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A novel approach to the computation of one-loop three- and four-point functions. III. The infrared divergent case

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This article is the third and last of a series presenting an alternative method for computing the one-loop scalar integrals. It extends the results of the first two articles to the infrared divergent case. This novel method enjoys a couple of interesting features as compared with the methods found in the literature. It directly proceeds in terms of the quantities driving algebraic reduction methods. It yields a simple decision tree based on the vanishing of internal masses and one-pinched kinematic matrices, which avoids a profusion of cases. Lastly, it extends to kinematics more general than the physical, e.g. collider processes, relevant at one loop. This last feature may be useful when considering the application of this method beyond one loop using generalized one-loop integrals as building blocks.

Subject Index B30, B57

1. Introduction

This article is the third of a triptych. The first one [1] presented a method exploiting a Stokes-type identity to compute "generalized" (in the sense of the underlying kinematics) one-loop three- and four-point scalar integrals for the real mass case. The second article [2] extended the results of the first paper to the case of general complex masses. The present article widens the results of Refs. [1] and [2] to the case where some internal masses are vanishing, leading to infrared divergences. We refer the reader to Ref. [3] for more details on the motivation of this work.

The scalar Feynman integrals for one-loop three- and four-point functions are all known and have been compiled in a useful article [4]. This article relies mainly on the results of other publications, especially the important work of Beenakker and Denner [5]. We should also mention Ref. [6], which provides a complete set of results for soft and/or collinear divergent four-point functions using different kinds of IR regulators. The purpose of the present article is to extend these results for more general kinematics beyond those relevant for collider processes at the one-loop order. Note that despite the fact that some internal masses may vanish, the others can be real or complex and we treat both cases in this article. The soft and collinear divergences are dealt with using dimensional regularization, $n = 4 - 2\,\varepsilon$, and doing an ε expansion.

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The outline of this article follows closely that of the preceding articles in Refs. [1] and [2]. We start by considering the three-point function I_3^n in a spacetime dimension shifted by a small amount from 4 to n. The kinematics leading to infrared divergences are discussed as a warm-up for Sect. 3. We successively present two variants of the method. The simplest variant, labelled the "direct way," is presented in Sect. 2.1. It is well suited for the three-point function, but cannot be extended to the case of the four-point function. Then, in Sect. 2.2, practical implementation of the results of the preceding subsection is discussed and some explicit examples are computed and compared to Ref. [4]. In Sect. 2.3 we present an alternative coined the "indirect way," easily applicable to the four-point case which is the subject of Sect. 3.

We first explain, in Sect. 3.1, how to extend the calculation of I_4^4 developed in Ref. [1] to the case where the infrared divergences are regulated in n dimensions. The net result is that the four-point scalar integral can be decomposed on sectors labelled by three indices, and a three-dimensional integral over the first octant of $\mathbb{R}^3$ is associated to each sector. Then, two cases are distinguished depending on the sectors. In the first case, presented in Sect. 3.2, the determinant of the one-pinched kinematical matrix $\mathcal{S}$ vanishes but not the internal mass associated with this sector: this case is met when a soft divergence appears. In the second case, presented in Sect. 3.3, both the determinant of the one-pinched $\mathcal{S}$ matrix and the internal mass associated with this sector vanish: this case is met when a collinear or a soft and collinear divergence shows up. In Sect. 3.4, the infrared divergent part of the scalar four-point integral is shown to be proportional to a three-point scalar integral, as it should be. Some explicit examples are given and compared to the results found in the literature. We then conclude.

Various appendices gather a number of utilities removed from the main text to facilitate its reading. Accordingly, in Appendix A we complete Appendix D of Ref. [1] and Appendix A of Ref. [2] by giving a missing case where the power of the integration variable is not an integer as required by dimensional regularization. Appendix B shows how to compute an integral appearing in the three-point case in closed form. Appendix C collects a bunch of integrals required to compute the three- and four-point functions having soft and/or collinear divergences in the case of general complex masses. Then, Appendix D goes through the examples given in Sect. 2.2 and explains in detail how the results obtained in the latter subsection can be found again from those derived in the "indirect way" case. Appendix E provides the way to compute the last integration in closed form in terms of dilogarithms for the case of infrared divergent integrals. It complements Appendix E of Ref. [1] and Appendix B of Ref. [2]. Lastly, Appendix F proves a tricky point used in Sect. 3: for the real mass case, the sign of the vanishing imaginary part of the denominator in the last integral can be safely changed.

2. Three-point function with infrared divergences

When some internal masses vanish, divergences of collinear or soft origin appear and the approach will be revisited. We regularize these divergences using dimensional regularization, shifting the dimension of the spacetime by a small positive amount from 4 to $n = 4 - 2\,\varepsilon$ with $\varepsilon < 0$. After performing the loop momentum integral, instead of Eq. (2.3) of Ref. [1] we get[1]

$$I_3^n = -\Gamma(1+\varepsilon) \int_0^1 \prod_{i=1}^{3} dz_i \, \delta\Big(1 - \sum_{i=1}^{3} z_i\Big) \left(-\frac{1}{2}\, z^{\mathrm{T}} \cdot \mathcal{S} \cdot z - i\,\lambda\right)^{-1-\varepsilon}. \tag{2.1}$$

[1] As in Refs. [1,2], we assume that the elements of the kinematic matrix $\mathcal{S}$ have been made dimensionless by an appropriate rescaling.

To appropriately shift the power of the denominator in Eq. (2.1) so as to apply the Stokes identity in Eq. (1.2) of Ref. [1] as we did in the massive case, we use the following modified integral representation instead of the identity in Eq. (2.9) of Ref. [1] (cf. Appendix B of Ref. [1]):

$$\frac{1}{D^{1+\varepsilon}} = \frac{\nu}{B(2-1/\nu, 1/\nu)} \int_0^{+\infty} \frac{d\xi}{(D+\xi^{\nu})^2},$$

with $\nu = 1/(1-\varepsilon)$. Instead of Eq. (2.10) of Ref. [1] we now get

$$I_3^n = -\,2^{1+\varepsilon}\,\frac{\Gamma(1+\varepsilon)}{1-\varepsilon}\,\frac{1}{B(1+\varepsilon,1-\varepsilon)} \int_0^{+\infty} d\xi \int_{\Sigma_{bc}} \frac{dx_b\,dx_c}{(D^{(a)}(x_b,x_c)+\xi^{\nu}-i\,\lambda)^2}. \tag{2.2}$$

We otherwise proceed as in Sect. 2.1 of Ref. [1]. The counterpart of Eq. (2.22) in Ref. [1] now reads

$$\begin{aligned} I_3^n = {} & 2^{\varepsilon}\,\frac{\Gamma(1+\varepsilon)}{1-\varepsilon}\,\frac{1}{B(1+\varepsilon,1-\varepsilon)} \\ & \times \sum_{i\in S_3} \frac{\overline{b}_i}{\det(G)} \int_0^{+\infty} \frac{d\xi}{\Delta_2-\xi^{\nu}+i\,\lambda} \int_0^1 \frac{dx}{D^{\{i\}(j)}(x)+\xi^{\nu}-i\,\lambda}, \end{aligned} \tag{2.3}$$

where $j \in S_3 \setminus \{i\}$ ($S_3 = \{1,2,3\}$). More precisely, we assume that j is chosen to be $1+(i$ modulo 3). Similarly to what we did for the three-point function in the massive case, one can also consider both a "direct way" and an "indirect way" in the IR case. We first focus on the "direct way," which provides a more straightforward and synthetic discussion of the various cases at hand. We then illustrate how these cases are involved in a few examples. The "indirect way" instead leads to a cumbersome split-up discussion. Notwithstanding this, the calculation of the four-point one-loop integral relying on the approach described in this article proceeds along the "indirect way" as we found no extension of the "direct way" approach in this case. In Refs. [7,8] it was shown on general grounds using the decomposition[2]

$$\det(\mathcal{S})\, I_4^n(\mathcal{S}) = \sum_{i=1}^{4} \overline{b}_i\, I_3^n(\mathcal{S}^{\{i\}}) - \det(G)\,(1-2\,\varepsilon)\, I_4^{n+2}(\mathcal{S}) \tag{2.4}$$

that the infrared structure of any IR divergent four-point one-loop integral is carried by IR-divergent three-point one-loop functions resulting from appropriate iterated pinchings. Therefore the comparison of the IR structures on both sides of Eq. (2.4) proceeds most conveniently via a term-by-term comparison using the three-point one-loop functions decomposed according to the "indirect way" as well. In anticipation, we hereby give the key ingredients to perform this comparison, as well as the general recombination of these "indirect way" ingredients into the more compact expression obtained from the "direct way," thereby checking their equivalence. The extensive collection of expressions computed in closed form which enable us to perform detailed case-by-case comparisons is gathered in Appendix D to lighten the presentation.

2.1. Direct way

Soft and/or collinear divergences are caused by some vanishing masses which make $\det(\mathcal{S})$ vanish so that $\Delta_2 = 0$, whereas the other internal masses may or may not vanish as well, and may even

[2] This decomposition has been discovered before and used for different purposes, see Refs. [9–12].

be complex. We will keep the $-i\,\lambda$ prescription, bearing in mind that it is ineffective in the case of complex masses.

Starting from Eq. (2.3) and performing the ξ integration using Eq. (A.4), we end up with

$$I_3^n = \frac{2^\varepsilon}{\varepsilon}\,\Gamma(1+\varepsilon) \sum_{i \in S_3} \frac{\overline{b}_i}{\det(G)} \int_0^1 dx \left(D^{\{i\}(j)}(x) - i\,\lambda\right)^{-1-\varepsilon}. \tag{2.5}$$

In the general case $D^{\{i\}(j)}(x)$ depends on two internal masses squared, m_j^2 and m_k^2, such that $m_j^2 = D^{\{i\}(j)}(0)/2 = \widetilde{D}_{ik}/2$ and $m_k^2 = D^{\{i\}(j)}(1)/2 = \widetilde{D}_{ij}/2$, cf. Sect. 2 of Ref. [1]. We introduced the label k, which is the only element of the complement of $\{i, j\}$ in S_3. With our assumption on j, this implies that $k \equiv 1 + ((i+1) \text{ modulo } 3)$. Let us focus on the function W given by

$$W\big(\det(G^{\{i\}}), \widetilde{D}_{ij}, \widetilde{D}_{ik}\big) = \frac{2^\varepsilon}{\varepsilon}\,\Gamma(1+\varepsilon) \int_0^1 dx \left(D^{\{i\}(j)}(x) - i\,\lambda\right)^{-1-\varepsilon}. \tag{2.6}$$

Remember that (cf. Eqs. (2.16), (2.17), and (2.18) of Ref. [1])

$$D^{\{i\}(j)}(x) = G^{\{i\}(j)}\,x^2 - 2\,V^{\{i\}(j)}\,x - C^{\{i\}(j)} \tag{2.7}$$

with

$$\begin{aligned} G^{\{i\}(j)} &= -\mathcal{S}_{kk} + 2\,\mathcal{S}_{kj} - \mathcal{S}_{jj} = \det(G^{\{i\}}), \\ V^{\{i\}(j)} &= \mathcal{S}_{kj} - \mathcal{S}_{jj} = \frac{1}{2}\big[\det(G^{\{i\}}) - \widetilde{D}_{ij} + \widetilde{D}_{ik}\big], \\ C^{\{i\}(j)} &= \mathcal{S}_{jj} = -\widetilde{D}_{ik}. \end{aligned} \tag{2.8}$$

The Gram matrix $G^{\{i\}(j)}$ is built from the one-pinched $\mathcal{S}$ matrix $\mathcal{S}^{\{i\}}$, and is a real matrix which depends only on a squared external momentum in the three-point case. Notice that in this case $G^{\{i\}(j)}$ is a 1×1 matrix and $V^{\{i\}(j)}$ a one-dimensional vector this explains the notation[3] used in Eqs. (2.7) and (2.8). Knowledge of $\det(G^{\{i\}})$, $\widetilde{D}_{ij}$, and $\widetilde{D}_{ik}$ fully determines the polynomial $D^{\{i\}(j)}(x)$. These two internal masses may or may not vanish, and hence three cases must be considered.

(a) Neither m_j^2 nor m_k^2 vanishes

We perform a Taylor expansion[4] of $W(\det(G^{\{i\}}), \widetilde{D}_{ij}, \widetilde{D}_{ik})$ in ε:

$$\begin{aligned} &W(\det(G^{\{i\}}), \widetilde{D}_{ij}, \widetilde{D}_{ik}) \\ &\quad = \frac{2^\varepsilon}{\varepsilon}\,\Gamma(1+\varepsilon) \left[\int_0^1 \frac{dx}{D^{\{i\}(j)}(x) - i\,\lambda} - \varepsilon \int_0^1 dx \frac{\ln\left(D^{\{i\}(j)}(x) - i\,\lambda\right)}{D^{\{i\}(j)}(x) - i\,\lambda}\right]. \end{aligned} \tag{2.9}$$

Note that x_1 and x_2, the two roots of $D^{\{i\}(j)}(x) - i\lambda$, are given by [cf. Eqs. (2.7) and (2.8)]

$$x_{\substack{1\\2}} = \frac{\det(G^{\{i\}}) - \widetilde{D}_{ij} + \widetilde{D}_{ik} \pm \sqrt{\mathcal{K}\big(\det(G^{\{i\}}), \widetilde{D}_{ij}, \widetilde{D}_{ik}\big) + i\,\lambda\,S_G}}{2\,\det(G^{\{i\}})}, \tag{2.10}$$

[3] Remember that $\mathcal{S}_{jk} = \mathcal{S}_{jk}^{\{i\}}$ for $j, k \neq i$.

[4] Here and below, only the terms in the ε-expansion providing the divergent and finite terms in the limit $\varepsilon \to 0$ are kept.

where $\mathcal{K}$ is the Källén function,

$$\mathcal{K}(x,y,z) = x^2 + y^2 + z^2 - 2\,x\,y - 2\,x\,z - 2\,y\,z, \tag{2.11}$$

and $S_G = \text{sign}(\det(G^{\{i\}}))$. Then, we introduce $J(x_1,x_2)$ and $K(x_1,x_2)$ defined by

$$K(x_1,x_2) = \int_0^1 dx \, \frac{1}{(x-x_1)\,(x-x_2)}, \tag{2.12}$$

$$J(x_1,x_2) = \int_0^1 dx \, \frac{\ln\left((x-x_1)\,(x-x_2)\right)}{(x-x_1)\,(x-x_2)}. \tag{2.13}$$

As will become clear in the forthcoming paragraph on the origin of infrared singularities, only the case with m_j^2 and m_k^2 both real matters in practice, which makes the explicit calculation of $J(x_1,x_2)$ somewhat simpler.[5] The latter is provided in Appendix B. The x integration in the function $K(x_1,x_2)$ straightforwardly gives

$$K(x_1,x_2) = \frac{1}{x_1 - x_2} \left[\ln\left(\frac{x_1-1}{x_1}\right) - \ln\left(\frac{x_2-1}{x_2}\right) \right]. \tag{2.14}$$

Thus, $W(\det(G^{\{i\}}), \widetilde{D}_{ij}, \widetilde{D}_{ik})$ reads

$$\begin{aligned} &W(\det(G^{\{i\}}), \widetilde{D}_{ij}, \widetilde{D}_{ik}) \\ &= \frac{1}{\varepsilon} \frac{\Gamma(1+\varepsilon)}{\det(G^{\{i\}})} \left\{ \left[1 - \varepsilon \, \ln\left(\frac{\det(G^{\{i\}})}{2} - i\,\lambda \right) \right] K(x_1,x_2) - \varepsilon \, J(x_1,x_2) \right\}. \end{aligned} \tag{2.15}$$

(b) One and only one of m_j^2 and m_k^2 vanishes

Let us assume that the vanishing internal mass is m_j^2. $D^{\{i\}(j)}(x)$ becomes

$$D^{\{i\}(j)}(x) = x \left(G^{\{i\}(j)}\, x - 2\, V^{\{i\}(j)} \right). \tag{2.16}$$

From Eqs. (2.6) and (2.8), $W(\det(G^{\{i\}}), \widetilde{D}_{ij}, 0)$ is thus of the form

$$W(\det(G^{\{i\}}), \widetilde{D}_{ij}, 0) = \frac{2^\varepsilon}{\varepsilon} \, \Gamma(1+\varepsilon) \int_0^1 dx\, x^{-1-\varepsilon} \, (a\,x + z)^{-1-\varepsilon}, \tag{2.17}$$

where $a = \det(G^{\{i\}})$ is real and $z = -\det(G^{\{i\}}) + \widetilde{D}_{ij} - i\,\lambda$ is complex. As z and $a\,x + z$ have imaginary parts of the same sign, $(a\,x+z)^{-1-\varepsilon}$ can be split as follows:

$$(a\,x + z)^{-1-\varepsilon} = z^{-1-\varepsilon} \left(1 + \frac{a}{z}\, x \right)^{-1-\varepsilon}.$$

The right-hand side of Eq. (2.17) involves the Gauss hypergeometric function ${}_2F_1$:

$$W(\det(G^{\{i\}}), \widetilde{D}_{ij}, 0) = -\frac{2^\varepsilon}{\varepsilon^2} \, \Gamma(1+\varepsilon)\, z^{-1-\varepsilon} \, {}_2F_1\left(1+\varepsilon, -\varepsilon; 1-\varepsilon; -\frac{a}{z} \right). \tag{2.18}$$

[5] With real masses, x_1 and x_2 in Eq. (2.10) have imaginary parts of opposite signs. This simplifies splittings and recombinations of logarithms of ratios in the explicit calculation of the function $J(x_1,x_2)$ computed in Appendix B.

We use the identity [13]

$$
{}_2F_1(a,b;c;w) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}\;{}_2F_1(a,b;a+b-c+1;1-w)
+ (1-w)^{c-a-b}\,\frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)}\;{}_2F_1(c-a,c-b;c-a-b+1;1-w)
$$

and the Pfaff identity

$$
{}_2F_1(a,b;c;w) = (1-w)^{-b}\,{}_2F_1\left(c-a,b;c;\frac{w}{w-1}\right)
$$

to rewrite

$$
\begin{aligned}
&{}_2F_1\left(1+\varepsilon,-\varepsilon;1-\varepsilon;-\frac{a}{z}\right)\\
&= 2\,\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,\left(-\frac{a}{z}\right)^{\varepsilon} - \left(\frac{a+z}{z}\right)^{-\varepsilon}\,\left(-\frac{a}{z}\right)^{2\,\varepsilon}\,{}_2F_1\left(-2\,\varepsilon,-\varepsilon;1-\varepsilon;\frac{a+z}{a}\right).
\end{aligned}
\tag{2.19}
$$

Performing a Taylor expansion in ε we get

$$
{}_2F_1\left(-2\,\varepsilon,-\,\varepsilon;1-\varepsilon;\tau\right) = 1 + 2\,\varepsilon^2\,\mathrm{Li}_2(\tau),
$$

and splitting $\ln((a+z)/z) = \ln(a+z) - \ln(z)$, we rewrite $W(\det(G^{\{i\}}),\widetilde{D}_{ij},0)$ as

$$
\begin{aligned}
W(\det(G^{\{i\}}),\widetilde{D}_{ij},0) = \frac{2^{\varepsilon}}{\varepsilon^2}\,\Gamma(1+\varepsilon)\,\frac{1}{z}\,\Bigg\{&(a+z)^{-\varepsilon}\,\left(-\frac{a}{z}\right)^{2\,\varepsilon}\left[1+2\,\varepsilon^2\,\mathrm{Li}_2\left(\frac{a+z}{a}\right)\right]\\
&- 2\,\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,(z)^{-\varepsilon}\,\left(-\frac{a}{z}\right)^{\varepsilon}\Bigg\}.
\end{aligned}
\tag{2.20}
$$

Making explicit that $z = -\det(G^{\{i\}}) + \widetilde{D}_{ij} - i\,\lambda$, $a+z = \widetilde{D}_{ij} - i\,\lambda$, we get

$$
\begin{aligned}
&W(\det(G^{\{i\}}),\widetilde{D}_{ij},0)\\
&= -\frac{2^{\varepsilon}}{\varepsilon^2}\,\Gamma(1+\varepsilon)\,\frac{1}{\det(G^{\{i\}})-\widetilde{D}_{ij}}\\
&\times\Bigg\{\left(\frac{\det(G^{\{i\}})}{\det(G^{\{i\}})-\widetilde{D}_{ij}+i\,\lambda}\right)^{2\,\varepsilon}\,\left(\widetilde{D}_{ij}-i\,\lambda\right)^{-\varepsilon}\,\left[1+2\,\varepsilon^2\,\mathrm{Li}_2\left(\frac{\widetilde{D}_{ij}-i\,\lambda}{\det(G^{\{i\}})}\right)\right]\\
&\quad - 2\,\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,\left(\frac{\det(G^{\{i\}})}{\det(G^{\{i\}})-\widetilde{D}_{ij}+i\,\lambda}\right)^{\varepsilon}\,\left(\widetilde{D}_{ij}-\det(G^{\{i\}})-i\,\lambda\right)^{-\varepsilon}\Bigg\}.
\end{aligned}
\tag{2.21}
$$

This formula is manifestly well behaved as $\widetilde{D}_{ij} \to 0$ ($m_k^2 \to 0$), yet it is not handy to expand around $\varepsilon = 0$. A more practical alternative may be obtained as follows. First, we use the identities relating $\mathrm{Li}_2(1-w)$, $\mathrm{Li}_2(w)$, and $\mathrm{Li}_2(1/w)$ to change the argument of the Li_2 function, and the following relations:

$$
\ln\left(\frac{a}{z}\right) = \ln\left(-\frac{a}{z}\right) - i\,\pi\,S(a\,z),
\tag{2.22}
$$

$$
\ln\left(\frac{a+z}{a}\right) = \ln\left(-\frac{a+z}{a}\right) + i\,\pi\,S(a\,z),
\tag{2.23}
$$

$$
\ln\left(-\frac{a+z}{a}\right) = \ln\left(\frac{a+z}{z}\right) - \ln\left(-\frac{a}{z}\right),
\tag{2.24}
$$

with

$$S(z) = \text{sign}\,(\text{Im}(z)), \tag{2.25}$$

so that the Li_2 function can be rewritten as

$$\text{Li}_2\left(\frac{a+z}{a}\right) = \text{Li}_2\left(-\frac{a}{z}\right) - \frac{\pi^2}{6} - \frac{1}{2}\,\ln^2\left(-\frac{a}{z}\right) + \ln\left(\frac{a+z}{z}\right)\,\ln\left(-\frac{a}{z}\right). \tag{2.26}$$

Secondly, we Taylor expand around $\varepsilon = 0$ the $(-a/z)$ terms in Eq. (2.20). We thus get

$$W(\det(G^{\{i\}}), \widetilde{D}_{ij}, 0) = -\frac{2^{\varepsilon}}{\varepsilon}\,\frac{\Gamma(1+\varepsilon)}{z}\left[\frac{2}{\varepsilon}\,(z)^{-\varepsilon} - \frac{1}{\varepsilon}\,(a+z)^{-\varepsilon} - 2\,\varepsilon\,\text{Li}_2\left(-\frac{a}{z}\right)\right], \tag{2.27}$$

i.e., making explicit z and a in terms of $\det(G^{\{i\}})$ and $\widetilde{D}_{ij}$,

$$\begin{aligned} W(\det(G^{\{i\}}), \widetilde{D}_{ij}, 0) = {} & \frac{1}{\varepsilon}\,\frac{\Gamma(1+\varepsilon)}{\det(G^{\{i\}}) - \widetilde{D}_{ij}} \\ & \times \left\{\frac{2}{\varepsilon}\left[\frac{1}{2}\left(\widetilde{D}_{ij} - \det(G^{\{i\}})\right) - i\,\lambda\right]^{-\varepsilon} - \frac{1}{\varepsilon}\left(\frac{\widetilde{D}_{ij}}{2} - i\,\lambda\right)^{-\varepsilon}\right. \\ & \left. - 2\,\varepsilon\,\text{Li}_2\left(\frac{\det(G^{\{i\}})}{\det(G^{\{i\}}) - \widetilde{D}_{ij} + i\,\lambda}\right)\right\}, \end{aligned} \tag{2.28}$$

which is both well behaved when $\widetilde{D}_{ij} \to 0$ ($m_k^2 \to 0$) and more compact.

(c) Both m_j^2 and m_k^2 vanish

The function $D^{\{i\}(j)}(x)$ becomes

$$D^{\{i\}(j)}(x) = -\,G^{\{i\}(j)}\,x\,(1-x) \tag{2.29}$$

and we immediately get

$$W(\det(G^{\{i\}}), 0, 0) = -\frac{1}{\varepsilon^2}\,\Gamma(1+\varepsilon)\,\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,\left(-\frac{\det(G^{\{i\}})}{2} - i\,\lambda\right)^{-1-\varepsilon}. \tag{2.30}$$

In the limit $\widetilde{D}_{ij} \to 0$ ($m_k^2 \to 0$), Eqs. (2.28) or (2.21) smoothly become Eq. (2.30), as expected.

2.2. *Practical implementation of the preceding cases and explicit examples*

The various cases reviewed above may or may not be involved in a specific computation because some coefficients weighing the $W(\det(G^{\{i\}}), \widetilde{D}_{ij}, \widetilde{D}_{ik})$ may vanish. In particular, as seen in Eq. (2.5), when $\Delta_2 = 0$, the three-point function in dimension $4 - 2\,\varepsilon$ is the sum of three two-point functions in dimension $2 - 2\,\varepsilon$. These two-point functions correspond to the three distinct pinchings of the internal propagators of the three-point function. At first sight, one should worry that some of these two-point functions in low dimensions may badly diverge due to a threshold singularity which is, however, not present in the three-point function! For example, one of the pinchings of a three-point function having IR/collinear singularities would lead to a two-point function with the external legs on the mass shell of one of the propagators whereas the other propagator is massless. This would lead to

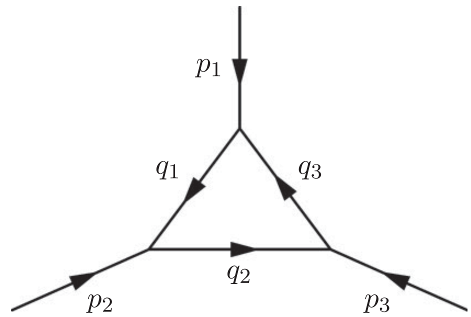

Fig. 1. The triangle picturing the one-loop three-point function.

a polynomial $D^{\{i\}(j)}(x) \propto x^2$ or $(1-x)^2$. Fortunately the corresponding $\overline{b}$ coefficients weighting such pathological terms identically vanish, and the discussion which follows, illustrated with examples, elucidates why this happens. Let us define $s_i = p_i^2$ with $i = 1, 2, 3$.

Case 1

A soft divergence occurs when the kinematic matrix $\mathcal{S}$ has a vanishing line (and corresponding column). This happens whenever a massless propagator connects two vertices in which external momenta enter on the mass shells of the two other propagators. As the external momenta are real, this case can occur only when the non-vanishing internal masses are real. Let us assume, cf. Fig. 1, that the internal mass squared m_1^2 vanishes whereas the external four-momenta p_1 and p_2 satisfy the mass shell conditions $s_1 = m_3^2$, $s_2 = m_2^2$. The $\mathcal{S}$ matrix has the following texture:

$$\mathcal{S}^{\,\text{soft}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2\,m_2^2 & s_3 - m_2^2 - m_3^2 \\ 0 & s_3 - m_2^2 - m_3^2 & -2\,m_3^2 \end{pmatrix}. \tag{2.31}$$

If one singles out row and column 1 in $\mathcal{S}^{\,\text{soft}}$, the two-component vector $V^{(1)}$ is readily seen to vanish and so do the coefficients $\overline{b}_2$ and $\overline{b}_3$ which are proportional to the two components of $(G^{(1)})^{-1} \cdot V^{(1)}$; thus, only $\overline{b}_1$ differs from zero, cf. Eq. (2.15) of Ref. [1].

To illustrate this point, let us consider the case where $m_1 = 0$, $m_2 = m_3 = m$, $s_1 = s_2 = m^2$, and s_3 is arbitrary. In this case, since $\overline{b}_1$ is the only non-vanishing coefficient, the polynomial $D^{\{1\}(2)}(x)$ involved in I_3^n corresponds to the one appearing in the two-point function obtained by pinching the internal line with four-momentum q_1 (cf. Fig. 1). This polynomial involves two masses (equal here), and this example corresponds to case (a) of the previous section. In this simple case, the two roots of the polynomial $D^{\{1\}(2)}(x)$ are given by

$$x_{1,2} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{4\,(m^2 - i\,\lambda)}{s_3}} \tag{2.32}$$

with the property

$$1 - x_1 = x_2.$$

Injecting this property into Eqs. (2.14) and (B.8), we get for the functions $J(x_1,x_2)$ and $K(x_1,x_2)$ in Eq. (2.15):

$$J(x_1,x_2) = \frac{2}{x_1 - x_2} \left\{ \mathrm{Li}_2\left(\frac{x_2}{x_1}\right) - \mathrm{Li}_2\left(\frac{x_1}{x_2}\right) + \ln\left(-\frac{x_2}{x_1}\right) \ln\left(-(x_1 - x_2)^2\right) \right\}, \tag{2.33}$$

$$K(x_1,x_2) = \frac{2}{x_1 - x_2} \ln\left(-\frac{x_2}{x_1}\right). \tag{2.34}$$

We check numerically that we recover the result of Ref. [4].

Case 2

A collinear divergence occurs when two internal masses vanish whereas the external four-momentum which enters into the vertex connecting the two adjacent massless propagators is lightlike (massless collinear splitting at this vertex). Note that the non-vanishing internal mass can be real or complex. Let us assume that the labels of the two massless propagators are 1 and 2, with $s_2 = 0$. The $\mathcal{S}$ matrix has the following texture:

$$\mathcal{S}^{\,\mathrm{coll}} = \begin{pmatrix} 0 & 0 & s_1 - m_3^2 \\ 0 & 0 & s_3 - m_3^2 \\ s_1 - m_3^2 & s_3 - m_3^2 & -2\, m_3^2 \end{pmatrix}. \tag{2.35}$$

If one singles out row and column 3 in $\mathcal{S}^{\,\mathrm{coll}}$, the Gram matrix $G^{(3)}$ and the vector $V^{(3)}$ read

$$G^{(3)} = \begin{pmatrix} 2\, s_1 & s_1 + s_3 \\ s_1 + s_3 & 2\, s_3 \end{pmatrix}, \tag{2.36}$$

$$V^{(3)} = \begin{pmatrix} s_1 + m_3^2 \\ s_3 + m_3^2 \end{pmatrix}. \tag{2.37}$$

A simple calculation yields

$$\sum_{i \in S_3 \setminus \{3\}} \left[\left(G^{(3)}\right)^{-1} \cdot V^{(3)} \right]_i = 1, \tag{2.38}$$

so that $\overline{b}_3$ given by Eq. (2.15) of Ref. [1] vanishes, whereas $\overline{b}_1$ and $\overline{b}_2$ generically differ from zero.

Let us compute I_3^n for this specific case. As $\overline{b}_3 = 0$, the relevant polynomials $D^{\{i\}(j)}(x)$ are those of the two-point functions obtained in the two pinching configurations (cf. Fig. 1) where either the internal line with four-momentum q_1 or the one with four-momentum q_2 is pinched. As these two lines are associated with vanishing masses and the third propagator is associated with a non-vanishing mass, the two polynomials both have one vanishing mass, which corresponds to case (b) of the previous section. Starting with Eq. (2.5), I_3^n reads

$$I_3^n = \frac{2^\varepsilon}{\varepsilon}\, \Gamma(1+\varepsilon) \left[\frac{\overline{b}_1}{\det(G)} \int_0^1 dx \left(D^{\{1\}(2)}(x) - i\,\lambda\right)^{-1-\varepsilon} + \frac{\overline{b}_2}{\det(G)} \int_0^1 dx \left(D^{\{2\}(3)}(x) - i\,\lambda\right)^{-1-\varepsilon} \right]. \tag{2.39}$$

It is better to change $x \leftrightarrow 1-x$ in the second integral in order to have a polynomial of the type in Eq. (2.16) and write I_3^n as

$$
\begin{aligned}
I_3^n &= \frac{2^\varepsilon}{\varepsilon}\,\Gamma(1+\varepsilon)\left[\frac{\overline{b}_1}{\det(G)}\int_0^1 dx\,\left(D^{\{1\}(2)}(x) - i\,\lambda\right)^{-1-\varepsilon}\right. \\
&\qquad\left. + \frac{\overline{b}_2}{\det(G)}\int_0^1 dx\,\left(D^{\{2\}(1)}(x) - i\,\lambda\right)^{-1-\varepsilon}\right] \\
&= \frac{\overline{b}_1}{\det(G)}\,W(\det(G^{\{1\}}), \widetilde{D}_{12}, 0) + \frac{\overline{b}_2}{\det(G)}\,W(\det(G^{\{2\}}), \widetilde{D}_{21}, 0).
\end{aligned}
\tag{2.40}
$$

Noting that $\overline{b}_1 = (s_3 - m_3^2)\,(s_1 - s_3)$ and $\overline{b}_2 = (s_1 - m_3^2)\,(s_3 - s_1)$, determining $\det(G^{\{1\}})$, $\det(G^{\{2\}})$, and $\widetilde{D}_{12}$ from the $\mathcal{S}$ matrix elements, cf. Eq. (2.8), and directly applying the result of Eq. (2.28), we get

$$
\begin{aligned}
I_3^n = \frac{\Gamma(1+\varepsilon)}{s_1 - s_3}\,\Bigg\{ &-\frac{1}{\varepsilon^2}\left[\left(-s_3 + m_3^2 - i\,\lambda\right)^{-\varepsilon} - \left(-s_1 + m_3^2 - i\,\lambda\right)^{-\varepsilon}\right] \\
&+ \mathrm{Li}_2\left(\frac{s_3}{s_3 - m_3^2 + i\,\lambda}\right) - \mathrm{Li}_2\left(\frac{s_1}{s_1 - m_3^2 + i\,\lambda}\right)\Bigg\}.
\end{aligned}
\tag{2.41}
$$

Using the Landen identity of Eq. (B.3) we recover the formula in Eq. (4.8) of Ref. [4] after some algebra.

Case 3

Both a soft and a collinear divergence may occur at the same time, thereby proceeding from both cases 1 and 2 above.

Let us take the example of case 2 and specify $s_1 = m_3^2$. The texture of the $\mathcal{S}$ matrix becomes

$$
\mathcal{S}^{\,\mathrm{cs}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s_3 - m_3^2 \\ 0 & s_3 - m_3^2 & -2\,m_3^2 \end{pmatrix}.
\tag{2.42}
$$

Only $\overline{b}_1$ does not vanish, so I_3^n reads simply

$$
\begin{aligned}
I_3^n &= \frac{2^\varepsilon}{\varepsilon}\,\Gamma(1+\varepsilon)\,\frac{\overline{b}_1}{\det(G)}\int_0^1 dx\,\left(D^{\{1\}(2)}(x) - i\,\lambda\right)^{-1-\varepsilon} \\
&= \frac{\overline{b}_1}{\det(G)}\,W(\det(G^{\{1\}}), \widetilde{D}_{12}, 0).
\end{aligned}
\tag{2.43}
$$

Using Eq. (2.28) and expressing $\det(G^{\{1\}})$ and $\widetilde{D}_{12}$ in terms of the $\mathcal{S}$ matrix elements, cf. Eq. (2.8), we get

$$I_3^n = \frac{\Gamma(1+\varepsilon)}{m_3^2 - s_3} \left\{ -\frac{1}{\varepsilon^2} \left(-s_3 + m_3^2 - i\,\lambda \right)^{-\varepsilon} + \frac{1}{2\,\varepsilon^2} \left(m_3^2 - i\,\lambda \right)^{-\varepsilon} + \mathrm{Li}_2\left(\frac{s_3}{s_3 - m_3^2 + i\,\lambda} \right) \right\}. \tag{2.44}$$

After some algebra, we recover[6] the formula in Eq. (4.11) of Ref. [4].

2.3. *Indirect way*

This subsection provides the calculation according to the "indirect way." The results presented are valid for both real and complex masses; unless explicitly specified the $i\,\lambda$ prescription is kept, bearing in mind that it is ineffective in the case of complex masses. Let us start from Eq. (2.3). The x integration is traded for a ρ integration in a way very similar to the four-dimensional case (see Sect. 2.2.2 of Ref. [1]) and we get

$$I_3^n = \sum_{i \in S_3} \frac{\overline{b}_i}{\det(G)} \sum_{j \in S_3 \setminus \{i\}} \frac{\overline{b}_j^{\{i\}}}{\det\left(G^{\{i\}}\right)} L_3^n\big(0, \Delta_1^{\{i\}}, \widetilde{D}_{ij}\big) \tag{2.45}$$

with

$$L_3^n\big(0, \Delta_1^{\{i\}}, \widetilde{D}_{ij}\big) = \kappa_{_{IR}} \int_0^{+\infty} \frac{d\xi}{\xi^\nu - i\,\lambda} \times \int_0^{+\infty} \frac{d\rho}{(\xi^\nu + \rho^2 - \Delta_1^{\{i\}} - i\,\lambda)(\xi^\nu + \rho^2 + \widetilde{D}_{ij} - i\,\lambda)^{1/2}} \tag{2.46}$$

and

$$\kappa_{_{IR}} = 2^\varepsilon \, \frac{\Gamma(1+\varepsilon)}{(1-\varepsilon)} \, \frac{1}{B(1+\varepsilon, 1-\varepsilon)}, \quad \nu = \frac{1}{1-\varepsilon}.$$

To handle the cases with soft and/or collinear divergences, the two relevant configurations are (1) $\widetilde{D}_{ij} \neq 0$ and (2) $\widetilde{D}_{ij} = 0$. Note that when both Δ_2 and $\Delta_1^{\{i\}}$ vanish, $L_3^n(0, 0, \widetilde{D}_{ij})$ is weighting a vanishing $\overline{b}$, and therefore it will not be considered (cf. Sect. 2.2).

The ρ integration can be done using Appendix A as in the four-dimensional case; as the outcome of this integration, we shall distinguish two cases depending on the sign of $\mathrm{Im}(\Delta_1^{\{i\}})$.

(1) $\widetilde{D}_{ij} \neq 0$

(1a) $\mathrm{Im}(\Delta_1^{\{i\}}) > 0$ This case covers real masses in particular. After the ρ integration, $L_3^n(0, \Delta_1^{\{i\}}, \widetilde{D}_{ij})$ becomes

$$L_3^n(0, \Delta_1^{\{i\}}, \widetilde{D}_{ij}) = \kappa_{_{IR}} \int_0^{+\infty} d\xi \int_0^1 dz \, \frac{1}{(\xi^\nu - i\,\lambda)(\xi^\nu - (1-z^2)\,\Delta_1^{\{i\}} + z^2\,\widetilde{D}_{ij} - i\,\lambda)}. \tag{2.47}$$

[6] In contrast to Eq. (2.44), Eq. (4.11) of Ref. [4] contains a factor $\Gamma^2(1-\varepsilon)/\,\Gamma(1-2\varepsilon)$ and an extra term $+\pi^2/12$ inside the brackets. These, however, cancel against each other when performing the ε-expansion at the appropriate order.

The ξ integration is performed first, using Eq. (A.4) of Appendix A, and $L_3^n(0, \Delta_1^{\{i\}}, \widetilde{D}_{ij})$ becomes

$$L_3^n(0, \Delta_1^{\{i\}}, \widetilde{D}_{ij}) = -\frac{2^\varepsilon}{\varepsilon}\,\Gamma(1+\varepsilon)\int_0^1 dz\big(z^2\,(\widetilde{D}_{ij} + \Delta_1^{\{i\}}) - \Delta_1^{\{i\}} - i\,\lambda\big)^{-1-\varepsilon}. \tag{2.48}$$

Expanding[7] the right-hand side of Eq. (2.48) around $\varepsilon = 0$ then gives

$$\begin{aligned} L_3^n(0, \Delta_1^{\{i\}}, \widetilde{D}_{ij}) = 2^\varepsilon\,\Gamma(1+\varepsilon)\Bigg[&-\frac{1}{\varepsilon}\int_0^1 \frac{dz}{z^2\,(\widetilde{D}_{ij} + \Delta_1^{\{i\}}) - \Delta_1^{\{i\}} - i\,\lambda} \\ &+ \int_0^1 dz\,\frac{\ln(z^2\,(\widetilde{D}_{ij} + \Delta_1^{\{i\}}) - \Delta_1^{\{i\}} - i\,\lambda)}{z^2\,(\widetilde{D}_{ij} + \Delta_1^{\{i\}}) - \Delta_1^{\{i\}} - i\,\lambda}\Bigg]. \end{aligned} \tag{2.49}$$

(1b) $\mathrm{Im}(\Delta_1^{\{i\}}) < 0$ This case occurs only with complex masses, in a way such that the $i\,\lambda$ prescriptions are overshadowed and ineffective; the latter are therefore dropped in this part. After ρ integration performed as in the four-dimensional case, we get, see Eq. (A.7) of Appendix A,

$$\begin{aligned} &L_3^n(0, \Delta_1^{\{i\}}, \widetilde{D}_{ij}) \\ &= -\kappa_{IR}\int_0^{+\infty}\frac{d\xi}{\xi^\nu - i\,\lambda}\Bigg[i\int_0^{+\infty}\frac{dz}{-\xi^\nu + \widetilde{D}_{ij}\,z^2 + (1+z^2)\,\Delta_1^{\{i\}}} \\ &\qquad + \int_1^{+\infty}\frac{dz}{\xi^\nu + \widetilde{D}_{ij}\,z^2 - (1-z^2)\,\Delta_1^{\{i\}}}\Bigg]. \end{aligned} \tag{2.50}$$

Using the identity in Eq. (A.4), $L_3^n(0, \Delta_1^{\{i\}}, \widetilde{D}_{ij})$ reads

$$\begin{aligned} L_3^n(0, \Delta_1^{\{i\}}, \widetilde{D}_{ij}) = \frac{2^\varepsilon}{\varepsilon}\,\Gamma(1+\varepsilon)\Bigg[&-i\int_0^{+\infty} dz\big(-z^2\,(\widetilde{D}_{ij} + \Delta_1^{\{i\}}) - \Delta_1^{\{i\}}\big)^{-1-\varepsilon} \\ &+ \int_1^{+\infty} dz\,\big(z^2\,(\widetilde{D}_{ij} + \Delta_1^{\{i\}}) - \Delta_1^{\{i\}}\big)^{-1-\varepsilon}\Bigg]. \end{aligned} \tag{2.51}$$

Expanding the terms in the square bracket in Eq. (2.51) around $\varepsilon = 0$ then gives

$$\begin{aligned} &L_3^n(0, \Delta_1^{\{i\}}, \widetilde{D}_{ij}) \\ &= 2^\varepsilon\,\Gamma(1+\varepsilon)\Bigg\{\frac{1}{\varepsilon}\Bigg[i\int_0^{+\infty}\frac{dz}{z^2\,(\widetilde{D}_{ij} + \Delta_1^{\{i\}}) + \Delta_1^{\{i\}}} + \int_1^{+\infty}\frac{dz}{z^2\,(\widetilde{D}_{ij} + \Delta_1^{\{i\}}) - \Delta_1^{\{i\}}}\Bigg] \\ &\qquad - i\int_0^{+\infty} dz\,\frac{\ln(-z^2\,(\widetilde{D}_{ij} + \Delta_1^{\{i\}}) - \Delta_1^{\{i\}})}{z^2\,(\widetilde{D}_{ij} + \Delta_1^{\{i\}}) + \Delta_1^{\{i\}}} \\ &\qquad - \int_1^{+\infty} dz\,\frac{\ln(z^2\,(\widetilde{D}_{ij} + \Delta_1^{\{i\}}) - \Delta_1^{\{i\}})}{z^2\,(\widetilde{D}_{ij} + \Delta_1^{\{i\}}) - \Delta_1^{\{i\}}}\Bigg\}. \end{aligned} \tag{2.52}$$

For Eqs. (2.49) and (2.52), the z integration can then be performed using Appendix E.

[7] Remember that only the terms in the ε-expansion which provide the divergent and finite terms in the limit $\varepsilon \to 0$ are kept.

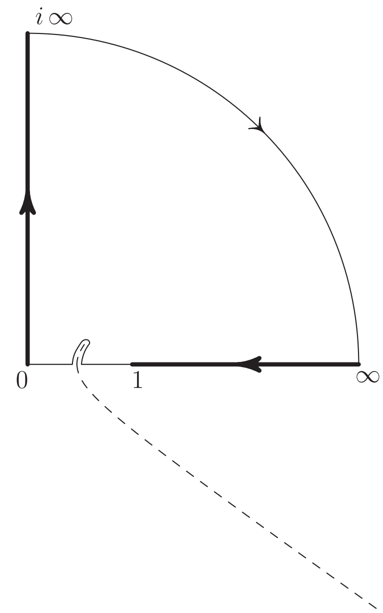

Fig. 2. Location of the relevant discontinuity cut $\mathcal{C}_{i,j}$ with respect to the two half straight lines $[0, +i\infty[$ and $[1, +\infty[$, and deformation of the contour $\widehat{(0,1)}$ partly wrapping the extremity of the cut.

Since the cuts of $(z^2\,(\widetilde{D}_{ij} + \Delta_1^{\{i\}}) - \Delta_1^{\{i\}})^{-1-\varepsilon}$ and $\ln(z^2\,(\widetilde{D}_{ij} + \Delta_1^{\{i\}}) - \Delta_1^{\{i\}})$ are the same, the discussion carried to extend the "indirect way" to the general complex mass case holds likewise here (cf. Sect. 2 of Ref. [2]). Equation (2.51) can be rewritten as

$$\begin{aligned} &L_3^n(0, \Delta_1^{\{i\}}, \widetilde{D}_{ij}) \\ &= -\frac{2^\varepsilon}{\varepsilon}\,\Gamma(1+\varepsilon)\,\left\{\int_0^{i\,\infty} + \int_{+\infty}^{1}\right\} dz\big(z^2\,(\widetilde{D}_{ij} + \Delta_1^{\{i\}}) - \Delta_1^{\{i\}}\big)^{-1-\varepsilon}. \end{aligned} \tag{2.53}$$

In Eq. (2.53) let us add a vanishing contribution along the "contour at ∞" in the north-east quadrant $\{\mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$ so as to concatenate the two contributions. The connected contour thus obtained can in turn be deformed into a finite contour $\widehat{(0,1)}_{i,j}$ stretched from 0 to 1 as pictured in Fig. 2, thereby unifying Eqs. (2.48) and (2.53):

$$L_3^n(0, \Delta_1^{\{i\}}, \widetilde{D}_{ij}) = -\frac{2^\varepsilon}{\varepsilon}\,\Gamma(1+\varepsilon)\int_{\widehat{(0,1)}_{i,j}} dz\big(z^2\,(\widetilde{D}_{ij} + \Delta_1^{\{i\}}) - \Delta_1^{\{i\}}\big)^{-1-\varepsilon}. \tag{2.54}$$

(2) $\widetilde{D}_{ij} = 0$
Here again we shall distinguish two cases depending on the sign of $\mathrm{Im}(\Delta_1^{\{i\}})$.

(2a) $\mathrm{Im}(\Delta_1^{\{i\}}) > 0$ This case covers real masses in particular. The calculation initially amounts to setting $\widetilde{D}_{ij} = 0$ in Eq. (2.48), which becomes

$$L_3^n(0, \Delta_1^{\{i\}}, 0) = -\frac{2^\varepsilon}{\varepsilon}\,\Gamma(1+\varepsilon)\,(-\Delta_1^{\{i\}} - i\,\lambda)^{-1-\varepsilon}\int_0^1 dz\,\big(1 - z^2\big)^{-1-\varepsilon}. \tag{2.55}$$

The integration over z is performed using Eq. (C.6), and $L_3^n(0, \Delta_1^{\{i\}}, 0)$ becomes

$$L_3^n(0, \Delta_1^{\{i\}}, 0) = \frac{1}{\varepsilon^2}\, \Gamma(1+\varepsilon)\, \frac{\Gamma(1-\varepsilon)^2}{\Gamma(1-2\,\varepsilon)}\, (-2\,\Delta_1^{\{i\}} - i\,\lambda)^{-1-\varepsilon}. \tag{2.56}$$

(2b) $\mathrm{Im}(\Delta_1^{\{i\}}) < 0$ Here again, this case occurs only with complex masses, with the $i\,\lambda$ prescriptions overshadowed and therefore dropped. The calculation initially amounts to setting $\widetilde{D}_{ij} = 0$ in Eq. (2.51), which becomes

$$\begin{aligned} L_3^n(0, \Delta_1^{\{i\}}, 0) = \frac{2^{\varepsilon}\,\Gamma(1+\varepsilon)}{\varepsilon} \Bigg[& -i\left(-\Delta_1^{\{i\}}\right)^{-1-\varepsilon} \int_0^{+\infty} dz\left(1+z^2\right)^{-1-\varepsilon} \\ & -\left(\Delta_1^{\{i\}}\right)^{-1-\varepsilon} \int_1^{+\infty} dz\left(z^2-1\right)^{-1-\varepsilon}\Bigg]. \end{aligned} \tag{2.57}$$

The z integrals are computed in Appendix C [Eqs. (C.4) and (C.5)]. Hence, for $L_3^n(0, \Delta_1^{\{i\}}, 0)$,

$$\begin{aligned} L_3^n(0, \Delta_1^{\{i\}}, 0) = & -\frac{2^{-\varepsilon}}{2\,\varepsilon^2}\, \Gamma(1+\varepsilon)\, \frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\, \frac{1}{\cos(\pi\,\varepsilon)} \\ & \times \left[i\,\sin(\pi\,\varepsilon)\left(-\Delta_1^{\{i\}}\right)^{-1-\varepsilon} + \left(\Delta_1^{\{i\}}\right)^{-1-\varepsilon} \right]. \end{aligned} \tag{2.58}$$

Since $\mathrm{Im}(\Delta_1^{\{i\}}) < 0$, $(\Delta_1^{\{i\}})^{-1-\varepsilon}$ may be rewritten as $-\,e^{i\,\pi\,\varepsilon}\,(-\,\Delta_1^{\{i\}})^{-1-\varepsilon}$, so that $L_3^n(0, \Delta_1^{\{i\}}, \widetilde{D}_{ij})$ simplifies into

$$L_3^n(0, \Delta_1^{\{i\}}, 0) = \frac{1}{\varepsilon^2}\, \Gamma(1+\varepsilon)\, \frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)} \left(-2\,\Delta_1^{\{i\}}\right)^{-1-\varepsilon}, \tag{2.59}$$

which coincides with Eq. (2.56).

2.4. *"Direct"–"indirect" equivalence*

We now show the equivalence between the "indirect way" and the "direct way" starting from the integral representation of $L_3^n(0, \Delta_1^{\{i\}}, \widetilde{D}_{ij})$ and disregarding the fact that some $\widetilde{D}_{ij}$ may or may not vanish. Thus, I_3^n now reads

$$\begin{aligned} I_3^n = & -\frac{2^{\varepsilon}}{\varepsilon}\, \Gamma(1+\varepsilon) \sum_{i\in S_3} \frac{\overline{b}_i}{\det(G)} \sum_{j\in S_3\setminus\{i\}} \frac{\overline{b}_j^{\{i\}}}{\det\left(G^{\{i\}}\right)} \\ & \times \int_{\widehat{(0,1)}_{i,j}} dz\left(z^2\,(\widetilde{D}_{ij} + \Delta_1^{\{i\}}) - \Delta_1^{\{i\}}\right)^{-1-\varepsilon}. \end{aligned} \tag{2.60}$$

Sticking to the general complex mass case, we perform the change of variable $s = \overline{b}_j^{\{i\}}\, z$ in such a way that the two integrands corresponding to the sum over j (at fixed i) in Eq. (2.60) are the same (use Eqs. (2.36) and (2.38) of Ref. [1]). Specifying the two elements of $S_3 \setminus \{i\}$ to be $k \equiv 1 + ((i+1)$ modulo 3) and $l \equiv 1 + (i$ modulo 3), the two integrals are concatenated into a single one integrated

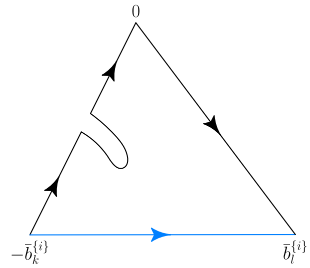

Fig. 3. Example of a contour deformation involving a triangle with one distorted side, for which no cut crosses the straight base $[-\overline{b}_k^{\{i\}}, \overline{b}_l^{\{i\}}]$.

along the contour $\mathcal{I}_{k,l}^{(i)} \equiv -\overline{b}_k^{\{i\}}\widehat{(0,1)}_{i,k} \cup \overline{b}_l^{\{i\}}\widehat{(0,1)}_{i,l}$ in the complex s-plane:

$$I_3^n = -\frac{2^\varepsilon}{\varepsilon}\,\Gamma(1+\varepsilon)\sum_{i\in S_3}\frac{\overline{b}_i}{\det(G)\,\det\left(G^{\{i\}}\right)} \times \int_{\mathcal{I}_{k,l}^{(i)}} ds \left(\frac{s^2+\det\left(\mathcal{S}^{\{i\}}\right)}{\det\left(G^{\{i\}}\right)}\right)^{-1-\varepsilon}. \tag{2.61}$$

As in the case $\Delta_2 \neq 0$ treated in Ref. [2], the contour $\mathcal{I}_{k,l}^{(i)}$ can be deformed into the straight line $\left[-\overline{b}_k^{\{i\}}, \overline{b}_l^{\{i\}}\right]$ as depicted in Fig. 3:

$$I_3^n = -\frac{2^\varepsilon}{\varepsilon}\,\Gamma(1+\varepsilon)\sum_{i\in S_3}\frac{\overline{b}_i}{\det(G)\,\det\left(G^{\{i\}}\right)} \times \int_{-\overline{b}_k^{\{i\}}}^{\overline{b}_l^{\{i\}}} ds \left(\frac{s^2+\det\left(\mathcal{S}^{\{i\}}\right)}{\det\left(G^{\{i\}}\right)}\right)^{-1-\varepsilon}. \tag{2.62}$$

Performing the change of variable $s = -\overline{b}_k^{\{i\}} - \det\left(G^{\{i\}}\right) u$ and following the procedure given by Eqs. (2.45)–(2.47) of Ref. [1] leads to

$$I_3^n = \frac{2^\varepsilon}{\varepsilon}\,\Gamma(1+\varepsilon)\sum_{i\in S_3}\frac{\overline{b}_i}{\det(G)}\int_0^1 du\left(D^{\{i\}\,(l)}(u)\right)^{-1-\varepsilon},$$

namely, Eq. (2.5).

It is instructive to recover the results of the "direct way" from those of the "indirect way" using for the latter the closed form formulae. These formulae can be obtained for the case $\widetilde{D}_{ij} \neq 0$ by using results of Appendix E and for $\widetilde{D}_{ij} = 0$ using Eq. (2.56). This exercise is performed in great detail in Appendix D.

3. Four-point function with infrared divergences

The one-loop four-point function is depicted in Fig. 4. In the case where infrared (IR) divergences appear, these can be regularized by dimensional regularization shifting the spacetime dimension $n = 4 - 2\varepsilon$ slightly above 4 ($\varepsilon < 0$). The Feynman parametrization of I_4^n reads

$$I_4^n = \Gamma\,(2+\varepsilon)\int_0^1\prod_{i=1}^4 dz_i\,\delta\!\left(1-\sum_{i=1}^4 z_i\right)\left(-\frac{1}{2}\,Z^{\mathrm{T}}\cdot\mathcal{S}\cdot Z - i\,\lambda\right)^{-2-\varepsilon}, \tag{3.1}$$

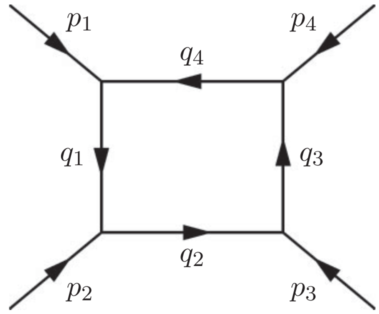

Fig. 4. The box picturing the one-loop four-point function.

where Z is a column four-vector whose components are the z_i. The power $-2-\varepsilon$ in Eq. (3.1) is not an integer; notwithstanding this, the tricks and techniques elaborated in Sect. 3 of Ref. [1] can be used with a slight adaptation. Let us sketch the different steps for this special case.

3.1. Computation of I_4^n

We make use of the identity in Eq. (2.26) of Ref. [1] to shift the power of the denominator in the integrand, choosing $\mu = 5/2$ and $\nu = 2/(1-2\varepsilon)$ so that I_4^n is recast as

$$\begin{aligned} I_4^n = & \frac{2^{3+\varepsilon}}{B(2+\varepsilon, 1/2-\varepsilon)} \frac{\Gamma(2+\varepsilon)}{(1-2\,\varepsilon)} \\ & \times \int_0^{+\infty} d\xi \int_{\Sigma_{bcd}} \frac{dx_b\, dx_c\, dx_d}{(D^{(a)}(x_b, x_c, x_d) + \xi^\nu - i\,\lambda)^{5/2}}. \end{aligned} \tag{3.2}$$

Step 1 is very similar to the case $n = 4$, and we get

$$\begin{aligned} I_4^n = & \frac{2^{3+\varepsilon}}{3\,B(2+\varepsilon, 1/2-\varepsilon)} \frac{\Gamma(2+\varepsilon)}{(1-2\,\varepsilon)} \\ & \times \sum_{i=1}^{4} \frac{\overline{b}_i}{\det(G)} \int_0^{+\infty} d\xi\, \frac{1}{\Delta_3 - \xi^\nu + i\,\lambda} \\ & \qquad \times \int_{\Sigma_{kl}} \frac{dx_k\, dx_l}{(D^{\{i\}\,(i')}(x_k, x_l) + \xi^\nu - i\,\lambda)^{3/2}}, \end{aligned} \tag{3.3}$$

with the same notational conventions as in Eqs. (3.6) and (3.17) of Sect. 3 in Ref. [1]. Likewise, steps 2 and 3 are identical to Sect. 3 of Ref. [1] except that the power of the variable ξ is ν instead of 2, and will not be repeated here. Regarding step 4, the integration is performed over the variables ξ, ρ, then σ in the corresponding $L_4^n(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk})$, now given by

$$\begin{aligned} L_4^n(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) = & \,\kappa \int_0^{+\infty} d\xi \int_0^{+\infty} d\rho \int_0^{+\infty} d\sigma\, \frac{1}{\xi^\nu - \Delta_3 - i\,\lambda} \\ & \times \frac{1}{\xi^\nu + \rho^2 - \Delta_2^{\{i\}} - i\,\lambda} \frac{1}{\xi^\nu + \rho^2 + \sigma^2 - \Delta_1^{\{i,j\}} - i\,\lambda} \\ & \times \frac{1}{(\widetilde{D}_{ijk} + \xi^\nu + \rho^2 + \sigma^2 - i\,\lambda)^{1/2}}, \end{aligned} \tag{3.4}$$

with

$$\kappa = \frac{2^{4+\varepsilon}}{3\,B(2+\varepsilon, 1/2-\varepsilon)\,B(3/2, 1/2)\,B(1, 1/2)} \frac{\Gamma(2+\varepsilon)}{(1-2\,\varepsilon)},$$

reminiscent of Eq. (3.35) of Sect. 3 in Ref. [1]. Infrared divergences in the four-point function are not dominant Landau-type singularities but subleading ones. Such an infrared divergence corresponds to the vanishing determinant det $(\mathcal{S}^{\{i\}})$ of some reduced $\mathcal{S}$ matrix (or some $\Delta_2^{\{i\}}$) associated with some three-point functions which are obtained from the four-point function considered by one pinching [7]. The various cases of vanishing kinematic matrices associated with three-point functions plagued with infrared soft or collinear singularities have been evoked in Sect. 2. Let us note that the method developed in Sect. 3 of Ref. [1] is still valid because we never divide by $\Delta_2^{\{i\}}$ per se but by $\Delta_2^{\{i\}} - \xi^{\nu} - \rho^2$. The divergences will show up when performing the integrations over the parameters of Eq. (3.4). Note also that if we face a case when IR divergences arise when there are some non-vanishing internal masses, only some of the contributions (let us call them sectors) "i, j, k," not all, are plagued with IR divergences; the other ones, which correspond to $\Delta_2^{\{i\}} \neq 0$, will be treated as in the $n = 4$ case. Also, in any IR-divergent sector "i, j, k" for which $\Delta_2^{\{i\}} = 0$, we do not have to sum over all three sub-sectors $j \in S_4 \setminus \{i\}$ because, as in the IR-divergent three-point function case, some of the $\overline{b}_i^{\{j\}}$ vanish.

Let us consider a sector "i, j, k" which diverges in the IR region. We have to distinguish two cases: (1) when $\Delta_2^{\{i\}} = 0$ whereas $\widetilde{D}_{ijk} \neq 0$, in which case there are only soft divergences; (2) when both $\Delta_2^{\{i\}} = 0$ and $\widetilde{D}_{ijk} = 0$, in which case there are collinear or both soft and collinear divergences. In this section we will treat both real and complex mass cases. Some of the cases correspond to real or complex masses, others to complex masses only. For those corresponding to real or complex masses, we keep the imaginary part $-i\,\lambda$, explicitly bearing in mind that with complex masses this $-i\,\lambda$ is ineffective. Let us stress in passing that we also keep an infinitesimal prescription $-i\,\lambda$ in the pole term where we put $\Delta_2^{\{i\}} = 0$.

3.2. $\Delta_2^{\{i\}} = 0$ *and* $\widetilde{D}_{ijk} \neq 0$

$$\begin{aligned} L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) = \kappa \int_0^{+\infty} d\xi \int_0^{+\infty} d\rho \int_0^{+\infty} d\sigma \, & \frac{1}{(\xi^{\nu} - \Delta_3 - i\,\lambda)\,(\xi^{\nu} + \rho^2 - i\,\lambda)} \\ \times & \frac{1}{(\xi^{\nu} + \rho^2 + \sigma^2 - \Delta_1^{\{i,j\}} - i\,\lambda)\,(\widetilde{D}_{ijk} + \xi^{\nu} + \rho^2 + \sigma^2 - i\,\lambda)^{1/2}}. \end{aligned} \tag{3.5}$$

In the complex mass case, $\mathrm{Im}(\Delta_3)$ and $\mathrm{Im}(\Delta_1^{\{i,j\}})$ have arbitrary signs and $\mathrm{Im}(\widetilde{D}_{ijk})$ is negative, so we have to distinguish between different cases.

Let us define the function $M_2(\xi^{\nu})$ as

$$L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) = \kappa \int_0^{+\infty} d\xi \, \frac{1}{\xi^{\nu} - \Delta_3 - i\,\lambda} M_2(\xi^{\nu}). \tag{3.6}$$

The integrations over σ and ρ are identical to those appearing in the massive cases, we thus borrow the results derived in Sect. 3.4 of Ref. [1] and in Sect. 3.1 of Ref. [2] with $\Delta_2^{\{i\}} = 0$, explicitly: [8]

[8] The integration over σ is of the "second kind," cf. Eqs. (A.6) and (A.7), while the integration over ρ is of the "first kind," cf. Eq. (A.1). For both integrations, the power ν appearing in Eqs. (A.1), (A.6), and (A.7) is taken equal to 2.

for $\mathrm{Im}(\Delta_1^{\{i,j\}}) \geq 0$,

$$M_2(\xi^\nu) = \frac{1}{2} B(1/2, 1/2) \int_0^1 \frac{du}{u^2\,(\widetilde{D}_{ijk} + \Delta_1^{\{i,j\}}) - \Delta_1^{\{i,j\}}} \times \left[\frac{1}{(\xi^\nu - i\,\lambda)^{1/2}} - \frac{1}{\left(\xi^\nu + u^2\,(\widetilde{D}_{ijk} + \Delta_1^{\{i,j\}}) - \Delta_1^{\{i,j\}} - i\,\lambda\right)^{1/2}} \right]; \tag{3.7}$$

for $\mathrm{Im}(\Delta_1^{\{i,j\}}) < 0$,

$$\begin{aligned} M_2(\xi^\nu) = -\frac{1}{2} B(1/2, 1/2) \Bigg\{ & i \int_0^{+\infty} \frac{du}{u^2\,(\widetilde{D}_{ijk} + \Delta_1^{\{i,j\}}) + \Delta_1^{\{i,j\}}} \\ & \times \left[\frac{1}{(\xi^\nu - i\,\lambda)^{1/2}} - \frac{1}{\left(\xi^\nu - u^2\,(\widetilde{D}_{ijk} + \Delta_1^{\{i,j\}}) - \Delta_1^{\{i,j\}}\right)^{1/2}} \right] \\ & + \int_1^{+\infty} \frac{du}{u^2\,(\widetilde{D}_{ijk} + \Delta_1^{\{i,j\}}) - \Delta_1^{\{i,j\}}} \\ & \times \left[\frac{1}{(\xi^\nu - i\,\lambda)^{1/2}} - \frac{1}{\left(\xi^\nu + u^2\,(\widetilde{D}_{ijk} + \Delta_1^{\{i,j\}}) - \Delta_1^{\{i,j\}}\right)^{1/2}} \right] \Bigg\}. \end{aligned} \tag{3.8}$$

The integration over ξ to obtain L_4^n is of the “second kind”— cf. Eqs. (A.6) and (A.7) with $\nu = 2/(1 - 2\,\varepsilon)$. Let us go through the different cases with respect to the sign of the imaginary part of Δ_3 and $\Delta_1^{\{i,j\}}$. Sticking with the notation of Sect. 3 of Ref. [1], we introduce

$$\begin{aligned} P_{ijk} &= \widetilde{D}_{ijk} + \Delta_1^{\{i,j\}}, \\ R_{ij} &= -\,\Delta_1^{\{i,j\}}, \\ T &= -\,\Delta_3. \end{aligned}$$

The strategy for computing the different integrals is the same for all cases, and very similar to Sect. 3 of Ref. [2]. We display the successive steps for the first case, and we give the final results for the three others.

3.2.1. $\mathrm{Im}(\Delta_3) > 0$, $\mathrm{Im}(\Delta_1^{\{i,j\}}) > 0$

We start with Eq. (3.7) for $M_2(\xi^\nu)$ and use Eq. (A.6) for the ξ integration to get

$$\begin{aligned} L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) = F(\varepsilon) \int_0^1 & \frac{du}{u^2\,P_{ijk} + R_{ij}} \\ & \times \left[\int_0^1 \frac{dz}{\left((1 - z^2)\,T - i\,\lambda\right)^{1+\varepsilon}} \right. \\ & \left. - \int_0^1 \frac{dz}{\left(z^2\,(u^2\,P_{ijk} + R_{ij}) + (1 - z^2)\,T - i\,\lambda\right)^{1+\varepsilon}} \right], \end{aligned} \tag{3.9}$$

where

$$F(\varepsilon) = 2^{1+\varepsilon}\,\Gamma(1 + \varepsilon). \tag{3.10}$$

To help the reader, we will define some steps in a similar way to Sect. 3.4 of Ref. [1].

Step 1 We change $u = \sqrt{y/x}$ and $z = \sqrt{x}$ and exchange the y and the x integration so that

$$L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) = -\frac{F(\varepsilon)}{4} \int_0^1 \frac{dy}{\sqrt{y}} \int_y^1 dx \, \frac{1}{y\,P_{ijk} + x\,R_{ij}} \times \left[\frac{1}{\left(y\,P_{ijk} + x\,(R_{ij} - T) + T - i\,\lambda\right)^{1+\varepsilon}} - \frac{1}{((1-x)\,T - i\,\lambda)^{1+\varepsilon}} \right]. \tag{3.11}$$

Step 2 We set $y = u^2$ and perform a partial fraction decomposition on the variable x to write $L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk})$ in the following form:

$$\begin{aligned} &L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) \\ &\quad = -\frac{F(\varepsilon)}{2} \int_0^1 \frac{du}{u^2\,P_{ijk} T + R_{ij}(T - i\,\lambda)} \int_{u^2}^1 dx \\ &\qquad \times \Bigg\{ (T - R_{ij}) \left[u^2\,P_{ijk} + x\,(R_{ij} - T) + T - i\,\lambda\right]^{-1-\varepsilon} \\ &\qquad - T\,\left[(1-x)\,T - i\,\lambda\right]^{-1-\varepsilon} + \frac{R_{ij}}{u^2\,P_{ijk} + x\,R_{ij}} \\ &\qquad \times \left[\left[u^2\,P_{ijk} + x\,(R_{ij} - T) + T - i\,\lambda\right]^{-\varepsilon} - \left[(1-x)\,T - i\,\lambda\right]^{-\varepsilon} \right] \Bigg\}. \end{aligned} \tag{3.12}$$

The last line of Eq. (3.12) provides a contribution of order ε only; for the computation of the one-loop four-point function it can thus be dropped.[9]

Step 3 Thus, ignoring these terms, the integration in x readily provides

$$\begin{aligned} L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) = \frac{2^\varepsilon}{T} \frac{\Gamma(1+\varepsilon)}{\varepsilon} \int_0^1 & \frac{du}{u^2\,P_{ijk} + R_{ij} - i\,\lambda\,\sigma_0} \\ & \times \left\{ \left[u^2\,(P_{ijk} + R_{ij} - T) + T - i\,\lambda\right]^{-\varepsilon} \right. \\ & \left. - \left[u^2\,P_{ijk} + R_{ij} - i\,\lambda\right]^{-\varepsilon} - (T - i\,\lambda)^{-\varepsilon}\,(1-u^2)^{-\varepsilon} \right\}, \end{aligned} \tag{3.13}$$

where we have introduced $\sigma_0 = \mathrm{sign}(R_{ij}/T)$ in the real mass case. In the complex mass case all "$-i\lambda$" and "$-i\sigma_0\,\lambda$" contour prescriptions are ineffective and irrelevant, and all three terms in Eq. (3.13) can be straightforwardly expanded in powers of ε. In contrast the real mass case requires a more

[9] Similar truncations of ε expansions will be performed everywhere throughout this section. From the perspective of computing generalized one-loop building blocks to be used in computations beyond one-loop, one might be led to keep further terms evanescent with ε to the appropriate order, whenever such terms would hit $1/\varepsilon$ poles generated by the extra integrations, cf. the introduction of [1].

cautious treatment, which we elaborate below. We note that when $u^2 \to -R_{ij}/P_{ijk}$,

$$u^2\,(P_{ijk}+R_{ij}-T)+T \to (T-R_{ij})\frac{\widetilde{D}_{ijk}}{P_{ijk}} \neq 0,$$
$$1-u^2 \to \frac{\widetilde{D}_{ijk}}{P_{ijk}} \neq 0,$$

i.e. the pole at $u^2 = -R_{ij}/P_{ijk}$ is distinct from the branch points of the first and third functions in the numerator. Therefore we can readily perform an expansion around $\varepsilon = 0$ for the first and third terms as

$$\frac{1}{\varepsilon}\left[u^2\,(P_{ijk}+R_{ij}-T)+T-i\,\lambda\right]^{-\varepsilon} = \frac{1}{\varepsilon} - \ln\left[u^2\,(P_{ijk}+R_{ij}-T)+T-i\,\lambda\right] + \mathcal{O}(\varepsilon),$$
$$\frac{1}{\varepsilon}\,(T-i\,\lambda)^{-\varepsilon}\,(1-u^2)^{-\varepsilon} = -\frac{1}{\varepsilon} - \ln(T-i\,\lambda) - \ln(1-u^2) + \mathcal{O}(\varepsilon).$$

The $1/\varepsilon$ poles cancel between these two contributions, leaving only logarithms. On the other hand, the contribution coming from the second term requires some care since pole and branch point coincide. If this singularity lies outside the integration region the contour prescription for the pole is irrelevant and can be dropped. If the singularity lies inside $[0,1]$ and $\sigma_0 = +$, the contour prescription for the pole and cut are the same, there is no pinching. The integral over u can be performed after an expansion in ε using Appendix E, cf. Eq. (E.8). If the singularity lies inside $[0,1]$ and $\sigma_0 = -$, however, a pinching occurs at the singular point in the limit $\lambda \to 0^+$. A too early expansion in powers of ε before performing the integration over u would lead to a divergence order by order in ε. On the other hand, we note that for $u^2 = -R_{ij}/P_{ijk}$ the numerator of the second term, i.e. the pole residue, is $(-\,i\,\lambda)^{-\varepsilon} \to 0$ with $\lambda \to 0^+$ for any fixed $\varepsilon < 0$. Therefore, up to terms vanishing $\propto \lambda^{-\varepsilon}$, we can make the replacement

$$\int_0^1 du\,\frac{\left[u^2\,P_{ijk}+R_{ij}-i\,\lambda\right]^{-\varepsilon}}{u^2\,P_{ijk}+R_{ij}+i\,\lambda} \to \int_0^1 du\,\frac{\left[u^2\,P_{ijk}+R_{ij}-i\,\lambda\right]^{-\varepsilon}}{u^2\,P_{ijk}+R_{ij}-i\,\lambda}.$$

Details are provided in Appendix F. Finally, we perform a partial expansion around $\varepsilon = 0$ to get

$$\begin{aligned} L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) &= \frac{2^\varepsilon}{T}\,\Gamma(1+\varepsilon) \\ &\times \left\{ \int_0^1 \frac{du}{u^2 P_{ijk}+R_{ij}-i\lambda\sigma_0}\left[\ln\left((T-i\lambda)(1-u^2)\right) - \ln\left(u^2(P_{ijk}+R_{ij}-T)+T-i\lambda\right)\right]\right. \\ &\left. -\frac{1}{\varepsilon}\int_0^1 \frac{du}{u^2\,P_{ijk}+R_{ij}-i\,\lambda} + \int_0^1 du\,\frac{\ln(u^2\,P_{ijk}+R_{ij}-i\,\lambda)}{u^2\,P_{ijk}+R_{ij}-i\,\lambda}\right\}. \end{aligned} \tag{3.14}$$

Note that the imaginary part of the argument of each logarithm in Eq. (3.14) keeps a constant sign when u spans $[0,1]$. This is obviously true for the real mass case, and in the case of complex masses this is easily verified keeping in mind the assumptions that $\mathrm{Im}(T) < 0$ and $\mathrm{Im}(R_{ij}) < 0$. For the sake of coherence with respect to Ref. [2], the pole residue contributions will be added and subtracted for the two logarithms making up the first term inside the curly brackets of Eq. (3.14), which is recast in the form

$$
\begin{aligned}
L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) &= \frac{2^\varepsilon}{T}\,\Gamma(1+\varepsilon) \\
&\times \Bigg\{ -\frac{1}{\varepsilon} \int_0^1 \frac{du}{u^2\,P_{ijk} + R_{ij} - i\,\lambda} + \int_0^1 du\, \frac{\ln(u^2\,P_{ijk} + R_{ij} - i\,\lambda)}{u^2\,P_{ijk} + R_{ij} - i\,\lambda} \\
&\quad + \int_0^1 \frac{du}{u^2\,P_{ijk} + R_{ij} - i\,\lambda\,\sigma_0} \\
&\quad \times \Bigg[\ln\big((T - i\lambda)\,(1-u^2)\big) - \ln\left(\frac{T\,(P_{ijk} + R_{ij})}{P_{ijk}} - i\,\lambda\right) \\
&\quad - \ln\big(u^2\,(P_{ijk} + R_{ij} - T) + T - i\,\lambda\big) + \ln\left(\frac{(P_{ijk} + R_{ij})\,(T - R_{ij})}{P_{ijk}} - i\,\lambda\right) \Bigg] \\
&\quad - \int_0^1 \frac{du}{u^2\,P_{ijk} + R_{ij} - i\,\lambda\,\sigma_0}\, \ln\left(\frac{T - R_{ij} - i\,\lambda\,\sigma_1}{T - i\,\lambda\,\sigma_1}\right) \Bigg\},
\end{aligned} \tag{3.15}
$$

with $\sigma_1 = \mathrm{sign}((P_{ijk} + R_{ij})/P_{ijk})$ for the real mass case and

$$
\begin{aligned}
L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) &= \frac{2^\varepsilon}{T}\,\Gamma(1+\varepsilon) \\
&\times \Bigg\{ \int_0^1 \frac{du}{u^2\,P_{ijk} + R_{ij}} \left[\ln\big(T\,(1-u^2)\big) - \ln\left(\frac{T\,(P_{ijk} + R_{ij})}{P_{ijk}}\right) \right] \\
&\quad - \int_0^1 \frac{du}{u^2\,P_{ijk} + R_{ij}} \left[\ln\big(u^2\,(P_{ijk} + R_{ij} - T) + T\big) - \ln\left(\frac{(P_{ijk} + R_{ij})\,(T - R_{ij})}{P_{ijk}}\right) \right] \\
&\quad - \frac{1}{\varepsilon} \int_0^1 \frac{du}{u^2\,P_{ijk} + R_{ij}} + \int_0^1 du\, \frac{\ln(u^2\,P_{ijk} + R_{ij})}{u^2\,P_{ijk} + R_{ij}} \\
&\quad - \int_0^1 \frac{du}{u^2\,P_{ijk} + R_{ij}} \left[\ln\left(\frac{T - R_{ij}}{T}\right) + \eta\left(\frac{T\,(P_{ijk} + R_{ij})}{P_{ijk}}, \frac{T - R_{ij}}{T}\right) \right] \Bigg\}
\end{aligned} \tag{3.16}
$$

for the complex mass case. The relevant integrals are given in Appendices E of this paper, E of Ref. [1], and B of Ref. [2].

Equation (3.16) matches Eq. (3.19) of Ref. [2] obtained in the corresponding general complex mass case 1(a), considering the latter in the limit $\mathrm{Re}(\Delta_2^{\{i\}}) = \mathrm{Re}(Q_i + T) \to 0$ while keeping an infinitesimal positive imaginary part $\mathrm{Im}(\Delta_2^{\{i\}}) = \lambda$. One then formally gets

$$
\begin{aligned}
\text{RHS of [2, Eq. (3.19)]} \to \frac{1}{T} &\Bigg\{ \int_0^1 \frac{du}{u^2\,P_{ijk} + R_{ij}} \Bigg[\ln\big(T\,(1-u^2)\big) - \ln\left(\frac{T\,(P_{ijk} + R_{ij})}{P_{ijk}}\right) \\
&- \ln\big(u^2\,(P_{ijk} + R_{ij} - T) + T\big) + \ln\left(\frac{(P_{ijk} + R_{ij})\,(T - R_{ij})}{P_{ijk}}\right) \Bigg] \\
&- \ln(Q_i + T) \int_0^1 \frac{du}{u^2\,P_{ijk} + R_{ij}} + \int_0^1 du\, \frac{\ln(u^2\,P_{ijk} + R_{ij})}{u^2\,P_{ijk} + R_{ij}} \\
&- \int_0^1 \frac{du}{u^2\,P_{ijk} + R_{ij}} \left[\ln\left(\frac{T - R_{ij}}{T}\right) + \eta\left(\frac{T\,(P_{ijk} + R_{ij})}{P_{ijk}}, \frac{T - R_{ij}}{T}\right) \right] \Bigg\},
\end{aligned}
$$

where the divergent term $\ln(Q_i + T)$ corresponds to the "dressed pole" $(2\,e^{-\gamma_E})^\varepsilon/\varepsilon$.

3.2.2. $\mathrm{Im}(\Delta_3) > 0$, $\mathrm{Im}(\Delta_1^{\{i,j\}}) < 0$

Compared to the previous case, one uses Eq. (3.8) for $M_2(\xi^\nu)$ and the ξ integration is carried out with the help of Eqs. (A.6) and (A.7) depending on the different terms. Then, performing the first step described in Sect. 3.2.1, we obtain

$$\begin{aligned}
&L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) \\
&\quad = -\frac{F(\varepsilon)}{4}\left\{ i \int_0^\infty \frac{dy}{\sqrt{y}} \int_0^1 \frac{dx}{y\,P_{ijk} - x\,R_{ij}}\,(T(1-x))^{-1-\varepsilon} \right.\\
&\qquad + e^{i\,\pi\,\varepsilon} \int_0^\infty \frac{dy}{\sqrt{y}} \int_0^\infty \frac{dx}{y\,P_{ijk} - x\,R_{ij}} \left(-y\,P_{ijk} + x\,R_{ij} - T\,(1+x)\right)^{-1-\varepsilon} \\
&\qquad + i \int_0^\infty \frac{dy}{\sqrt{y}} \int_1^\infty \frac{dx}{y\,P_{ijk} - x\,R_{ij}} \left(-y\,P_{ijk} + x\,R_{ij} + T\,(1-x)\right)^{-1-\varepsilon} \\
&\qquad + \int_1^\infty \frac{dy}{\sqrt{y}} \int_0^1 \frac{dx}{y\,P_{ijk} + x\,R_{ij}} \left[(T\,(1-x))^{-1-\varepsilon} \right.\\
&\qquad\qquad \left. - \left(y\,P_{ijk} + x\,R_{ij} + T\,(1-x)\right)^{-1-\varepsilon}\right] \\
&\qquad + \int_0^1 \frac{dy}{\sqrt{y}} \int_0^y \frac{dx}{y\,P_{ijk} + x\,R_{ij}} \left[(T\,(1-x))^{-1-\varepsilon} \right.\\
&\qquad\qquad \left.\left. - \left(y\,P_{ijk} + x\,R_{ij} + T\,(1-x)\right)^{-1-\varepsilon}\right]\right\}.
\end{aligned} \tag{3.17}$$

We set $y = u^2$, perform a partial fraction decomposition on the variable x, and expand around $\varepsilon = 0$. The x integration is readily done, and $L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk})$ is written as

$$\begin{aligned}
&L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) \\
&\quad = -\frac{2^\varepsilon}{T}\,\Gamma(1+\varepsilon)\left\{ -\frac{T^{-\varepsilon}}{\varepsilon}\left[i \int_0^{+\infty} \frac{du}{u^2\,P_{ijk} - R_{ij}} + \int_1^{+\infty} \frac{du}{u^2\,P_{ijk} + R_{ij}} \right] \right.\\
&\qquad + i \int_0^{+\infty} \frac{du}{u^2\,P_{ijk} - R_{ij}} \left[\ln\left(\frac{u^2\,P_{ijk} - R_{ij}}{u^2\,P_{ijk}}\right) - \ln\left(\frac{R_{ij} - T}{R_{ij}}\right)\right] \\
&\qquad - \int_0^{+\infty} \frac{du}{u^2\,P_{ijk} + R_{ij}} \left[\ln\left(\frac{-u^2\,P_{ijk}}{R_{ij}}\right) - \ln\left(\frac{P_{ijk}\,u^2 + T}{T - R_{ij}}\right)\right] \\
&\qquad + \int_1^{+\infty} \frac{du}{u^2\,P_{ijk} + R_{ij}} \ln\left(\frac{u^2\,P_{ijk} + R_{ij}}{u^2\,P_{ijk} + T}\right) \\
&\qquad + \int_0^1 \frac{du}{u^2\,P_{ijk} + R_{ij}} \left[\ln\left(\frac{u^2\,(P_{ijk} + R_{ij} - T) + T}{u^2\,P_{ijk} + T}\right) \right.\\
&\qquad\qquad \left.\left. - \ln(1-u^2)\right]\right\}.
\end{aligned} \tag{3.18}$$

In this case we have $\mathrm{Im}(R_{ij}) > 0$, $\mathrm{Im}(P_{ijk}) < 0$, and $\mathrm{Im}(R_{ij} - T) > 0$. Furthermore,

- $u^2\,P_{ijk} - R_{ij} = (1+u^2)\,\Delta_1^{\{i,j\}} + u^2\,\widetilde{D}_{ijk}$, and thus $\mathrm{Im}(u^2\,P_{ijk} - R_{ij}) < 0$ when $u \in [0, \infty[$;
- $u^2\,P_{ijk} + R_{ij} = u^2\,\widetilde{D}_{ijk} + (u^2 - 1)\,\Delta_1^{\{i,j\}}$, and thus $\mathrm{Im}(u^2\,P_{ijk} + R_{ij}) < 0$ when $u \in [1, \infty[$;

- $u^2\,P_{ijk} + T = u^2\,(\widetilde{D}_{ijk} + \Delta_1^{\{i,j\}}) - \Delta_3$, and thus $\mathrm{Im}(u^2\,P_{ijk} + T) < 0$ when $u \in [0, \infty[$;
- $u^2\,(P_{ijk} + R_{ij} - T) + T = u^2\,\widetilde{D}_{ijk} - (1 - u^2)\,\Delta_3$, and thus $\mathrm{Im}(u^2\,(P_{ijk} + R_{ij} - T) + T) < 0$ when $u \in [0, 1]$.

Rearrangements and simplifications similar to those done in the massive case 2(a) of Ref. [2] can be performed, which lead to the following alternative expression:

$$
\begin{aligned}
L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) &= \frac{2^\varepsilon}{T}\,\Gamma(1+\varepsilon) \\
&\times \Bigg\{ -\frac{1}{\varepsilon} \int_{0\;1} \frac{du}{u^2\,P_{ijk} + R_{ij}} \quad + \quad \int_{0\;1} du\,\frac{\ln(u^2\,P_{ijk} + R_{ij})}{u^2\,P_{ijk} + R_{ij}} \\
&\quad + \int_0^1 \frac{du}{u^2\,P_{ijk} + R_{ij}} \Bigg[\ln\big(T\,(1-u^2)\big) - \ln\left(\frac{T\,(P_{ijk} + R_{ij})}{P_{ijk}}\right) \\
&\quad - \ln\big(u^2\,(P_{ijk} + R_{ij} - T) + T\big) + \ln\left(\frac{(P_{ijk} + R_{ij})\,(T - R_{ij})}{P_{ijk}}\right) \\
&\quad - \eta\left(\frac{T\,(P_{ijk} + R_{ij})}{P_{ijk}}, \frac{T - R_{ij}}{T}\right)\Bigg] - \int_{0\;1} \frac{du}{u^2\,P_{ijk} + R_{ij}}\,\ln\left(\frac{T - R_{ij}}{T}\right)\Bigg\}. \qquad (3.19)
\end{aligned}
$$

The contour $_{0\;1}$ can be deformed into a contour $\widehat{(0,1)}^{+}$ stretched from 0 to 1 and which eventually wraps from above the cut of $\ln(u^2\,P_{ijk} + R_{ij})$ emerging from the branch point $u_0 = \sqrt{-R_{ij}/P_{ijk}}$, whenever the latter lies in the "north-east" quadrant $\{\mathrm{Re}(u) > 0, \mathrm{Im}(u) > 0\}$.

Equation (3.19) can be compared with Eq. (3.37) of Ref. [2] obtained in the corresponding general complex mass case 2(a) considered in the limit $\mathrm{Re}(\Delta_2^{\{i\}}) = \mathrm{Re}(Q_i + T) \to 0$ while keeping an infinitesimal positive imaginary part $\mathrm{Im}(\Delta_2^{\{i\}}) = \lambda$. Whereas it appeared convenient to formulate Eq. (3.37) of Ref. [2] in terms of manifestly vanishing pole residues, this is no longer the case for Eq. (3.19) since the pole has also become the branch point of $\ln(u^2\,P_{ijk} + (R_{ij} + Q_i + T))$ in the limit $(Q_i + T) \to 0$. For the purpose of comparison, the η functions introduced in Eq. (3.37) of Ref. [2] containing $Q_i + T$ will thus be made explicit in terms of constant logarithms, part of which then cancel against the constant logarithms which were subtracted so as to build the explicitly vanishing pole residues. We get

$$
\begin{aligned}
\text{RHS of [2, Eq. (3.37)]} &\to \frac{1}{T} \\
&\times \Bigg\{ -\ln(Q_i + T) \int_{\widehat{(0,1)}^{+}} \frac{du}{u^2\,P_{ijk} + R_{ij}} \quad + \quad \int_{\widehat{(0,1)}^{+}} du\,\frac{\ln(u^2\,P_{ijk} + R_{ij})}{u^2\,P_{ijk} + R_{ij}} \\
&\quad + \int_0^1 \frac{du}{u^2\,P_{ijk} + R_{ij}} \Bigg[\ln\big(T\,(1-u^2)\big) - \ln\left(\frac{T\,(P_{ijk} + R_{ij})}{P_{ijk}}\right) \\
&\quad - \ln\big(u^2\,(P_{ijk} + R_{ij} - T) + T\big) + \ln\left(\frac{(P_{ijk} + R_{ij})\,(T - R_{ij})}{P_{ijk}}\right) \\
&\quad - \eta\left(\frac{T\,(P_{ijk} + R_{ij})}{P_{ijk}}, \frac{T - R_{ij}}{T}\right)\Bigg] - \int_{\widehat{(0,1)}^{+}} \frac{du}{u^2\,P_{ijk} + R_{ij}}\,\ln\left(\frac{T - R_{ij}}{T}\right)\Bigg\},
\end{aligned}
$$

where the divergent term $\ln(Q_i + T)$ corresponds to the "dressed pole" $(2\,e^{-\gamma_E})^\varepsilon/\varepsilon$.

3.2.3. $\mathrm{Im}(\Delta_3) < 0$, $\mathrm{Im}(\Delta_1^{\{i,j\}}) > 0$

In this case we start with Eq. (3.7) for $M_2(\xi^\nu)$ and use Eq. (A.7) for all the ξ integrations. We give the result for $L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk})$ after the intermediate step 1 of Sect. 3.2.1:

$$
\begin{aligned}
&L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) \\
&\quad = \frac{F(\varepsilon)}{4} \left\{ i\, e^{-i\pi\varepsilon} \int_0^\infty \frac{dy}{\sqrt{y}} \int_y^\infty \frac{dx}{y\,P_{ijk} + x\,R_{ij}} \left[\left(y\,P_{ijk} + x\,R_{ij} - T\,(1+x) \right)^{-1-\varepsilon} \right.\right. \\
&\qquad\qquad \left. - \left(-T\,(1+x)\right)^{-1-\varepsilon} \right] \\
&\qquad + \int_1^\infty \frac{dy}{\sqrt{y}} \int_y^\infty \frac{dx}{y\,P_{ijk} + x\,R_{ij}} \left[\left(y\,P_{ijk} + x\,R_{ij} + T\,(1-x) \right)^{-1-\varepsilon} - \left(T\,(1-x)\right)^{-1-\varepsilon} \right] \\
&\qquad + \int_0^1 \frac{dy}{\sqrt{y}} \int_1^\infty \frac{dx}{y\,P_{ijk} + x\,R_{ij}} \left[\left(y\,P_{ijk} + x\,R_{ij} + T\,(1-x) \right)^{-1-\varepsilon} \right. \\
&\qquad\qquad \left.\left. - \left(T\,(1-x)\right)^{-1-\varepsilon} \right] \right\}.
\end{aligned}
\tag{3.20}
$$

Then, performing steps 2 and 3 above yields

$$
\begin{aligned}
&L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) \\
&\quad = \frac{2^\varepsilon}{T}\,\Gamma(1+\varepsilon) \left\{ -\frac{(-T)^{-\varepsilon}}{\varepsilon} \int_0^1 \frac{du}{u^2\,P_{ijk} + R_{ij}} \right. \\
&\qquad + i \int_0^{+\infty} \frac{du}{u^2\,P_{ijk} - R_{ij}} \left[\ln\left(\frac{u^2\,(P_{ijk} + R_{ij} - T) - T}{R_{ij} - T} \right) - \ln\left(u^2 + 1\right) \right] \\
&\qquad + \int_1^{+\infty} \frac{du}{u^2\,P_{ijk} + R_{ij}} \left[\ln\left(\frac{u^2\,(P_{ijk} + R_{ij} - T) + T}{R_{ij} - T} \right) - \ln\left(u^2 - 1\right) \right] \\
&\qquad \left. + \int_0^1 \frac{du}{u^2\,P_{ijk} + R_{ij}} \ln\left(\frac{u^2\,P_{ijk} + R_{ij}}{R_{ij} - T} \right) \right\}.
\end{aligned}
\tag{3.21}
$$

In this case we have $\mathrm{Im}(R_{ij} - T) < 0$. Furthermore,

- $u^2\,P_{ijk} + R_{ij} = u^2\,\widetilde{D}_{ijk} - (1-u^2)\,\Delta_1^{\{i,j\}}$, and thus $\mathrm{Im}(u^2\,P_{ijk} + R_{ij}) < 0$ when $u \in [0,1]$;
- $u^2\,(P_{ijk} + R_{ij} - T) - T = u^2\,\widetilde{D}_{ijk} + (1+u^2)\,\Delta_3$, and thus $\mathrm{Im}(u^2\,(P_{ijk} + R_{ij} - T) - T) < 0$ when $u \in [0,\infty[$;
- $u^2\,(P_{ijk} + R_{ij} - T) + T = u^2\,\widetilde{D}_{ijk} + (u^2-1)\,\Delta_3$, and thus $\mathrm{Im}(u^2\,(P_{ijk} + R_{ij} - T) + T) < 0$ when $u \in [1,\infty[$.

Rearrangements and simplifications similar to those done in the massive case 1(c) of Ref. [2] can be performed, leading to the following alternative expression:

$$
\begin{aligned}
L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) &= \frac{2^\varepsilon}{T}\,\Gamma(1+\varepsilon) \\
&\times \Bigg\{ -\frac{1}{\varepsilon} \int_0^1 \frac{du}{u^2\,P_{ijk} + R_{ij}} \quad + \quad \int_0^1 du\, \frac{\ln(u^2\,P_{ijk} + R_{ij})}{u^2\,P_{ijk} + R_{ij}} \\
&\quad + \int \frac{du}{u^2\,P_{ijk} + R_{ij}} \Bigg[\ln\big(T\,(1-u^2)\big) - \ln\left(\frac{T\,(P_{ijk} + R_{ij})}{P_{ijk}}\right) \\
&\qquad - \ln\big(u^2\,(P_{ijk} + R_{ij} - T) + T\big) + \ln\left(\frac{(P_{ijk} + R_{ij})\,(T - R_{ij})}{P_{ijk}}\right) \\
&\qquad - \eta\left(\frac{T\,(P_{ijk} + R_{ij})}{P_{ijk}}, \frac{T - R_{ij}}{T}\right)\Bigg] - \int_0^1 \frac{du}{u^2\,P_{ijk} + R_{ij}}\,\ln\left(\frac{T - R_{ij}}{T}\right)\Bigg\}. \qquad (3.22)
\end{aligned}
$$

The deformation of the contour into a contour $\widehat{(0,1)}^{+}$ stretched from 0 to 1 may eventually wrap from above the cut of $\ln(u^2\,(P_{ijk} + R_{ij} - T) + T)$ emerging from the branch point $u_0 = \sqrt{-T/(P_{ijk} + R_{ij} - T)}$, whenever the latter lies in the "north-east" quadrant.

Similar to the previous case, Eq. (3.22) matches Eq. (3.33) of Ref. [2] obtained in the corresponding general complex mass case 1(c) considered in the limit $\mathrm{Re}(\Delta_2^{\{i\}}) = \mathrm{Re}(Q_i + T) \to 0$ while keeping an infinitesimal positive imaginary part $\mathrm{Im}(\Delta_2^{\{i\}}) = \lambda$.

3.2.4. $\mathrm{Im}(\Delta_3) < 0$, $\mathrm{Im}(\Delta_1^{\{i,j\}}) < 0$

One uses Eq. (3.8) for $M_2(\xi^\nu)$ and the ξ integration is carried out with the help of Eqs. (A.6) and (A.7), depending on the terms. Then, step 1 of Sect. 3.2.1 leads to

$$
\begin{aligned}
&L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) \\
&= \frac{F(\varepsilon)}{4}\Bigg\{ i\,(-T)^{-1-\varepsilon} \int_0^\infty \frac{dy}{\sqrt{y}} \int_1^\infty \frac{dx}{y\,P_{ijk} - x\,R_{ij}}\,(x-1)^{-1-\varepsilon} \\
&\qquad - e^{-i\,\pi\,\varepsilon}\,(-T)^{-1-\varepsilon} \int_0^\infty \frac{dy}{\sqrt{y}} \int_0^\infty \frac{dx}{y\,P_{ijk} - x\,R_{ij}}\,(1+x)^{-1-\varepsilon} \\
&\qquad + i \int_0^\infty \frac{dy}{\sqrt{y}} \int_0^1 \frac{dx}{y\,P_{ijk} - x\,R_{ij}}\,\big(-y\,P_{ijk} + x\,R_{ij} + T\,(1-x)\big)^{-1-\varepsilon} \\
&\qquad - i\,e^{-i\,\pi\,\varepsilon} \int_0^\infty \frac{dy}{\sqrt{y}} \int_0^y \frac{dx}{y\,P_{ijk} + x\,R_{ij}}\Big[\big(y\,P_{ijk} + x\,R_{ij} - T\,(1+x)\big)^{-1-\varepsilon} \\
&\qquad\qquad - (-T\,(1+x))^{-1-\varepsilon}\Big] \\
&\qquad - \int_1^\infty \frac{dy}{\sqrt{y}} \int_1^y \frac{dx}{y\,P_{ijk} + x\,R_{ij}}\Big[\big(y\,P_{ijk} + x\,R_{ij} + T\,(1-x)\big)^{-1-\varepsilon} \\
&\qquad\qquad - (T\,(1-x))^{-1-\varepsilon}\Big]\Bigg\}. \qquad (3.23)
\end{aligned}
$$

At the end of step 5 we get

$$
\begin{aligned}
&L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) \\
&\quad = -\frac{2^\varepsilon}{T}\,\Gamma(1+\varepsilon)\left\{-\frac{(-T)^{-\varepsilon}}{\varepsilon}\left[i\int_0^{+\infty}\frac{du}{u^2\,P_{ijk}-R_{ij}} + \int_1^{+\infty}\frac{du}{u^2\,P_{ijk}+R_{ij}}\right]\right. \\
&\qquad + i\int_0^{+\infty}\frac{du}{u^2\,P_{ijk}-R_{ij}}\left[\ln\left(\frac{R_{ij}-u^2\,P_{ijk}}{R_{ij}}\right) - \ln\left(\frac{P_{ijk}\,u^2-T}{u^2\,P_{ijk}}\right)\right] \\
&\qquad - i\int_0^{+\infty}\frac{du}{u^2\,P_{ijk}-R_{ij}}\left[\ln\left(\frac{u^2\,(P_{ijk}+R_{ij}-T)-T}{u^2\,P_{ijk}-T}\right) - \ln\left(u^2+1\right)\right] \\
&\qquad - \int_1^{+\infty}\frac{du}{u^2\,P_{ijk}+R_{ij}}\left[\ln\left(\frac{u^2\,(P_{ijk}+R_{ij}-T)+T}{u^2\,P_{ijk}+R_{ij}}\right) - \ln(u^2-1)\right] \\
&\qquad \left. - \int_0^{+\infty}\frac{du}{u^2\,P_{ijk}+R_{ij}}\,\ln\left(\frac{-u^2\,P_{ijk}}{R_{ij}}\right)\right\}.
\end{aligned}
\tag{3.24}
$$

In this case we have $\mathrm{Im}(R_{ij}) > 0$, $\mathrm{Im}(P_{ijk}) < 0$. Furthermore,

- $u^2\,P_{ijk} - R_{ij} = u^2\,\widetilde{D}_{ijk} + (u^2+1)\,\Delta_1^{\{i,j\}}$, and thus $\mathrm{Im}(u^2\,P_{ijk} - R_{ij}) < 0$ when $u \in [0,\infty[$;
- $u^2\,P_{ijk} + R_{ij} = u^2\,\widetilde{D}_{ijk} + (u^2-1)\,\Delta_1^{\{i,j\}}$, and thus $\mathrm{Im}(u^2\,P_{ijk} + R_{ij}) < 0$ when $u \in [1,\infty[$;
- $u^2\,P_{ijk} - T = u^2\,(\widetilde{D}_{ijk} + \Delta_1^{\{i,j\}}) + \Delta_3$, and thus $\mathrm{Im}(u^2\,P_{ijk} - T) < 0$ when $u \in [0,\infty[$;
- $u^2\,(P_{ijk} + R_{ij} - T) - T = u^2\,\widetilde{D}_{ijk} + (u^2+1)\,\Delta_3$, and thus $\mathrm{Im}(u^2\,(P_{ijk} + R_{ij} - T) - T) < 0$ when $u \in [0,\infty[$;
- $u^2\,(P_{ijk} + R_{ij} - T) + T = u^2\,\widetilde{D}_{ijk} + (u^2-1)\,\Delta_3$, and thus $\mathrm{Im}(u^2\,(P_{ijk} + R_{ij} - T) + T) < 0$ when $u \in [1,\infty[$.

Rearrangements and simplifications similar to those done in the massive case 2(c) of Ref. [2] can be performed, leading to the following alternative expression:

$$
\begin{aligned}
L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) &= \frac{2^\varepsilon}{T}\,\Gamma(1+\varepsilon)\int_{0\ 1}\frac{du}{u^2\,P_{ijk}+R_{ij}} \\
&\quad \times\left\{-\frac{1}{\varepsilon} + \ln(u^2 P_{ijk} + R_{ij}) + \ln\left(T(1-u^2)\right) - \ln\left(\frac{T(P_{ijk}+R_{ij})}{P_{ijk}}\right)\right. \\
&\qquad - \ln\left(u^2\,(P_{ijk}+R_{ij}-T)+T\right) + \ln\left(\frac{(P_{ijk}+R_{ij})\,(T-R_{ij})}{P_{ijk}}\right) \\
&\qquad \left. - \eta\left(\frac{T\,(P_{ijk}+R_{ij})}{P_{ijk}}, \frac{T-R_{ij}}{T}\right) - \ln\left(\frac{T-R_{ij}}{T}\right)\right\}.
\end{aligned}
\tag{3.25}
$$

As in the previous cases, the contours $_{0\ 1}$ can be deformed into contours $\widehat{(0,1)}^{+}_{1,2}$ stretched from 0 to 1. They eventually wrap from above the cuts of $\ln(u^2\,P_{ijk} + R_{ij})$ emerging from the branch point $\sqrt{-R_{ij}/P_{ijk}}$ and of $\ln(u^2\,(P_{ijk}+R_{ij}-T)+T)$ emerging from the branch point $\sqrt{-T/(P_{ijk}+R_{ij}-T)}$, respectively, whenever either or both of these branch points lie in the "north-east" quadrant.

Similar to the previous case, Eq. (3.25) matches Eq. (3.41) of Ref. [2] obtained in the corresponding general complex mass case 2(c) considered in the limit $\mathrm{Re}(\Delta_2^{\{i\}}) = \mathrm{Re}(Q_i + T) \to 0$ while keeping an infinitesimal positive imaginary part $\mathrm{Im}(\Delta_2^{\{i\}}) = \lambda$.

3.3. $\Delta_2^{\{i\}} = 0$ *and* $\widetilde{D}_{ijk} = 0$

In this case we have $P_{ijk} = -R_{ij}$ and $Q_i = -T$, so that Eqs. (3.7) and (3.8) become

for $\mathrm{Im}(\Delta_1^{\{i,j\}} > 0)$,

$$M_2(\xi^\nu) = \frac{1}{2} B(1/2, 1/2) \int_0^1 \frac{dz}{\Delta_1^{\{i,j\}}\,(z^2-1)} \times \left[\frac{1}{(\xi^\nu - i\,\lambda)^{1/2}} - \frac{1}{\left(\xi^\nu + \Delta_1^{\{i,j\}}\,(z^2-1) - i\,\lambda\right)^{1/2}} \right]; \tag{3.26}$$

for $\mathrm{Im}(\Delta_1^{\{i,j\}} < 0)$,

$$\begin{aligned} M_2(\xi^\nu) = -\frac{1}{2} B(1/2, 1/2) \Bigg\{ & i \int_0^{+\infty} \frac{dz}{\Delta_1^{\{i,j\}}\,(z^2+1)} \\ & \times \left[\frac{1}{(\xi^\nu - i\,\lambda)^{1/2}} - \frac{1}{\left(\xi^\nu - \Delta_1^{\{i,j\}}\,(1+z^2)\right)^{1/2}} \right] \\ & + \int_1^{+\infty} \frac{dz}{\Delta_1^{\{i,j\}}\,(z^2-1)} \\ & \times \left[\frac{1}{(\xi^\nu - i\,\lambda)^{1/2}} - \frac{1}{\left(\xi^\nu + \Delta_1^{\{i,j\}}\,(z^2-1)\right)^{1/2}} \right] \Bigg\}. \end{aligned} \tag{3.27}$$

Here again, the ξ integration will be of the type in Eqs. (A.6) or (A.7) (cf. Appendix A). Let us go through the different cases according to the signs of $\mathrm{Im}(\Delta_3)$ and $\mathrm{Im}(\Delta_1^{\{i,j\}})$.

3.3.1. $\mathrm{Im}(\Delta_3) > 0$, $\mathrm{Im}(\Delta_1^{\{i,j\}}) > 0$

We proceed along the two first steps described in Sect. 3.2.1. We borrow the result obtained in Eq. (3.11), set $P_{ijk} = -R_{ij}$, and make the change of variables $x = y + (1-y)\,v$. The quantity $L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0)$ now reads

$$L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0) = -\frac{F(\varepsilon)}{4\,R_{ij}} \int_0^1 \frac{dv}{v} \left[\frac{1}{[v\,R_{ij} + (1-v)\,T\ - i\,\lambda]^{1+\varepsilon}} - \frac{1}{[(1-v)\,T\ - i\,\lambda]^{1+\varepsilon}} \right] \times \int_0^1 dy\, y^{-1/2}\,(1-y)^{-1-\varepsilon}. \tag{3.28}$$

The integrals over y and v are unnested and are computed easily using Eqs. (C.3), (C.6), and (C.10). Notice that $\mathrm{Im}(v\,R_{ij} + (1-v)\,T\ - i\,\lambda)$ never changes its sign when v spans [0, 1]; this is obviously true in the real mass case, and due to the fact that $\mathrm{Im}(T)$ and $\mathrm{Im}(R_{ij})$ have imaginary parts of the same sign in the complex mass case. Inserting the explicit results for the y and v integrals, we get

$$L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0) = \frac{1}{2}\,\Gamma(1+\varepsilon)\,\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,\frac{1}{R_{ij}\,T} \times \left[\frac{1}{\varepsilon^2}\,(2\,R_{ij} - i\,\lambda)^{-\varepsilon} + \mathrm{Li}_2\left(\frac{T - R_{ij}}{T - i\,\lambda}\right) - \frac{\pi^2}{6} \right]. \tag{3.29}$$

Equation (3.29) explicitly displays the singularity of $L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0)$ when $\varepsilon \to 0$. In Eq. (3.29) we use the property $\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \pi^2/6 - \ln(z)\ \ln(1-z)$ to obtain the following alternative form suitable for further comparisons:

$$
\begin{aligned}
L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0) = {} & \frac{1}{2}\,\Gamma(1+\varepsilon)\,\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,\frac{1}{R_{ij}\,T} \\
& \times \left\{ \frac{1}{\varepsilon^2}\,(2\,R_{ij} - i\lambda)^{-\varepsilon} - \mathrm{Li}_2\left(\frac{R_{ij} - i\lambda}{T - i\lambda}\right) \right. \\
& \left. - \left[\ln\left(R_{ij} - i\lambda\right) - \ln\left(T - i\lambda\right)\right] \ln\left(\frac{T - R_{ij}}{T - i\lambda}\right) \right\}.
\end{aligned} \tag{3.30}
$$

3.3.2. $\mathrm{Im}(\Delta_3) > 0$, $\mathrm{Im}(\Delta_1^{\{i,j\}}) < 0$

The starting point is Eq. (3.17) with $P_{ijk} = -R_{ij}$. In the first line of Eq. (3.17) we rescale $y = x\,u^2$ so that the double integral factorizes into a product of two unnested integrals over u and x. The integrals of the second and third lines of Eq. (3.17) yield no divergences when $\varepsilon \to 0$, so we take $\varepsilon = 0$ in them. We then make the change of variables $x = y - (y-1)\,v$ in the penultimate line, and $x = y - (1-y)\,v$ in the last line. We obtain

$$
\begin{aligned}
& L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0) \\
& = \frac{F(\varepsilon)}{4\,R_{ij}} \left\{ i\,T^{-1-\varepsilon} \int_0^{+\infty} \frac{2\,du}{1+u^2} \int_0^1 \frac{dx}{\sqrt{x}}\,(1-x)^{-1-\varepsilon} \right. \\
& \quad + \int_0^{+\infty} \frac{dy}{\sqrt{y}} \int_0^{+\infty} \frac{dx}{(y+x)\ ((x+y)\,R_{ij} - T\,(1+x))} \\
& \quad + i \int_0^{+\infty} \frac{dy}{\sqrt{y}} \int_1^{+\infty} \frac{dx}{(y+x)\ ((x+y)\,R_{ij} + T\,(1-x))} \\
& \quad + \int_1^{+\infty} \frac{dy}{\sqrt{y}}\,(y-1)^{-1-\varepsilon} \int_1^{\frac{y}{y-1}} \frac{dv}{v} \left[T^{-1-\varepsilon}\,(v-1)^{-1-\varepsilon} - \left(-v\,R_{ij} - T\,(1-v)\right)^{-1-\varepsilon} \right] \\
& \quad \left. + \int_0^1 \frac{dy}{\sqrt{y}}\,(1-y)^{-1-\varepsilon} \int_0^{\frac{y}{1-y}} \frac{dv}{v} \left[T^{-1-\varepsilon}\,(1+v)^{-1-\varepsilon} - \left(-v\,R_{ij} + T\,(1+v)\right)^{-1-\varepsilon} \right] \right\}.
\end{aligned} \tag{3.31}
$$

Let us compute the different terms. Using Eq. (C.3), the first integral is readily given by

$$
\int_0^{+\infty} \frac{2\,du}{1+u^2} \int_0^1 \frac{dx}{\sqrt{x}}\,(1-x)^{-1-\varepsilon} = \pi\,B\left(\frac{1}{2}, -\varepsilon\right). \tag{3.32}
$$

In the second and third lines of Eq. (3.31), the x integration is easily performed after a partial fraction decomposition on the x variable. We get

$$\int_0^{+\infty} \frac{dy}{\sqrt{y}} \int_0^{+\infty} \frac{dx}{(y+x)\,((x+y)\,R_{ij} - T\,(1+x))}$$
$$= \frac{1}{T} \int_0^{+\infty} \frac{dy}{\sqrt{y}} \frac{1}{y-1} \left[\ln\left(\frac{y\,R_{ij} - T}{R_{ij} - T} \right) - \ln(y) \right], \tag{3.33}$$

$$\int_0^{+\infty} \frac{dy}{\sqrt{y}} \int_1^{+\infty} \frac{dx}{(y+x)\,((x+y)\,R_{ij} + T\,(1-x))}$$
$$= \frac{1}{T} \int_0^{+\infty} \frac{dy}{\sqrt{y}} \frac{1}{y+1} \ln\left(\frac{R_{ij}}{R_{ij} - T} \right). \tag{3.34}$$

We write the last two integrals of Eq. (3.31) as

$$\int_1^{+\infty} \frac{dy}{\sqrt{y}} (y-1)^{-1-\varepsilon} \int_1^{\frac{y}{y-1}} \frac{dv}{v} \left[T^{-1-\varepsilon}\,(v-1)^{-1-\varepsilon} - \left(-v\,R_{ij} - T\,(1-v)\right)^{-1-\varepsilon} \right]$$
$$= \int_1^{+\infty} \frac{dy}{\sqrt{y}} (y-1)^{-1-\varepsilon} \left[E_1(y) - E_1(1^+) \right] \; + \; E_1(1^+)\, B\left(\frac{1}{2} + \varepsilon, -\varepsilon \right) \tag{3.35}$$

$$\int_0^1 \frac{dy}{\sqrt{y}} (1-y)^{-1-\varepsilon} \int_0^{\frac{y}{1-y}} \frac{dv}{v} \left[T^{-1-\varepsilon}\,(1+v)^{-1-\varepsilon} - \left(-v\,R_{ij} + T\,(1+v)\right)^{-1-\varepsilon} \right]$$
$$= \int_0^1 \frac{dy}{\sqrt{y}} (1-y)^{-1-\varepsilon} \left[E_2(y) - E_2(1^-) \right] \; + \; E_2(1^-)\, B\left(\frac{1}{2}, -\varepsilon \right), \tag{3.36}$$

with

$$E_1(y) = \int_1^{\frac{y}{y-1}} \frac{dv}{v} \left[T^{-1-\varepsilon}\,(v-1)^{-1-\varepsilon} - \left(-v\,R_{ij} - T\,(1-v)\right)^{-1-\varepsilon} \right], \tag{3.37}$$

$$E_2(y) = \int_0^{\frac{y}{1-y}} \frac{dv}{v} \left[T^{-1-\varepsilon}\,(1+v)^{-1-\varepsilon} - \left(-v\,R_{ij} + T\,(1+v)\right)^{-1-\varepsilon} \right]. \tag{3.38}$$

When $y \to 1^+$, the upper bound of the integral defining the function $E_1(y)$ goes to $+\infty$ and likewise for $E_2(y)$ when $y \to 1^-$. The quantities $E_1(y) - E_1(1^+)$ and $E_2(y) - E_2(1^-)$ are given by integrals between $y/(y-1)$ and $+\infty$ and between $y/(1-y)$ and $+\infty$, respectively. As, in $E_1(y) - E_1(1^+)$, y is greater than 1 and so is the lower bound $y/(y-1)$, the singular support $v = 1$ of the distribution $(v-1)^{-1-\varepsilon}$ lies outside the range of integration, and thus the first term of the right-hand side of Eq. (3.37) can be taken at $\varepsilon = 0$. In $E_2(y) - E_2(1^-)$, the integrand is non-singular and the first term of the right-hand side of Eq. (3.38) can also be taken at $\varepsilon = 0$. These two terms give

$$\int_1^{+\infty} \frac{dy}{\sqrt{y}} (y-1)^{-1-\varepsilon} \left[E_1(y) - E_1(1^+) \right] = -\frac{1}{T} \int_1^{+\infty} \frac{dy}{\sqrt{y}} \frac{1}{y-1} \ln\left(\frac{y\,R_{ij} - T}{R_{ij} - T} \right), \tag{3.39}$$

$$\int_0^1 \frac{dy}{\sqrt{y}} (1-y)^{-1-\varepsilon} \left[E_2(y) - E_2(1^-) \right] = -\frac{1}{T} \int_0^1 \frac{dy}{\sqrt{y}} \frac{1}{y-1} \ln\left(\frac{y\,R_{ij} - T}{R_{ij} - T} \right). \tag{3.40}$$

The combined right-hand sides of Eqs. (3.39) and (3.40) cancel against the first term in Eq. (3.33). The expressions of $E_1(1^+)$ and $E_2(1^-)$ are computed using Eqs. (C.13) and (C.15):

$$\begin{aligned} E_1(1^+) &= -K_4(T-R_{ij},-T) \\ &= \frac{1}{T}\left\{-\frac{1}{\varepsilon}\left(-R_{ij}\right)^{-\varepsilon}+\ln(T)-\ln(T-R_{ij})\right. \\ &\qquad\left.+\varepsilon\left[\mathrm{Li}_2\left(\frac{R_{ij}}{T}\right)+\ln\left(-R_{ij}\right)\,\ln\left(\frac{T-R_{ij}}{T}\right)\right]\right\}, \end{aligned} \tag{3.41}$$

$$\begin{aligned} E_2(1^-) &= -K_3(T-R_{ij},T) \\ &= \frac{1}{T}\ln\left(\frac{T-R_{ij}}{T}\right)\left[1-\varepsilon\,\ln(T)\right]. \end{aligned} \tag{3.42}$$

Putting everything together, we get

$$\begin{aligned} &L_4^n(\Delta_3,0,\Delta_1^{\{i,j\}},0) \\ &= \frac{F(\varepsilon)}{4\,R_{ij}\,T}\left\{i\,\pi\,B\left(\frac{1}{2},-\varepsilon\right)(1-\varepsilon\ln(T))\right. \\ &\qquad -\int_0^{+\infty}\frac{dy}{\sqrt{y}}\,\frac{1}{y-1}\,\ln(y)+i\int_0^{+\infty}\frac{dy}{\sqrt{y}}\,\frac{1}{y+1}\,\ln\left(\frac{R_{ij}}{R_{ij}-T}\right) \\ &\qquad +B\left(\frac{1}{2},-\varepsilon\right)\left[\ln\left(\frac{T-R_{ij}}{T}\right)(1-\varepsilon\,\ln(T))\right] \\ &\qquad +B\left(\frac{1}{2}+\varepsilon,-\varepsilon\right)\left[-\frac{1}{\varepsilon}\left(-R_{ij}\right)^{-\varepsilon}+\ln(T)-\ln(T-R_{ij})\right. \\ &\qquad\qquad\left.\left.+\ \varepsilon\left[\mathrm{Li}_2\left(\frac{R_{ij}}{T}\right)+\ln\left(-R_{ij}\right)\,\ln\left(\frac{T-R_{ij}}{T}\right)\right]\right]\right\}. \end{aligned} \tag{3.43}$$

Extracting the Euler Beta functions from Eqs. (C.2), (C.5), (C.3), and (C.6), and using the fact that $\ln(-R_{ij})=\ln(R_{ij})-i\,\pi$, Eq. (3.43) can be cast in the form

$$\begin{aligned} &L_4^n(\Delta_3,0,\Delta_1^{\{i,j\}},0) \\ &= \frac{1}{2}\,\Gamma(1+\varepsilon)\,\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,\frac{1}{R_{ij}\,T} \\ &\quad\times\left\{\frac{1}{\varepsilon^2}\left(2\,R_{ij}\right)^{-\varepsilon}-\mathrm{Li}_2\left(\frac{R_{ij}}{T}\right)-\left[\ln\left(R_{ij}\right)-\ln\left(T\right)\right]\,\ln\left(\frac{T-R_{ij}}{T}\right)\right\}. \end{aligned} \tag{3.44}$$

Equations (3.44) and (3.30) have the same analytic expression.

3.3.3. $\mathrm{Im}(\Delta_3)<0$, $\mathrm{Im}(\Delta_1^{\{i,j\}})>0$

We start with Eq. (3.20) with $P_{ijk}=-R_{ij}$. We then set $x=y+(1+y)\,v$ in the first integral of Eq. (3.20), $x=y+(y-1)\,v$ in the second integral, and $x=y+(1-y)\,v$ in the third one, and we get

$$
\begin{aligned}
L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0) = \frac{F(\varepsilon)}{4R_{ij}} \Bigg\{ & i\mathrm{e}^{-i\pi\varepsilon} \int_0^{+\infty} \frac{dy}{\sqrt{y}} (1+y)^{-1-\varepsilon} \\
& \times \int_0^{+\infty} \frac{dv}{v} \left[\frac{1}{[vR_{ij} - (1+v)T]^{1+\varepsilon}} - \frac{1}{[-(1+v)\,T]^{1+\varepsilon}} \right] \\
+ & \int_1^{+\infty} \frac{dy}{\sqrt{y}} (y-1)^{-1-\varepsilon} \\
& \times \int_0^{+\infty} \frac{dv}{v} \left[\frac{1}{[vR_{ij} - (1+v)T]^{1+\varepsilon}} - \frac{1}{[-(1+v)T]^{1+\varepsilon}} \right] \\
+ & \int_0^{1} \frac{dy}{\sqrt{y}} (1-y)^{-1-\varepsilon} \\
& \times \int_1^{+\infty} \frac{dv}{v} \left[\frac{1}{[vR_{ij} + (1-v)T]^{1+\varepsilon}} - \frac{1}{[(1-v)\,T]^{1+\varepsilon}} \right] \Bigg\}.
\end{aligned}
\tag{3.45}
$$

Here again, the integrals over y and v are unnested. Since the signs of $\mathrm{Im}(R_{ij})$ and $\mathrm{Im}(T)$ are mutually opposite, let us note that the imaginary parts of each of the terms raised to the power $1+\varepsilon$ in denominators in the v integrals remain constant over the corresponding ranges of integration over v. The first two integrals on v of Eq. (3.45) are given by Eq. (C.13) with $A = R_{ij} - T$ and $B = -T$, while the last one is given by Eq. (C.15) with $A' = R_{ij} - T$ and $B' = T$. As for the y integration, they can be read from Eqs. (C.1)–(C.6). All the ingredients combine into:

$$
\begin{aligned}
& L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0) \\
& \quad = \frac{1}{2}\,\Gamma(1+\varepsilon)\,\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,\frac{1}{R_{ij}\,T} \\
& \quad \times \left\{ \frac{1}{\varepsilon^2}\,(2\,R_{ij})^{-\varepsilon} - \mathrm{Li}_2\left(\frac{R_{ij}}{T}\right) - \left[\ln\left(R_{ij}\right) - \ln\left(-\,T\right) - i\,\pi\right] \ln\left(\frac{T - R_{ij}}{T}\right) \right\}.
\end{aligned}
\tag{3.46}
$$

In the present case $\mathrm{Im}(-\,R_{ij})$ and $\mathrm{Im}(T) > 0$, so that $\ln(-\,T) = \ln(T) - i\,\pi$ and thus Eq. (3.46) can be rewritten as

$$
\begin{aligned}
L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0) = {} & \frac{1}{2}\,\Gamma(1+\varepsilon)\,\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,\frac{1}{R_{ij}\,T} \\
& \times \left\{ \frac{1}{\varepsilon^2}\,(2\,R_{ij})^{-\varepsilon} - \mathrm{Li}_2\left(\frac{R_{ij}}{T}\right) - \left[\ln\left(R_{ij}\right) - \ln\left(T\right)\right] \ln\left(\frac{T - R_{ij}}{T}\right) \right\},
\end{aligned}
\tag{3.47}
$$

i.e. again, the same analytic form as the previous two cases, cf. Eqs. (3.30) and (3.44).

3.3.4. $\mathrm{Im}(\Delta_3) < 0$, $\mathrm{Im}(\Delta_1^{\{i,j\}}) < 0$

The starting point is now Eq. (3.23) with $P_{ijk} = -R_{ij}$. In the first two integrals we rescale $y = x\,u^2$ to factorize the double integral into a product of two unnested integrals over u and x. Using the results given in Eqs. (C.1) and (C.2), these unnested integrals are readily performed to yield

$$
\begin{aligned}
& L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0) \\
&= \frac{F(\varepsilon)}{4\,R_{ij}} \left\{ i\,\pi\ (-T)^{-1-\varepsilon} \left[-i\,\mathrm{e}^{-i\,\pi\,\varepsilon}\, B\left(\frac{1}{2}, \frac{1}{2} + \varepsilon\right) - B\left(\frac{1}{2} + \varepsilon, -\varepsilon\right) \right] \right. \\
&\quad - i \int_0^{+\infty} \frac{dy}{\sqrt{y}} \int_0^1 \frac{dx}{y+x} \left(R_{ij}\,(x+y) + T\,(1-x) \right)^{-1-\varepsilon} \\
&\quad + i\,\mathrm{e}^{-i\,\pi\,\varepsilon} \int_0^{+\infty} \frac{dy}{\sqrt{y}} \int_0^y \frac{dx}{y-x} \\
&\qquad \times \left[\left(-R_{ij}\,(y-x) - T\,(1+x) \right)^{-1-\varepsilon} - (-T)^{-1-\varepsilon}\ (1+x)^{-1-\varepsilon} \right] \\
&\quad + \int_1^{+\infty} \frac{dy}{\sqrt{y}} \int_1^y \frac{dx}{y-x} \\
&\qquad \left. \times \left[\left(-R_{ij}\,(y-x) + T\,(1-x) \right)^{-1-\varepsilon} - (-T)^{-1-\varepsilon}\ (x-1)^{-1-\varepsilon} \right] \right\}.
\end{aligned}
\tag{3.48}
$$

In the three remaining integrals in Eq. (3.48), the first two remain finite when $\varepsilon \to 0$ and are thus computed in this limit, which yields

$$
\begin{aligned}
& \int_0^{+\infty} \frac{dy}{\sqrt{y}} \int_0^1 \frac{dx}{y+x} \left(R_{ij}\,(x+y) + T\,(1-x) \right)^{-1} \\
&\qquad = \frac{1}{T} \int_0^{+\infty} \frac{dy}{\sqrt{y}} \frac{1}{1+y} \left[\ln\left(\frac{y\,R_{ij} + T}{R_{ij}} \right) - \ln(y) \right],
\end{aligned}
\tag{3.49}
$$

$$
\begin{aligned}
& \int_0^{+\infty} \frac{dy}{\sqrt{y}} \int_0^y \frac{dx}{y-x} \left[\left(-R_{ij}\,(y-x) - T\,(1+x) \right)^{-1-\varepsilon} - (-T)^{-1}\ (1+x)^{-1} \right] \\
&\qquad = \frac{1}{T} \int_0^{+\infty} \frac{dy}{\sqrt{y}} \frac{1}{1+y}\ \ln\left(\frac{y\,R_{ij} + T}{T} \right).
\end{aligned}
\tag{3.50}
$$

Making the change of variable $u = 1/y$ we readily see that

$$
\int_0^{+\infty} \frac{dy}{\sqrt{y}} \frac{\ln(y)}{1+y} \ = \ - \int_0^{+\infty} \frac{du}{\sqrt{u}} \frac{\ln(u)}{1+u} \ = \ 0;
$$

the combination $\{-\,i \times \text{Eq. (3.49)} +\ i \times \text{Eq. (3.50)}\}$ thus gives $i\,\pi\ \ln(R_{ij}/T)$. In the third integral, we make the change of variable $x = y - (y-1)\,v$ to get

$$
\int_1^{+\infty} \frac{dy}{\sqrt{y}} \,(y-1)^{-1-\varepsilon} \int_0^1 \frac{dv}{v} \left[\left(-v\,R_{ij} + T\,(v-1) \right)^{-1-\varepsilon} - (-T)^{-1-\varepsilon}\ (1-v)^{-1-\varepsilon} \right].
$$

The v integration is performed using the result of $K_2(-R_{ij}, -T)$, cf. Eq. (C.10). After some algebra, Eq. (3.48) thus reads

$$
\begin{aligned}
& L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0) \\
&\quad = \frac{1}{2}\,\Gamma(1+\varepsilon)\, \frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\, \frac{1}{R_{ij}\,T} \left[\frac{1}{\varepsilon^2}\ \left(2\,R_{ij} \right)^{-\varepsilon} + \mathrm{Li}_2\left(\frac{T - R_{ij}}{T} \right) - \frac{\pi^2}{6} \right].
\end{aligned}
\tag{3.51}
$$

Equation (3.51) is identical to Eq. (3.29); since $\mathrm{Im}(R_{ij})$ and $\mathrm{Im}(T)$ have the same sign as in Sect. 3.3.1, Eq. (3.51) can be recast into a form identical to eq. (3.30):

$$\begin{aligned} & L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0) \\ & \quad = \frac{1}{2}\,\Gamma(1+\varepsilon)\,\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,\frac{1}{R_{ij}\,T} \\ & \quad \times \left\{\frac{1}{\varepsilon^2}\,\left(2\,R_{ij}\right)^{-\varepsilon} - \mathrm{Li}_2\left(\frac{R_{ij}}{T}\right) - \left[\ln\left(R_{ij}\right) - \ln\left(T\right)\right]\,\ln\left(\frac{T-R_{ij}}{T}\right)\right\}. \end{aligned} \tag{3.52}$$

In summary, in all four cases $L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0)$ takes the same analytical form. Compared with the multiplicity of forms met in the general complex mass case, and still with the diverse cases met in Sect. 3.2, this simplification comes from the coalescence of the pole and branch points all at the value 1, which is the end-point singularity causing the appearance of the soft and collinear singularity in all four cases.

3.4. Consistency checks and explicit examples

In Refs. [7,8] the infrared structure of any IR-divergent N-point one-loop integral was shown to be carried by IR-divergent three-point one-loop functions resulting from appropriate iterated pinchings. In the following this feature is explicitly verified for the formulae obtained in this article for the four-point functions, when compared with those for the three-point functions, where the latter are formulated most conveniently according to the so-called "indirect way" for this purpose.

In the case of purely soft divergence, i.e. whenever some $\Delta_2^{\{i\}} = 0$ whereas $\widetilde{D}_{ijk} \neq 0$, the various cases of Eqs. (3.14), (3.19), (3.22), and (3.25) can be encompassed in one single formula. For this purpose let us introduce the following notation, where for any complex Q we denote $Q_{\mathrm{R}} \equiv \mathrm{Re}(Q)$ and $Q_{\mathrm{I}} \equiv \mathrm{Im}(Q)$.

$$\int_{\widetilde{(0,1)}} du\,F(u) = \begin{cases} \int_0^{-i\,S_A\,\infty} du\,F(u) + \int_{+\infty}^{1} du\,F(u) & \text{if} \quad 0 < -B_{\mathrm{I}}/A_{\mathrm{I}} < 1 \\ & \text{and} \quad A_{\mathrm{I}}\,[A_{\mathrm{R}}\,B_{\mathrm{I}} - A_{\mathrm{I}}\,B_{\mathrm{R}}] > 0, \\ \int_0^1 du\,F(u) & \qquad \text{otherwise,} \end{cases} \tag{3.53}$$

with $S_A = \mathrm{sign}(A_{\mathrm{I}})$, whether

$$F(u) \;=\; \frac{\ln(A\,u^2+B) - \ln(A\,u_0^2+B)}{u^2-u_0^2} \quad \text{with} \quad u_0^2 \neq -\frac{B}{A}$$

or

$$F(u) \;=\; (A\,u^2+B)^{-1-\varepsilon}.$$

The condition "$0 < -B_{\mathrm{I}}/A_{\mathrm{I}} < 1$ and $A_{\mathrm{I}}\,[A_{\mathrm{R}}\,B_{\mathrm{I}} - A_{\mathrm{I}}\,B_{\mathrm{R}}] > 0$" is the condition for the discontinuity cuts of $\ln(A\,u^2+B)$ to cross the real axis between 0 and 1 (cf. Appendix D of Ref. [2]). This enables us to rewrite $L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk})$ in a generic way in the various cases as

$$\begin{aligned} L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) &= \frac{2^\varepsilon}{T}\,\Gamma(1+\varepsilon) \\ &\times \Bigg\{ -\frac{1}{\varepsilon} \int_{\widetilde{(0,1)}} du \, \left(u^2 P_{ijk} + R_{ij}\right)^{-1-\varepsilon} - U\left(\Delta_3, \Delta_1^{\{ij\}}, \widetilde{D}_{ijk}\right) \\ &\quad + \int_{\widetilde{(0,1)}} \frac{du}{u^2 P_{ijk} + R_{ij}} \left[\ln\left(T\,(1-u^2)\right) - \ln\left(\frac{T\,(P_{ijk}+R_{ij})}{P_{ijk}}\right) \right. \\ &\quad \left. - \ln\left(u^2\,(P_{ijk}+R_{ij}-T)+T\right) + \ln\left(\frac{(P_{ijk}+R_{ij})\,(T-R_{ij})}{P_{ijk}}\right) \right] \\ &\quad - \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij}} \left[\eta\left(\frac{T\,(P_{ijk}+R_{ij})}{P_{ijk}}, \frac{T-R_{ij}}{T}\right) + \ln\left(\frac{T-R_{ij}}{T}\right) \right] \Bigg\}, \end{aligned} \tag{3.54}$$

with

$$\begin{aligned} &U(\Delta_3, \Delta_1^{\{ij\}}, \widetilde{D}_{ijk}) \\ &= \begin{cases} 0 & \text{if} \quad \mathrm{Im}(\Delta_3) > 0 \ \mathrm{Im}(\Delta_1^{\{ij\}}) > 0, \\ \int_{\Gamma+} \frac{du}{u^2 P_{ijk}+R_{ij}} \ln\left(\frac{T-R_{ij}}{T}\right) & \text{if} \quad \mathrm{Im}(\Delta_3) > 0 \ \mathrm{Im}(\Delta_1^{\{ij\}}) < 0, \\ \int_{\Gamma+} \frac{du}{u^2 P_{ijk}+R_{ij}} \eta\left(\frac{T\,(P_{ijk}+R_{ij})}{P_{ijk}}, \frac{T-R_{ij}}{T}\right) & \text{if} \quad \mathrm{Im}(\Delta_3) < 0 \ \mathrm{Im}(\Delta_1^{\{ij\}}) > 0, \\ \int_{\Gamma+} \frac{du}{u^2 P_{ijk}+R_{ij}} \left[\ln\left(\frac{T-R_{ij}}{T}\right) \right. & \\ \qquad \left. + \, \eta\left(\frac{T\,(P_{ijk}+R_{ij})}{P_{ijk}}, \frac{T-R_{ij}}{T}\right) \right] & \text{if} \quad \mathrm{Im}(\Delta_3) < 0 \ \mathrm{Im}(\Delta_1^{\{ij\}}) < 0, \end{cases} \end{aligned} \tag{3.55}$$

where the contour Γ^+ is the closed contour encircling the "north-east" quadrant *clockwise* (cf. Sect. 3.1 of Ref. [2]). Depending on the location of the cuts of $\left(u^2 P_{ijk} + R_{ij}\right)^{-1-\varepsilon}$, the first term of Eq. (3.54) is the same as those which appear in Eqs. (2.48) or (2.51); $L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk})$ can thus be written as

$$L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) = \frac{1}{T}\, L_3^n(0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) + \tilde{L}_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}), \tag{3.56}$$

with

$$\begin{aligned} &\tilde{L}_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}) \\ &= \frac{1}{T} \Bigg\{ \int_{\widetilde{(0,1)}} \frac{du}{u^2 P_{ijk} + R_{ij}} \left[\ln\left(T\,(1-u^2)\right) - \ln\left(\frac{T\,(P_{ijk}+R_{ij})}{P_{ijk}}\right) \right. \\ &\quad \left. - \ln\left(u^2\,(P_{ijk}+R_{ij}-T)+T\right) + \ln\left(\frac{(P_{ijk}+R_{ij})\,(T-R_{ij})}{P_{ijk}}\right) \right] \\ &\quad - \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij}} \left[\eta\left(\frac{T\,(P_{ijk}+R_{ij})}{P_{ijk}}, \frac{T-R_{ij}}{T}\right) + \ln\left(\frac{T-R_{ij}}{T}\right) \right] \\ &\quad - U\left(\Delta_3, \Delta_1^{\{ij\}}, \widetilde{D}_{ijk}\right) \Bigg\}, \end{aligned} \tag{3.57}$$

and $L_3^n(0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk})$ is given by Eq. (2.48) or Eq. (2.53) depending on the sign of the imaginary part of $\Delta_1^{\{i,j\}}$.

For the cases where $\Delta_2^{\{i\}} = 0$ and $\widetilde{D}_{ijk} = 0$, a unique formula for $L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0)$ was found above, whatever the signs of $\mathrm{Im}(\Delta_3)$ and $\mathrm{Im}(\Delta_1^{\{ij\}})$. The decomposition of the form in Eq. (3.56) stills holds: the IR-divergent part of $L_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0)$ is the same as in the three-point case, cf. Eqs. (2.56) and (2.59), whereas now,

$$\tilde{L}_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0) = -\frac{1}{2\, R_{ij}\, T} \left[\mathrm{Li}_2\left(\frac{R_{ij}}{T}\right) + \left[\ln(R_{ij}) - \ln(T)\right] \ln\left(\frac{T - R_{ij}}{T}\right)\right]. \tag{3.58}$$

Coming back to I_4^n and using the results of Sect. 2.3, we have

$$\begin{aligned} I_4^n = & \sum_{i \in S_4} \frac{\overline{b}_i}{\det(G)} \sum_{j \in S_4 \backslash \{i\}} \frac{\overline{b}_j^{\{i\}}}{\det(G^{\{i\}})} \frac{W\big(\det(G^{\{i,j\}}), \widetilde{D}_{ijk}, \widetilde{D}_{ijl}\big)}{T} \\ & + \sum_{i \in S_4} \frac{\overline{b}_i}{\det(G)} \sum_{j \in S_4 \backslash \{i\}} \frac{\overline{b}_j^{\{i\}}}{\det(G^{\{i\}})} \sum_{k \in S_4 \backslash \{i,j\}} \frac{\overline{b}_k^{\{i,j\}}}{\det(G^{\{i,j\}})} \tilde{L}_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk}), \end{aligned} \tag{3.59}$$

where $l \in S_4 \backslash \{i,j,k\}$. The first term on the right-hand side of Eq. (3.59) is nothing but the combination of the three-point functions as in Refs. [7,8] decomposed according to the so-called "direct way" made explicit in Sect.. 2.1. Note that when $\widetilde{D}_{ijk} = 0$, the quantity $\tilde{L}_4^n(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0)$, which does not actually depend on k, factors out from the sum over k, which then yields trivially -1. The function $W(\det(G^{\{i,j\}}), \widetilde{D}_{ijk}, \widetilde{D}_{ijl})$ has been defined by Eq. (2.6). Its arguments have one extra subscript, tracing back the extra pinching which was involved compared with the three-point case. We recap here the different results concerning this function:

$$W\big(\det(G^{\{i,j\}}), \widetilde{D}_{ijk}, \widetilde{D}_{ijl}\big) = \frac{2^\varepsilon}{\varepsilon}\, \Gamma(1+\varepsilon) \int_0^1 dx \left(D^{\{i,j\}(k)}(x) - i\,\lambda\right)^{-1-\varepsilon}, \tag{3.60}$$

where

$$D^{\{i,j\}(k)}(x) = G^{\{i,j\}(k)}\, x^2 - 2\, V^{\{i,j\}(k)}\, x - C^{\{i,j\}(k)}, \tag{3.61}$$

with

$$\begin{aligned} G^{\{i,j\}(k)} &= -\mathcal{S}_{ll} + 2\,\mathcal{S}_{kl} - \mathcal{S}_{kk} = \det(G^{\{i,j\}}), \\ V^{\{i,j\}(k)} &= \mathcal{S}_{kl} - \mathcal{S}_{kk} = \frac{1}{2}\left[\det(G^{\{i,j\}}) - \widetilde{D}_{ijk} + \widetilde{D}_{ijl}\right], \\ C^{\{i,j\}(k)} &= \mathcal{S}_{kk} = -\widetilde{D}_{ijl}. \end{aligned} \tag{3.62}$$

Note that $W(\det(G^{\{i,j\}}), \widetilde{D}_{ijk}, \widetilde{D}_{ijl})$ is symmetric under the exchange of i and j.

- If $\widetilde{D}_{ijk}$ and $\widetilde{D}_{ijl}$ both differ from zero, $W\big(\det(G^{\{i,j\}}), \widetilde{D}_{ijk}, \widetilde{D}_{ijl}\big)$ is given by Eq. (2.15).
- If only $\widetilde{D}_{ijl}$ vanishes, $W\big(\det(G^{\{i,j\}}), \widetilde{D}_{ijk}, 0\big)$ is read from Eq. (2.28).
- If $\widetilde{D}_{ijk}$ and $\widetilde{D}_{ijl}$ both vanish, $W\big(\det(G^{\{i,j\}}), 0, 0\big)$ is given by Eq. (2.30).

For practical purposes let us stress that we do not have to compute a dedicated formula for each IR four-point case as in Ref. [4], where 16 cases were distinguished. Indeed, for each contribution labelled by the index i, we merely distinguish two cases: either $\Delta_2^{\{i\}} \neq 0$, for which we use the generic formula suited to the massive case, or $\Delta_2^{\{i\}} = 0$, for which we use the appropriate formula suited

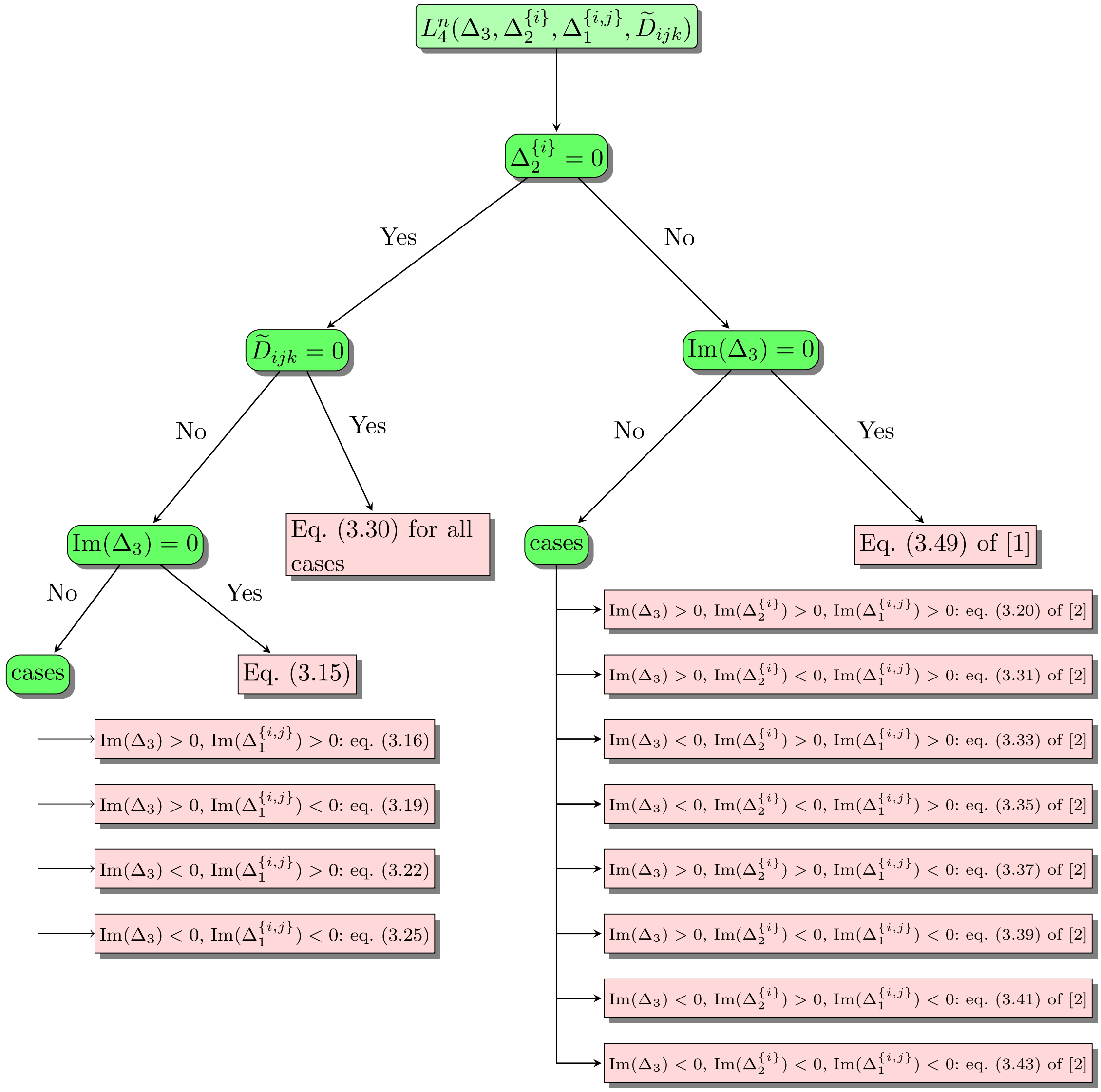

Fig. 5. Decision tree to compute $L_4^n(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \widetilde{D}_{ijk})$ for a given sector labelled by i, j, and k.

to the IR case at hand. The massive case is split depending on the vanishing of $\mathrm{Im}(\Delta_3)$ (namely, if one internal mass squared has an imaginary part different from zero). The IR case is also divided in two cases: $\widetilde{D}_{ijk} = 0$ and $\widetilde{D}_{ijk} \neq 0$. The latter is further separated according to the fact that one or several internal masses squared have a non-vanishing imaginary part. All these cases are depicted on the decision tree presented in Fig. 5; for each case the appropriate formula is given. Notice that when $\mathrm{Im}(\Delta_3) \neq 0$, it may appear that $\Delta_2^{\{i\}}$ and/or $\Delta_1^{\{i,j\}}$ are real; in this case they must be understood as having a "$+i\,\lambda$" prescription. The expense paid by the present method is a possible proliferation of dilogarithms, a counteraction against which would require some extra work. This point will be commented on in more detail in the examples studied below.

3.4.1. Two opposite external masses

In this case all internal masses are zero and two opposite external legs (say 1 and 3) have non-lightlike four-momenta, the two others being lightlike, i.e. $p_1^2 \neq 0, p_2^2 = 0, p_3^2 \neq 0$, and $p_4^2 = 0$ (our convention is depicted in Fig. 4). The texture of the $\mathcal{S}$ matrix is[10]

$$\mathcal{S} = \begin{pmatrix} 0 & 0 & s_{23} & s_1 \\ 0 & 0 & s_3 & s_{12} \\ s_{23} & s_3 & 0 & 0 \\ s_1 & s_{12} & 0 & 0 \end{pmatrix}, \tag{3.63}$$

with $s_i = p_i^2$ and $s_{ij} = (p_i + p_j)^2$ (all the momenta are taken ingoing). All the contributions i are such that $\Delta_2^{\{i\}} = 0$ and $\widetilde{D}_{ijk} = 0$. In this case, $R_{ij} = -\Delta_1^{\{i,j\}}$ is symmetric under the exchange of i and j. The four-point function is given, cf. Eq. (3.29), by

$$\begin{aligned} I_4^n = {} & \frac{1}{2\,\det(\mathcal{S})}\,\Gamma(1+\varepsilon)\,\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)} \sum_{i \in S_4} \sum_{j > i} \frac{1}{\Delta_1^{\{i,j\}}} \left(\frac{\overline{b}_i\,\overline{b}_j^{\{i\}}}{\det(G^{\{i\}})} + \frac{\overline{b}_j\,\overline{b}_i^{\{j\}}}{\det(G^{\{j\}})} \right) \\ & \times \left[\frac{1}{\varepsilon^2} \left(-2\,\Delta_1^{\{i,j\}} \right)^{-\varepsilon} + \mathrm{Li}_2\left(\frac{T - R_{ij}}{T - i\,\lambda} \right) - \frac{\pi^2}{6} \right]. \end{aligned} \tag{3.64}$$

Due to the hollow texture of the reduced $\mathcal{S}$ matrices, a bunch of $\overline{b}_j^{\{i\}}$ coefficients vanish. Equation (3.64) thus simplifies into

$$\begin{aligned} I_4^n = {} & \frac{1}{2\,\det(\mathcal{S})}\,\Gamma(1+\varepsilon)\,\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)} \sum_{i=1}^{2} \sum_{j=3}^{4} \frac{1}{\Delta_1^{\{i,j\}}} \left(\frac{\overline{b}_i\,\overline{b}_j^{\{i\}}}{\det(G^{\{i\}})} + \frac{\overline{b}_j\,\overline{b}_i^{\{j\}}}{\det(G^{\{j\}})} \right) \\ & \times \left[\frac{1}{\varepsilon^2} \left(-2\,\Delta_1^{\{i,j\}} \right)^{-\varepsilon} + \mathrm{Li}_2\left(1 - \frac{R_{ij} - i\,\lambda}{T - i\,\lambda} \right) - \frac{\pi^2}{6} \right]. \end{aligned} \tag{3.65}$$

Expressing the $\overline{b}_i$ and $\overline{b}_j^{\{i\}}$ coefficients as well as the various determinants as functions of the $\mathcal{S}$ matrix elements, we get

$$\begin{aligned} I_4^n = {} & \Gamma(1+\varepsilon)\,\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,\frac{2}{d} \\ & \Bigg\{ (-s_1 - i\,\lambda)^{-\varepsilon} + (-s_3 - i\,\lambda)^{-\varepsilon} - (-s_{12} - i\,\lambda)^{-\varepsilon} - (-s_{23} - i\,\lambda)^{-\varepsilon} \\ & \quad - \mathrm{Li}_2\left(1 - \frac{s_{12} + i\,\lambda}{d/\Sigma + i\,\lambda} \right) - \mathrm{Li}_2\left(1 - \frac{s_{23} + i\,\lambda}{d/\Sigma + i\,\lambda} \right) \\ & \quad + \mathrm{Li}_2\left(1 - \frac{s_1 + i\,\lambda}{d/\Sigma + i\,\lambda} \right) + \mathrm{Li}_2\left(1 - \frac{s_3 + i\,\lambda}{d/\Sigma + i\,\lambda} \right) \Bigg\}, \end{aligned} \tag{3.66}$$

with

$$d = s_1\,s_3 - s_{12}\,s_{23}, \tag{3.67}$$

$$\Sigma = s_1 + s_3 - s_{12} - s_{23}. \tag{3.68}$$

Equation (3.68) involves four dilogarithms, as in Refs. [14,15]. The arguments of the dilogarithms in Eq. (3.66) vs. in Ref. [14] are seemingly different, namely Ref. [14] involves $\mathrm{Li}_2(1 - (w + i\,\lambda)/(d/\Sigma))$

[10] Remember that the elements of the $\mathcal{S}$ matrix are defined by $\mathcal{S}_{ij} = (q_i - q_j)^2 - m_i^2 - m_j^2$, where the internal momenta q_i of the Feynman diagram depicted in Fig. 4 are such that $q_2 - q_1 = p_2$, $q_3 - q_2 = p_3$, $q_4 - q_3 = p_4$, and $q_1 - q_4 = p_1$.

whereas $\mathrm{Li}_2(1-(w+i\,\lambda)/(d/\Sigma+i\,\lambda))$ appears in Eq. (3.66), where w stands for s_1, s_3, s_{12}, or s_{23}. However, the $i\,\lambda$ prescriptions matter only when the real parts of the arguments of the dilogarithms are greater than 1, i.e. whenever $w\,\Sigma/d < 0$, in which case the signs of $(d/\Sigma)-w$ and of Σ/d, which respectively control the signs of the $i\lambda$ prescriptions in either case, are the same: the two results in Eq. (3.66) and in Ref. [14] are actually identical.

This is to be compared with the formula given in Ref. [4]. This was taken from Ref. [15] and involves five dilogarithms instead of four. The authors of Ref. [15] used the so-called Mantel identity, which entails nine dilogarithms, to prove that the four-dilogarithm and five-dilogarithm results are actually equivalent. The Mantel identity happens to be a corollary of the Hill identity, the former being derived by applying the latter three times to some suitable combinations of variables.[11] The continuation of the Hill identity to any two arbitrary complex variables, however, requires additional combinations of η functions handling the mismatch between the various discontinuities of the dilogarithms involved, and these η functions are often skipped in the literature.[12] An even busier modification is then required for the Mantel identity. This drove of η functions makes the analytical check of the equivalence between the four-dilogarithm and five-dilogarithm expressions extremely awkward in general, and to our understanding this drove of η functions was not accounted for in Ref. [15]. We did perform numerical tests accounting for these η functions, which verified the equivalence for the configurations probed.

3.4.2. A simple case with one internal mass

In this example, with the convention depicted in Fig. 4, propagator 3 (with four-momentum q_3) is taken massive, the others being massless, the two external legs 1 and 2 have lightlike four-momenta, that of the external leg 3 is on the mass shell of propagator 3, and the external leg 4 has a non-lightlike four-momentum; i.e. $p_1^2 = p_2^2 = 0$, $p_3^2 = m_3^2$, and $p_4^2 \neq 0$. With the same notation as the preceding example, the texture of the $\mathcal{S}$ matrix is

$$\mathcal{S} = \begin{pmatrix} 0 & 0 & s_{23}-m_3^2 & 0 \\ 0 & 0 & 0 & s_{12} \\ s_{23}-m_3^2 & 0 & -2\,m_3^2 & s_4-m_3^2 \\ 0 & s_{12} & s_4-m_3^2 & 0 \end{pmatrix}. \tag{3.69}$$

In this case, the sector $i=1$ has no soft or collinear divergence and is computed using the massive formula. The other sectors correspond to the cases $\Delta_2^{\{i\}}=0$, $\widetilde{D}_{ijk}\neq 0$ and $\Delta_2^{\{i\}}=0$, $\widetilde{D}_{ijk}=0$. The four-point amplitude can be cast in a divergent part and a finite one. The divergent part is given by

$$\left(I_4^n\right)_{\text{div}} = \sum_{i\in S_4\setminus\{1\}} \frac{\overline{b}_i}{\det(G)} \sum_{j\in S_4\setminus\{i\}} \frac{\overline{b}_j^{\{i\}}}{\det(G^{\{i\}})}\, \frac{W\left(\det(G^{\{i,j\}}), \widetilde{D}_{ijk}, \widetilde{D}_{ijl}\right)}{T}. \tag{3.70}$$

[11] See Ref. [16], Chap. 1, pp. 2–3 and Chap. 2, pp. 17–18.
[12] See, however, Ref. [17].

As several $\overline{b}_j^{\{i\}}$ vanish due to the hollow texture of reduced $\mathcal{S}$ matrices, we actually have to compute[13]

$$\begin{aligned}\left(I_4^n\right)_{\text{div}} = \frac{\overline{b}_2}{\det(\mathcal{S})}&\left[\frac{\overline{b}_1^{\{2\}}}{\det(G^{\{2\}})}\,W\Big(\det(G^{\{1,2\}}),\widetilde{D}_{124},0\Big) + \frac{\overline{b}_4^{\{2\}}}{\det(G^{\{2\}})}\,W\Big(\det(G^{\{2,4\}}),\widetilde{D}_{124},0\Big)\right]\\ &+\frac{\overline{b}_3}{\det(\mathcal{S})}\,\frac{\overline{b}_1^{\{3\}}}{\det(G^{\{3\}})}\,W\Big(\det(G^{\{1,3\}}),0,0\Big)\\ &+\frac{\overline{b}_4}{\det(\mathcal{S})}\,\frac{\overline{b}_2^{\{4\}}}{\det(G^{\{4\}})}\,W\Big(\det(G^{\{2,4\}}),\widetilde{D}_{124},0\Big),\end{aligned} \tag{3.71}$$

where $W\Big(\det(G^{\{1,2\}}),\widetilde{D}_{124},0\Big)$ and $W\Big(\det(G^{\{2,4\}}),\widetilde{D}_{124},0\Big)$ are given by Eq. (2.28), and $W\Big(\det(G^{\{1,3\}}),0,0\Big)$ by Eq. (2.30). We get:

$$\begin{aligned}\left(I_4^n\right)_{\text{div}} = \frac{1}{\varepsilon}\,\Gamma(1+\varepsilon)\,&\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,\frac{1}{s_{12}\,(s_{23}-m_3^2)}\\ &\left[\frac{2}{\varepsilon}\,\left(-s_{23}+m_3^2-i\,\lambda\right)^{-\varepsilon}+\frac{1}{\varepsilon}\,\left(-s_{12}-i\,\lambda\right)^{-\varepsilon}\right.\\ &-\frac{1}{\varepsilon}\,\left(-s_4+m_3^2-i\,\lambda\right)^{-\varepsilon}-\frac{1}{2\,\varepsilon}\left(m_3^2-i\,\lambda\right)^{-\varepsilon}\\ &\left.+\varepsilon\,\text{Li}_2\left(\frac{s_4}{s_4-m_3^2+i\,\lambda}\right)-2\,\varepsilon\,\text{Li}_2\left(\frac{s_{23}}{s_{23}-m_3^2+i\,\lambda}\right)+\varepsilon\,\frac{\pi^2}{12}\right].\end{aligned} \tag{3.72}$$

Expanding Eq. (3.72) in ε, we recover the results of Eq. (4.27) in Ref. [4] taken from Ref. [18] for the terms proportional to $1/\varepsilon^2$ and $1/\varepsilon$. Concerning the finite part, we obtain a host of terms for which it is cumbersome to verify analytically that they do reduce to the finite part of Eq. (4.27) of Ref. [4]. We verified numerically that they do indeed.

4. Summary and outlook

In this article we have presented an extension of a novel approach developed in the companion articles of Refs. [1,2] to the case of vanishing internal masses involving soft and/or collinear divergences. For this latter case, the method remains very similar to the massive cases: the three- and four-point functions are split into "sectors" whose coefficients are expressed in terms of algebraic kinematical invariants involved in reduction algorithms. Each "sector" may diverge or not when the IR regulator is sent to zero, yielding to a simple decision tree to compute the relevant integrals. This avoids the computation of the numerous different integrals over Feynman parameters, as is usually done in the literature. This extension also applies to general kinematics beyond those relevant for one-loop collider processes, offering a potential application to the calculation of two-loop processes using one-loop (generalized) N-point functions as building blocks, as discussed in the introduction of Ref. [1].

One drawback of the present method is the proliferation of dilogarithms in the expression of the four-point function computed in closed form. This requires some extra work to be better apprehended, in order to counteract it. But as the method used here is the same as in the real mass case, up to slight

[13] We will use the following properties: $\det(G^{\{i,j\}})$ is symmetric under the permutation $i \leftrightarrow j$, and $\widetilde{D}_{ijk}$ is symmetric under any permutation of the set $\{i,j,k\}$.

modifications, any solution found for the latter case can be applied in the infrared-divergent case. This issue will be addressed in a future article.

The last goal is to provide the generalized one-loop building blocks entering as integrands in the computation of two-loop three- and four-point functions by means of an extra numerical double integration. In this respect, let us mention that the expansion around $\varepsilon = 0$ of the results given in this article has been truncated in order to keep only the divergent and the constant terms. This is sufficient for any one-loop computation, but may not be enough for two-loop applications of the method. This article already contains a lot of results, so the expansion around $\varepsilon = 0$ at the necessary orders is postponed to a future work.

In memoriam

Various ideas and techniques used in this work were initiated by Prof. Shimizu after a visit to LAPTh. He explained to us his ideas about the numerical computation of scalar two-loop three- and four-point functions, he shared his notes partly in English, partly in Japanese with us, and he encouraged us to push this project forward. J.Ph. G. would like to thank Shimizu-sensei for giving him a taste of the Japanese culture and for his kindness.

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Appendix A. Two basic integrals

In what follows, A and B are assumed dimensionless and complex valued, the signs of their real parts are unknown, and the signs of their imaginary parts may or may not be the same.

A.1. First kind

The computation of the three- and four-point functions in a spacetime of arbitrary dimensions involves the following extension of the case treated in Appendix D of Ref. [1]:

$$K(\nu) = \int_0^\infty \frac{d\xi}{(\xi^\nu + A)\,(\xi^\nu + B)}. \tag{A.1}$$

After partial fraction decomposition, the right-hand side of Eq. (A.1) becomes

$$K(\nu) = \frac{1}{B - A} \int_0^\infty d\xi \left[\frac{1}{\xi^\nu + A} - \frac{1}{\xi^\nu + B}\right]. \tag{A.2}$$

Let us assume that $\nu > 1$, such that the right-hand side of Eq. (A.1) can be split into the difference of two convergent integrals at infinity, which can be separately computed using Appendix B of Ref. [1] with $\mu = 1$. $K(\nu)$ thus reads

$$K(\nu) = \frac{1}{B - A}\,\frac{1}{\nu}\,B\left(1 - \frac{1}{\nu}, \frac{1}{\nu}\right)\left[A^{\frac{1}{\nu}-1} - B^{\frac{1}{\nu}-1}\right] \tag{A.3}$$

regardless of the signs of $\mathrm{Im}(A)$ vs. $\mathrm{Im}(B)$. In the case of the three-point function $\nu = 1/(1 - \varepsilon)$, which is slightly less than 1 for $\varepsilon < 0$, and the right-hand side of Eq. (A.3) can be analytically continued in ν as long as $\nu \neq -1/n$ or $\nu \neq 1/(n+1)$ with n an arbitrary positive integer. So, the result of Eq. (A.3) can be used in the case where the dimension of the spacetime is shifted by a small positive amount from $n = 4$ to $n = 4 - 2\,\varepsilon$. We also note that the limit $\nu \to 1$ of $K(\nu)$ leads to Eq. (D4) of Ref. [1].

A practical case met in Sect. 2 is $A = -i\,\lambda$ and B remaining an arbitrary complex number with an imaginary part different from 0. In the limit $\lambda \to 0^+$, Eq. (A.3) becomes

$$\int_0^{+\infty} \frac{d\xi}{(\xi^\nu - i\,\lambda)\,(\xi^\nu + B)} = -\frac{1}{\varepsilon}\,(1-\varepsilon)\,\Gamma(1+\varepsilon)\,\Gamma(1-\varepsilon)\,B^{-1-\varepsilon}. \tag{A.4}$$

This is the well-known fact that the two limits $\lambda \to 0^+$ and $\varepsilon \to 0$ do not commute.

A.2. Second kind

The most general case for the integral

$$J(\nu) = \int_0^{+\infty} \frac{d\xi}{(\xi^\nu + A)\,\sqrt{\xi^\nu + B}} \tag{A.5}$$

has been treated in Appendix A of Ref. [2]; let us just recap the results. Two cases can be distinguished according to the signs of the imaginary parts of A and B.

(1) $\mathrm{Im}(A)$ and $\mathrm{Im}(B)$ of the same sign:

$$J(\nu) = \frac{1}{\nu}\,B\left(\frac{3}{2} - \frac{1}{\nu}, \frac{1}{\nu}\right) \int_0^1 dz\, \left((1-z^2)\,A + z^2\,B\right)^{-3/2+1/\nu}. \tag{A.6}$$

(2) $\mathrm{Im}(A)$ and $\mathrm{Im}(B)$ of opposite signs:

$$\begin{aligned} J(\nu) = &-\frac{1}{\nu}\,B\left(\frac{3}{2} - \frac{1}{\nu}, \frac{1}{\nu}\right) \\ &\times \left[e^{-i\,S_B\,\pi/\nu} \int_0^{+\infty} dz\, \left(B\,z^2 - (1+z^2)\,A\right)^{-3/2+1/\nu} \right. \\ &\left. + \int_1^{+\infty} dz\, \left(B\,z^2 + (1-z^2)\,A\right)^{-3/2+1/\nu} \right], \end{aligned} \tag{A.7}$$

with $S_B = \mathrm{sign}\,(\mathrm{Im}\,(B))$.

Remember that the two cases (1) and (2) can be reunified by seeing Eq. (A.7) as an analytic continuation in A of Eq. (A.6), which possibly requires a deformation of the contour [0, 1] originally drawn along the real axis in Eq. (A.6); see Appendix A of Ref. [2] for more details.

Appendix B. The function $J(x_1, x_2)$

This appendix computes the function

$$J(x_1, x_2) = \int_0^1 dx\, \frac{\ln\left((x-x_1)\,(x-x_2)\right)}{(x-x_1)\,(x-x_2)}.$$

Using partial fraction decomposition, $J(x_1,x_2)$ can be written as

$$
\begin{aligned}
&J(x_1,x_2)\\
&=\frac{1}{x_1-x_2}\left[\int_0^1 dx\,\frac{\ln(x-x_1)}{x-x_1}-\int_0^1 dx\,\frac{\ln(x-x_2)}{x-x_2}\right.\\
&\qquad+\int_0^1 dx\,\frac{\ln(x-x_2)-\ln(x_1-x_2)}{x-x_1}\;-\;\int_0^1 dx\,\frac{\ln(x-x_1)-\ln(x_2-x_1)}{x-x_2}\\
&\qquad\left.+\ln(x_1-x_2)\int_0^1 dx\,\frac{dx}{x-x_1}\;-\;\ln(x_2-x_1)\int_0^1 dx\,\frac{dx}{x-x_2}\right].
\end{aligned}
\tag{B.1}
$$

With the help of Appendix E of Ref. [1] (see also Appendix B of Ref. [19]), we immediately get

$$
\begin{aligned}
\int_0^1 dx\,\frac{\ln(x-x_1)-\ln(x_2-x_1)}{x-x_2}&=R'(x_1,x_2)\\
&=\mathrm{Li}_2\left(\frac{x_2}{x_2-x_1}\right)-\mathrm{Li}_2\left(\frac{x_2-1}{x_2-x_1}\right)\\
&\quad+\eta\left(-x_1,\frac{1}{x_2-x_1}\right)\ln\left(\frac{x_2}{x_2-x_1}\right)\\
&\quad-\eta\left(1-x_1,\frac{1}{x_2-x_1}\right)\ln\left(\frac{x_2-1}{x_2-x_1}\right),
\end{aligned}
\tag{B.2}
$$

where $R'(x_1,x_2)$ is given by Eq. (E15) of Ref. [1][14] Since x_1 and x_2 have imaginary parts of opposite signs, all the η functions vanish in Eq. (B.2). Then, using the Landen identity,

$$
\mathrm{Li}_2(z)+\mathrm{Li}_2\left(\frac{z}{z-1}\right)=-\frac{1}{2}\ln^2(1-z),
\tag{B.3}
$$

we can rewrite Eq. (B.2) as

$$
\begin{aligned}
&\int_0^1 dx\,\frac{\ln(x-x_1)-\ln(x_2-x_1)}{x-x_2}\\
&\qquad=\mathrm{Li}_2\left(\frac{x_2-1}{x_1-1}\right)-\mathrm{Li}_2\left(\frac{x_2}{x_1}\right)\\
&\qquad\quad+\frac{1}{2}\left[\ln^2(x_1-1)-\ln^2(x_1)-2\ln(x_1-x_2)\ln\left(\frac{x_1-1}{x_1}\right)\right].
\end{aligned}
\tag{B.4}
$$

[14] The subtlety discussed in Appendix E of Ref. [1] does not show up here because x_1 and x_2 have imaginary parts of opposite signs.

Substituting into Eq. (B.1) and easily computing the remaining x integrals, we get

$$\begin{aligned} &J(x_1,x_2)\\ &= \frac{1}{x_1-x_2}\Bigg\{\ \frac{1}{2}\left[\ln^2(1-x_1)-\ln^2(-x_1)\right]-\frac{1}{2}\left[\ln^2(1-x_2)-\ln^2(-x_2)\right]\\ &\qquad + \mathrm{Li}_2\left(\frac{x_1-1}{x_2-1}\right)-\mathrm{Li}_2\left(\frac{x_1}{x_2}\right)-\mathrm{Li}_2\left(\frac{x_2-1}{x_1-1}\right)+\mathrm{Li}_2\left(\frac{x_2}{x_1}\right)\\ &\qquad + \frac{1}{2}\left[\ln^2(x_2-1)\ -\ln^2(x_2)-2\ \ln(x_2-x_1)\ \ln\left(\frac{x_2-1}{x_2}\right)\right]\\ &\qquad - \frac{1}{2}\left[\ln^2(x_1-1)\ -\ln^2(x_1)-2\ \ln(x_1-x_2)\ \ln\left(\frac{x_1-1}{x_1}\right)\right]\\ &\qquad + \ln(x_1-x_2)\ \ln\left(\frac{x_1-1}{x_1}\right)-\ln(x_2-x_1)\ \ln\left(\frac{x_2-1}{x_2}\right)\ \Bigg\}. \end{aligned} \tag{B.5}$$

The identity

$$\ln^2(1-x_1)-\ln^2(-x_1)-\ln^2(x_1-1)+\ln^2(x_1)=-2\,i\,\pi\,S(x_1)\ \ln\left(\frac{x_1-1}{x_1}\right),$$

where $S(x_1)$ is the sign of the imaginary part of x_1, allows us to simplify Eq. (B.5):

$$\begin{aligned} J(x_1,x_2) = \frac{1}{x_1-x_2}\Bigg\{&\mathrm{Li}_2\left(\frac{x_1-1}{x_2-1}\right)-\mathrm{Li}_2\left(\frac{x_1}{x_2}\right)-\mathrm{Li}_2\left(\frac{x_2-1}{x_1-1}\right)+\mathrm{Li}_2\left(\frac{x_2}{x_1}\right)\\ &+\left[2\ \ln(x_1-x_2)-i\,\pi\,S(x_1)\right]\ \ln\left(\frac{x_1-1}{x_1}\right)\\ &-\left[2\ \ln(x_2-x_1)-i\,\pi\,S(x_2)\right]\ \ln\left(\frac{x_2-1}{x_2}\right)\Bigg\}. \end{aligned} \tag{B.6}$$

Then, we use the identity relating $\mathrm{Li}_2(z)$ and $\mathrm{Li}_2(1/z)$ [13], and also the relation

$$2\ \ln(x_1-x_2)-i\,\pi\,S(x_1)=2\ \ln(x_2-x_1)-i\,\pi\,S(x_2)=\ln\left(-(x_1-x_2)^2\right), \tag{B.7}$$

whose validity relies on the fact that x_1 and x_2 have imaginary parts of opposite signs. This yields

$$\begin{aligned} &J(x_1,x_2)\\ &= \frac{1}{x_1-x_2}\Bigg\{\ 2\,\mathrm{Li}_2\left(\frac{x_1-1}{x_2-1}\right)-2\,\mathrm{Li}_2\left(\frac{x_1}{x_2}\right)+\frac{1}{2}\ \ln^2\left(-\frac{x_1-1}{x_2-1}\right)-\frac{1}{2}\ \ln^2\left(-\frac{x_1}{x_2}\right)\\ &\qquad +\ \ln\left(-(x_1-x_2)^2\right)\ \left[\ln\left(\frac{x_1-1}{x_1}\right)-\ln\left(\frac{x_2-1}{x_2}\right)\right]\ \Bigg\}. \end{aligned} \tag{B.8}$$

Appendix C. Herbarium of utilitarian integrals

This appendix collects a bunch of integrals appearing in the three- and four-point cases to help the reader.

The first series appears under the following forms and can be computed in terms of the Euler Beta function:

$$\int_0^{+\infty} dz\,\left(1+z^2\right)^{-1-\varepsilon} = \frac{1}{2}\int_0^{+\infty}\frac{dy}{\sqrt{y}}\,(1+y)^{-1-\varepsilon} = \frac{1}{2}\,B\left(\frac{1}{2},\frac{1}{2}+\varepsilon\right), \tag{C.1}$$

$$\int_1^{+\infty} dz\,\left(z^2-1\right)^{-1-\varepsilon} = \frac{1}{2}\int_1^{+\infty}\frac{dy}{\sqrt{y}}\,(y-1)^{-1-\varepsilon} = \frac{1}{2}\,B\left(\frac{1}{2}+\varepsilon,-\varepsilon\right), \tag{C.2}$$

$$\int_0^{1} dz\,\left(1-z^2\right)^{-1-\varepsilon} = \frac{1}{2}\int_0^{1}\frac{dy}{\sqrt{y}}\,(1-y)^{-1-\varepsilon} = \frac{1}{2}\,B\left(\frac{1}{2},-\varepsilon\right). \tag{C.3}$$

Using the duplication formula for the Gamma functions [13], the z integrals computed in closed form read

$$\int_0^{+\infty} dz\,\left(1+z^2\right)^{-1-\varepsilon} = \frac{\tan(\pi\,\varepsilon)}{2\varepsilon}\,\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,2^{-2\,\varepsilon}, \tag{C.4}$$

$$\int_1^{+\infty} dz\,\left(z^2-1\right)^{-1-\varepsilon} = -\frac{1}{2\,\varepsilon}\,\frac{1}{\cos(\pi\,\varepsilon)}\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,2^{-2\,\varepsilon}, \tag{C.5}$$

$$\int_0^{1} dz\,\left(1-z^2\right)^{-1-\varepsilon} = -\frac{1}{2\,\varepsilon}\,2^{-2\,\varepsilon}\,\frac{\Gamma(1-\varepsilon)^2}{\Gamma(1-2\,\varepsilon)}. \tag{C.6}$$

For the four-point functions in the real mass case, or in the complex mass case when $\text{sign}(\text{Im}(R_{ij})) = \text{sign}(\text{Im}(T))$, the following integral needs to be evaluated:

$$K_2(R_{ij},T) = \int_0^1 \frac{dv}{v}\left[\frac{1}{[v\,R_{ij}+(1-v)\,T\,-i\,\lambda]^{1+\varepsilon}} - \frac{1}{[(1-v)\,T\,-i\,\lambda]^{1+\varepsilon}}\right]. \tag{C.7}$$

Note that in both cases, $\text{Im}(v\,R_{ij}+(1-v)\,T\,-i\,\lambda)$ has a constant sign when v spans $[0,1]$. After a partial fraction decomposition with respect to the variable v, $K_2(R_{ij},T)$ can be written as

$$\begin{aligned} K_2(R_{ij},T) = \frac{1}{T}\int_0^1 dv\,\Bigg\{&-(R_{ij}-T)\,\left[R_{ij}\,v+T\,(1-v)-i\,\lambda\right]^{-1-\varepsilon} - (T-i\,\lambda)^{-\varepsilon}\,(1-v)^{-1-\varepsilon} \\ &+\frac{1}{v}\left[\left(R_{ij}\,v+T\,(1-v)-i\,\lambda\right)^{-\varepsilon} - (T-i\,\lambda)^{-\varepsilon}\right] \\ &+\frac{(T-i\,\lambda)^{-\varepsilon}}{v}\,\left[1-(1-v)^{-\varepsilon}\right]\Bigg\}. \end{aligned} \tag{C.8}$$

The first two terms of Eq. (C.8), which yield a $1/\varepsilon$ pole, are integrated in closed form, whereas the last three terms of Eq. (C.8), which are not divergent, can be expanded around $\varepsilon = 0$ up to order ε:

$$\begin{aligned} K_2(R_{ij},T) = \frac{1}{T}\Bigg[\;&\frac{1}{\varepsilon}\,(R_{ij}-i\,\lambda)^{-\varepsilon} \\ &-\;\varepsilon\int_0^1\frac{dv}{v}\,\left[\ln\left(R_{ij}\,v+T\,(1-v)-i\,\lambda\right)-\ln\left(T-i\,\lambda\right)\right] \\ &+\;\varepsilon\,(T-i\,\lambda)^{-\varepsilon}\int_0^1\frac{dv}{v}\,\ln(1-v)\;\Bigg]. \end{aligned} \tag{C.9}$$

Since $\mathrm{sign}(\mathrm{Im}(R_{ij}\, v + T\,(1-v) - i\,\lambda)) = \mathrm{sign}(\mathrm{Im}(T - i\,\lambda))$ when $v \in [0,1]$, the logarithms in the first integral can be combined together. The last integration is performed explicitly to give

$$K_2(R_{ij}, T) = \frac{1}{T}\left[\frac{1}{\varepsilon}\,(R_{ij} - i\,\lambda)^{-\varepsilon} + \varepsilon\,\mathrm{Li}_2\left(\frac{T - R_{ij}}{T - i\,\lambda}\right) - \varepsilon\,\frac{\pi^2}{6}\right]. \tag{C.10}$$

With respect to the preceding case, two new integrals show up when $\mathrm{sign}(\mathrm{Im}(T)) \neq \mathrm{sign}(\mathrm{Im}(R_{ij}))$:

$$K_3(A, B) = \int_0^{+\infty} \frac{dv}{v}\,\left[(A\,v + B)^{-1-\varepsilon} - (B\,(1+v))^{-1-\varepsilon}\right], \tag{C.11}$$

$$K_4(A', B') = \int_1^{+\infty} \frac{dv}{v}\,\left[(A'\,v + B')^{-1-\varepsilon} - (B'\,(1-v))^{-1-\varepsilon}\right], \tag{C.12}$$

where $\mathrm{sign}(\mathrm{Im}(A\,v + B))$ [resp. $\mathrm{sign}(\mathrm{Im}(A'\,v + B'))$] keeps a constant sign when v spans $[0, +\infty[$ (resp. $[1, +\infty[$). To compute $K_3(A, B)$, we expand the right-hand side of Eq. (C.11) around $\varepsilon = 0$ to get

$$K_3(A, B) = \frac{1}{B}\,\ln\left(\frac{B}{A}\right)[1 - \varepsilon\,\ln(B)]. \tag{C.13}$$

To compute $K_4(A', B')$, we expand $(A'\,v + B')^{-1-\varepsilon}$ in Eq. (C.12) around $\varepsilon = 0$, keeping in mind that $\mathrm{sign}(\mathrm{Im}(A' + B')) = \mathrm{sign}(\mathrm{Im}(A'))$, and we get

$$\begin{aligned} K_4(A', B') = \frac{1}{B'}\Bigg\{ &-\frac{1}{\varepsilon}(-B')^{\varepsilon} + \left(\ln(A' + B') - \ln(A')\right) \\ &+ \varepsilon\left[\mathrm{Li}_2\left(-\frac{B'}{A'}\right) - \frac{\pi^2}{6} - \frac{1}{2}\left(\ln^2(A' + B') - \ln^2(A')\right)\right]\Bigg\}. \end{aligned} \tag{C.14}$$

With the additional assumption that $\mathrm{sign}(\mathrm{Im}(A')) = -\mathrm{sign}(\mathrm{Im}(B'))$, and after some algebra, $K_4(A', B')$ can be recast in an alternative, more useful, form:

$$\begin{aligned} K_4(A', B') = \frac{1}{B'}\Bigg\{ &-\frac{1}{\varepsilon}(A' + B')^{\varepsilon} + \left(\ln(-B') - \ln(A')\right) \\ &+ \varepsilon\left[\mathrm{Li}_2\left(\frac{A' + B'}{B'}\right) + \ln\left(A' + B'\right)\,\ln\left(-\frac{A'}{B'}\right)\right]\Bigg\}. \end{aligned} \tag{C.15}$$

Notice that this additional assumption is always fulfilled in the cases met.

Appendix D. Detailed comparisons with the "direct way" (Sect. 2)

Our present goal is to check that the "indirect way" leads to the same results for infrared-divergent three-point functions as the "direct way." By explicitly performing the sum over the j index in Eq. (2.45), we recover the results of Sect. 2. This part is not necessary for the understanding of the method proposed in this article and can be skipped in a first reading.

D.1. Real mass case

Let us start with the real mass case. We successively revisit the examples examined in Sect. (2.2).

Occurrence of a soft divergence Let us recap the texture of the $\mathcal{S}$ matrix [cf. Eq. (2.31)]:

$$\mathcal{S} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2\,m_2^2 & s_3 - m_2^2 - m_3^2 \\ 0 & s_3 - m_2^2 - m_3^2 & -2\,m_3^2 \end{pmatrix}; \tag{D.1}$$

thus $\det(\mathcal{S}) = 0$. Let us single out row and column 1. We readily see that the vector $V^{(1)}$ vanishes, and so do the coefficients $\overline{b}_2$ and $\overline{b}_3$; thus $\overline{b}_1 = \det(G)$ since the coefficients $\overline{b}_i$ fulfill $\sum_{i \in S_3} \overline{b}_i = \det(G)$. The three-point function is thus given by [cf. Eq. (2.45)]

$$I_3^n = \frac{\overline{b}_2^{\{1\}}}{\det(G^{\{1\}})}\, L_3^n\big(0, \Delta_1^{\{1\}}, \widetilde{D}_{12}\big) + \frac{\overline{b}_3^{\{1\}}}{\det(G^{\{1\}})}\, L_3^n\big(0, \Delta_1^{\{1\}}, \widetilde{D}_{13}\big). \tag{D.2}$$

The only relevant reduced $\mathcal{S}$ matrix is

$$\mathcal{S}^{\{1\}} = \begin{pmatrix} -2\,m_2^2 & s_3 - m_2^2 - m_3^2 \\ s_3 - m_2^2 - m_3^2 & -2\,m_3^2 \end{pmatrix}, \tag{D.3}$$

whose determinant $\det(\mathcal{S}^{\{1\}}) = -\mathcal{K}(s_3, m_2^2, m_3^2)$ involves the Källén function given by Eq. (2.11). The associated Gram "matrix" degenerates into a single scalar,

$$G^{\{1\}(2)} = \det(G^{\{1\}}) = (2\,s_3). \tag{D.4}$$

One easily reads the $\overline{b}_j^{\{1\}}$ coefficients from the reduced Gram matrix $G^{\{1\}(2)}$ and the vector $V^{\{1\}(2)}$ (cf. Eq. (2.29) of Ref. [1]):

$$\overline{b}_2^{\{1\}} = m_2^2 - m_3^2 - s_3, \quad \overline{b}_3^{\{1\}} = m_3^2 - m_2^2 - s_3. \tag{D.5}$$

Since $\widetilde{D}_{12} = 2\,m_3^2$ and $\widetilde{D}_{13} = 2\,m_2^2$, the quantities $L_3^n(0, \Delta_1^{\{1\}}, \widetilde{D}_{12})$ and $L_3^n(0, \Delta_1^{\{1\}}, \widetilde{D}_{13})$ are given by Eq. (2.49). The $\widetilde{D}_{12} + \Delta_1^{\{1\}}$, $\widetilde{D}_{13} + \Delta_1^{\{1\}}$, and $\Delta_1^{\{1\}}$ terms are given by Eqs. (2.36) and (2.38) of Ref. [1]:

$$\widetilde{D}_{12} + \Delta_1^{\{1\}} = \frac{(s_3 + m_3^2 - m_2^2)^2}{2\,s_3}, \tag{D.6}$$

$$\widetilde{D}_{13} + \Delta_1^{\{1\}} = \frac{(s_3 + m_2^2 - m_3^2)^2}{2\,s_3}, \tag{D.7}$$

$$\Delta_1^{\{1\}} = \frac{\mathcal{K}(s_3, m_2^2, m_3^2)}{2\,s_3}. \tag{D.8}$$

The roots of the denominator of Eq. (2.49) are such that

$$(\overline{z}^{12})^2 = \frac{\Delta_1^{\{1\}} + i\,\lambda}{\widetilde{D}_{12} + \Delta_1^{\{1\}}} = \frac{\mathcal{K}(s_3, m_2^2, m_3^2) + i\,\lambda\,\sigma_s}{(s_3 + m_3^2 - m_2^2)^2}, \tag{D.9}$$

$$(\overline{z}^{13})^2 = \frac{\Delta_1^{\{1\}} + i\,\lambda}{\widetilde{D}_{13} + \Delta_1^{\{1\}}} = \frac{\mathcal{K}(s_3, m_2^2, m_3^2) + i\,\lambda\,\sigma_s}{(s_3 + m_2^2 - m_3^2)^2}, \tag{D.10}$$

with $\sigma_s = \text{sign}(s_3)$. We specify

$$\overline{z}^{12} = \frac{\sqrt{\mathcal{K}(s_3, m_2^2, m_3^2) + i\,\lambda\,\sigma_s}}{s_3 + m_3^2 - m_2^2}, \quad \overline{z}^{13} = \frac{\sqrt{\mathcal{K}(s_3, m_2^2, m_3^2) + i\,\lambda\,\sigma_s}}{s_3 + m_2^2 - m_3^2}. \tag{D.11}$$

We introduce two new quantities, $\tilde{x}_1$ and $\tilde{x}_2$, which are the two roots of the equation $D^{\{1\}\,(2)}(x) = 0$ appearing in the "direct way":

$$\tilde{x}_1 = \frac{s_3 + m_2^2 - m_3^2 + \sqrt{\mathcal{K}(s_3, m_2^2, m_3^2) + i\,\lambda\,\sigma_s}}{2\,s_3}, \tag{D.12}$$

$$\tilde{x}_2 = \frac{s_3 + m_2^2 - m_3^2 - \sqrt{\mathcal{K}(s_3, m_2^2, m_3^2) + i\,\lambda\,\sigma_s}}{2\,s_3}. \tag{D.13}$$

Notice that the quantities $1-\tilde{x}_1$ and $1-\tilde{x}_2$ are the roots of the equation $D^{\{1\}\,(3)}(x) = 0$. The quantities $\bar{z}^{12}$, one of the roots of the equation $(\widetilde{D}_{12} + \Delta_1^{\{1\}})\,z^2 - \Delta_1^{\{1\}} - i\,\lambda = 0$, and $\bar{z}^{13}$, a root of the equation $(\widetilde{D}_{13} + \Delta_1^{\{1\}})\,z^2 - \Delta_1^{\{1\}} - i\,\lambda = 0$, can be related to the roots $\tilde{x}_1$ and $\tilde{x}_2$ by the following relations:

$$\bar{z}^{12} = \frac{\tilde{x}_1 - \tilde{x}_2}{2 - \tilde{x}_1 - \tilde{x}_2}, \quad \bar{z}^{13} = \frac{\tilde{x}_1 - \tilde{x}_2}{\tilde{x}_1 + \tilde{x}_2}. \tag{D.14}$$

Putting things together, I_3^n can be written as

$$\begin{aligned} I_3^n = {} & -\frac{2^\varepsilon\,\Gamma(1+\varepsilon)}{2\,\sqrt{\mathcal{K}(s_3, m_2^2, m_3^2) + i\,\lambda\,\sigma_s}} \left\{ -\frac{1}{\varepsilon}\left[\ln\left(\frac{\bar{z}^{12}-1}{\bar{z}^{12}+1}\right) + \ln\left(\frac{\bar{z}^{13}-1}{\bar{z}^{13}+1}\right)\right] \right. \\ & \left. + \bar{H}_{0,1}\big(\widetilde{D}_{12} + \Delta_1^{\{1\}}, -\Delta_1^{\{1\}} - i\,\lambda\big) + \bar{H}_{0,1}\big(\widetilde{D}_{13} + \Delta_1^{\{1\}}, -\Delta_1^{\{1\}} - i\,\lambda\big) \right\}, \end{aligned} \tag{D.15}$$

with

$$\bar{H}_{0,1}\,(A, B) = 2\,A\,\sqrt{-\frac{B}{A}}\,H_{0,1}\,(A, B), \tag{D.16}$$

which is effectively the content between the curly brackets of Eq. (E.8) in Appendix E. Expressing all the arguments of the logarithms and dilogarithms in Eq. (E.8) in terms of $\tilde{x}_1$ and $\tilde{x}_2$, we get

$$\begin{aligned} I_3^n = & -\frac{2^\varepsilon\,\Gamma(1+\varepsilon)}{(\tilde{x}_1 - \tilde{x}_2)\,\det(G^{\{1\}})} \\ & \times \left\{ -\frac{1}{\varepsilon}\left[\ln\left(-\frac{1-\tilde{x}_1}{1-\tilde{x}_2}\right) + \ln\left(-\frac{\tilde{x}_2}{\tilde{x}_1}\right)\right] \right. \\ & + \ln\left(-\frac{1-\tilde{x}_1}{1-\tilde{x}_2}\right)\left[\ln\left(\frac{(2-\tilde{x}_1-\tilde{x}_2)^2}{4}\,\det(G^{\{1\}}) + i\,\lambda\,\sigma_s\right) \right. \\ & \qquad \left. + \frac{1}{2}\left(\ln\left(\frac{4\,(1-\tilde{x}_1)\,(1-\tilde{x}_2)}{(2-\tilde{x}_1-\tilde{x}_2)^2}\right) + \ln\left(-\frac{4\,(\tilde{x}_1-\tilde{x}_2)^2}{(2-\tilde{x}_1-\tilde{x}_2)^2}\right)\right)\right] \\ & + \ln\left(-\frac{\tilde{x}_2}{\tilde{x}_1}\right)\left[\ln\left(\frac{(\tilde{x}_1+\tilde{x}_2)^2}{4}\,\det(G^{\{1\}}) + i\,\lambda\,\sigma_s\right) \right. \\ & \qquad \left. + \frac{1}{2}\left(\ln\left(\frac{4\,\tilde{x}_1\,\tilde{x}_2}{(\tilde{x}_1+\tilde{x}_2)^2}\right) + \ln\left(-\frac{4\,(\tilde{x}_1-\tilde{x}_2)^2}{(\tilde{x}_1+\tilde{x}_2)^2}\right)\right)\right] \\ & \left. + \mathrm{Li}_2\left(\frac{1-\tilde{x}_2}{\tilde{x}_1-\tilde{x}_2}\right) - \mathrm{Li}_2\left(\frac{\tilde{x}_1-1}{\tilde{x}_1-\tilde{x}_2}\right) + \mathrm{Li}_2\left(\frac{\tilde{x}_1}{\tilde{x}_1-\tilde{x}_2}\right) - \mathrm{Li}_2\left(-\frac{\tilde{x}_2}{\tilde{x}_1-\tilde{x}_2}\right)\right\}. \end{aligned} \tag{D.17}$$

As $\mathrm{Im}(\tilde{x}_1)$ and $\mathrm{Im}(\tilde{x}_2)$ have opposite signs, Eq. (D.17) can be rearranged as

$$
\begin{aligned}
I_3^n = &- \frac{2^\varepsilon\,\Gamma(1+\varepsilon)}{(\tilde{x}_1-\tilde{x}_2)\,\det(G^{\{1\}})} \\
&\times \left\{ -\frac{1}{\varepsilon}\left[\ln\left(\frac{\tilde{x}_1-1}{\tilde{x}_1}\right) - \ln\left(\frac{\tilde{x}_2-1}{\tilde{x}_2}\right)\right]\right. \\
&\quad + \left[\ln\left(\det(G^{\{1\}}) + i\,\lambda\,\sigma_s\right) + \frac{1}{2}\ln\left(-(\tilde{x}_1-\tilde{x}_2)^2\right)\right]\left[\ln\left(\frac{\tilde{x}_1-1}{\tilde{x}_1}\right) - \ln\left(\frac{\tilde{x}_2-1}{\tilde{x}_2}\right)\right] \\
&\quad + \frac{1}{2}\ln\left(-\frac{1-\tilde{x}_1}{1-\tilde{x}_2}\right)\ln\left((1-\tilde{x}_1)(1-\tilde{x}_2)\right) + \frac{1}{2}\ln\left(-\frac{\tilde{x}_2}{\tilde{x}_1}\right)\ln(\tilde{x}_1\,\tilde{x}_2) \\
&\quad - \frac{1}{2}\ln^2\left(\frac{\tilde{x}_1-1}{\tilde{x}_1-\tilde{x}_2}\right) + \frac{1}{2}\ln^2\left(\frac{1-\tilde{x}_2}{\tilde{x}_1-\tilde{x}_2}\right) - \frac{1}{2}\ln^2\left(\frac{-\tilde{x}_2}{\tilde{x}_1-\tilde{x}_2}\right) + \frac{1}{2}\ln^2\left(\frac{\tilde{x}_1}{\tilde{x}_1-\tilde{x}_2}\right) \\
&\quad \left. + 2\,\mathrm{Li}_2\left(\frac{\tilde{x}_1-1}{\tilde{x}_2-1}\right) + \frac{1}{2}\ln^2\left(\frac{1-\tilde{x}_2}{\tilde{x}_1-1}\right) - 2\,\mathrm{Li}_2\left(\frac{\tilde{x}_1}{\tilde{x}_2}\right) - \frac{1}{2}\ln^2\left(-\frac{\tilde{x}_2}{\tilde{x}_1}\right)\right\}.
\end{aligned} \tag{D.18}
$$

Using the relation between $\ln(z)$ and $\ln(-z)$, after some algebra the quantity

$$
\begin{aligned}
E = &\ \frac{1}{2}\ln\left(-\frac{1-\tilde{x}_1}{1-\tilde{x}_2}\right)\ln\left((1-\tilde{x}_1)(1-\tilde{x}_2)\right) + \frac{1}{2}\ln\left(-\frac{\tilde{x}_2}{\tilde{x}_1}\right)\ln(\tilde{x}_1\,\tilde{x}_2) \\
&- \frac{1}{2}\ln^2\left(\frac{\tilde{x}_1-1}{\tilde{x}_1-\tilde{x}_2}\right) + \frac{1}{2}\ln^2\left(\frac{1-\tilde{x}_2}{\tilde{x}_1-\tilde{x}_2}\right) - \frac{1}{2}\ln^2\left(\frac{-\tilde{x}_2}{\tilde{x}_1-\tilde{x}_2}\right) + \frac{1}{2}\ln^2\left(\frac{\tilde{x}_1}{\tilde{x}_1-\tilde{x}_2}\right)
\end{aligned}
$$

can be rewritten as

$$
\begin{aligned}
E &= \frac{1}{2}\left[2\ln(\tilde{x}_1-\tilde{x}_2) - i\,\pi\,S(\tilde{x}_1)\right]\left[\ln\left(\frac{\tilde{x}_1-1}{\tilde{x}_1}\right) - \ln\left(\frac{\tilde{x}_2-1}{\tilde{x}_2}\right)\right] \\
&= \frac{1}{2}\ln\left(-(\tilde{x}_1-\tilde{x}_2)^2\right)\left[\ln\left(\frac{\tilde{x}_1-1}{\tilde{x}_1}\right) - \ln\left(\frac{\tilde{x}_2-1}{\tilde{x}_2}\right)\right],
\end{aligned} \tag{D.19}
$$

with $S(\tilde{x}_1) = \mathrm{sign}(\mathrm{Im}(\tilde{x}_1))$. Substituting Eq. (D.19) into Eq. (D.18), we end up with

$$
\begin{aligned}
I_3^n = &- \frac{2^\varepsilon\,\Gamma(1+\varepsilon)}{(\tilde{x}_1-\tilde{x}_2)\,\det(G^{\{1\}})} \\
&\times \left\{ -\frac{1}{\varepsilon}\left[\ln\left(\frac{\tilde{x}_1-1}{\tilde{x}_1}\right) - \ln\left(\frac{\tilde{x}_2-1}{\tilde{x}_2}\right)\right]\right. \\
&\quad + \left[\ln\left(\det(G^{\{1\}}) + i\,\lambda\,\sigma_s\right) + \ln\left(-(\tilde{x}_1-\tilde{x}_2)^2\right)\right]\left[\ln\left(\frac{\tilde{x}_1-1}{\tilde{x}_1}\right) - \ln\left(\frac{\tilde{x}_2-1}{\tilde{x}_2}\right)\right] \\
&\quad \left. + 2\,\mathrm{Li}_2\left(\frac{\tilde{x}_1-1}{\tilde{x}_2-1}\right) + \frac{1}{2}\ln^2\left(\frac{1-\tilde{x}_2}{\tilde{x}_1-1}\right) - 2\,\mathrm{Li}_2\left(\frac{\tilde{x}_1}{\tilde{x}_2}\right) - \frac{1}{2}\ln^2\left(-\frac{\tilde{x}_2}{\tilde{x}_1}\right)\right\} \\
= &- \frac{2^\varepsilon\,\Gamma(1+\varepsilon)}{\varepsilon\,\det(G^{\{1\}})}\left\{\left[1 - \varepsilon\,\ln\left(\det(G^{\{1\}}) + i\,\lambda\,\sigma_s\right)\right]K(\tilde{x}_1,\tilde{x}_2) - \varepsilon\,J(\tilde{x}_1,\tilde{x}_2)\right\},
\end{aligned} \tag{D.20}
$$

where $K(\tilde{x}_1,\tilde{x}_2)$ is given by Eq. (2.14) and $J(\tilde{x}_1,\tilde{x}_2)$ by Eq. (B.8). Last, we note that the prescription "$+i\,\lambda\,\sigma_s$" in Eq. (D.20) can be replaced by "$-i\,\lambda$" as it matters only when $\det(G^{\{1\}}) < 0$. Thus, Eq. (D.20) is nothing but Eq. (2.15): the indirect and direct ways do indeed lead to the same result.

Occurrence of a collinear divergence We recap the texture of the $\mathcal{S}$ matrix in this case [cf. Eq. (2.35)]:

$$\mathcal{S} = \begin{pmatrix} 0 & 0 & s_1 - m_3^2 \\ 0 & 0 & s_3 - m_3^2 \\ s_1 - m_3^2 & s_3 - m_3^2 & -2\, m_3^2 \end{pmatrix}. \tag{D.21}$$

Obviously, we have $\det(\mathcal{S}) = 0$. As explained in Sect. 2, Eqs. (2.36) and (2.37), the coefficient $\overline{b}_3$ vanishes whereas $\overline{b}_2$ and $\overline{b}_1$ are different from zero. So, the three-point function is given by

$$\begin{aligned} I_3^n = & \frac{\overline{b}_1}{\det(G)} \left[\frac{\overline{b}_2^{\{1\}}}{\det(G^{\{1\}})} L_3^n\big(0, \Delta_1^{\{1\}}, \widetilde{D}_{12}\big) + \frac{\overline{b}_3^{\{1\}}}{\det(G^{\{1\}})} L_3^n\big(0, \Delta_1^{\{1\}}, \widetilde{D}_{13}\big) \right] \\ & + \frac{\overline{b}_2}{\det(G)} \left[\frac{\overline{b}_1^{\{2\}}}{\det(G^{\{2\}})} L_3^n\big(0, \Delta_1^{\{2\}}, \widetilde{D}_{21}\big) + \frac{\overline{b}_3^{\{2\}}}{\det(G^{\{2\}})} L_3^n\big(0, \Delta_1^{\{2\}}, \widetilde{D}_{23}\big) \right]. \end{aligned} \tag{D.22}$$

As $\widetilde{D}_{12} = \widetilde{D}_{21} = 2\, m_3^2$ and $\widetilde{D}_{13} = \widetilde{D}_{23} = 0$, $L_3^n(0, \Delta_1^{\{1\}}, \widetilde{D}_{12})$ and $L_3^n(0, \Delta_1^{\{2\}}, \widetilde{D}_{12})$ are given by Eq. (2.49) whereas $L_3^n(0, \Delta_1^{\{1\}}, \widetilde{D}_{13})$ and $L_3^n(0, \Delta_1^{\{2\}}, \widetilde{D}_{23})$ are given by Eq. (2.56).

Let us first focus on the first term of the right-hand side of Eq. (D.22). The relevant reduced $\mathcal{S}$ matrix is

$$\mathcal{S}^{\{1\}} = \begin{pmatrix} 0 & s_3 - m_3^2 \\ s_3 - m_3^2 & -2\, m_3^2 \end{pmatrix}, \tag{D.23}$$

whose determinant is $\det(\mathcal{S}^{\{1\}}) = -(s_3 - m_3^2)^2$. The 1×1 associated Gram matrix is $G^{\{1\}(2)} = (2\, s_3)$, and the reduced $\bar{b}$ coefficients are given by

$$\frac{\overline{b}_2^{\{1\}}}{\det(G^{\{1\}})} = -\frac{s_3 + m_3^2}{2\, s_3}, \quad \frac{\overline{b}_3^{\{1\}}}{\det(G^{\{1\}})} = -\frac{s_3 - m_3^2}{2\, s_3}, \tag{D.24}$$

whereas

$$\Delta_1^{\{1\}} = \frac{(s_3 - m_3^2)^2}{2\, s_3}, \quad \widetilde{D}_{12} + \Delta_1^{\{1\}} = \frac{(s_3 + m_3^2)^2}{2\, s_3}. \tag{D.25}$$

The square of the root of the polynomial $(\widetilde{D}_{12} + \Delta_1^{\{1\}})\, z^2 - \Delta_1^{\{1\}} - i\, \lambda$ is

$$\bar{z}^2 = \frac{(s_3 - m_3^2)^2}{(s_3 + m_3^2)^2} + i\, \lambda\, \sigma_s, \tag{D.26}$$

with $\sigma_s = \text{sign}(s_3)$. Since

$$\sqrt{\bar{z}^2} = \left| \frac{s_3 - m_3^2}{s_3 + m_3^2} \right| + i\, \lambda\, \sigma_s,$$

a handier choice for further manipulations is instead

$$\bar{z} = \frac{s_3 - m_3^2}{s_3 + m_3^2} + i\, \lambda\, \sigma_s\, \sigma_r, \tag{D.27}$$

where $\sigma_r = \text{sign}((s_3 - m_3^2)/(s_3 + m_3^2))$. Let us define

$$\Sigma_3^n(s_3) = \frac{\overline{b}_2^{\{1\}}}{\det(G^{\{1\}})} L_3^n\big(0, \Delta_1^{\{1\}}, \widetilde{D}_{12}\big) + \frac{\overline{b}_3^{\{1\}}}{\det(G^{\{1\}})} L_3^n\big(0, \Delta_1^{\{1\}}, \widetilde{D}_{13}\big), \tag{D.28}$$

so we get

$$\begin{aligned}\Sigma_3^n(s_3) = {} & \frac{2^\varepsilon\,\Gamma(1+\varepsilon)}{2\,(m_3^2 - s_3)} \\ & \times \Bigg[-\frac{1}{\varepsilon}\,\ln\left(-\frac{m_3^2}{s_3} + i\,\lambda\,\sigma_s\,\sigma_r\right) + \bar{H}_{0,1}\big(\widetilde{D}_{12} + \Delta_1^{\{1\}}, -\Delta_1^{\{1\}} - i\,\lambda\big) \\ & \qquad - \frac{1}{\varepsilon^2}\,\frac{\Gamma(1-\varepsilon)^2}{\Gamma(1-2\,\varepsilon)}\left(-\frac{2\,(s_3 - m_3^2)^2}{s_3} - i\,\lambda\right)^{-\varepsilon}\Bigg].\end{aligned} \tag{D.29}$$

Using the definition of $\bar{H}_{0,1}(X, Y)$, cf. Eq. (D.16), and Eq. (E.8), we have

$$\begin{aligned}& \bar{H}_{0,1}(\widetilde{D}_{12} + \Delta_1^{\{1\}}, -\Delta_1^{\{1\}} - i\,\lambda) \\ & = \text{Li}_2\left(\frac{s_3}{s_3 - m_3^2} - i\,\lambda\sigma_s\,\sigma_r\right) - \text{Li}_2\left(-\frac{m_3^2}{s_3 - m_3^2} + i\,\lambda\,\sigma_s\,\sigma_r\right) \\ & \quad + \ln\left(-\frac{m_3^2}{s_3} + i\,\lambda\,\sigma_s\,\sigma_r\right)\left[\ln\left(\frac{(s_3 + m_3^2)^2}{2\,s_3} + i\,\lambda\,\sigma_s\right) + \frac{1}{2}\,\ln\left(\frac{4\,s_3\,m_3^2}{(s_3 + m_3^2)^2} - i\,\lambda\sigma_s\right)\right. \\ & \qquad \left. + \frac{1}{2}\,\ln\left(-\frac{4\,(s_3 - m_3^2)^2}{(s_3 + m_3^2)^2} - i\,\lambda\,\sigma_s\right)\right],\end{aligned} \tag{D.30}$$

which can be rewritten as

$$\begin{aligned}& \bar{H}_{0,1}\big(\widetilde{D}_{12} + \Delta_1^{\{1\}}, -\Delta_1^{\{1\}} - i\,\lambda\big) \\ & = \ln\left(-\frac{m_3^2}{s_3} + i\,\lambda\,\sigma_s\,\sigma_r\right)\left[-\ln\left(2\,s_3 - i\,\lambda\,\sigma_s\right) + \frac{1}{2}\,\ln\big(4\,s_3\,m_3^2 - i\,\lambda\sigma_s\big)\right. \\ & \qquad \left. + \frac{1}{2}\,\ln\big(-4\,(s_3 - m_3^2)^2 - i\,\lambda\,\sigma_s\big)\right] \\ & \quad + 2\,\text{Li}_2\left(\frac{s_3}{s_3 - m_3^2} - i\,\lambda\sigma_s\,\sigma_r\right) - \frac{\pi^2}{6} \\ & \quad + \ln\left(-\frac{m_3^2}{s_3 - m_3^2} + i\,\lambda\,\sigma_s\,\sigma_r\right)\,\ln\left(\frac{s_3}{s_3 - m_3^2} - i\,\lambda\sigma_s\,\sigma_r\right).\end{aligned} \tag{D.31}$$

Putting Eq. (D.31) into Eq. (D.29), the $\ln(2)$ drop out and we end with

$$\begin{aligned}\Sigma_3^n(s_3) = {} & \frac{\Gamma(1+\varepsilon)}{2\,(m_3^2 - s_3)} \\ & \times \left\{ -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\left[\ln\left(-\frac{(s_3 - m_3^2)^2}{s_3} - i\,\lambda\right) - \ln\left(-\frac{m_3^2}{s_3} + i\,\lambda\,\sigma_s\,\sigma_r\right)\right]\right. \\ & \quad - \frac{1}{2}\ln^2\left(-\frac{(s_3 - m_3^2)^2}{s_3} - i\,\lambda\right) + 2\,\mathrm{Li}_2\left(\frac{s_3}{s_3 - m_3^2} - i\,\lambda\sigma_s\,\sigma_r\right) \\ & \quad + \ln\left(-\frac{m_3^2}{s_3} + i\,\lambda\,\sigma_s\,\sigma_r\right)\left[-\ln\left(s_3 - i\,\lambda\,\sigma_s\right)\right. \\ & \quad \left. + \frac{1}{2}\ln\left(s_3\,m_3^2 - i\,\lambda\sigma_s\right) + \frac{1}{2}\ln\left(-(s_3 - m_3^2)^2 - i\,\lambda\,\sigma_s\right)\right] \\ & \quad \left. + \ln\left(-\frac{m_3^2}{s_3 - m_3^2} + i\,\lambda\,\sigma_s\,\sigma_r\right)\ln\left(\frac{s_3}{s_3 - m_3^2} - i\,\lambda\sigma_s\,\sigma_r\right)\right\}. \end{aligned} \tag{D.32}$$

$\Sigma_3^n(s_3)$ can be shown to be equal to the following quantity:

$$\begin{aligned}\Upsilon_3^n(s_3) = {} & \frac{\Gamma(1+\varepsilon)}{2\,(m_3^2 - s_3)}\left\{-\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\left[2\,\ln\left(-s_3 + m_3^2 - i\,\lambda\right) - \ln\left(m_3^2 - i\,\lambda\right)\right]\right. \\ & \left. - \ln^2\left(-s_3 + m_3^2 - i\,\lambda\right) + \frac{1}{2}\ln^2\left(m_3^2 - i\,\lambda\right) + 2\,\mathrm{Li}_2\left(\frac{s_3}{s_3 - m_3^2 + i\,\lambda}\right)\right\}. \end{aligned} \tag{D.33}$$

To show that, one has to distinguish the following cases: (1) $s_3 < -m_3^2$, (2) $-m_3^2 < s_3 < 0$, (3) $0 < s_3 < m_3^2$, and (4) $m_3^2 < s_3$. For each of them, some tedious algebra performed on the right-hand side of Eq. (D.32) shows that the two results are equal. Let us discuss in detail only the case $s_3 > m_3^2$, for which $\sigma_s = \sigma_r = +$. The argument of the dilogarithm in Eq. (D.32) has a real part greater than 1, hence

$$\mathrm{Li}_2\left(\frac{s_3}{s_3 - m_3^2} - i\,\lambda\sigma_s\,\sigma_r\right) = \mathrm{Li}_2\left(\frac{s_3}{s_3 - m_3^2} - i\,\lambda\right) = \mathrm{Li}_2\left(\frac{s_3}{s_3 - m_3^2 + i\,\lambda}\right).$$

The logarithms in Eq. (D.32) can be modified in such a way that their arguments are ratios of positive quantities; $\Sigma_3^n(s_3)$ thus reads

$$\begin{aligned}\Sigma_3^n(s_3) = {} & \frac{\Gamma(1+\varepsilon)}{2\,(m_3^2 - s_3)} \\ & \left\{-\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\left[\ln\left(\frac{(s_3 - m_3^2)^2}{s_3}\right) - \ln\left(\frac{m_3^2}{s_3}\right) - 2\,i\,\pi\right]\right. \\ & - \frac{1}{2}\left(\ln\left(\frac{(s_3 - m_3^2)^2}{s_3}\right) - i\,\pi\right)^2 + 2\,\mathrm{Li}_2\left(\frac{s_3}{s_3 - m_3^2 + i\,\lambda}\right) \end{aligned}$$

$$
\begin{aligned}
&+ \left(\ln \left(\frac{m_3^2}{s_3} \right) + i\,\pi \right) \\
&\quad \times \left[- \ln (s_3) + \frac{1}{2} \, \left(\ln \left(s_3\, m_3^2 \right) + \ln \left((s_3 - m_3^2)^2 \right) - i\,\pi \right) \right] \\
&+ \left(\ln \left(\frac{m_3^2}{s_3 - m_3^2} \right) + i\,\pi \right) \ln \left(\frac{s_3}{s_3 - m_3^2} \right) \Bigg\}.
\end{aligned} \tag{D.34}
$$

Splitting logarithms of ratios, we get

$$
\begin{aligned}
\Sigma_3^n(s_3) = \frac{\Gamma(1+\varepsilon)}{2\,(m_3^2 - s_3)} & \left\{ - \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \left[2 \, \left(\ln \left(s_3 - m_3^2 \right) - i\,\pi \right) - \ln \left(m_3^2 \right) \right] \right. \\
& - \left(\ln^2 \left(s_3 - m_3^2 \right) - 2\,i\,\pi \, \ln \left(s_3 - m_3^2 \right) - \pi^2 \right) \\
& \left. + \frac{1}{2} \, \ln^2 (m_3^2) + 2 \, \mathrm{Li}_2 \left(\frac{s_3}{s_3 - m_3^2 + i\,\lambda} \right) \right\}.
\end{aligned} \tag{D.35}
$$

In Eq. (D.35), for the case at hand, we recognize Eq. (D.33). Similar handling can be performed in the other three cases to reach the same conclusion. We thus conclude that

$$
\begin{aligned}
\Sigma_3^n(s_3) &= \Upsilon_3(s_3) \\
&= \frac{\Gamma(1+\varepsilon)}{(m_3^2 - s_3)} \left\{ - \frac{1}{\varepsilon^2} \, \left(-s_3 + m_3^2 - i\,\lambda \right)^{-\varepsilon} + \frac{1}{2\,\varepsilon^2} \, \left(m_3^2 - i\,\lambda \right)^{-\varepsilon} \right. \\
&\qquad \left. + \mathrm{Li}_2 \left(\frac{s_3}{s_3 - m_3^2 + i\,\lambda} \right) \right\}.
\end{aligned} \tag{D.36}
$$

So long for the first term of Eq. (D.22). The second term can be obtained from the first one by replacing s_3 by s_1. The coefficients $\overline{b}_1$ and $\overline{b}_2$, as well as $\det(G)$, are easily extracted from the $\mathcal{S}$ matrix, cf. Eq. (D.21):

$$
\begin{aligned}
\overline{b}_1 \; = \; (s_3 - m_3^2)\,(s_1 - s_3), \quad \overline{b}_2 \; = \; (s_1 - m_3^2)\,(s_3 - s_1), \\
\det(G) = -(s_1 - s_3)^2.
\end{aligned} \tag{D.37}
$$

Thus, we finally get

$$
\begin{aligned}
I_3^n &= \frac{\overline{b}_1}{\det(G)} \, \Sigma_3^n(s_3) + \frac{\overline{b}_2}{\det(G)} \, \Sigma_3^n(s_1) \\
&= \frac{\Gamma(1+\varepsilon)}{(s_1 - s_3)} \left\{ - \frac{1}{\varepsilon^2} \left[\left(-s_3 + m_3^2 - i\,\lambda \right)^{-\varepsilon} - \left(-s_1 + m_3^2 - i\,\lambda \right)^{-\varepsilon} \right] \right. \\
&\qquad \left. + \, \mathrm{Li}_2 \left(\frac{s_3}{s_3 - m_3^2 + i\,\lambda} \right) - \mathrm{Li}_2 \left(\frac{s_1}{s_1 - m_3^2 + i\,\lambda} \right) \right\},
\end{aligned}
$$

which coincides with Eq. (2.41): the direct and indirect ways lead to the same result.

Concomitant occurrence of soft and collinear divergences Here again, we start with a recap of the texture of the $\mathcal{S}$ matrix:[15]

$$\mathcal{S} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s_3 \\ 0 & s_3 & 0 \end{pmatrix}. \tag{D.38}$$

Obviously, $\det(\mathcal{S}) = 0$. As in the first example, the coefficients $\overline{b}_2$ and $\overline{b}_3$ vanish whereas $\overline{b}_1 = \det(G)$, and, as $\widetilde{D}_{12} = \widetilde{D}_{13} = 0$, $L(0, \Delta_1^{\{1\}}, 0)$ is given by Eq. (2.56), thus

$$I_3^n = \frac{\overline{b}_2^{\{1\}}}{\det(G^{\{1\}})} \, L_3^n\big(0, \Delta_1^{\{1\}}, 0\big) + \frac{\overline{b}_3^{\{1\}}}{\det(G^{\{1\}})} \, L_3^n\big(0, \Delta_1^{\{1\}}, 0\big). \tag{D.39}$$

The relevant reduced $\mathcal{S}$ matrix is

$$\mathcal{S}^{\{1\}} = \begin{pmatrix} 0 & s_3 \\ s_3 & 0 \end{pmatrix}, \tag{D.40}$$

whose determinant is $\det(\mathcal{S}^{\{1\}}) = -s_3^2$. The 1×1 associated Gram "matrix" is $G^{\{1\}(2)} = (2\, s_3)$, and the reduced $\bar{b}$ coefficients are given by

$$\frac{\overline{b}_2^{\{1\}}}{\det(G^{\{1\}})} = \frac{\overline{b}_3^{\{1\}}}{\det(G^{\{1\}})} \; = \; -\frac{1}{2}, \tag{D.41}$$

whereas

$$\Delta_1^{\{1\}} = \frac{s_3}{2}. \tag{D.42}$$

We thus obtain

$$\begin{aligned} I_3^n &= -L_3^n\big(0, \Delta_1^{\{1\}}, 0\big) \\ &= -\frac{1}{\varepsilon^2}\, \Gamma(1+\varepsilon)\, \frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)} \, (-s_3 - i\,\lambda)^{-1-\varepsilon} \\ &= W\big(\det(G^{\{1\}}), 0, 0\big), \end{aligned}$$

cf. Eqs. (2.5), (2.6), and (2.30), so the direct and indirect ways lead to the same results.

D.2. Complex mass case

We now treat the complex mass case. As discussed in Sect. 2, the only relevant case is the collinear case where the non-vanishing internal mass, say m_3^2, is complex:[16] $m_3^2 = m_{\mathrm{R}}^2 + i\, m_{\mathrm{I}}^2$ with m_{R}^2 and m_{I}^2 real and $m_{\mathrm{R}}^2 > 0$, $m_{\mathrm{I}}^2 < 0$.[17] The $\mathcal{S}$ matrix is given by Eq. (D.21), and I_3^n by Eq. (D.22). As in the real

[15] Contrary to example 3 in Sect. 2.2, we choose to set $m_3^2 = 0$ because, for the purpose of this appendix, a non-vanishing mass does not bring anything new with respect to the previous case.

[16] We follow the convention of Appendix C of Ref. [2].

[17] One could be tempted here to recover the real mass case results by setting $m_{\mathrm{I}}^2 = -\lambda$. Doing that could lead to wrong formulae because, when deriving the complex mass case, we have already assumed that $|m_{\mathrm{I}}^2| \gg \lambda$ and so dropped some $i\,\lambda$ terms.

mass case, let us focus on the first line of Eq. (D.22); the second line can be obtained by changing s_3 in s_1. To compute $L(0, \Delta_1^{\{1\}}, \widetilde{D}_{12})$ and $L(0, \Delta_1^{\{1\}}, \widetilde{D}_{13})$, we have to determine which formulae to use, depending on the sign of $\mathrm{Im}(\Delta_1^{\{1\}})$. The quantity $\Delta_1^{\{1\}}$ is given by Eq. (D.25), which reads

$$\Delta_1^{\{1\}} = \frac{1}{2\, s_3} \left((s_3 - m_{\mathrm{R}}^2)^2 - m_{\mathrm{I}}^4 - 2\, i\, m_{\mathrm{I}}^2\, (s_3 - m_{\mathrm{R}}^2) \right). \tag{D.43}$$

When $(s_3 - m_{\mathrm{R}}^2)/s_3 > 0$, i.e. either $s_3 > m_{\mathrm{R}}^2$ or $s_3 < 0$, $L(0, \Delta_1^{\{1\}}, \widetilde{D}_{12})$ is given by Eq. (2.49) as in the real mass case, and when $(s_3 - m_{\mathrm{R}}^2)/s_3 < 0$, i.e. $0 < s_3 < m_{\mathrm{R}}^2$, $L(0, \Delta_1^{\{1\}}, \widetilde{D}_{12})$ is given by Eq. (2.52). As discussed previously, there is no such dichotomy for $L(0, \Delta_1^{\{1\}}, \widetilde{D}_{13})$, which is given by Eq. (2.56).

Let us first consider $s_3 > m_{\mathrm{R}}^2$ or $s_3 < 0$. We define $\bar{z} = (s_3 - m_3^2)/(s_3 + m_3^2)$, reminiscent of Eq. (D.27), and $\Sigma_3^n(s_3)$ is given by Eq. (D.32), in which the infinitesimal imaginary parts $\propto \lambda$ are dropped out except for arguments of logarithms which do not depend on m_3^2, namely

$$\begin{aligned}
\Sigma_3^n(s_3) = \frac{\Gamma(1+\varepsilon)}{2\,(m_3^2 - s_3)} \Bigg\{ & -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \left[\ln\left(-\frac{(s_3 - m_3^2)^2}{s_3} \right) - \ln\left(-\frac{m_3^2}{s_3} \right) \right] \\
& - \frac{1}{2} \ln^2\left(-\frac{(s_3 - m_3^2)^2}{s_3} \right) + 2\,\mathrm{Li}_2\left(\frac{s_3}{s_3 - m_3^2} \right) \\
& + \ln\left(-\frac{m_3^2}{s_3} \right) \\
& \times \left[-\ln\left(s_3 - i\,\lambda\,\sigma_s\right) + \frac{1}{2} \left[\ln\left(s_3\, m_3^2\right) + \ln\left(-(s_3 - m_3^2)^2\right) \right] \right] \\
& + \ln\left(-\frac{m_3^2}{s_3 - m_3^2} \right) \ln\left(\frac{s_3}{s_3 - m_3^2} \right) \Bigg\}.
\end{aligned} \tag{D.44}$$

Logarithms of products and ratios in Eq. (D.44) are further split. For this purpose we use Eq. (E.1) as well as

$$\ln\left((s_3 - m_3^2)^2 \right) = 2\, \ln(s_3 - m_3^2) - 2\, i\, \pi\, \theta\left(-s_3 + m_{\mathrm{R}}^2 \right)$$

and, for any real a and complex b,

$$\begin{aligned}
\ln\left(\frac{b}{a} \right) &= -\ln\left(a + i\,\lambda\, S(b)\right) + \ln(b), \\
\ln(a\, b) &= \quad \ln\left(a - i\,\lambda\, S(b)\right) + \ln(b).
\end{aligned}$$

We also use the notation $S(z) = \mathrm{sign}\,(\mathrm{Im}(z))$. Let us note that $S\left((s_3 - m_3^2)^2 \right) = \mathrm{sign}(s_3 - m_{\mathrm{R}}^2) \equiv \sigma_p$, and that $\ln(s_3 - i\,\lambda\,\sigma_s)$ is equivalent to $\ln(s_3 + i\,\lambda)$. Then, to compactify Eq. (D.44), we take advantage of the following relations:

$$\begin{aligned}
\ln\left(-s_3 \pm i\,\lambda\right) &= \ln\left(|s_3|\right) \pm i\,\pi\,\theta(s_3),\\
\ln\left(s_3 \pm i\,\lambda\right) &= \ln\left(|s_3|\right) \pm i\,\pi\,\theta(-s_3),\\
\theta(\pm s_3) &= \frac{(1 \pm \sigma_s)}{2},\\
\theta(-s_3 + m_{\mathrm{R}}^2) &= \frac{1-\sigma_p}{2}.
\end{aligned} \tag{D.45}$$

Equation (D.44) can be rewritten as

$$\begin{aligned}
\Sigma_3^n(s_3) = \frac{\Gamma(1+\varepsilon)}{2\,(m_3^2 - s_3)} \Bigg\{ &-\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\left[2\,\left(\ln\left(s_3 - m_3^2\right) - i\,\pi\right) - \ln\left(m_3^2\right)\right]\\
&- \frac{1}{2}\left[2\,\ln\left(s_3 - m_3^2\right) - i\,\pi\,\left(1 - \frac{\sigma_p}{2} + \frac{\sigma_p\,\sigma_s}{2}\right)\right]^2\\
&+ 2\,\mathrm{Li}_2\left(\frac{s_3}{s_3 - m_3^2}\right) + \left(\ln\left(m_3^2\right) + i\,\pi\,\frac{1+\sigma_s}{2}\right)\\
&\times \left[\frac{1}{2}\,\ln\left(m_3^2\right) + \ln\left(s_3 - m_3^2\right) - i\,\pi\,(3 - \sigma_s)\right]\\
&+ \left(\ln\left(m_3^2\right) - \ln\left(s_3 - m_3^2\right) + i\,\pi\right)\,\left(i\,\pi\,\frac{1-\sigma_s}{2} - \ln\left(s_3 - m_3^2\right)\right)\\
&+ i\,\frac{\pi}{2}\,\ln\left(|s_3|\right)\,(1-\sigma_s)\,(1+\sigma_p)\Bigg\}.
\end{aligned} \tag{D.46}$$

Let us treat the case where $s_3 > m_{\mathrm{R}}^2$. We have $\sigma_p = +$ and $\sigma_s = +$. By expanding Eq. (D.46), we get

$$\begin{aligned}
\Sigma_3^n(s_3) = \frac{\Gamma(1+\varepsilon)}{2\,(m_3^2 - s_3)} \Bigg\{ &-\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\left[2\,\left(\ln\left(s_3 - m_3^2\right) - i\,\pi\right) - \ln\left(m_3^2\right)\right]\\
&- \left(\ln^2\left(s_3 - m_3^2\right) - 2\,i\,\pi\,\ln\left(s_3 - m^2\right) - \pi^2\right)\\
&+ \frac{1}{2}\,\ln\left(m_3^2\right) + 2\,\mathrm{Li}_2\left(\frac{s_3}{s_3 - m_3^2}\right)\Bigg\}.
\end{aligned} \tag{D.47}$$

As $\ln(s_3 - m_3^2) - i\,\pi = \ln(-s_3 + m_3^2)$, we recover $\Upsilon_3^n(s_3)$ given by Eq. (D.33). The same exercise can easily be done for the case $s_3 < 0$, leading to the same conclusion.

In the case where $0 < s_3 < m_{\mathrm{R}}^2$, $L(0, \Delta_1^{\{1\}}, \widetilde{D}_{12})$ is given by Eq. (2.52), i.e.

$$\begin{aligned}
L(0, \Delta_1^{\{1\}}, \widetilde{D}_{12}) = \frac{2^{\varepsilon}\,\Gamma(1+\varepsilon)}{2\,(\widetilde{D}_{12} + \Delta_1^{\{1\}})\,\bar{z}} \Bigg[&\frac{1}{\varepsilon}\left(i\,\pi\,S(i\,\bar{z}) + \ln\left(\frac{1+\bar{z}}{1-\bar{z}}\right)\right)\\
&- \bar{H}_{0,\infty}(-\widetilde{D}_{12} - \Delta_1^{\{1\}}, -\Delta_1^{\{1\}})\\
&- \bar{H}_{1,\infty}(\widetilde{D}_{12} + \Delta_1^{\{1\}}, -\Delta_1^{\{1\}})\Bigg],
\end{aligned} \tag{D.48}$$

where $\bar{z} = \sqrt{\Delta_1^{\{1\}}/(\widetilde{D}_{12} + \Delta_1^{\{1\}})}$. We have chosen for the root of the equation $(\widetilde{D}_{12} + \Delta_1^{\{1\}})\,z^2 + \Delta_1^{\{1\}} = 0$, $\tilde{z} = i\,\bar{z}$. The function $\bar{H}_{1,\infty}(x,y)$ is given by the expression in curly brackets of Eq. (E.15), and

$\bar{H}_{0,\infty}(x,y)$ by the expression in square brackets in Eq. (E.19) of Appendix E, i.e.

$$\bar{H}_{0,\infty}(-\widetilde{D}_{12}-\Delta_1^{\{1\}},-\Delta_1^{\{1\}}) = i\,\pi\,S(i\,\bar{z})\,\Big[2\,\ln(2\,i\,\bar{z}) + \ln\big(-\widetilde{D}_{12}-\Delta_1^{\{1\}}\big) + \eta\big(-\widetilde{D}_{12}-\Delta_1^{\{1\}},\bar{z}^2\big)\Big] + \pi^2, \tag{D.49}$$

$$\bar{H}_{1,\infty}(\widetilde{D}_{12}+\Delta_1^{\{1\}},-\Delta_1^{\{1\}}) = \ln\left(\frac{1+\bar{z}}{1-\bar{z}}\right)\left[\ln\big(\widetilde{D}_{12}+\Delta_1^{\{1\}}\big) + \frac{1}{2}\,\ln\big(1-\bar{z}^2\big) + \ln(2\,\bar{z}) + \eta\left(\widetilde{D}_{12}+\Delta_1^{\{1\}},\frac{\widetilde{D}_{12}}{\widetilde{D}_{12}+\Delta_1^{\{1\}}}\right)\right] + \frac{\pi^2}{2} - i\,\pi\,S(\bar{z})\,\ln\left(\frac{\bar{z}+1}{2\,\bar{z}}\right) - \mathrm{Li}_2\left(\frac{\bar{z}+1}{2\,\bar{z}}\right) + \mathrm{Li}_2\left(\frac{\bar{z}-1}{2\,\bar{z}}\right). \tag{D.50}$$

The quantities $\widetilde{D}_{12}+\Delta_1^{\{1\}}$ and $\Delta_1^{\{1\}}$ can be expressed in terms of s_3 and m_3^2 with Eq. (D.25). The expression obtained contains the ratio $(s_3-m_3^2)/(s_3+m_3^2)$ given by

$$\frac{s_3-m_3^2}{s_3+m_3^2} = \frac{s_3^2-m_{\mathrm{R}}^4-m_{\mathrm{I}}^4-2\,i\,m_{\mathrm{I}}^2\,s_3}{(s_3+m_{\mathrm{R}}^2)^2+m_{\mathrm{I}}^4}, \tag{D.51}$$

which has a negative real part and a positive imaginary part for $0 < s_3 < m_{\mathrm{R}}^2$. This implies that

$$S\,(i\,\bar{z}) = -1, \quad S\,(\bar{z}) = +1. \tag{D.52}$$

In addition, since $\mathrm{Im}((s_3-m_3^2)^2) = -2\,m_{\mathrm{I}}^2\,(s_3-m_{\mathrm{R}}^2) < 0$ and $\mathrm{Im}((s_3+m_3^2)^2) = 2\,m_{\mathrm{I}}^2\,(s_3+m_{\mathrm{R}}^2) < 0$, one can show that

$$\eta\left(\frac{(s_3+m_3^2)^2}{2\,s_3},\frac{4\,m_3^2\,s_3}{(s_3+m_3^2)^2}\right) = \eta\left(-\frac{(s_3+m_3^2)^2}{2\,s_3},\frac{(s_3-m_3^2)^2}{(s_3+m_3^2)^2}\right) = 0.$$

Putting all these things together, $L(0,\Delta_1^{\{1\}},\widetilde{D}_{12})$ becomes

$$\begin{aligned} L(0,\Delta_1^{\{1\}},\widetilde{D}_{12}) &= \frac{2^{\varepsilon}\,\Gamma(1+\varepsilon)\,s_3}{(s_3+m_3^2)\,(s_3-m_3^2)}\left\{\frac{1}{\varepsilon}\left[\ln\left(\frac{s_3}{m_3^2}\right)-i\,\pi\right] - \frac{10\,\pi^2}{6} + i\,\pi\,\ln\left(\frac{s_3}{s_3-m_3^2}\right)\right.\\ &\quad + i\,\pi\left[2\,\ln\left(2\,i\,\frac{s_3-m_3^2}{s_3+m_3^2}\right) + \ln\left(-\frac{(s_2-m_3^2)^2}{2\,s_3}\right)\right]\\ &\quad + 2\,\mathrm{Li}_2\left(\frac{s_3}{s_3-m_3^2}\right) + \ln\left(-\frac{m_3^2}{s_3-m_3^2}\right)\ln\left(\frac{s_3}{s_3-m_3^2}\right)\\ &\quad - \ln\left(\frac{s_3}{m_3^2}\right)\left[\ln\left(\frac{(s_3+m_3^2)^2}{2\,s_3}\right) + \frac{1}{2}\,\ln\left(\frac{4\,m_3^2\,s_3}{(s_3+m_3^2)^2}\right)\right.\\ &\quad \left.\left. + \ln\left(2\,\frac{s_3-m_3^2}{s_3+m_3^2}\right)\right]\right\}. \end{aligned} \tag{D.53}$$

Substituting Eq. (D.53) into Eq. (D.28) with the explicit values for $\overline{b}_2^{\{1\}}$, $\overline{b}_3^{\{1\}}$, and $\det(G^{\{1\}})$, and using Eq. (2.56) for $L(0, \Delta_1^{\{1\}}, \widetilde{D}_{13})$, we get

$$\begin{aligned}\Sigma_3^n(s_3) = \frac{\Gamma(1+\varepsilon)}{2\,(m_3^2 - s_3)} \Bigg\{ & -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\left[\ln\left(\frac{s_3}{m_3^2}\right) + \ln\left(-\frac{(s_3-m_3^2)^2}{s_3}\right) - i\,\pi\right] \\ & - \frac{1}{2}\ln^2\left(-\frac{(s_3-m_3^2)^2}{s_3}\right) - \frac{3\,\pi^2}{2} + i\,\pi\,\ln\left(\frac{s_3}{s_3-m_3^2}\right) \\ & + i\,\pi\left[2\,\ln\left(i\,\frac{s_3-m_3^2}{s_3+m_3^2}\right) + \ln\left(-\frac{(s_2-m_3^2)^2}{s_3}\right)\right] - \ln\left(\frac{s_3}{m_3^2}\right) \\ & \times\left[\ln\left(\frac{(s_3+m_3^2)^2}{s_3}\right) + \frac{1}{2}\ln\left(\frac{m_3^2\,s_3}{(s_3+m_3^2)^2}\right) + \ln\left(\frac{s_3-m_3^2}{s_3+m_3^2}\right)\right] \\ & + 2\,\mathrm{Li}_2\left(\frac{s_3}{s_3-m_3^2}\right) + \ln\left(-\frac{m_3^2}{s_3-m_3^2}\right)\ln\left(\frac{s_3}{s_3-m_3^2}\right)\Bigg\}. \end{aligned} \tag{D.54}$$

Keeping in mind that $0 < s_3 < m_R^2$, we split the logarithms and expand the terms to end with

$$\begin{aligned}\Sigma_3^n(s_3) = \frac{\Gamma(1+\varepsilon)}{2\,(m_3^2 - s_3)} \Bigg\{ & -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\left[2\,\ln\left(s_3-m_3^2\right) - 2\,i\,\pi - \ln\left(m_3^2\right)\right] \\ & - \left(\ln^2\left(s_3-m_3^2\right) - \pi^2 - 2\,i\,\pi\,\ln\left(s_3-m_3^2\right)\right) \\ & + \frac{1}{2}\ln^2\left(m_3^2\right) + 2\,\mathrm{Li}_2\left(\frac{s_3}{s_3-m_3^2}\right)\Bigg\}, \end{aligned} \tag{D.55}$$

and, using $\ln(s_3 - m_3^2) = \ln(-s_3 + m_3^2) + i\,\pi$, we again recover Eq. (D.33). We note that the same formula holds for both $\mathrm{Im}(\Delta_1^{\{1\}}) > 0$, i.e. either $s_3 < 0$ or $s_3 > m_R^2$, and for $\mathrm{Im}(\Delta_1^{\{1\}}) < 0$, i.e. $0 < s_3 < m_R^2$. This is because in the last case, the integration contour $\int_0^{+i\infty} + \int_{+\infty}^1$ can actually be deformed into $\int_0^1$, i.e. Eq. (2.52) can be deformed into Eq. (2.49) by means of the Cauchy theorem. Indeed, when $0 < s_3 < m_R^2$, $\mathrm{Im}(z^2\,(\widetilde{D}_{12} + \Delta_1^{\{1\}}) - \Delta_1^{\{1\}})$ never vanishes as z spans the real interval $[0, 1]$ and hence the cut of $\ln(z^2\,(\widetilde{D}_{12} + \Delta_1^{\{1\}}) - \Delta_1^{\{1\}})$ in the half-plane $\{\mathrm{Re}(z) > 0\}$ entirely lies inside the "south-east" quadrant $\{\mathrm{Re}(z) > 0, \mathrm{Im}(z) < 0\}$.

$\Sigma_3^n(s_1)$ is read from Eq. (D.55) by replacing s_3 by s_1, and the coefficients $\overline{b}_1$ and $\overline{b}_2$ as well as $\det(G)$ are still given by Eq. (D.37). So, in the complex mass case the same result is obtained as in the real mass case for I_3^n, and this leads to the conclusion that for the case of complex masses, the "direct way" and the "indirect way" also coincide.

Appendix E. Basic integrals in terms of dilogarithms and logarithms

In the presence of vanishing internal masses, specific integrals of the type

$$H = \int_b^a du\,\frac{\ln(A\,u^2 + B)}{A\,u^2 + B}$$

for the contour $[0, 1]$, as well as the two other contours $[0, +\infty[$ and $[1, +\infty[$, are involved.

Here, we compute all the above types of integrals successively. The presentation is ordered according to the integration contours (a, b) considered. We last provide an extra load of backup integrals.

This appendix often makes use of the identity

$$\ln(z) = \ln(-z) + i\,\pi\,S(z), \tag{E.1}$$

where $S(z)$ is given by Eq. (2.25).

E.1. H-type integrals for the IR case

E.1.1. First kind

$$H_{0,1}(A,B) = \int_0^1 du\, \frac{\ln(A\,u^2 + B)}{A\,u^2 + B}.$$

The cases of real masses (A real) and of complex masses ($\text{Im}(A) \neq 0$) are treated all at once, considering A and B both complex yet such that sign(Im$(A\,u^2 + B)$ is kept constant when u spans the range [0, 1], as is always the case for all our needs (cf. Sects. 2 and 3). We write

$$H_{0,1}(A,B) = \frac{1}{A} \int_0^1 du\, \frac{C_A + \ln(u^2 - \bar{u}^2)}{u^2 - \bar{u}^2}, \tag{E.2}$$

where

$$C_A = \ln(A - i\,\lambda\, S(-\bar{u}^2)) \quad \text{if } \text{Im}(A) = 0, \tag{E.3}$$

$$C_A = \ln(A) + \eta(A, -\bar{u}^2) \quad \text{otherwise}, \tag{E.4}$$

and $\bar{u}^2 = -B/A$. The η function is given by Eq. (E6) of Ref. [1]. The term $\ln(u^2 - \bar{u}^2)$ can be split without the η function since $\bar{u}$ and $-\bar{u}$ have imaginary parts of opposite signs. Performing a partial fraction decomposition, we get

$$\begin{aligned} H_{0,1}(A,B) = \frac{1}{2\,A\,\bar{u}} \Bigg[& C_A \int_0^1 du \left(\frac{1}{u - \bar{u}} - \frac{1}{u + \bar{u}} \right) \\ & + \int_0^1 du\, \frac{\ln(u - \bar{u})}{u - \bar{u}} - \int_0^1 du\, \frac{\ln(u - \bar{u})}{u + \bar{u}} \\ & + \int_0^1 du\, \frac{\ln(u + \bar{u})}{u - \bar{u}} - \int_0^1 du\, \frac{\ln(u + \bar{u})}{u + \bar{u}} \Bigg]. \end{aligned} \tag{E.5}$$

We can rearrange the terms of the right-hand side of Eq. (E.5) in the following way:

$$\begin{aligned} H_{0,1}(A,B) = \frac{1}{2\,A\,\bar{u}} \Bigg\{ & C_A \left[\ln\left(\frac{\bar{u} - 1}{\bar{u}}\right) - \ln\left(\frac{\bar{u} + 1}{\bar{u}}\right) \right] \\ & + \frac{1}{2} \left[\ln^2(1 - \bar{u}) - \ln^2(-\bar{u}) - \ln^2(1 + \bar{u}) + \ln^2(\bar{u}) \right] \\ & + \int_0^1 du\, \frac{\ln(u + \bar{u}) - \ln(2\,\bar{u})}{u - \bar{u}} \quad + \quad \int_0^1 du\, \frac{\ln(2\,\bar{u})}{u - \bar{u}} \\ & - \int_0^1 du\, \frac{\ln(u - \bar{u}) - \ln(-2\,\bar{u})}{u + \bar{u}} - \int_0^1 du\, \frac{\ln(-2\,\bar{u})}{u + \bar{u}} \quad \Bigg\}. \end{aligned} \tag{E.6}$$

Using Eq. (E.1) we can write Eq. (E.6) as

$$H_{0,1}(A,B) = \frac{1}{2\,A\,\bar{u}} \left\{ \ln\left(\frac{\bar{u}-1}{\bar{u}+1}\right) \left[C_A + \frac{1}{2}\,[\ln(\bar{u}-1) + \ln(\bar{u}+1)] + i\,\pi\,S(-\bar{u}) + \ln(2\,\bar{u}) \right] \right.$$
$$\left. + R'(-\bar{u},\bar{u}) \right\}, \tag{E.7}$$

where the function R' is defined in Eq. (E11) of Ref. [1].[18] Using Eq. (E15) of Ref. [1] with $y = -\bar{u}$ and $z = \bar{u}$ and rearranging the term in square brackets, we get

$$H_{0,1}(A,B) = \frac{1}{2\,A\,\bar{u}} \left\{ \ln\left(\frac{\bar{u}-1}{\bar{u}+1}\right) \left[C_A + \frac{1}{2}\left[\ln\left(1-\bar{u}^2\right) + \ln\left(-4\,\bar{u}^2\right)\right] \right] \right.$$
$$\left. + \operatorname{Li}_2\left(\frac{\bar{u}+1}{2\,\bar{u}}\right) - \operatorname{Li}_2\left(\frac{\bar{u}-1}{2\,\bar{u}}\right) \right\}. \tag{E.8}$$

E.1.2. Second kind

With complex masses we need to also compute

$$H_{1,\infty}(A,B) = \int_1^{\infty} du\, \frac{\ln(A\,u^2 + B)}{A\,u^2 + B}, \tag{E.9}$$

where A and B are complex yet such that $\operatorname{sign}(\operatorname{Im}(A\,u^2 + B))$ is kept constant while u spans $[1,+\infty[$. The quantity $H_{1,\infty}(A,B)$ can be written as

$$H_{1,\infty}(A,B) = \frac{1}{A} \int_1^{\infty} du\, \frac{C'_A + \ln(u^2 - \bar{u}^2)}{u^2 - \bar{u}^2}, \tag{E.10}$$

where $\bar{u}^2 = -B/A$ and C'_A is given by

$$C'_A = \ln(A) + \eta(A, 1-\bar{u}^2). \tag{E.11}$$

We perform a partial fraction decomposition and, writing $H_{1,\infty}(A,B)$ as a sum of terms which are individually divergent when $u \to \infty$, we face a situation similar to the one met in Sect. B.2 of Ref. [2]. We proceed likewise, introducing a large u-cut-off Λ and write $H_{1,\infty}(A,B)$ as

$$H_{1,\infty}(A,B) = \frac{1}{2\,A\,\bar{u}} \lim_{\Lambda\to+\infty} \mathcal{H}^{\Lambda}_{1,\infty}(A,B),$$

where

$$\mathcal{H}^{\Lambda}_{1,\infty}(A,B) = \left\{ C'_A \int_1^{\Lambda} du \left[\frac{1}{u-\bar{u}} - \frac{1}{u+\bar{u}} \right] \right.$$
$$+ \int_1^{\Lambda} du\, \frac{\ln(u-\bar{u})}{u-\bar{u}} - \int_1^{\Lambda} du\, \frac{\ln(u+\bar{u})}{u+\bar{u}}$$
$$+ \int_1^{\Lambda} du\, \frac{\ln(u+\bar{u}) - \ln(2\,\bar{u})}{u-\bar{u}} \quad + \quad \ln(2\,\bar{u}) \int_1^{\Lambda} \frac{du}{u-\bar{u}}$$
$$\left. - \int_1^{\Lambda} du\, \frac{\ln(u-\bar{u}) - \ln(-2\,\bar{u})}{u+\bar{u}} - \ln(-2\,\bar{u}) \int_1^{\Lambda} \frac{du}{u+\bar{u}} \right\}. \tag{E.12}$$

[18] The subtlety discussed in Ref. [1] does not appear in this case.

We express $\mathcal{H}^{\Lambda}_{1,\infty}(A,B)$ in terms of the function $R^{\Lambda}(y,z)$ defined by Eq. (B.9) of Ref. [2]:

$$\mathcal{H}^{\Lambda}_{1,\infty}(A,B) = \Bigg\{ \left[C'_A + \ln(2\,\bar{u})\right] \ln\left(\frac{\Lambda-\bar{u}}{1-\bar{u}}\right) - \left[C'_A + \ln(-2\,\bar{u})\right] \ln\left(\frac{\Lambda+\bar{u}}{1+\bar{u}}\right) + \frac{1}{2}\left[\ln^2(\Lambda-\bar{u}) - \ln^2(1-\bar{u})\right] - \frac{1}{2}\left[\ln^2(\Lambda+\bar{u}) - \ln^2(1+\bar{u})\right] + R^{\Lambda}(-\bar{u},\bar{u}) - R^{\Lambda}(\bar{u},-\bar{u}) \Bigg\}. \tag{E.13}$$

Using Eq. (B.14) of Ref. [2] for the R^{Λ} terms, we take the limit $\Lambda \to \infty$. The terms proportional to $\ln^2(\Lambda)$ and those proportional to $\ln(\Lambda)$ drop out, and we get

$$H_{1,\infty}(A,B) = \frac{1}{2\,A\,\bar{u}} \Bigg\{ \left[C'_A + \ln(-2\,\bar{u})\right] \ln(1+\bar{u}) - \left[C'_A + \ln(2\,\bar{u})\right] \ln(1-\bar{u}) + \frac{1}{2}\left[\ln^2(1+\bar{u}) - \ln^2(1-\bar{u}) + \ln^2(2\,\bar{u}) - \ln^2(-2\,\bar{u})\right] - \mathrm{Li}_2\left(\frac{\bar{u}+1}{2\,\bar{u}}\right) + \mathrm{Li}_2\left(\frac{\bar{u}-1}{2\,\bar{u}}\right) \Bigg\}. \tag{E.14}$$

Noting that

$$\ln\left(\frac{1+\bar{u}}{1-\bar{u}}\right) = \ln(1+\bar{u}) - \ln(1-\bar{u}),$$

Eq. (E.14) becomes, after some algebra,

$$H_{1,\infty}(A,B) = \frac{1}{2\,A\,\bar{u}} \Bigg\{ \ln\left(\frac{1+\bar{u}}{1-\bar{u}}\right) \left[C'_A + \frac{1}{2}\ln(1-\bar{u}^2) + \ln(2\,\bar{u})\right] + \frac{\pi^2}{2} - i\,\pi\,S(\bar{u})\,\ln\left(\frac{\bar{u}+1}{2\,\bar{u}}\right) - \mathrm{Li}_2\left(\frac{\bar{u}+1}{2\,\bar{u}}\right) + \mathrm{Li}_2\left(\frac{\bar{u}-1}{2\,\bar{u}}\right) \Bigg\}. \tag{E.15}$$

We remark that the same combination of dilogarithms, up to a sign, appears in Eq. (E.15) and in $H_{0,1}(A,B)$, so that we can rewrite $H_{1,\infty}$ as

$$H_{1,\infty}(A,B) = -\,H_{0,1}(A,B) + \frac{1}{2\,A\,\bar{u}} \Bigg\{ i\,\pi\,S(\bar{u})\,\left[2\,\ln(2\,\bar{u}) + C_A\right] + \pi^2 + \left[\eta(A,-\bar{u}^2) - \eta(A,1-\bar{u}^2)\right] \ln\left(\frac{1+\bar{u}}{1-\bar{u}}\right) \Bigg\}. \tag{E.16}$$

E.1.3. Third kind

With complex masses a third kind of integrals also has to be considered:

$$H_{0,\infty}(A,B) = \int_0^{\infty} du\,\frac{\ln(A\,u^2+B)}{A\,u^2+B}, \tag{E.17}$$

where A and B are complex yet such that $\mathrm{sign}(\mathrm{Im}(A\,u^2+B))$ is kept constant while u spans $[0,+\infty[$. The quantity $H_{0,\infty}(A,B)$ can be split as

$$H_{0,\infty}(A,B) = H_{0,1}(A,B) + H_{1,\infty}(A,B). \tag{E.18}$$

From Eq. (E.16), and remembering that the assumption on the sign of $\mathrm{Im}(A\,u^2 + B)$ implies that $\eta(A, -\bar{u}^2) = \eta(A, 1 - \bar{u}^2)$, we immediately get

$$H_{0,\infty}(A,B) = \frac{1}{2\,A\,\bar{u}}\Big[i\,\pi\,S(\bar{u})\,\left[2\,\ln(2\,\bar{u}) + C_A\right] + \pi^2\Big]. \tag{E.19}$$

As happened for $K^C_{0,\infty}(A,B)$ (cf. Appendix B of Ref. [2]), $H_{0,\infty}(A,B)$ contains only logarithmic terms.

E.2. An extra load of backup integrals

We also need the following load of simpler integrals:

$$W_1(u_0^2) = \int_0^1 du\,\frac{\ln(1-u^2)}{u^2 - u_0^2}, \tag{E.20}$$

$$W_2(u_0^2) = \int_0^1 du\,\frac{\ln(u)}{u^2 - u_0^2}, \tag{E.21}$$

$$W_3(u_0^2) = \int_1^\infty du\,\frac{\ln(u^2-1)}{u^2 - u_0^2}, \tag{E.22}$$

$$W_4(u_0^2) = \int_0^\infty du\,\frac{\ln(u)}{u^2 - u_0^2}, \tag{E.23}$$

$$W_5(u_0^2) = \int_0^\infty du\,\frac{\ln(u^2+1)}{u^2 + u_0^2}. \tag{E.24}$$

For all these integrals, u_0^2 is assumed to be a complex number; this is indeed the case because these integrals appear in the computation of the four-point function in the IR case where u_0^2 is either a complex number with an imaginary part $\propto \lambda$ or a genuine complex number.

One might compute these integrals using specific values for A and B in $K^R_{0,1}(A,B)$, $K^C_{0,1}(A,B)$, $K^C_{1,\infty}(A,B)$, and $K^C_{0,\infty}(A,B)$ given in Appendices E of Ref. [1] and B of Ref. [2]; however, these integrals are simple enough to be computed directly (we verified that the results can be retrieved using the K-type integrals after some transformation on the dilogarithms). We give here the results of these integrals without any details:

$$W_1(u_0^2) = \frac{1}{2\,u_0}\left[\mathrm{Li}_2\left(\frac{2}{1+u_0}\right) - \mathrm{Li}_2\left(\frac{2}{1-u_0}\right) - 2\,\ln(2)\,\ln\left(\frac{u_0+1}{u_0-1}\right)\right], \tag{E.25}$$

$$W_2(u_0^2) = \frac{1}{2\,u_0}\left[\mathrm{Li}_2\left(\frac{1}{u_0}\right) - \mathrm{Li}_2\left(-\frac{1}{u_0}\right)\right], \tag{E.26}$$

$$W_3(u_0^2) = -W_1(u_0^2) + \frac{1}{2\,u_0}\,i\,S(u_0)\,\pi\,\ln\left(1-u_0^2\right), \tag{E.27}$$

$$W_4(u_0^2) = \frac{1}{4\,u_0}\,i\,S(u_0)\,\pi\,\ln(-u_0^2), \tag{E.28}$$

$$W_5(u_0^2) = \frac{\pi}{u_0}\,\ln(1+u_0), \tag{E.29}$$

with $u_0 = \sqrt{u_0^2}$.

Appendix F. Change of contour prescription for the pole in the IR four-point integral

This appendix legitimates the replacement

$$\int_0^1 du\, \frac{\left[u^2 P_{ijk} + R_{ij} - i\,\lambda\right]^{-\varepsilon}}{u^2 P_{ijk} + R_{ij} + i\,\lambda} \to \int_0^1 du\, \frac{\left[u^2 P_{ijk} + R_{ij} - i\,\lambda\right]^{-\varepsilon}}{u^2 P_{ijk} + R_{ij} - i\,\lambda} \tag{F.1}$$

when $\lambda \to 0^+$ for fixed $0 < -\varepsilon \ll 1$, whenever P_{ijk} and R_{ij} are both real with $0 < -R_{ij}/P_{ijk} < 1$. Intuitively, this replacement is based on the fact that for any function $f(u)$ analytic along $[0,1]$ and $0 < u_0 < 1$,

$$\int_0^1 du\, \frac{f(u)}{u - u_0 - i\,\lambda} - \int_0^1 du\, \frac{f(u)}{u - u_0 + i\,\lambda} \to 2i\,\pi\, f(u_0) \;\text{ when }\; \lambda \to 0^+,$$

which vanishes if u_0 is a zero of $f(u)$. However, the situation is made trickier when $f(u) = (u - u_0)^{-\varepsilon}$ and thus has a branch point at $u = u_0$ and a cut running along part of the interval of integration. To make the above argument apply, one could think of shifting the branch point and cut by a contour prescription $-i\,a\lambda$ with $a > 1$ so as to pass either above or below the pole while remaining on the same side of the cut. We would then get a residue value $\propto \lambda^{-\varepsilon}$ vanishing in the limit $\lambda \to 0^+$, keeping $\varepsilon < 0$ fixed. The "hierarchised lambdalogy" underpinning this disentanglement of pole from branch point may look awkward to the rigorous reader; let us therefore back up this hand-waving argument more rigorously as follows.

We first perform a partial fraction decomposition of the pole term:

$$\begin{aligned}&\frac{1}{u^2 P_{ijk} + R_{ij} + i\,s\,\lambda}\\ &= \frac{1}{P_{ijk}}\, \frac{1}{2\sqrt{-R_{ij}/P_{ijk}}} \left[\frac{1}{u - (u_0 - i\,s\lambda')} - \frac{1}{u + (u_0 - i\,s\lambda')}\right],\end{aligned} \tag{F.2}$$

where $s = \pm$, $u_0 = \sqrt{-R_{ij}/P_{ijk}}$ is assumed in $]0, 1[$, and $\lambda' = \lambda/(2u_0 P_{ijk})$.[19] We focus on the pole at $+u_0$ in the decomposition of Eq. (F.2): since the pole at $-u_0$ involved in the second term lies outside the integration region, the contour prescription for this pole is irrelevant and so is the corresponding pole term in the discussion. We then study the legitimacy of the replacement

$$\int_0^1 du\, \frac{\left[u^2 P_{ijk} + R_{ij} - i\,a\lambda\right]^{-\varepsilon}}{u - u_0 + i\,\lambda'} \to \int_0^1 du\, \frac{\left[u^2 P_{ijk} + R_{ij} - i\,a\lambda\right]^{-\varepsilon}}{u - u_0 - i\,\lambda'} \tag{F.3}$$

(a being positive yet kept arbitrary). We consider $\delta \equiv$ "left-hand side minus right-hand side" of Eq. (F.3), which can be written as

$$\delta = \int_0^1 \frac{du\,(2\,i\,\lambda')}{(u - u_0)^2 + \lambda'^2} \left[P_{ijk}\,\left((u - u_0)(u + u_0) - i\,(2u_0 a)\,\lambda'\right)\right]^{-\varepsilon}.$$

We make the change of variable $(u - u_0) = |\lambda'|\,v$, and δ reads

$$\delta = 2\,i\,\sigma\,\left(\frac{\lambda}{2u_0}\right)^{-\varepsilon} \int_{-u_0/|\lambda'|}^{(1-u_0)/|\lambda'|} \frac{dv}{v^2 + 1} \left[\sigma\,v\left(\sigma\lambda' v + 2u_0\right) - i(2u_0 a)\right]^{-\varepsilon} \tag{F.4}$$

[19] We keep track of this multiplicative change to control various normalizations in the reasoning so as to check the independence of the conclusion with respect to any assumption of "hierarchised lambdalogy."

(where $\sigma = \text{sign}(\lambda') = \text{sign}(P_{ijk})$). For any $b > 1$ and $|\lambda'|$ small enough we have, for all real v,

$$\left|\sigma v(\sigma\lambda' v + 2u_0) - i(2u_0 a)\right| < \left[v^4 + (2u_0)^2 b\, v^2 + (2u_0 a)^2\right]^{1/2},$$

and the integral

$$\int_{-\infty}^{+\infty} dv\, \frac{[v^4 + (2u_0)^2 b\, v^2 + (2u_0 a)^2]^{-\varepsilon/2}}{v^2 + 1}$$

is convergent when $-\varepsilon > 0$ is small enough. This provides an "integrable hat" for the application of Lebesgue's theorem of dominated convergence. When $\lambda \to 0^+$ keeping $-\varepsilon > 0$ fixed and small enough, the integral in Eq. (F.4) has the limit

$$\mathcal{L} = (2u_0)^{-\varepsilon} \int_{-\infty}^{+\infty} \frac{dv}{v^2 + 1}\, (\sigma v - ia)^{-\varepsilon},$$

which is finite regardless of $a > 0$ and ε small enough (its actual value, readily computable using the residue theorem, is irrelevant for the conclusion). We thus see that $\delta \sim \mathcal{O}(\lambda^{-\varepsilon})$, as anticipated.

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