ \boldsymbol{M}_{0+}^{[m]t} & M_{00}^{[m]} & \boldsymbol{M}_{0-}^{[m]t} \\ \mathbb{M}_{-+}^{[m]} & \boldsymbol{M}_{-0}^{[m]} & \mathbb{M}_{--}^{[m]} \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{\eta}}_{+}^{[m]} \\ \tilde{\eta}^{(0)[m]} \\ \tilde{\boldsymbol{\eta}}_{-}^{[m]} \end{pmatrix} = \vec{p}^2 \begin{pmatrix} \tilde{\boldsymbol{\eta}}_{+}^{[m]} \\ \tilde{\eta}^{(0)[m]} \\ \tilde{\boldsymbol{\eta}}_{-}^{[m]} \end{pmatrix}, \tag{5.30}$$

where

$$\begin{aligned}
\mathbb{M}_{++}^{[m]} &\equiv E^2 - 2\mathbb{C}_{+}^{[m]}E + \left(\mathbb{C}_{+}^{[m]2} + \boldsymbol{C}_{0-}^{[m]}\boldsymbol{C}_{0+}^{[m]t} + \mathbb{C}_{-}^{[m]2}\right) - \left[\mathbb{C}_{+}^{[m]}, \mathbb{M}\right] - \mathbb{M}^2, \\
\mathbb{M}_{+-}^{[m]} &\equiv -2\mathbb{C}_{-}^{[m]}E + \left\{\mathbb{C}_{+}^{[m]} + \mathbb{M}, \mathbb{C}_{-}^{[m]}\right\} + \boldsymbol{C}_{0-}^{[m]}\boldsymbol{C}_{0-}^{[m]t}, \\
\mathbb{M}_{-+}^{[m]} &\equiv -2\mathbb{C}_{-}^{[m]}E + \left\{\mathbb{C}_{+}^{[m]} - \mathbb{M}, \mathbb{C}_{-}^{[m]}\right\} + \boldsymbol{C}_{0+}^{[m]}\boldsymbol{C}_{0+}^{[m]t}, \\
\mathbb{M}_{--}^{[m]} &\equiv E^2 - 2\mathbb{C}_{+}^{[m]}E + \left(\mathbb{C}_{+}^{[m]2} + \boldsymbol{C}_{0+}^{[m]}\boldsymbol{C}_{0-}^{[m]t} + \mathbb{C}_{-}^{[m]2}\right) + \left[\mathbb{C}_{+}^{[m]}, \mathbb{M}\right] - \mathbb{M}^2, \\
M_{00}^{[m]} &\equiv E^2 - 2C_{0,0}^{[m]}E + C_{0,0}^{[m]2} + \boldsymbol{C}_{0+}^{[m]t}\boldsymbol{C}_{0-}^{[m]} + \boldsymbol{C}_{0-}^{[m]t}\boldsymbol{C}_{0+}^{[m]},
\end{aligned} \tag{5.31}$$

and

$$\begin{aligned}
\boldsymbol{M}_{+0}^{[m]} &\equiv -\left(\boldsymbol{C}_{0+}^{[m]} + \boldsymbol{C}_{0-}^{[m]}\right)E + \mathbb{C}_{-}^{[m]}\boldsymbol{C}_{0-}^{[m]} + \boldsymbol{C}_{0-}^{[m]}C_{0,0}^{[m]} + \left(\mathbb{C}_{+}^{[m]} + \mathbb{M}\right)\boldsymbol{C}_{0+}^{[m]}, \\
\boldsymbol{M}_{0+}^{[m]t} &\equiv -\left(\boldsymbol{C}_{0+}^{[m]t} + \boldsymbol{C}_{0-}^{[m]t}\right)E + \boldsymbol{C}_{0+}^{[m]t}\mathbb{C}_{-}^{[m]} + C_{0,0}^{[m]}\boldsymbol{C}_{0+}^{[m]t} + \boldsymbol{C}_{0-}^{[m]t}\left(\mathbb{C}_{+}^{[m]} - \mathbb{M}\right), \\
\boldsymbol{M}_{0-}^{[m]t} &\equiv -\left(\boldsymbol{C}_{0+}^{[m]t} + \boldsymbol{C}_{0-}^{[m]t}\right)E + \boldsymbol{C}_{0-}^{[m]t}\mathbb{C}_{-}^{[m]} + C_{0,0}^{[m]}\boldsymbol{C}_{0-}^{[m]t} + \boldsymbol{C}_{0+}^{[m]t}\left(\mathbb{C}_{+}^{[m]} + \mathbb{M}\right), \\
\boldsymbol{M}_{-0}^{[m]} &\equiv -\left(\boldsymbol{C}_{0+}^{[m]} + \boldsymbol{C}_{0-}^{[m]}\right)E + \mathbb{C}_{-}^{[m]}\boldsymbol{C}_{0+}^{[m]} + \boldsymbol{C}_{0+}^{[m]}C_{0,0}^{[m]} + \left(\mathbb{C}_{+}^{[m]} - \mathbb{M}\right)\boldsymbol{C}_{0-}^{[m]}.
\end{aligned} \tag{5.32}$$

By diagonalizing the matrix on the right-hand side of Eq. (5.30), we can obtain the dispersion relations.

Let us consider a case in which ω is sufficiently smaller than $\omega_{\max}$ in order to see how the dispersion relation for the lowest mode is distorted by the spin of the vortex. As we have seen in Sect. 3.4, $\beta(\rho)$ is exponentially small in such a case, and we can expand Eq. (5.30) in terms of the elements of the matrix $C_{k,l}^{[m]}$. We can immediately see that the contributions from the off-diagonal elements in Eq. (5.30) to the eigenvalues are $\mathcal{O}(\beta^2)$. Hence, at $\mathcal{O}(\beta)$, the dispersion relation for the lowest mode is read off as

$$E^2 - 2C_{0,0}^{[m]}E - \vec{p}^2 = \mathcal{O}(\beta^2). \tag{5.33}$$

When $\vec{p}^2 \ll C_{0,0}^{[m]2}$, this is reduced to[12]

$$E \simeq 2C_{0,0}^{[m]} + \frac{\vec{p}^2}{2C_{0,0}^{[m]}}, \tag{5.34}$$

which indicates that the lowest mode has a tiny but nonvanishing mass $C_{0,0}^{[m]}$. As we mentioned in the previous section, the degeneracy in the KK spectrum is resolved due to the m-dependence of the effective masses.

When ω is close to $\omega_{\max}$, we have to take into account the mixing with the higher KK modes by diagonalizing the matrix in Eq. (5.30).

In the static limit ($\omega \to 0$), the 4D Lorentz symmetry is recovered and Eq. (5.30) provides ordinary relativistic dispersion relations:

$$\left\{ \left(E^2 - \vec{p}^2\right) \mathbf{1} - \begin{pmatrix} \mathbb{M}^2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbb{M}^2 \end{pmatrix} \right\} \begin{pmatrix} \tilde{\boldsymbol{\eta}}_+^{[m]} \\ \tilde{\eta}^{(0)[m]} \\ \tilde{\boldsymbol{\eta}}_-^{[m]} \end{pmatrix} = \mathbf{0}. \tag{5.35}$$

Hence, the lowest modes become massless and chiral. In this limit, either $h_{\mathrm{R}\pm}^{(0)[m]}(\rho, \theta)$ or $h_{\mathrm{L}\pm}^{(0)[m]}(\rho, \theta)$ vanish depending on the sign of n, as shown in Appendix D.

6. Summary

We have considered a situation that the 3-brane where we live is spinning in extra-dimensional space. The ANO vortex in the Abelian–Higgs model does not have a degree of freedom to rotate the vortex configuration without an energy cost, so the stationary spinning solution does not exist. We have extended the model by adding an extra charged scalar so that an extra U(1) global symmetry appears, and the stationary spinning vortex solution is allowed. We find that the vortex profile has a nontrivial dependence on the angular velocity in the field space ω only in a limited region, and there is an upper limit on ω. Thus the vortex configuration should be parametrized by the angular momentum for the rotation in the extra-dimensional space, rather than ω. In contrast to the ANO vortex, the U(1) gauge symmetry is not restored at the core of the vortex due to the nonvanishing background of the second scalar ϕ_2.

The spin of the vortex violates the 4D Lorentz symmetry in the effective theory. Due to the nonvanishing background of the temporal component of the gauge field A_0, the dotted and the undotted spinor indices become indistinguishable. Hence each KK fermionic mode resides in both 4D chiral components, and they are described by two-component spinors of the unbroken SO(3). The dispersion relations are also modified by the spin of the vortex (or the vacuum expectation value, VEV, of A_0). In particular, the zero-modes are mixed with higher KK modes due to the spin, and obtain nonvanishing masses. We should also note that the vortex spin resolves the degeneracy in the KK spectrum, which exists in the static vortex case.

There are many directions in which we should proceed. We would like to generalize the situation by considering various kinds of vortices in various models, and extract universal properties of the spinning vortices. If we extend the model in a supersymmetric way, we can also discuss the SUSY-breaking effects in the 4D effective theory induced by the spin of a BPS vortex. The vortex in

[12] The other solution $E \simeq -\frac{\vec{p}^2}{2C_{0,0}^{[m]}}$ corresponds to the annihilation mode of the anti-particle. (See footnote 9.)

motion on the compact space is also an intriguing subject. In this paper, we have only considered classical motion. However, the spinning vortex may radiate some particles by a quantum effect and lose energy. In such a case, the vortex solution is no longer stationary, but the angular velocity will slow down, and the configuration will be reduced to be static. It would be interesting to pursue this process and study how it affects the cosmological history. In addition, we have neglected gravity in our analysis to simplify the discussion, but it is important to investigate the effects of spin on the 4D cosmological evolution in 6D gravitational theories. We will discuss these issues in subsequent papers.

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Appendix A. Vacuum structure of the model in Sect. 3

Here we summarize the vacuum structure of the model (3.1).[13] The minimization conditions of the potential U are

$$
\begin{aligned}
0 &= \frac{\partial U}{\partial \phi_1^*} = \lambda_1 \phi_1 \left(|\phi_1|^2 - v_1^2\right) + \gamma \phi_1 |\phi_2|^2, \\
0 &= \frac{\partial U}{\partial \phi_2^*} = \lambda_2 \phi_2 \left(|\phi_2|^2 - v_2^2\right) + \gamma |\phi_1|^2 \phi_2.
\end{aligned} \tag{A.1}
$$

By solving these, we find the following stationary points of U:

1. $\phi_1 = \phi_2 = 0$
2. $|\phi_1| = v_1$ and $\phi_2 = 0$
3. $\phi_1 = 0$ and $|\phi_2| = v_2$
4. $|\phi_1|^2 = \dfrac{\lambda_2 \left(\lambda_1 v_1^2 - \gamma v_2^2\right)}{\lambda_1 \lambda_2 - \gamma^2}$ and $|\phi_2|^2 = \dfrac{\lambda_1 \left(\lambda_2 v_2^2 - \gamma v_1^2\right)}{\lambda_1 \lambda_2 - \gamma^2}$
   This solution is possible only when

$$
\begin{aligned}
(\lambda_1 \lambda_2 - \gamma^2)(\lambda_1 v_1^2 - \gamma v_2^2) > 0, \\
(\lambda_1 \lambda_2 - \gamma^2)(\lambda_2 v_2^2 - \gamma v_1^2) > 0.
\end{aligned} \tag{A.2}
$$

In order to investigate the stability of the vacua, we divide the complex scalar fields as

$$
\phi_1 = \phi_{1\mathrm{R}} + i\phi_{1\mathrm{I}}, \qquad \phi_2 = \phi_{2\mathrm{R}} + i\phi_{2\mathrm{I}}, \tag{A.3}
$$

[13] See Ref. [15] for a similar setup.

and evaluate the Hessian matrix,

$$H(U) = \begin{pmatrix} \dfrac{\partial^2 U}{\partial \phi_{1R}^2} & \dfrac{\partial^2 U}{\partial \phi_{1R}\partial \phi_{1I}} & \dfrac{\partial^2 U}{\partial \phi_{1R}\partial \phi_{2R}} & \dfrac{\partial^2 U}{\partial \phi_{1R}\partial \phi_{2I}} \\ \dfrac{\partial^2 U}{\partial \phi_{1I}\partial \phi_{1R}} & \dfrac{\partial^2 U}{\partial \phi_{1I}^2} & \dfrac{\partial^2 U}{\partial \phi_{1I}\partial \phi_{2R}} & \dfrac{\partial^2 U}{\partial \phi_{1I}\partial \phi_{2I}} \\ \dfrac{\partial^2 U}{\partial \phi_{2R}\partial \phi_{1R}} & \dfrac{\partial^2 U}{\partial \phi_{2R}\partial \phi_{1I}} & \dfrac{\partial^2 U}{\partial \phi_{2R}^2} & \dfrac{\partial^2 U}{\partial \phi_{2R}\partial \phi_{2I}} \\ \dfrac{\partial^2 U}{\partial \phi_{2I}\partial \phi_{1R}} & \dfrac{\partial^2 U}{\partial \phi_{2I}\partial \phi_{1I}} & \dfrac{\partial^2 U}{\partial \phi_{2I}\partial \phi_{2R}} & \dfrac{\partial^2 U}{\partial \phi_{2I}^2} \end{pmatrix}. \tag{A.4}$$

For stationary point 1, the Hessian is

$$H(U) = \mathrm{diag}\left(-2\lambda_1 v_1^2, -2\lambda_1 v_1^2, -2\lambda_2 v_2^2, -2\lambda_2 v_2^2\right). \tag{A.5}$$

Thus this is a local maximum.

For stationary point 2, we choose $\phi_1 = v_1$ without loss of generality. Then, we obtain

$$H(U) = \mathrm{diag}\left(4\lambda_1 v_1^2, 0, -2\lambda_2 v_2^2 + 2\gamma v_1^2, -2\lambda_2 v_2^2 + 2\gamma v_1^2\right). \tag{A.6}$$

Thus, we have the NG-mode for the ϕ_{1I}-direction, and the other modes are massive when $\gamma v_1^2 > \lambda_2 v_2^2$. The potential value at this vacuum is

$$U_{\mathrm{pt2}} = \frac{\lambda_2 v_2^4}{2} + U_0. \tag{A.7}$$

For stationary point 3, we choose $\phi_2 = v_2$ without loss of generality. Then we obtain

$$H(U) = \mathrm{diag}\left(-2\lambda_1 v_1^2 + 2\gamma v_2^2, -2\lambda_1 v_1^2 + 2\gamma v_2^2, 4\lambda_2 v_2^2, 0\right). \tag{A.8}$$

Thus, we have the NG-mode for the ϕ_{2I}-direction, and the other modes are massive when $\gamma v_2^2 > \lambda_1 v_1^2$. The potential value at this vacuum is

$$U_{\mathrm{pt3}} = \frac{\lambda_1 v_1^4}{2} + U_0. \tag{A.9}$$

In stationary point 4, we choose

$$\phi_1 = \sqrt{\frac{\lambda_2(\lambda_1 v_1^2 - \gamma v_2^2)}{\lambda_1\lambda_2 - \gamma^2}}, \qquad \phi_2 = \sqrt{\frac{\lambda_1(\lambda_2 v_2^2 - \gamma v_1^2)}{\lambda_1\lambda_2 - \gamma^2}}, \tag{A.10}$$

without loss of generality. Then we obtain

$$H(U) = \frac{4}{\lambda_1\lambda_2 - \gamma^2}\begin{pmatrix} \lambda_1\lambda_2\Lambda_1 & 0 & \gamma\sqrt{\lambda_1\lambda_2\Lambda_1\Lambda_2} & 0 \\ 0 & 0 & 0 & 0 \\ \gamma\sqrt{\lambda_1\lambda_2\Lambda_1\Lambda_2} & 0 & \lambda_1\lambda_2\Lambda_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A.11}$$

where

$$\Lambda_1 \equiv \lambda_1 v_1^2 - \gamma v_2^2, \qquad \Lambda_2 \equiv \lambda_2 v_2^2 - \gamma v_1^2. \tag{A.12}$$

Thus, we have two massless modes, which correspond to the NG-modes for the breakings of the U(1) gauge and U(1) global symmetries. This stationary point is a local minimum iff

$$\det\left\{\frac{4}{\lambda_1\lambda_2-\gamma^2}\begin{pmatrix}\lambda_1\lambda_2\Lambda_1 & \gamma\sqrt{\lambda_1\lambda_2\Lambda_1\Lambda_2}\\ \gamma\sqrt{\lambda_1\lambda_2\Lambda_1\Lambda_2} & \lambda_1\lambda_2\Lambda_2\end{pmatrix}\right\}=\frac{16\lambda_1\lambda_2\Lambda_1\Lambda_2}{\lambda_1\lambda_2-\gamma^2} \tag{A.13}$$

is positive. Combined with the condition (A.2), this indicates that $\Lambda_1 > 0$ and $\Lambda_2 > 0$. Namely, points 2 or 3 and point 4 cannot be local minima simultaneously. The potential value at this vacuum is

$$U_{\rm pt4}=-\frac{\gamma}{2(\lambda_1\lambda_2-\gamma^2)}\left(\gamma\lambda_1v_1^4-2\lambda_1\lambda_2v_1^2v_2^2+\gamma\lambda_2v_2^4\right)+U_0. \tag{A.14}$$

In the case that points 2 and 3 are local minima, i.e., $\Lambda_2 < 0$ and $\Lambda_1 < 0$, we find that

$$\frac{\lambda_1}{\gamma}<\frac{v_2^2}{v_1^2}<\frac{\gamma}{\lambda_2}, \tag{A.15}$$

which indicates that

$$\gamma^2 > \lambda_1\lambda_2. \tag{A.16}$$

In fact, the interaction parametrized by γ prevents both scalars ϕ_1 and ϕ_2 having nonvanishing VEVs.

When the conditions

$$\gamma v_1^2 > \lambda_2 v_2^2, \qquad \lambda_2 v_2^4 < \lambda_1 v_1^4 \tag{A.17}$$

are satisfied, point 2 becomes a global minimum of the potential.

Appendix B. Derivation of Eq. (4.8)

We separate the mode functions in Eq. (4.5) as

$$h_\Phi^{(K)}(\rho,\theta)=b_\Phi^{(K)}(\rho)c_\Phi^{(K)}(\theta). \tag{B.1}$$

Substituting this into the mode function (4.6), we obtain

$$\begin{aligned}\Gamma^{(K)}(\rho)&\equiv-\frac{\rho^2}{b_\Phi^{(K)}}\left\{\partial_\rho^2+\frac{1}{\rho}\partial_\rho+\tilde{\omega}^2\beta^2-\frac{n^2\alpha^2}{\rho^2}-M_\Phi^2-\tilde{\kappa}_1f_1^2-\tilde{\kappa}_2\xi f_2^2+\tilde{m}_K^2\right\}b_\Phi^{(K)}\\ &=\frac{1}{c_\Phi^{(K)}}\left(\partial_\theta^2-2in\alpha\partial_\theta\right)c_\Phi^{(K)}.\end{aligned} \tag{B.2}$$

Note that $\Gamma^{(K)}(\rho)$ is a function of only ρ. Since $c_\Phi^{(K)}(\theta)$ is a function of only θ and $\alpha(\rho)$ has a nontrivial ρ-dependence, this equation holds only when

$$\begin{aligned}\frac{\partial_\theta c_\Phi^{(K)}(\theta)}{c_\Phi^{(K)}(\theta)}&=s_1,\\ \frac{\partial_\theta^2 c_\Phi^{(K)}(\theta)}{c_\Phi^{(K)}(\theta)}&=s_2,\\ \Gamma^{(K)}(\rho)&=s_2-2in\alpha(\rho)s_1,\end{aligned} \tag{B.3}$$

where s_1 and s_2 are constants. The first two equations are solved as

$$c_\Phi^{(K)}(\theta) = N_c e^{s_1\theta}, \qquad s_2 = s_1^2. \tag{B.4}$$

Since the mode function should be single-valued, we find that $s_1 = im$, where m is an integer. Thus, the last equation in Eq. (B.3) becomes

$$\Gamma^{(K)}(\rho) + m^2 - 2nm\alpha(\rho) = 0. \tag{B.5}$$

This is the same as Eq. (4.9). By choosing the normalization of $c_\Phi^{(K)}(\theta)$ as $N_c = 1$, we obtain Eq. (4.8).

Here note that the integer m (or the constant s_1) can have different values for a given value of $\tilde{m}_K$ in Eq. (B.2). This indicates that there is a degeneracy in the spectrum and m labels that degeneracy. Hence we can write the KK label K by two integers k and m, where k labels different KK masses.

Appendix C. Notations

Here we collect the notations for the fermions. For two-component spinors, we basically follow the notations of Ref. [16].

The 6D gamma matrices Γ^M are chosen as

$$\Gamma^\mu = \begin{pmatrix} \mathbf{0}_4 & \gamma^\mu \\ \gamma^\mu & \mathbf{0}_4 \end{pmatrix}, \qquad \Gamma^4 = \begin{pmatrix} \mathbf{0}_4 & i\gamma_5 \\ i\gamma_5 & \mathbf{0}_4 \end{pmatrix}, \qquad \Gamma^5 = \begin{pmatrix} \mathbf{0}_4 & \mathbf{1}_4 \\ -\mathbf{1}_4 & \mathbf{0}_4 \end{pmatrix}, \tag{C.1}$$

where $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \mathrm{diag}(\mathbf{1}_2, -\mathbf{1}_2)$. They satisfy

$$\left\{\Gamma^M, \Gamma^N\right\} = -2\eta^{MN}, \tag{C.2}$$

where $\eta_{MN} = \mathrm{diag}(-1, 1, 1, 1, 1, 1)$ is the 6D Minkowski metric. The 4D gamma matrices γ^μ are decomposed as

$$\gamma^\mu = \begin{pmatrix} \mathbf{0}_2 & \sigma^\mu \\ \bar{\sigma}^\mu & \mathbf{0}_2 \end{pmatrix}. \tag{C.3}$$

The 6D chirality matrix Γ_7 is defined as

$$\Gamma_7 \equiv -\Gamma^0\Gamma^1 \cdots \Gamma^5 = \begin{pmatrix} \mathbf{1}_4 & \mathbf{0}_4 \\ \mathbf{0}_4 & -\mathbf{1}_4 \end{pmatrix}. \tag{C.4}$$

The 6D Weyl fermions $\Psi_\pm$ are expressed by

$$\Psi_+ = \begin{pmatrix} \psi_+ \\ \mathbf{0}_4 \end{pmatrix}, \qquad \Psi_- = \begin{pmatrix} \mathbf{0}_4 \\ \psi_- \end{pmatrix}, \tag{C.5}$$

where the four-component spinors $\psi_\pm$ are further decomposed as

$$\psi_\pm = \begin{pmatrix} \chi_{\pm\alpha} \\ \bar{\zeta}_\pm^{\dot{\alpha}} \end{pmatrix}. \tag{C.6}$$

Appendix D. Number of zero-modes

Here we focus on the zero-mode solutions of the mode equations in Eq. (5.17).[14]

Appendix D.1. In the presence of Yukawa coupling

The mode equations for the zero-modes $b^{(0)[m]}_{\mathrm{R}\pm}(\rho)$ and $b^{(0)[m]}_{\mathrm{L}\pm}(\rho)$ are

$$\left\{\partial_\rho + \frac{q_+ n\alpha(\rho) - m}{\rho}\right\} b^{(0)[m]}_{\mathrm{R}+} - \tilde{y}_1 f_1(\rho) b^{(0)[m]}_{\mathrm{R}-} = 0,$$
$$\left\{\partial_\rho - \frac{q_- n\alpha(\rho) - m - n - 1}{\rho}\right\} b^{(0)[m]}_{\mathrm{R}-} - \tilde{y}_1 f_1(\rho) b^{(0)[m]}_{\mathrm{R}+} = 0,$$
$$\left\{\partial_\rho - \frac{q_+ n\alpha(\rho) - m - 1}{\rho}\right\} b^{(0)[m]}_{\mathrm{L}+} + \tilde{y}_1 f_1(\rho) b^{(0)[m]}_{\mathrm{L}-} = 0,$$
$$\left\{\partial_\rho + \frac{q_- n\alpha(\rho) - m - n}{\rho}\right\} b^{(0)[m]}_{\mathrm{L}-} + \tilde{y}_1 f_1(\rho) b^{(0)[m]}_{\mathrm{L}+} = 0. \tag{D.1}$$

Note that $b^{(0)[m]}_{\mathrm{R}\pm}(\rho)$ and $b^{(0)[m]}_{\mathrm{L}\pm}(\rho)$ are decoupled in these equations. Thus they can be parametrized by independent labels for m. Here we will write them as $b^{(0)[m]}_{\mathrm{R}\pm}(\rho)$ and $b^{(0)[m']}_{\mathrm{L}\pm}(\rho)$ to emphasize this point.

Let us consider the behavior for $\rho \gg 1$. Since $f_1(\rho), \alpha(\rho) \simeq 1$ in this region and the terms proportional to ρ^{-1} are neglected, the asymptotic forms of the mode functions are[15]

$$b^{(0)[m]}_{\mathrm{R}\pm}(\rho),\ b^{(0)[m']}_{\mathrm{L}\pm}(\rho) \sim e^{-|\tilde{y}_1|\rho}. \tag{D.2}$$

Next we consider the behavior around the vortex core $\rho \ll 1$. Using Eq. (3.15), the solutions of Eq. (D.1) are expressed as

$$b^{(0)[m]}_{\mathrm{R}+}(\rho) \simeq A^{[m]}_{\mathrm{R}+}\rho^{m} + B^{[m]}_{\mathrm{R}+}\rho^{-m+|n|-n},$$
$$b^{(0)[m]}_{\mathrm{R}-}(\rho) \simeq A^{[m]}_{\mathrm{R}-}\rho^{m+|n|+1} + B^{[m]}_{\mathrm{R}-}\rho^{-m-n-1},$$
$$b^{(0)[m']}_{\mathrm{L}+}(\rho) \simeq A^{[m']}_{\mathrm{L}+}\rho^{-m'-1} + B^{[m']}_{\mathrm{L}+}\rho^{m'+|n|+n+1},$$
$$b^{(0)[m']}_{\mathrm{L}-}(\rho) \simeq A^{[m']}_{\mathrm{L}-}\rho^{-m'+|n|} + B^{[m']}_{\mathrm{L}-}\rho^{m'+n}, \tag{D.3}$$

where $A^{[m]}_{\mathrm{R}\pm}$, $B^{[m]}_{\mathrm{R}\pm}$, $A^{[m']}_{\mathrm{L}\pm}$, and $B^{[m']}_{\mathrm{L}\pm}$ are constants. The regularity at the origin requires that all the powers in Eq. (D.3) should be non-negative. This leads to the following constraints on m and m':

Case of $\boldsymbol{n > 0}$:There is no solution for m, and

$$-n \le m' \le -1. \tag{D.4}$$

Thus we have n left-handed zero-modes.

Case of $\boldsymbol{n < 0}$:There is no solution for m', and

$$0 \le m \le |n| - 1. \tag{D.5}$$

[14] Note that these solutions are not the mass eigenstates except for the static case ($\omega = 0$). The "zero-modes" in this section denote the modes with zero-eigenvalue for the mode equations in Eq. (5.17).

[15] The solution proportional to $e^{|\tilde{y}_1|\rho}$ is non-normalizable, and is excluded.

Thus we have $|n|$ right-handed zero-modes.

In either case, $|n|$ 4D chiral fermions are obtained in the effective theory [17].

Appendix D.2. In the absence of Yukawa coupling

Next, we consider the case in which the fermions do not couple to the scalar fields. In this case, the four equations in Eq. (5.9) are decoupled and can be solved independently. We can separate the variables by assuming that

$$\begin{aligned} h_{\mathrm{R}+}^{(0)[m_+]}(\rho,\theta) &= b_{\mathrm{R}+}^{(0)[m_+]}(\rho)e^{im_+\theta}, \\ h_{\mathrm{R}-}^{(0)[m_-]}(\rho,\theta) &= b_{\mathrm{R}-}^{(0)[m_-]}(\rho)e^{im_-\theta}, \\ h_{\mathrm{L}+}^{(0)[m'_+]}(\rho,\theta) &= b_{\mathrm{L}+}^{(0)[m'_+]}(\rho)e^{im'_+\theta}, \\ h_{\mathrm{L}-}^{(0)[m'_-]}(\rho,\theta) &= b_{\mathrm{L}-}^{(0)[m'_-]}(\rho)e^{im'_-\theta}, \end{aligned} \tag{D.6}$$

where $m_\pm$ and $m'_\pm$ are integers. The solutions are

$$\begin{aligned} b_{\mathrm{R}\pm}^{(0)[m_\pm]}(\rho) &= C_{\mathrm{R}\pm}^{[m_\pm]} \exp\left\{\pm\int_1^\rho d\rho'\ \frac{-q_\pm n\alpha(\rho') + m_\pm}{\rho'}\right\}, \\ b_{\mathrm{L}\pm}^{(0)[m'_\pm]}(\rho) &= C_{\mathrm{L}\pm}^{[m'_\pm]} \exp\left\{\mp\int_1^\rho d\rho'\ \frac{-q_\pm n\alpha(\rho') + m'_\pm}{\rho'}\right\}, \end{aligned} \tag{D.7}$$

where $C_{\mathrm{R}\pm}^{[m_\pm]}$ and $C_{\mathrm{L}\pm}^{[m'_\pm]}$ are normalization constants. In a region of $\rho \gg 1$, they behave as

$$\begin{aligned} b_{\mathrm{R}\pm}^{(0)[m_\pm]}(\rho) &\simeq C_{\mathrm{R}\pm}^{[m_\pm]} \exp\{\pm(-q_\pm n + m_\pm)\ln\rho\} = C_{\mathrm{R}\pm}^{[m_\pm]}\rho^{\pm(-q_\pm n + m_\pm)}, \\ b_{\mathrm{L}\pm}^{(0)[m'_\pm]}(\rho) &\simeq C_{\mathrm{L}\pm}^{[m'_\pm]} \exp\left\{\mp(-q_\pm n + m'_\pm)\ln\rho\right\} = C_{\mathrm{L}\pm}^{[m'_\pm]}\rho^{\mp(-q_\pm n + m'_\pm)}. \end{aligned} \tag{D.8}$$

The normalization conditions require that

$$1 \pm 2(-q_\pm n + m_\pm) < -1, \qquad 1 \mp 2(-q_\pm n + m'_\pm) < -1. \tag{D.9}$$

In a region of $\rho \ll 1$, Eq. (D.7) is approximated as

$$\begin{aligned} b_{\mathrm{R}\pm}^{(0)[m_\pm]}(\rho) &\simeq C_{\mathrm{R}\pm}^{[m_\pm]} \exp(\pm m_\pm \ln\rho) = C_{\mathrm{R}\pm}^{[m_\pm]}\rho^{\pm m_\pm}, \\ b_{\mathrm{L}\pm}^{(0)[m'_\pm]}(\rho) &\simeq C_{\mathrm{L}\pm}^{[m'_\pm]} \exp\left(\mp m'_\pm \ln\rho\right) = C_{\mathrm{L}\pm}^{[m'_\pm]}\rho^{\mp m'_\pm}. \end{aligned} \tag{D.10}$$

Hence, the regularity at the origin requires that

$$\pm m_\pm \geq 0, \qquad \mp m'_\pm \geq 0. \tag{D.11}$$

From Eqs. (D.9) and (D.11), the integers $m_\pm$ and $m'_\pm$ are constrained as

$$\begin{aligned} &0 \leq m_+ < q_+ n - 1, \qquad q_- n + 1 < m_- \leq 0, \\ &q_+ n + 1 < m'_+ \leq 0, \qquad 0 \leq m'_- < q_- n - 1. \end{aligned} \tag{D.12}$$

Thus, the number of zero-modes depends on the charges $q_\pm$, in contrast to the previous case. In the absence of Yukawa interactions, it is determined by the charge and the flux threading the extra-dimensional space, as the index theorem insists [18,19]. However, because the extra-dimensional space is non-compact in our model, some of the zero-modes are non-normalizable and are dropped in the spectrum. So the number of zero-modes is smaller than that of the compact case.

When we turn on the Yukawa coupling y_1 or y_2, some of the zero-modes allowed in this subsection obtain masses via the Yukawa coupling and are decoupled at low energies. The number of remaining zero-modes only depends on the vortex number n and is independent of the charges [17], as we saw in the previous subsection.

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