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A meta-analysis of neutron lifetime measurements

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We calculate the median as well as weighted mean central estimates for the neutron lifetime from a subset of measurements compiled in the 2019 update of the Particle Data Group (PDG). We then reconstruct the error distributions for the residuals using three different central estimates and then check for consistency with a Gaussian distribution. We find that although the error distributions using the weighted mean as well as median estimate are consistent with a Gaussian distribution, the Student's t and Cauchy distribution provide a better fit. This median statistic estimate of the neutron lifetime from these measurements is given by 881.5 ± 0.47 seconds. This can be used as an alternate estimate of the neutron lifetime. We also note that the discrepancy between beam and bottle-based measurements using median statistics of the neutron lifetime persists with a significance between $4\,\sigma$ and $8\,\sigma$, depending on which combination of measurements is used.

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1. Introduction

The precise measurement and theoretical estimate of the neutron lifetime is of paramount importance for both particle physics and astrophysics [1,2]. The current weighted average of seven neutron lifetime measurements, reported in the 2019 version of the Particle Data Group [3] (PDG, hereafter)[1] using the seven best measurements is 879.4 ± 0.6 seconds. At face value, the weighted mean error from these measurements is equal to 0.4 seconds. Therefore, the reduced χ^2 value for a constant neutron lifetime is equal to 14.6 for six degrees of freedom, corresponding to a p-value of 0.023 [4]. If we define the significance as the number of standard deviations a Gaussian variable would fluctuate in one direction corresponding to this p-value, then the observed p-value corresponds to a $2\,\sigma$ [5] discrepancy for a constant value of the neutron lifetime. Therefore, the PDG has scaled the weighted mean error by a scale factor equal to $\sqrt{\chi^2/\nu}$, where ν is the total degrees of freedom. With this multiplicative scale factor of 1.6, the total error is now equal to the reported value of 0.6 seconds. Therefore, the subset of neutron lifetime measurements vetted by the PDG are inconsistent with a constant value at $2\,\sigma$ significance.

The theoretical neutron lifetime is a function of the axial vector to vector coupling ratio as well as the CKM matrix element V_{ub} [6,7]. The most recent theoretical estimate of the neutron lifetime is between 875.3 and 891.2 seconds, within $3\,\sigma$ [6]. Theoretical uncertainties in the neutron lifetime calculation, and expected improvements in the near future, have recently been reviewed in Ref. [7].

Neutron lifetime measurement techniques can be broadly classified into two types: "bottle"- and "beam"-based measurements. In the bottle method, ultra-cold neutrons are stored in a container

[1] At the time of writing, the 2019 PDG update on neutron lifetime measurements is only available online at http://pdg.lbl.gov/2019/listings/rpp2019-list-n.pdf. The published version [3] contains listings from 2018.

(which consists of either some bottle or a trap), and the neutron lifetime is measured by fitting the surviving neutrons to a decaying exponential. In the beam method, on the other hand, the numbers of neutrons and protons are produced from β-decay, and the lifetime is obtained from the neutron decay rate. More details about these techniques can be found in Refs. [1,2].

However, there is a long-standing discrepancy between these two methods used for neutron lifetime measurements [8]. As of 2018, the current value from two beam experiments [9,10] included in the 2018 edition of PDG[2] is equal to 888 ± 2.0 seconds [6], and the same from five bottle experiments [11–15] is equal to 879.6 ± 0.6 seconds [6]. This is a formally a $4\,\sigma$ discrepancy, and as pointed out in Fornal and Grienstein [6] (F18 hereafter) could either be evidence of uncontrolled systematics or could point to new physics. Another possibility, however, not mentioned in the above works, is that the measurements could contain non-Gaussian errors, and consequently the weighted mean cannot be used as the central estimate.

The central estimate of the neutron lifetime mentioned in PDG, as well as all other works which analyze this discrepancy, has been obtained from a weighted average of all the measurements. The central estimate of a quantity using weighted measurements makes the following main assumptions [16]: (i) individual data points are statistically independent and contain no systematic effects; (ii) the errors are Gaussianly distributed. If any of the measurements contain catastrophic outliers or unaccounted systematic effects, then the second assumption is automatically violated. In that case, the weighted mean can produce extremely biased results. On the other hand, median statistics do not incorporate the individual measurement errors, and hence are unaffected by the presence of a few outliers. Secondly, even if the errors are not correctly estimated, as shown using simulations of Zeldovich's thought experiment involving watches [17], a median estimate gives a more robust estimate. Even if a dataset is drawn from a distribution with infinite variance such as a Cauchy distribution, the median is a more robust central estimate [16]. Many additional pitfalls in using the weighted mean as a central estimate, and how using the median value ameliorates these problems, can be found in Refs. [16,17] and the references therein. The only assumption used for median-statistic-based estimates is that the measurements are independent and free of systematic errors.

In the last decade, Ratra and collaborators have shown that the error distributions for a whole slew of astrophysical and cosmological measurements are inconsistent with a Gaussian distribution [16–26]. The datasets they explored for this purpose included measurements of H_0 [20], lithium-7 measurements [21] (see also Ref. [27]), distance to the Large Magellanic Cloud [22], the distance to the Galactic Center [28], deuterium abundance [25], etc. For each of these datasets, they have fitted the data to a variety of probability distributions. From all these studies, they inferred that the error distribution is non-Gaussian. Consequently, they have argued that median statistics should be used for the central estimates of these parameters instead of the weighted mean [16,17]. To the best of our knowledge, no one has investigated the Gaussianity of the neutron lifetime measurements (or for that matter any other datasets in PDG). The importance of doing such tests has been stressed in a number of works [16,23,24,29]. Due to the non-Gaussanity of the error residuals for the aforementioned astrophysical datasets, median statistics have been used to obtain central estimates of some of these quantities such as the Hubble constant [16,17,19], Newton's gravitational constant [17], the mean matter density [18], and other cosmological parameters [23]. Alternately, one can use the method recently proposed by Cowan, where the uncertainty in the systematic errors has been modeled using probabilistic distributions [30].

[2] These two measurements are not used for the neutron lifetime estimate by the 2019 PDG edition.

Given the importance of the physics implications of these discrepancies in the neutron lifetime measurements, and to obtain a more robust estimate, which can be easily compared with the theoretical estimate, we revisit the issue of checking for non-Gaussianity of the errors and to obtain a more robust central estimate from the vetted measurements in PDG. The outline of this manuscript is as follows. The dataset used for our analysis is described in Sect. 2. Our analysis procedure and results are described in Sect. 3. We discuss the discrepacy between beam- and bottle-based measurements in Sect. 4. We conclude in Sect. 5.

2. Neutron lifetime data

We briefly review the neutron lifetime measurements used for this analysis. The 2019 edition of PDG lists a total of 27 measurements from 1972 to the present. From these measurements, only seven have been used by the PDG to obtain the central estimate. Using these seven measurements, a weighted mean central value of 879.4 ± 0.6 s was estimated, wherein the error has been rescaled by a factor of 1.6. All of these are bottle-based experiments. The corresponding value from the 2018 PDG edition was 880.2 ± 1.0 s, with five of them been bottle based and two beam based. The remaining measurements were ignored either because the error bars for some of the pre-1980 measurements were large, or if the results from the old measurements were reanalyzed, and lastly because some of the measurements were withdrawn. However, a few measurements have also been culled without any explanation. For our analysis, we also include all older measurements, except if they were reanalyzed or withdrawn. We also include one additional measurement [31], which was not included in either the 2018 or 2019 PDG. In all, we have collected a total of 19 measurements for our analysis, which are tabulated in Table 1. We note that in addition to these direct experimental measurements of neutron lifetime, there are also cosmological constraints on the measurements of neutron lifetime [32]. We do not include them in our analysis, however, as these results are model dependent, and not direct experimental measurements.

3. Analysis

The first step in analyzing the Gaussianity of the error measurements of a dataset is to obtain a central estimate using the available data. For this analysis, we use all 19 measurements tabulated in Table 1. We do not check for Gaussianity of the beam- and bottle-based measurements separately, as the total number of data points in each category is too small for a robust test. However, once the number of measurements in each category grows, this should also be tested to check for systematics in each category. We note that in Ref. [25] a similar analysis was done using 15 deuterium abundance measurements. Similar to the works by Ratra et al. (e.g. Ref. [25], P18 hereafter), we consider two central estimates: the weighted mean and the median.

The median value (τ_{med}) corresponds to the 50% percentile value, for which half of the data points are below and half above. The standard deviation of the median depends upon the distribution it is sampled from. A number of methods have been proposed in the literature to calculate the sample variance of the median [33–35]. For this work, to estimate the 68% confidence interval on the median we use the methodology in P18, based on Ref. [16], as the estimate is made using only the data and is independent of the sampling distribution. The weighted mean central value (τ_{wm}) using the observed neutron lifetime measurements (τ_i) is given by [36]

Table 1. Summary of the 19 measurements used for the analysis. PDG18 refers to the 2018 published version of PDG, and PDG19 refers to the 2019 online update. The last eight are listed in PDG, but not used to calculate the weighted mean neutron lifetime by either PDG edition [3]. The first three measurements are used only in the 2019 edition to calculate the weighted average. The two beam-based measurements [9,10] are only used for the 2018 PDG estimate.

Reference	Neutron lifetime (secs)	Type	Comment
Ezhov 18 [37]	$878.3 \pm 1.6 \pm 1.0$	Bottle	Only in PDG19
Serebrov 17 [38]	$881.5 \pm 0.7 \pm 0.6$	Bottle	Only in PDG19
Pattie 17 [39]	$877.7 \pm 0.7 + 0.4/-0.2$	Bottle	Only in PDG19
Leung 16 [31]	887 ± 39	Bottle	Neither PDG18 nor PDG19
Arzumanov 15 [15]	880.2 ± 1.2	Bottle	PDG
Yue 13 [10]	$887.7 \pm 1.2 \pm 1.9$	Beam	Only in PDG18
Steyerl 12 [14]	$882.5 \pm 1.4 \pm 1.5$	Bottle	PDG
Pichlmaier 10 [13]	$880.7 \pm 1.3 \pm 1.2$	Bottle	PDG
Serebrov 05 [12]	$878.5 \pm 0.7 \pm 0.3$	Bottle	PDG
Byrne 96 [9]	$889.2 \pm 3.0 \pm 3.8$	Beam	Only in PDG18
Mampe 93 [11]	882.6 ± 2.7	Bottle	PDG
Alfikmenov 90 [40]	888.4 ± 2.9	Bottle	PDG (but not used)
Kossakowski 89 [41]	$878 \pm 27 \pm 14$	Beam	PDG (but not used)
Paul 89 [42]	877 ± 10	Bottle	PDG (but not used)
Last 88 [43]	$876 \pm 10 \pm 19$	Beam	PDG (but not used)
Spivak 88 [44]	891 ± 9	Beam	PDG (but not used)
Kosvintsev 86 [45]	903 ± 13	Bottle	PDG (but not used)
Kosvintsev 86 [45]	875 ± 95	Bottle	PDG (but not used)
Christensen 72 [46]	918 ± 14	Beam	PDG (but not used)

$$\tau_{\rm wm} = \frac{\sum\limits_{i=1}^{N} \tau_i/\sigma_i^2}{\sum\limits_{i=1}^{N} 1/\sigma_i^2}, \tag{1}$$

where σ_i denotes the total error in each measurement. The total weighted mean error is given by

$$\sigma_{\rm M}^2 = \frac{1}{\sum\limits_{i=1}^{N} 1/\sigma_i^2}. \tag{2}$$

From the measurements in Table 1, the weighted mean estimate is found to be $\tau_{\rm wm} = 879.97 \pm 0.39$ seconds, and the median estimate is calculated to be $\tau_{\rm med} = 881.5 \pm 0.47$ seconds.

3.1. Error distributions

Once we have a central estimate for the neutron lifetime ($\tau_{\rm CE}$) using one of the above three methods, we calculate the residual error using [25,28]

$$N_{\sigma_i} = \frac{\tau_i - \tau_{\rm CE}}{\sqrt{\sigma_i^2 + \sigma_{\rm CE}^2}}. \tag{3}$$

In the above equation, $\sigma_{\rm CE}$ is the error in the central estimate and σ_i is the error in the individual measurement. Similar to Refs. [25,26,28], we denote our error distribution for the median ($\tau_{\rm med}$) and the weighted mean ($\tau_{\rm wm}$) calculated from Eq. 3 by $N_{\sigma_i}^{\rm med}$ and $N_{\sigma_i}^{\rm wm+}$ respectively. If the central estimate is determined from the weighted mean, one must also account for correlations, and the modified version of the error distribution that accounts for these correlations is given by [28]

$$N_{\sigma_i}^{\mathrm{wm}-} = \frac{\tau_i - \tau_{\mathrm{CE}}}{\sqrt{\sigma_i^2 - \sigma_{\mathrm{CE}}^2}}. \tag{4}$$

Each of these three sets of $|N_\sigma|$ histograms is then symmetrized around zero. We then fit the symmetrized histogram of $|N_{\sigma_i}|$ to multiple probability distributions as described in the next subsection.

3.2. Fits to probability distributions

We fit the symmetrized histograms for each of the $|N_\sigma|$ to a Gaussian distribution as well as to variants of Gaussian distributions, such as Cauchy, Laplacian, and Student's *t* distribution, to see which of these is most compatible with the data. This is similar in spirit to recent works by Ratra et al., such as P18 and references therein. We briefly review this procedure; more details can be found in P18.

The Gaussian distribution we consider has zero mean and standard deviation equal to unity:

$$P(N) = \frac{1}{\sqrt{2\pi}} \exp(-|N|^2/2). \tag{5}$$

The second distribution we consider is the Laplacian distribution, which has a sharp peak and longer tails than a Gaussian distribution and is described by

$$P(N) = \frac{1}{2} \exp(-|N|). \tag{6}$$

The third distribution we will use is the Cauchy or Lorentz distribution. It has longer and thicker tails compared to a Gaussian distribution. It is described by

$$P(N) = \frac{1}{\pi(1 + |N|^2)}. \tag{7}$$

Finally, we use the Student's *t* distribution characterized by *n* (which is sometimes referred to as "degrees of freedom") and is given by

$$P(N) = \frac{\Gamma[(n+1)/2]}{\sqrt{\pi n}\,\Gamma(n/2)(1 + |N|^2/n)^{(n+1)/2}}. \tag{8}$$

For $n = 1$ the Student's *t* distribution is same as the Cauchy distribution, and it is equal to the Gaussian distribution for $n = \infty$. For our analysis we vary *n* from 2 to 2000. Note that the Student's *t* distribution for the error residuals can be obtained by modeling the error in systematic errors as a gamma distribution [30].

In addition to comparing the error distributions to the probability distribution functions (PDFs) in Eqs. 5–8, which mainly depend on $|N|$, we also compare to these distributions after replacing N by N/S, where S is an arbitrary scale factor, which we vary from 0.001 to 2.5 in steps of size 0.01.

The comparison is done using the one-sample unbinned Kolmogorov–Smirnov (K–S) test [47]. The K–S test is based on the D statistic, which measures the maximum distance between two cumulative distributions. The K–S test is widely used in both astrophysics and particle physics to compare a dataset to a wide range of probability distributions, as it is agnostic to the distribution against which it is being tested, and does not depend on the size of the sample. Furthermore, critical values based

Table 2. Probabilities from the K–S test for various distributions using the observed neutron lifetime measurements.*

	Median (τ_{med})			Weighted mean (τ_{wm+})			Weighted mean (τ_{wm-})		
Distribution	S	p	n	S	p	n	S	p	n
Gaussian	1	0.299		1	0.327		1	0.186	
	1.317	0.875		1.378	0.974		1.562	0.958	
Laplacian	1	0.771		1	0.691		1	0.556	
	1.214	0.983		1.304	0.996		1.428	0.996	
Cauchy	1	0.878		1	0.908		1	0.925	
	0.786	0.997		0.817	0.982		0.85	0.980	
Student's t	1	0.954	2	1	0.928	2	1	0.267	2
	1.021	0.966	2	1.091	0.989	2	1.201	0.987	2

* S: The scale factor (other than 1) which maximizes p. p: The p-value that the data is derived from the PDF. n: The value n in the Student's t distribution.

upon the D statistic have been calculated in the literature and can be easily computed for any value of D. This test is also invariant to reparameterization of the data. The one-sample K–S test can therefore serve as a goodness-of-fit test. Although some concerns have been raised regarding incorrect usage of the K–S test in the astrophysics literature, as well as other caveats and limitations of this test [48], these do not apply in our case, and hence we use the K–S test to evaluate the compatibility of the error residuals with various distributions. In this case, the two distributions are the error histograms and the parent PDF to which it is compared. From the D statistic, the K–S test also provides a p-value, whose analytic formula can be found in any statistics work [25,47]. For this work we used the `scipy` module in `Python` for the computations. The higher the p-value, the more similar the two distributions, whereas a low p-value indicates an inconsistency between the distributions. Our results for the comparison with all four distributions are summarized in Table 2.

We find that for all three estimates the Gaussian distribution is not the best fit, unless the scale factor is different from unity. The data are much more consistent with the Cauchy or Student's t distribution. However, none of the p-values for the Gaussian distribution are small enough to reject the null hypothesis.

4. Discrepancy between beam and bottle measurements

We now quantify the significance of the discrepancy between beam- and bottle-based experiments using central estimates based on the median statistics. We do this analysis using three different combinations of datasets for beam- and bottle-based experiments. A summary of these comparisons can be found in Table 3.

We first use the same data points as in F18 [6], who argued for a $4.4\,\sigma$ discrepancy. We obtain a median estimate using the same bottle-based experiments considered in F18 [11–15], and compare the same with the beam-based experiments therein [9,10]. The median lifetime of the five bottle-based experiments along with the $1\,\sigma$ median error bar is given by 880.7 ± 1.3 seconds. The corresponding lifetime for the two beam-based experiments considered in F18 is 888.45 seconds. Since it is not possible to obtain a median error estimate with just two measurements, we do not quote its $1\,\sigma$ median uncertainty. The results do not change even after including the two additional bottle-based measurements [38,39] not used for their average. Therefore, considering the median statistics estimates, the discrepancy is about $6\,\sigma$.

Table 3. Summary of the significance of the discrepancies between beam and bottle-based measurements using median statistics. The first column refers to the datasets used. The second and third columns contain the median statistics estimate of the neutron lifetime(τ_N) using bottle and beam-based measurements respectively, using 1σ median error bars obtained using the procedure in Ref. [16]. The last column indicates the statistical significance of the discrepancy.

Dataset	τ_N (bottle-based) (s)	τ_N (beam-based) (s)	Discrepancy
F18 [6]	880.7 ± 1.3	888.45	$6\,\sigma$
Data from Table 1	880.7 ± 1.2	888.45 ± 1.65	$3.79\,\sigma$
Data from Table 1 with errors < 10 s	880.2 ± 1.1	889.2	$8.2\,\sigma$

If we do this comparison by including all the measurements in Table 3, the median lifetime for all the bottle-based experiments is equal to 880.7 ± 1.2 seconds. The corresponding number for all the beam-based experiments is 888.45 ± 1.65 seconds. Therefore, comparing the median estimates between the beam- and bottle-based measurements amounts to a $3.79\,\sigma$ discrepancy.

If we then redo this comparison for the subset of the measurements in Table 1 having a total error less than 10 seconds, the median central estimate for all bottle-based experiments is 880.2 ± 1.1 s. Since the total number of beam-based measurements in Table 1 is a very small number (three), we can only obtain a central estimate, which is equal to 889.2 seconds. Therefore, the total discrepancy is about $8.2\,\sigma$.

Hence, we infer that the discrepancy between beam- and bottle-based measurements persists, even when median statistics is used for the central estimate of the neutron lifetime.

5. Conclusions

There has been a long-standing discrepancy in the literature related to neutron lifetime measurements between two different techniques, viz. bottle- and beam-based methods. As of 2019, the current discrepancy is about $4\,\sigma$ [6]. To get some insight into these issues, we carried out an extensive meta-analysis of the vetted neutron lifetime measurements compiled in the literature. We first used a compilation of 19 measurements of the neutron lifetime and their corresponding errors listed in the 2019 edition of PDG [3] (cf. Table 1), in order to ascertain the non-Gaussianity of the residuals and to obtain a central estimate. The error distributions were analyzed in the same way as previously done for a variety of astrophysical measurements by Ratra et al. [25,26,28]. For this purpose, the central estimate was obtained using both the weighted mean (with and without correlations) and the median value. The median estimate does not incorporate the errors in the neutron lifetime. We then fitted these residuals to four distributions, viz. Gaussian, Laplace, Cauchy, and Student's t distribution. The resulting fits are tabulated in Table 2.

We conclude from these observations that none of the p-values (obtained using all three central estimates) are small enough to reject the Gaussian distribution for the error residuals. However, the Student's t and Cauchy distributions provide a more robust fit than the Gaussian distribution.

Therefore, more data points are necessary to robustly determine if the error residuals are consistent with a Gaussian distribution. Nevertheless, it would be a useful exercise to obtain the central estimate of the neutron lifetime with median statistics, and to check if the discrepancy between beam- and bottle-based measurements persists using median statistics. The median value along with $1\,\sigma$ error bars using the 19 measurements that we obtained is given by 881.5 ± 0.47 seconds. This estimate is complementary to the PDG-based result obtained using weighted mean statistics, which includes the addition of an ad hoc scale factor. This value can be used as an alternate estimate of the observed neutron lifetime, and used for comparison with the theoretical estimate, which is currently between 875.3 and 891.2 seconds within $3\,\sigma$ [6]. Furthermore, this median value provides an alternate central estimate of the neutron lifetime, which can be used for comparison with theoretical estimates.

We then used the median estimate to evaluate the statistical significance of the discrepancy between beam- and bottle-based measurements. When we use the same measurements as in F18, the discrepancy exacerbates to $6\,\sigma$. If we consider all the measurements in Table 1, the discrepancy becomes $3.8\,\sigma$ ($8.2\,\sigma$), depending on whether we include (exclude) measurements in this, with the total error less than 10 seconds.

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