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$B-L$ extension of the standard model with Q_6 symmetry

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Abstract

We propose a renormalizable $B-L$ Standard Model (SM) extension based on Q_6 symmetry which accommodates fermion mass and mixing parameters. Both normal and inverted neutrino mass orderings, the smallness of the active neutrino masses are generated through type I seesaw mechanism. The obtained physical parameters are well consistent with the global fit of neutrino oscillation [1] while the quark masses are in good agreement with the recent experimental data [2]. The model also predicts effective neutrino mass parameters of $\langle m_{ee}\rangle = 3.23\times 10^{-3}$ eV, $m_\beta = 8.92\times 10^{-3}$ eV for normal hierarchy (NH) and $\langle m_{ee}\rangle = m_\beta = 4.96\times 10^{-2}$ eV for inverted hierarchy (IH) which are all well consistent with the recent experiments.

1. Introduction

Despite its great successes, the Standard Model (SM) fails to provide a convincing explanation of the fermion mass hierarchy and mixing angles. A lot of experiments show that neutrinos have tiny masses, of about 8 orders of magnitude much smaller than the electron mass. In addition, the absolute neutrino mass scale as well as the sign of Δm^2_{31} is still unknown. Furthermore, quark

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mixing angles are small, whereas two of the leptonic mixing angles are large and the other is Cabibbo sized, which are not explained in the context of the SM. All these unexplained issues within the context of the SM, suggest that new physics have to be invoked to address the fermion puzzle of the SM.

Among the possible extensions of the SM, the extension with an extra $U(1)_{B-L}$ (baryon number minus lepton number) gauge symmetry [3–23] is a promising model since the simplest possibility is to introduce three right-handed neutrinos that we need to incorporate the neutrino mass in the SM. In this model, the masses of new boson Z' and three right-handed neutrinos are generated when the $B-L$ gauge symmetry is broken [9,10], and other phenomena including dark matter [11,13,16,18,19,22], the muon anomalous magnetic moment [12], inflation [17], leptogenesis [20,23], gravitational wave radiation [21], etc., are explained, however, this model by itself cannot give the fermion mixing matrices.

The best-fit values for neutrino mass squared splittings, leptonic mixing angles and the Dirac CP violating phase, with Super-Kamiokande atmospheric (wSK-atm) neutrino data, are given in Ref. [1]:

$$\begin{aligned}
&\Delta m^2_{21} = 7.39\times 10^{-5}\ \mathrm{eV}^2, && \sin^2\theta_{12} = 0.310,\ \ \sin^2\theta_{23} = 0.582,\\
&\Delta m^2_{31} = 2.525\times 10^{-3}\ \mathrm{eV}^2\ (\mathrm{NH}), && \sin^2\theta_{13} = 0.02240\ (\mathrm{NH}),\ \ \delta = 217^\circ\ (\mathrm{NH}),\\
&\Delta m^2_{32} = -2.512\times 10^{-3}\ \mathrm{eV}^2\ (\mathrm{IH}), && \sin^2\theta_{13} = 0.02263\ (\mathrm{IH}),\ \ \delta = 280^\circ\ (\mathrm{IH}),
\end{aligned} \tag{1}$$

while the 3σ CL ranges on the magnitude of the elements of the leptonic mixing matrix are [1]:

$$\left|U^{3\sigma}_{\text{wSK-atm}}\right| = \begin{pmatrix} 0.797\to 0.842 & 0.518\to 0.585 & 0.143\to 0.156\\ 0.235\to 0.484 & 0.458\to 0.671 & 0.647\to 0.781\\ 0.304\to 0.531 & 0.497\to 0.699 & 0.607\to 0.747 \end{pmatrix}. \tag{2}$$

Besides that, the best-fit results for the magnitudes of all nine CKM elements as well as quark masses have now been determined with high accuracy [2]

$$\left|V^{exp}_{\text{CKM}}\right| = \begin{pmatrix} 0.97446 & 0.22452 & 0.00365\\ 0.22438 & 0.97359 & 0.04214\\ 0.00896 & 0.04133 & 0.999105 \end{pmatrix}, \tag{3}$$

and m_u, m_d are very small compared to other quark masses:

$$\begin{aligned}
&m_u = 2.16\ \mathrm{MeV},\ \ m_c = 1.27\ \mathrm{GeV},\ \ m_t = 172.9\ \mathrm{GeV},\\
&m_d = 4.67\ \mathrm{MeV},\ \ m_s = 93\ \mathrm{MeV},\ \ \ m_b = 4.18\ \mathrm{GeV}.
\end{aligned} \tag{4}$$

The difference of the lepton and quark mixing angles in Eqs. (1)-(3) has stimulated works on discrete symmetries which has shown many outstanding advantages in explaining the observed pattern of SM fermionic masses and mixing angles. There have been many models based on discrete symmetries, see for instance, Refs. [24–39]. However, in those works, the fermion masses and mixings are generated from non-renormalizable interactions or at loop levels.

In this work, the first generation of leptons is put in $\underline{1}''$ while the first generation of quarks are put in $\underline{1}'''$ and the two others of leptons and quarks are put in $\underline{2}$ under Q_6 symmetry. We note that Q_6 symmetry has not been considered before in this kind of model.[1] Q_6 is the smallest

[1] In this model, fermion masses and mixing angles are generated from renormalizable Yukawa interactions. Both S_3 and Q_6 groups contain representations $\underline{1}$, $\underline{1}'$ and $\underline{2}$, however, their singlet/doublet components are combined in different

Table 1
The fermion content of the model and the charge assignment ($\alpha = 2, 3$).

	$\mathrm{SU(3)}_c$	$\mathrm{SU(2)}_L$	$\mathrm{U(1)}_Y$	$\mathrm{U(1)}_{B-L}$	Q_6	Z_4
ψ_{1L}	$\mathbf{1}$	$\mathbf{2}$	$-1/2$	-1	$\underline{1}''$	i
$\psi_{\alpha L}$	$\mathbf{1}$	$\mathbf{2}$	$-1/2$	-1	$\underline{2}$	i
l_{1R}	$\mathbf{1}$	$\mathbf{1}$	-1	-1	$\underline{1}''$	i
$l_{\alpha R}$	$\mathbf{1}$	$\mathbf{1}$	-1	-1	$\underline{2}$	i
ν_{1R}	$\mathbf{1}$	$\mathbf{1}$	0	-1	$\underline{1}''$	i
$\nu_{\alpha R}$	$\mathbf{1}$	$\mathbf{1}$	0	-1	$\underline{2}$	i
Q_{1L}	$\mathbf{3}$	$\mathbf{2}$	$1/6$	$1/3$	$\underline{1}'''$	i
$Q_{\alpha L}$	$\mathbf{3}$	$\mathbf{2}$	$1/6$	$1/3$	$\underline{2}$	$-i$
u_{1R}	$\mathbf{3}$	$\mathbf{1}$	$2/3$	$1/3$	$\underline{1}'''$	i
$u_{\alpha R}$	$\mathbf{3}$	$\mathbf{1}$	$2/3$	$1/3$	$\underline{2}$	$-i$
d_{1R}	$\mathbf{3}$	$\mathbf{1}$	$-1/3$	$1/3$	$\underline{1}'''$	i
$d_{\alpha R}$	$\mathbf{3}$	$\mathbf{1}$	$-1/3$	$1/3$	$\underline{2}$	$-i$

group of the binary dihedral group Q_N with $N = 6$ possesses two complex one-dimensional representations ($\underline{1}'', \underline{1}'''$). It has the 12 elements divided into six conjugacy classes with $\underline{1}$, $\underline{1}'$, $\underline{1}''$, $\underline{1}'''$, $\underline{2}$ and $\underline{2}'$ as its six irreducible representations. We will work in the basic in which the two-dimensional representation $\underline{2}$ of Q_6 is pseudo-real while $\underline{2}'$ is a real representation, $\underline{2}^*(1^*, 2^*) = \underline{2}(-2^*, 1^*)$, $\underline{2}'^*(1^*, 2^*) = \underline{2}'(1^*, 2^*)$ and their basic tensor product rules are

$$\underline{2}(1,2) \otimes \underline{2}(1,2) = \underline{1}(12-21) \oplus \underline{1}'(12+21) \oplus \underline{2}' \begin{pmatrix} 11 \\ 22 \end{pmatrix}, \tag{5}$$

$$\underline{2}'(1,2) \otimes \underline{2}'(1,2) = \underline{1}(12+21) \oplus \underline{1}'(12-21) \oplus \underline{2}' \begin{pmatrix} 11 \\ 22 \end{pmatrix}. \tag{6}$$

The rules to conjugate of the representations of Q_6 and its Clebsch–Gordan coefficients are briefly presented in Appendix A.

This paper is organized as follows. In section 2 we present a simple SM extension by adding U_{B-L} and Q_6 symmetries. In Section 3 we present the necessary lepton sectors of the model and introduce necessary Higgs fields responsible masses for the leptons. Section 4 deals with quark masses and mixings. We make conclusions in Sec. 5.

2. The model

The electroweak sector of the SM is supplemented by a gauge symmetry $\mathrm{U(1)}_{B-L}$, two discrete symmetries Q_6 and Z_4. In addition, three right-handed neutrinos, three $SU(2)_L$ doublets H', H'' with $B-L=0$ respectively put in $\underline{1}'$, $\underline{2}'$ under Q_6 and three flavons φ, ϕ with $B-L=2$ respectively put in $\underline{1}'$, $\underline{2}'$ under Q_6 are introduced to the SM model particle content. The particle content of the model thus is given in Table 1.

The charged lepton masses can arise from the couplings of $\bar{\psi}_{(1,\alpha)L} l_{(1,\alpha)R}$ to scalars, where under $SU(3)_c \times SU(2)_L \times U(1)_Y \times U(1)_{B-L} \times Q_6 \times Z_4$ symmetry, $\bar{\psi}_{1L} l_{1R}$ transforms as

ways. Furthermore, both Q_4 and Q_6 possesses a pseudo real representation $\underline{2}$ with the rules to conjugate, $\underline{2}^*(1^*, 2^*) = \underline{2}(-2^*, 1^*)$, but Q_6 contains a pair of complex representations $\underline{1}'', \underline{1}'''$ while all one-dimensional representations of Q_4 are real. This makes Q_6 group has some advantages compared to the others and the model under consideration is completely different from previous works.

Table 2
Scalar content of the model and the charge assignment.

	SU(3)$_c$	SU(2)$_L$	U(1)$_Y$	U(1)$_{B-L}$	Q_6	Z_4
H	**1**	**2**	1/2	0	$\underline{1}$	1
H'	**1**	**2**	1/2	0	$\underline{1}'$	1
H''	**1**	**2**	1/2	0	$\underline{2}'$	−1
φ	**1**	**1**	0	2	$\underline{1}'$	−1
ϕ	**1**	**1**	0	2	$\underline{2}'$	−1

$(\mathbf{1}, \mathbf{2}, -1/2, 0, \underline{1}, 1)$ which implies that in order to generate the charged-lepton mass matrix, one needs one Q_6 singlet transforming as $(\mathbf{1}, \mathbf{2}, 1/2, 0, \underline{1}, 1)$ to build an invariant under all given symmetries. Similarly, we have $\bar{\psi}_{1L} l_{\alpha R}, \bar{\psi}_{\alpha L} l_{1R} \sim (\mathbf{1}, \mathbf{2}, 1/2, 0, \underline{2}, 1)$ and $\bar{\psi}_{\alpha L} l_{\alpha R} \sim (\mathbf{1}, \mathbf{2}, 1/2, 0, \underline{1} + \underline{1}' + \underline{2}', 1)$. Therefore, to generate masses for the charged leptons, we need two $SU(2)_L$ scalar fields H and H' as given in Table 2.

In neutrino sector, for two $SU(2)_L$ doublets (H, H'), available interactions are $\bar{\psi}_{1L} \widetilde{H} \nu_{1R}$, $\left(\bar{\psi}_{\alpha L} \nu_{\alpha R}\right)_{\mathbf{1}} \widetilde{H}$ and $\left(\bar{\psi}_{\alpha L} \nu_{\alpha R}\right)_{\mathbf{1}'} \widetilde{H'}$ however they only generate Dirac mass term. To generate Majorana mass term for neutrinos we will therefore introduce two new $SU(2)_L$ singlets φ, ϕ with $B - L = 2$ respectively put in $\underline{1}'$ and $\underline{2}'$ under Q_6, coupling to $\bar{\nu}^c_{\alpha R} \nu_{\alpha R}$ and $\bar{\nu}^c_{1R} \nu_{\alpha R}$, responsible for the realistic neutrino masses and mixings.

In the quarks sector, with the known scalar fields H, H', φ, ϕ, available interactions are $\bar{Q}_{1L} \widetilde{H} u_{1R}$, $(\bar{Q}_{\alpha L} u_{\alpha R})_{\underline{1}} \widetilde{H}$, $(\bar{Q}_{\alpha L} u_{\alpha R})_{\underline{1}'} \widetilde{H'}$, $\bar{Q}_{1L} H d_{1R}$, $(\bar{Q}_{\alpha L} d_{\alpha R})_{\underline{1}} H$ and $(\bar{Q}_{\alpha L} d_{\alpha R})_{\underline{1}'} H'$ which can generate masses for the up and down quarks. In this case, the quark mass matrices are diagonal and, consequently, the CKM mixing matrix is equal to identity matrix. However, as we know, the quark mixing matrix is given in Eq. (3) which is a little difference from unit matrix, and it adheres to the following approximate pattern[2]:

$$|V_{\mathrm{CKM}}| \sim \begin{pmatrix} 0.97446 & 0.22452 & 0 \\ 0.22438 & 0.97359 & 0 \\ 0 & 0 & 0.99911 \end{pmatrix}. \tag{7}$$

To obtain the quark mixing form in Eq. (7) we propose an additional $SU(2)_L$ doublet H'' put in $\underline{2}$ under Q_6 coupling to $\bar{Q}_{1L} u_{\alpha R}, \bar{Q}_{\alpha L} u_{1R}, \bar{Q}_{1L} d_{\alpha R}, \bar{Q}_{\alpha L} d_{1R}$, respectively, contributing to the elements (12), (21) and (13), (31) of the up and down quark mass matrices.

With the particle content shown in Tables 1 and 2, the following relevant Yukawa terms[3] for the quark and lepton sector invariant under all symmetries of the model arise:

$$\begin{aligned}
-\mathcal{L}^l_Y &= h_1 \bar{\psi}_{1L} H l_{1R} + h_2 \left(\bar{\psi}_{\alpha L} l_{\alpha R}\right)_{\mathbf{1}} H + h_3 \left(\bar{\psi}_{\alpha L} l_{\alpha R}\right)_{\mathbf{1}'} H' \\
&+ \frac{x_1}{2} \bar{\psi}_{1L} \widetilde{H} \nu_{1R} + \frac{x_2}{2} \left(\bar{\psi}_{\alpha L} \nu_{\alpha R}\right)_{\mathbf{1}} \widetilde{H} + \frac{x_3}{2} \left(\bar{\psi}_{\alpha L} \nu_{\alpha R}\right)_{\mathbf{1}'} \widetilde{H'} + \\
&+ \frac{y_1}{2} \left(\bar{\nu}^c_{\alpha R} \nu_{\alpha R}\right)_{\underline{2}'} \phi + \frac{y_2}{2} \left(\bar{\nu}^c_{\alpha R} \nu_{1R} + \bar{\nu}^c_{1R} \nu_{\alpha R}\right)_{\underline{2}'} \phi + \frac{y_3}{2} \left(\bar{\nu}^c_{\alpha R} \nu_{\alpha R}\right)_{\underline{1}'} \varphi + \mathrm{H.c}, \\
-\mathcal{L}^q_Y &= h_{1u} \bar{Q}_{1L} \widetilde{H} u_{1R} + h_{2u} (\bar{Q}_{\alpha L} u_{\alpha R})_{\underline{1}} \widetilde{H} + h_{3u} (\bar{Q}_{\alpha L} u_{\alpha R})_{\underline{1}'} \widetilde{H'} \\
&+ h_{4u} \bar{Q}_{1L} (\widetilde{H}'' u_{\alpha R})_{\underline{1}'''} + h_{5u} (\bar{Q}_{\alpha L} \widetilde{H}'')_{\underline{1}''} u_{1R}
\end{aligned} \tag{8}$$

[2] This is a good approximation for the realistic quark mixing matrix, which implies that the mixings among the quarks are small.

[3] The following interactions $\bar{\psi}_{\alpha L} H'' l_{\alpha R}$, $\bar{\psi}_{\alpha L} \bar{H}'' \nu_{\alpha R}$ and $\bar{Q}_{\alpha L} H'' d_{\alpha R}$ are forbidden because of the Z_4 symmetry.

$$+ h_{1d}\bar{Q}_{1L}Hd_{1R} + h_{2d}(\bar{Q}_{\alpha L}d_{\alpha R})_{\underline{1}}H + h_{3d}(\bar{Q}_{\alpha L}d_{\alpha R})_{\underline{1}'}H' + h_{4d}\bar{Q}_{1L}(H''d_{\alpha R})_{\underline{1}'''} + h_{5d}(\bar{Q}_{\alpha L}H''))_{\underline{1}''}d_{1R} + \text{H.c.} \quad (9)$$

Since there are many Yukawa couplings in Higgs potential and it is easy to arrange a suitable Higgs potential, in order to generate the remarkable fermion mixing pattern, we choose the VEVs of scalar fields as follows:

$$\langle H\rangle = v, \quad \langle H'\rangle = v', \quad \langle H''\rangle = (v'',\ v''), \quad \langle\varphi\rangle = v_\varphi, \quad \langle\phi\rangle = (v_\phi,\ v_\phi). \quad (10)$$

Indeed, the general potential invariant under any group takes the form[4]:

$$\begin{aligned} V_{\text{total}} &= V(H) + V(H') + V(H'') + V(\varphi) + V(\phi) \\ &+ V(HH') + V(HH'') + V(H\varphi) + V(H\phi) + V(H'H'') \\ &+ V(H'\varphi) + V(H'\phi) + V(H''\varphi) + V(H''\phi) + V(\varphi\phi), \end{aligned} \quad (11)$$

where:

$$\begin{aligned} V(H) &= \mu_H^2 H^\dagger H + \lambda^H (H^\dagger H)^2, \quad V(H') = V(H\to H') \\ V(H'') &= \mu_{H''}^2 (H''^\dagger H'')_{\underline{1}} + \lambda_1^{H''}(H''^\dagger H'')_{\underline{1}}(H''^\dagger H'')_{\underline{1}} \\ &+ \lambda_2^{H''}(H''^\dagger H'')_{\underline{1}'}(H''^\dagger H'')_{\underline{1}'} + \lambda_3^{H''}(H''^\dagger H'')_{\underline{2}'}(H''^\dagger H'')_{\underline{2}'}, \\ V(\varphi) &= V(H\to\varphi), \quad V(\phi) = V(H''\to\phi), \\ V(H,H') &= \lambda_1^{HH'}(H^\dagger H)_{\underline{1}}(H'^\dagger H')_{\underline{1}} + \lambda_2^{HH'}(H^\dagger H')_{\underline{1}'}(H'^\dagger H)_{\underline{1}'}, \\ V(H,H'') &= \lambda_1^{HH''}(H^\dagger H)_{\underline{1}}(H''^\dagger H'')_{\underline{1}} + \lambda_2^{HH''}(H^\dagger H'')_{\underline{2}'}(H''^\dagger H)_{\underline{2}'}, \\ V(H,\varphi) &= V(H,H'\to\varphi), \quad V(H,\phi) = V(H,H''\to\phi), \\ V(H',H'') &= V(H\to H',H''), \quad V(H',\varphi) = V(H\to\varphi,H'), \\ V(H',\phi) &= V(H',H''\to\phi), \quad V(H'',\varphi) = V(\varphi\to H,H''), \\ V(H'',\phi) &= \lambda_1^{H''\phi}(H''^\dagger H'')_{\underline{1}}(\phi^\dagger\phi)_{\underline{1}} + \lambda_2^{H''\phi}(H''^\dagger\phi)_{\underline{1}'}(\phi^\dagger H'')_{\underline{1}'} \\ &+ \lambda_3^{H''\phi}(H''^\dagger\phi)_{\underline{2}'}(\phi^\dagger H'')_{\underline{2}'}, \quad V(\varphi,\phi) = V(H\to\varphi,\phi). \end{aligned} \quad (12)$$

The flavons H, H', H'', φ and ϕ with their VEVs aligned in Eq. (10) is an solution from the minimization conditions of V_{total}. To see this obviously, in the system of minimization equations, let us put $v_{H''_1} = v_{H''_2} = v''$, $v_{\phi_1} = v_{\phi_2} = v_\phi$ and $v^* = v, v'^* = v', v''^* = v'', v_\varphi^* = v_\varphi, v_\phi^* = v_\phi$, which reduces to

$$\mu_H^2 + 2\lambda^H v^2 + (\lambda_1^{HH'} + \lambda_2^{HH'})v'^2 + 2(\lambda_1^{HH''} + \lambda_2^{HH''})v''^2 + (\lambda_1^{H\varphi} + \lambda_2^{H\varphi})v_\varphi^2 + 2(\lambda_1^{H\phi} + \lambda_2^{H\phi})v_\phi^2 = 0, \quad (13)$$

$$\mu_{H'}^2 + 2\lambda^{H'} v'^2 + (\lambda_1^{HH'} + \lambda_2^{HH'})v^2 + 2(\lambda_1^{H'H''} + \lambda_2^{H'H''})v''^2 + (\lambda_1^{H'\varphi} + \lambda_2^{H'\varphi})v_\varphi^2 + 2(\lambda_1^{H'\phi} + \lambda_2^{H'\phi})v_\phi^2 = 0, \quad (14)$$

$$\mu_{H''}^2 + (\lambda_1^{HH''} + \lambda_2^{HH''})v^2 + (\lambda_1^{H'H''} + \lambda_2^{H'H''})v'^2 + 2(2\lambda_1^{H''} + \lambda_2^{H''})v''^2 + (\lambda_1^{H''\varphi} + \lambda_2^{H''\varphi})v_\varphi^2 + [2(\lambda_1^{H''\phi} + \lambda_2^{H''\phi}) + \lambda_3^{H''\phi}]v_\phi^2 = 0, \quad (15)$$

[4] Here, we denote $V(X\to X_1, Y\to Y_1,\cdots) \equiv V(X,Y,\cdots)|_{\{X=X_1,Y=Y_1,\cdots\}}$.

$$\mu_\varphi^2+2\lambda^\varphi v_\varphi^2+(\lambda_1^{H\varphi}+\lambda_2^{H\varphi})v^2+(\lambda_1^{H'\varphi}+\lambda_2^{H'\varphi})v'^2 \\ +2(\lambda_1^{H''\varphi}+\lambda_2^{H''\varphi})v''^2+2(\lambda_1^{\varphi\phi}+\lambda_2^{\varphi\phi})v_\phi^2=0 \tag{16}$$

$$\mu_\phi^2+(\lambda_1^{HH'}+\lambda_2^{HH'})v^2+2\lambda^{H'}v'^2+2(\lambda_1^{H'H''}+\lambda_2^{H'H''})v''^2 \\ +(\lambda_1^{H'\varphi}+\lambda_2^{H'\varphi})v_\varphi^2+2(\lambda_1^{H'\phi}+\lambda_2^{H'\phi})v_\phi^2=0. \tag{17}$$

The system of equations from (13) to (17) always give the solution $(v, v', v'', v_\varphi, v_\phi)$ as chosen in Eq. (10). It is also noted that this aligned is only one solution to have the desirable results.

In models with more than one $SU(2)_L$ Higgs doublet, as in the present model, the Flavor Changing Neutral Current processes exist in the Higgs sector, however, they can be suppressed by adding discrete symmetries which have been presented in Refs. [40–47] thus we do not consider here.

3. Lepton masses and mixings

From the charged lepton Yukawa interactions given in Eq. (8) and using the product rules of the Q_6 group shown in the Appendix A, we can rewrite the Yukawa interactions in charged lepton sector in the form:

$$-\mathcal{L}_Y^{clep}=h_1\bar{\psi}_{1L}Hl_{1R}-h_2\left(\bar{\psi}_{2L}l_{2R}+\bar{\psi}_{3L}l_{3R}\right)H+h_3\left(\bar{\psi}_{2L}l_{2R}-\bar{\psi}_{3L}l_{3R}\right)H'+\text{H.c.} \tag{18}$$

With the help of Eq. (10), the Lagrangian mass term of the charged leptons can be written in the form:

$$-\mathcal{L}_l^{\text{mass}}=(\bar{l}_{1L},\bar{l}_{2L},\bar{l}_{3L})M_l(l_{1R},l_{2R},l_{3R})^T+H.c, \tag{19}$$

where

$$M_l=\text{diag}\left(h_1v,\ -h_2v+h_3v',\ -h_2v-h_3v'\right)\equiv\text{diag}\left(m_e,\ m_\mu\ m_\tau\right), \tag{20}$$

which has the diagonal form. Thus, the charged lepton diagonalization matrices are $U_{lL}=U_{lR}=1$ and the lepton mixing matrix depends on only that of the neutrinos. By comparing Eq. (20) with the experimental values given in Ref. [2] for masses of the charged leptons: $m_e\simeq 0.51099$ MeV, $m_\mu\simeq 105.65837$ MeV, $m_\tau\simeq 1776.86$ MeV, we get $h_1v=5.11\times10^5$ eV, $h_2v=-9.41\times10^8$ eV, $h_3v'=-8.36\times10^8$ eV.

Regarding the neutrino sector, from the lepton Yukawa terms given in Eq. (8), we get:

$$-\mathcal{L}_Y^\nu=\frac{x_1}{2}\bar{\psi}_{1L}\widetilde{H}\nu_{1R}-\frac{x_2}{2}\left(\bar{\psi}_{2L}\nu_{2R}+\bar{\psi}_{3L}\nu_{3R}\right)\widetilde{H}+\frac{x_3}{2}\left(\bar{\psi}_{2L}\nu_{2R}-\bar{\psi}_{3L}\nu_{3R}\right)\widetilde{H}' \\ +\frac{y_1}{2}(\bar{\nu}^c_{2R}\nu_{2R}\phi_2+\bar{\nu}^c_{3R}\nu_{3R}\phi_1)+\frac{y_2}{2}\left[(\bar{\nu}^c_{1R}\nu_{2R}+\bar{\nu}^c_{2R}\nu_{1R})\phi_2+(\bar{\nu}^c_{1R}\nu_{3R}+\bar{\nu}^c_{3R}\nu_{1R})\phi_1\right] \\ +\frac{y_3}{2}\left(\bar{\nu}^c_{2R}\nu_{3R}+\bar{\nu}^c_{3R}\nu_{2R}\right)\varphi+\text{H.c.} \tag{21}$$

With the VEVs in Eq. (10), we get the Dirac neutrino mass matrix (M_D) and the right-handed Majorana neutrino mass matrix (M_R) as follows

$$M_D=\text{diag}\left(a_D,-b_D+c_D,-b_D-c_D\right),\quad M_R=\begin{pmatrix} 0 & b_R & b_R \\ b_R & a_R & c_R \\ b_R & c_R & a_R \end{pmatrix}, \tag{22}$$

where

$$a_D = x_1 v^*, \quad b_D = x_2 v^*, \quad c_D = x_3 v'^*, \quad a_R = y_1 v_\phi, \quad b_R = y_2 v_\phi, \quad c_R = y_3 v_\varphi. \tag{23}$$

The effective neutrino mass matrix, obtained through the type-I seesaw mechanism, reads:

$$M_{\text{eff}} = -M_D^T M_R^{-1} M_D = M_0 + \Delta M, \tag{24}$$

where

$$M_0 = \begin{pmatrix} a_0 & b_0 & b_0 \\ b_0 & -c_0 & c_0 \\ b_0 & c_0 & -c_0 \end{pmatrix}, \ \Delta M = \begin{pmatrix} 0 & -a_1 & a_1 \\ -a_1 & b_1 - c_1 & -c_1 \\ a_1 & -c_1 & -b_1 - c_1 \end{pmatrix}, \tag{25}$$

with

$$a_0 = \frac{a_D^2 (a_R + c_R)}{2b_R^2}, \quad b_0 = \frac{a_D b_D}{2b_R}, \quad c_0 = \frac{b_D^2}{2(a_R - c_R)}, \tag{26}$$

$$a_1 = \frac{a_D c_D}{2b_R}, \quad b_1 = \frac{b_D c_D}{a_R - c_R}, \quad c_1 = \frac{c_D^2}{2(a_R - c_R)}. \tag{27}$$

The first matrix in Eq. (24) due to the contribution of H, ϕ and φ providing the $\mu - \tau$ mixing form, i.e., can approximately fit the experimental data given Eq. (1) with a small deviation for θ_{13} while H' contributes to the second term (ΔM) will take the role for such a deviation for θ_{13}, non-maximal θ_{23} mixing and being responsible for the CP violating phase in the lepton sector. Thus, in this work we consider the contribution of H' as a small perturbation and terminating the theory at the first order. Furthermore, a_1, b_1 are proportional to c_D while c_1 is proportional to c_D^2, i.e., c_1 is a second order approximation parameter, thus will be ignored, and ΔM in Eq. (25) reduces to

$$\Delta M = \begin{pmatrix} 0 & -a_1 & a_1 \\ -a_1 & b_1 & 0 \\ a_1 & 0 & -b_1 \end{pmatrix}, \tag{28}$$

which starts from the first order of the perturbation. It is noteworthy that the presence of H' is very important because in the case there is no contribution of H' the resulting mass matrix cannot fit the data in Eq. (1).

The matrix M_0 in Eq. (25) has three eigenvalues as follows,

$$m_{1,2} = \frac{1}{2}\left(a_0 \mp \sqrt{a_0^2 + 8b_0^2}\right), \quad m_3^0 = -2c_0, \tag{29}$$

and the corresponding lepton mixing matrix takes the form:

$$\mathbb{U}_0 = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\frac{\sin\theta}{\sqrt{2}} & \frac{\cos\theta}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{\sin\theta}{\sqrt{2}} & \frac{\cos\theta}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \tag{30}$$

where

$$\theta = \arctan\left(\mathcal{K}/\sqrt{2}\right), \ \mathcal{K} = (a_0 - m_1)/b_0 = m_2/b_0, \tag{31}$$

with a_0, b_0 and $m_{1,2}$ are respectively defined in Eqs. (26) and (29).

At the first order of perturbation theory, the matrix ΔM in Eq. (28) does not effect on neutrino mass eigenvalues but it changes the corresponding eigenvectors. The physical perturbed leptonic mixing matrix are obtained as:

$$\mathbb{U} = \mathbb{U}_0 + \Delta\mathbb{U}, \tag{32}$$

where $\mathbb{U}_0$ is defined by Eq. (30) and $\Delta\mathbb{U}$ has the following entries:

$$\begin{aligned}
&\Delta\mathbb{U}_{11}=\Delta\mathbb{U}_{12}=0,\\
&\Delta\mathbb{U}_{13}=\frac{(\sqrt{2}a_1\sin\theta-b_1\cos\theta)\sin\theta}{m_2-m_3}+\frac{(\sqrt{2}a_1\cos\theta+b_1\sin\theta)\cos\theta}{m_1-m_3},\\
&\Delta\mathbb{U}_{21}=\frac{a_1\cos\theta+b_1\sin\theta/\sqrt{2}}{m_3-m_1},\quad \Delta\mathbb{U}_{22}=\frac{a_1\sin\theta-b_1\cos\theta/\sqrt{2}}{m_3-m_2},\\
&\Delta\mathbb{U}_{23}=\frac{(\sqrt{2}a_1\sin\theta-b_1\cos\theta)\cos\theta}{\sqrt{2}(m_2-m_3)}-\frac{(\sqrt{2}a_1\cos\theta+b_1\sin\theta)\sin\theta}{\sqrt{2}(m_1-m_3)},\\
&\Delta\mathbb{U}_{31}=\frac{a_1\cos\theta+b_1\sin\theta/\sqrt{2}}{m_1-m_3},\quad \Delta\mathbb{U}_{32}=\frac{a_1\sin\theta-b_1\cos\theta/\sqrt{2}}{m_2-m_3},\quad \Delta\mathbb{U}_{33}=\Delta\mathbb{U}_{23}.
\end{aligned} \tag{33}$$

In the three-neutrino framework, lepton mixing angles can be defined via the elements of the neutrino mixing matrix:

$$t_{12}=\frac{|\mathbb{U}_{12}|}{|\mathbb{U}_{11}|},\; t_{23}=\frac{|\mathbb{U}_{23}|}{|\mathbb{U}_{33}|},\; s_{13}=\mathbb{U}_{13}e^{i\delta}, \tag{34}$$

where $t_{12}=s_{12}/c_{12}$, $t_{23}=s_{23}/c_{23}$, $c_{ij}=\cos\theta_{ij}$, $s_{ij}=\sin\theta_{ij}$ with θ_{12}, θ_{23} and θ_{13}, respectively, being the solar angle, atmospheric angle and the reactor angle and δ is the Dirac CP violating phase.

From Eqs. (30), (32), (33) and (34), we get a solution:

$$a_1=\frac{b_1(m_1-m_2)\cos\theta\sin\theta+(m_1-m_3)(m_2-m_3)s_{13}e^{-i\delta}}{\sqrt{2}\left[(m_2-m_3)\cos^2\theta+(m_1-m_3)\sin^2\theta\right]}, \tag{35}$$

$$\theta=\theta_{12}, \tag{36}$$

$$b_1=\begin{cases}\frac{\alpha+\sqrt{\beta}}{2(1-t_{23}^2)} & \text{for NH},\\ \frac{\alpha-\sqrt{\beta}}{2(1-t_{23}^2)} & \text{for IH},\end{cases} \tag{37}$$

where

$$\begin{aligned}
\alpha&=2[(m_2-m_3)\cos^2\theta+(m_1-m_3)\sin^2\theta](1+t_{23}^2)\\
&\quad+(m_1-m_2)\sin(2\theta)s_{13}(1-t_{23}^2)\cos\delta,\\
\beta&=16\left[(m_2-m_3)\cos^2\theta+(m_1-m_3)\sin^2\theta\right]^2t_{23}^2\\
&\quad-(m_1-m_2)^2\sin^2(2\theta)s_{13}^2(t_{23}^2-1)^2\sin^2\delta.
\end{aligned} \tag{38}$$

Although the global analysis in Ref. [48] shows a hint in favor of the NH over the inverted one at more than 3σ and the global analysis in Ref. [1] obtain a preference for NH at about 2σ, however, it is the fact that the neutrino mass spectrum is currently unknown and it can be NH or

Table 3
The model parameters in the case $c_0 = 2.55 \times 10^{-2}$ eV in NH.

Parameters	The derived values
a_1	$(3.52 - 2.58i)10^{-3}$ eV
b_1	3.43×10^{-3} eV
m_1	9.43×10^{-3} eV
m_2	1.28×10^{-2} eV
m_3	5.10×10^{-2} eV

IH depending on the sign of Δm^2_{31} [2]. In the model under consideration, both NH and IH can be found in which the model parameters are in good agreement with the global analysis in Ref. [1].

3.1. Normal spectrum

By taking the best-fit values for neutrino mass squared splittings, leptonic mixing angles and the Dirac CP violating phase as given in Eq. (1), we get a solution[5]:

$$a_0 = 0.5\sqrt{-0.0198 + 32c_0^2 + 2\sqrt{\alpha_N}}\ [\mathrm{eV}], \quad b_0 = -0.354i\,\sqrt[4]{\alpha_N}\ [\mathrm{eV}], \tag{39}$$

where

$$\alpha_N = 9.8 \times 10^{-5} - 0.317c_0^2 + 256c_0^4\ [\mathrm{eV}^4], \tag{40}$$

which depend only on c_0. In the case $c_0 = 2.55 \times 10^{-2}$ eV the other parameters are found in Table 3. The sum of neutrino mass is found to be $\sum_{i=1}^{3} m_i^N = 7.32 \times 10^{-2}$ eV which is well consistent with the strongest bound from cosmology, $\sum m_\nu < 0.078$ eV [49]. At present there exist various bounds for $\sum m_\nu$. For instance, it has the upper limits [49] $\sum m_\nu < 0.152$ eV in the minimal $\Lambda CDM + \sum m_\nu$ model, $\sum m_\nu < 0.118$ eV by adding the high-l polarization data [49], $\sum m_\nu < 0.101$ eV is achieved in NPDDE model, $\sum m_\nu < 0.093$ eV in the NPDDE+r model and the most aggressive constraint is $\sum m_\nu < 0.078$ eV in NPDDE+r with the R16 prior. The magnitude of the elements of the leptonic mixing matrix in Eq. (32) then takes the form:

$$\left|\mathbb{U}^N\right| = \begin{pmatrix} 0.831 & 0.557 & 0.150 \\ 0.296 & 0.587 & 0.765 \\ 0.499 & 0.590 & 0.649 \end{pmatrix}, \tag{41}$$

which is consistent with the constraint on the absolute values of the entries of the lepton mixing matrix given in Ref. [1].

3.2. Inverted spectrum

Similar to the normal spectrum, taking the best-fit values of neutrino oscillation parameters for IH as given in Eq. (1), we get a solution[6]:

[5] There are eight solutions for a_0, b_0 however they give the same value of Δm^2_{ij} and the same absolute values of $m_{1,2,3}$ thus we only consider in detail the solution in Eqs. (39) and (40).

[6] There are eight solutions for b_0, c_0 however they give the same value for Δm^2_{ij} and the same absolute values of $m_{1,2,3}$ thus we only consider in detail the solution in Eqs. (42) and (43).

Table 4
The model parameters in the case $c_0 = -2.5 \times 10^{-4}$ eV in IH.

Parameters	The derived values
a_1	$(0.927 + 5.14i)10^{-3}$ eV
b_1	-4.09×10^{-3} eV
c_1	-5.57×10^{-10} eV
m_1^I	4.94×10^{-2} eV
m_2^I	5.01×10^{-2} eV
m_3^I	5.0×10^{-4} eV
$\sum_{i=1}^3 m_i^I$	0.10 eV

$$a_0 = 0.5\sqrt{0.0198 + 32c_0^2 + 2\sqrt{\alpha_I}}\ [\text{eV}],\quad b_0 = -0.354i\sqrt[4]{\alpha_I}\ [\text{eV}], \tag{42}$$

where

$$\alpha_I = 9.8 \times 10^{-5} + 0.317c_0^2 + 256c_0^4\ [\text{eV}^4], \tag{43}$$

which depend only on c_0. In the case $c_0 = -2.5 \times 10^{-4}$ eV the other parameters are found in Table 4. The magnitude of the leptonic mixing matrix in Eq. (32) takes the form:

$$\left|\mathbb{U}^I\right| = \begin{pmatrix} 0.831 & 0.557 & 0.150 \\ 0.387 & 0.532 & 0.765 \\ 0.420 & 0.649 & 0.649 \end{pmatrix}, \tag{44}$$

which is consistent with the constraint on the absolute values of the entries of the lepton mixing matrix given in Ref. [1]. The result on the sum of three active neutrino masses $\sum_{i=1}^3 m_i^I$ in Table 4 is above the strongest bound from cosmology, $\sum m_\nu < 0.078$ eV [49], however, it is well consistent with the other bounds in various cosmological scenarios [49], such as, $\sum m_\nu < 0.152$ in the minimal $\Lambda CDM + \sum m_\nu$ model, $\sum m_\nu < 0.118$ eV in the minimal $\Lambda CDM + \sum m_\nu$ model with the high-l polarization data, $\sum m_\nu < 0.305$ eV in the DDE model with TT + BAO + PAN + $\tau 0p055$, $\sum m_\nu < 0.305$ eV in the DDE model with TT + BAO + PAN + $\tau 0p055$, $\sum m_\nu < 0.247$ eV in the DDE model with TTTEEE + BAO + PAN + R16 + $\tau 0p055$ and $\sum m_\nu < 0.101$ eV in NPDDE model with TTTEEE + BAO + PAN + $\tau 0p055$.

3.3. Effective neutrino mass parameters

The effective neutrino masses governing the beta decay and neutrinoless double beta decay has the forms, $m_\beta = \left(\sum_{i=1}^3 |U_{ei}|^2 m_i^2\right)^{1/2}$, $\langle m_{ee}\rangle = \left|\sum_{i=1}^3 U_{ei}^2 m_i\right|$, where U_{ei} $(i = 1, 2, 3)$ are the PMNS leptonic mixing matrix elements and m_i correspond to the masses of three light neutrinos. In the NH, $m_1 \ll m_2 \sim m_3$ thus $m_1 \equiv m^N_{light}$ is the lightest neutrino mass while in the IH, $m_3 \ll m_1 \sim m_2$ thus $m_3 \equiv m^I_{light}$ is the lightest neutrino mass. m_{light}, m_β and $\langle m_{ee}\rangle$ as functions of c_0 are plotted in Fig. 1 (Fig. 2) for NH (IH).

By using the model parameters obtained in subsections 3.1 and 3.2, we get the following values:

$$\langle m_{ee}\rangle = \begin{cases} 1.08 \times 10^{-2}\ \text{eV} & \text{for} \quad \text{NH}, \\ 4.96 \times 10^{-2}\ \text{eV} & \text{for} \quad \text{IH}, \end{cases} \tag{45}$$

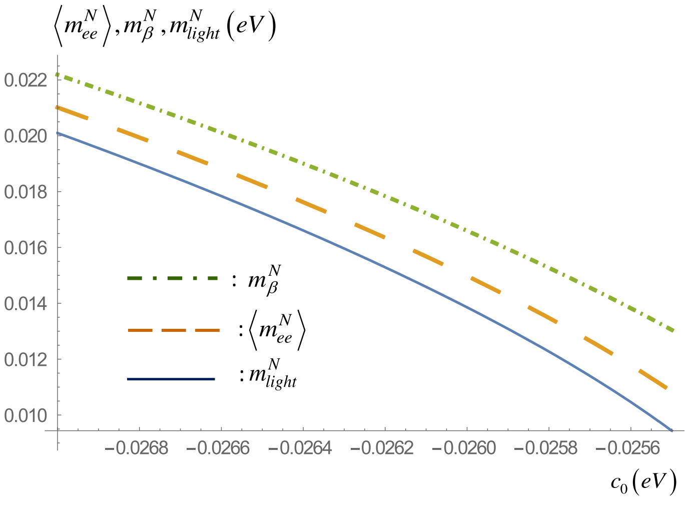

Fig. 1. $\langle m_{ee}^N \rangle$, m_β^N and m_{light}^N as functions of c_0 with $c_0 \in (-2.70, -2.55)10^{-2}$ eV in the NH.

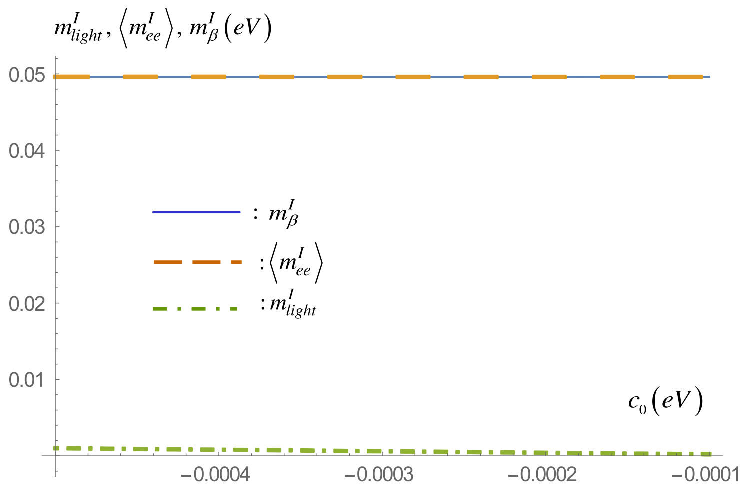

Fig. 2. $\langle m_{ee}^I \rangle$, m_β^I and m_{light}^I as functions of c_0 with $c_0 \in (-5.0, -1.0)10^{-4}$ eV in the IH.

and

$$m_\beta = \begin{cases} 1.30 \times 10^{-2}\ \text{eV} & \text{for} \quad \text{NH}, \\ 4.96 \times 10^{-2}\ \text{eV} & \text{for} \quad \text{IH}. \end{cases} \tag{46}$$

The Jarlskog invariant which controls the size of CP violation takes the values [2]:

$$J_{CP} = \text{Im}(\mathbb{U}_{23}\mathbb{U}_{13}^*\mathbb{U}_{12}\mathbb{U}_{22}^*) = \begin{cases} -2.45 \times 10^{-2}\ \text{eV} & \text{for} \quad \text{NH}, \\ -3.27 \times 10^{-2}\ \text{eV} & \text{for} \quad \text{IH}. \end{cases} \tag{47}$$

The resulting effective neutrino mass parameters in Eqs. (45) and (46), for both normal and inverted hierarchies, are below all the upper bounds arising from present $0\nu\beta\beta$ decay experiments, such as, KamLAND-Zen [50] $\langle m_{ee}\rangle < 0.05 \div 0.16$ eV, GERDA [51] $\langle m_{ee}\rangle < 0.12 \div 0.26$ eV, MAJORANA [52] $\langle m_{ee}\rangle < 0.24 \div 0.53$ eV, EXO [53–55] $\langle m_{ee}\rangle < 0.17 \div 0.49$ eV, CUORE [56,57] $\langle m_{ee}\rangle < 0.11 \div 0.5$ eV.

4. Quark mass and mixing

From the quark Yukawa interactions given in Eq. (9) and using the product rules of the Q_6 group shown in the Appendix A, the Yukawa interactions in quarks sector is rewritten in the form:

$$\begin{aligned}
-\mathcal{L}_q &= h_{1u}\bar{Q}_{1L}\widetilde{H}u_{1R} - h_{2u}(\bar{Q}_{2L}\tilde{H}u_{2R} + \bar{Q}_{3L}\tilde{H}u_{3R}) + h_{3u}(\bar{Q}_{2L}\tilde{H}'u_{2R} - \bar{Q}_{3L}\tilde{H}'u_{3R})\\
&\quad + h_{4u}(\bar{Q}_{1L}\widetilde{H}''_1 u_{2R} + \bar{Q}_{1L}\widetilde{H}''_2 u_{3R}) - h_{5u}(\bar{Q}_{2L}\widetilde{H}''_2 u_{1R} + \bar{Q}_{3L}\widetilde{H}''_1 u_{1R})\\
&\quad + h_{1d}\bar{Q}_{1L}Hd_{1R} - h_{2d}(\bar{Q}_{2L}Hd_{2R} + \bar{Q}_{3L}Hd_{3R}) + h_{3d}(\bar{Q}_{2L}H'd_{2R} - \bar{Q}_{3L}H'd_{3R})\\
&\quad + h_{4d}(\bar{Q}_{1L}H''_1 d_{2R} + \bar{Q}_{1L}H''_2 d_{3R}) - h_{5d}(\bar{Q}_{2L}H''_2 d_{1R} + \bar{Q}_{3L}H''_1 d_{1R}) + \text{H.c.} \qquad (48)
\end{aligned}$$

With the VEV alignments of H, H' and H'' in Eq. (10), the mass Lagrangian of quarks reads

$$\begin{aligned}
-\mathcal{L}_q^{\text{mass}} &= h_{1u}v_H^*\bar{u}_{1L}u_{1R} + (h_{3u}v_{H'}^* - h_{2u}v_H^*)\bar{u}_{2L}u_{2R} - (h_{2u}v_H^* + h_{3u}v_{H'}^*)\bar{u}_{3L}u_{3R}\\
&\quad + h_{4u}v_{H''}^*(\bar{u}_{1L}u_{2R} + \bar{u}_{1L}u_{3R}) - h_{5u}v_{H''}^*(\bar{u}_{2L}u_{1R} + \bar{u}_{3L}u_{1R})\\
&\quad + h_{1d}v_H\bar{d}_{1L}d_{1R} + (h_{3d}v_{H'} - h_{2d}v_H)\bar{d}_{2L}d_{2R} - (h_{2d}v_H + h_{3d}v_{H'})\bar{d}_{3L}d_{3R}\\
&\quad + h_{4d}v_{H''}(\bar{u}_{1L}d_{2R} + \bar{u}_{1L}d_{3R}) - h_{5d}v_{H''}(\bar{u}_{2L}d_{1R} + \bar{u}_{3L}d_{1R}) + \text{H.c.}\\
&\equiv (\bar{u}_{1L}, \bar{u}_{2L}, \bar{u}_{3L})\mathbb{M}_u(u_{1R}, u_{2R}, u_{3R})^T + (\bar{d}_{1L}, \bar{d}_{2L}, \bar{d}_{3L})\mathbb{M}_d(d_{1R}, d_{2R}, d_{3R})^T\\
&\quad + \text{H.c.}, \qquad (49)
\end{aligned}$$

where $\mathbb{M}_u$ and $\mathbb{M}_d$ take the form:

$$\mathbb{M}_u = \mathbb{M}_u^0 + \Delta\mathbb{M}_u, \quad \mathbb{M}_d = \mathbb{M}_d^0 + \Delta\mathbb{M}_d, \qquad (50)$$

$$\mathbb{M}_u^0 = \begin{pmatrix} a_{1u} & 0 & 0 \\ 0 & -a_{2u}+a_{3u} & 0 \\ 0 & 0 & -a_{2u}-a_{3u} \end{pmatrix}, \quad \Delta\mathbb{M}_u = \begin{pmatrix} 0 & a_{4u} & a_{4u} \\ -a_{5u} & 0 & 0 \\ -a_{5u} & 0 & 0 \end{pmatrix}, \qquad (51)$$

$$\mathbb{M}_d^0 = \begin{pmatrix} a_{1d} & 0 & 0 \\ 0 & -a_{2d}+a_{3d} & 0 \\ 0 & 0 & -a_{2d}-a_{3d} \end{pmatrix}, \quad \Delta\mathbb{M}_d = \begin{pmatrix} 0 & a_{4d} & a_{4d} \\ -a_{5d} & 0 & 0 \\ -a_{5d} & 0 & 0 \end{pmatrix}, \qquad (52)$$

with

$$\begin{aligned}
&a_{iu} = h_{iu}v_H^*, \quad a_{3u} = h_{3u}v_{H'}^*, \quad a_{ku} = h_{ku}v_{H''}^*,\\
&a_{id} = h_{id}v_H, \quad a_{3d} = h_{3d}v_{H'}, \quad a_{kd} = h_{kd}v_{H''}\ (i = 1, 2; k = 4, 5). \qquad (53)
\end{aligned}$$

It is noted that, in Eqs. (51) and (52), $\mathbb{M}_{u,d}^0$ due to the contribution of two $SU(2)_L$ scalar doublet (H, H') while $\Delta\mathbb{M}_{u,d}$ come from the contribution of the $SU(2)$ scalar doublet H'' only. In the case without contribution of H'', $\Delta\mathbb{M}_{u,d}$ will vanish and the quark mass matrices $\mathbb{M}_{u,d}$ in Eq. (50) reduces to $\mathbb{M}_{u,d}^0$ and the quark mixing matrix U_{CKM} being the identity which was ruled out by the recent data. As we known, the quark mixing angles are very small, thus implying a quark mixing matrix close to the identity matrix. Hence, second terms in Eq. (50), which is

Table 5
The best-fit results for quark masses and mixing angles taken from Refs. [2,58].

m_u (MeV)	m_c (GeV)	m_t (GeV)	m_d (MeV)	m_s (MeV)	m_b (GeV)	$\sin\theta_{12}^{(q)}$	$\sin\theta_{23}^{(q)}$	$\sin\theta_{13}^{(q)}$	$\delta^{(q)}$
2.16	1.27	172.9	4.67	93.0	4.18	0.225	0.0420	0.003675	68°

proportional to the contribution of H'', will play the role of perturbation, for reproducing the realistic quark mixing. At the first order of perturbation theory, the matrices $\Delta\mathbb{M}_{u,d}$ have no effect on eigenvalues but they change the corresponding eigenvectors. The quark masses are obtained:

$$\begin{aligned} &m_u = a_{1u}, \quad m_c = -a_{2u} + a_{3u}, \quad m_t = -a_{2u} - a_{3u}, \\ &m_d = a_{1d}, \quad m_s = -a_{2d} + a_{3d}, \quad m_b = -a_{2d} - a_{3d}, \end{aligned} \tag{54}$$

and the corresponding perturbed eigenstates in quark sector are:

$$U_L^{u,d} = U_R^{u,d} = \begin{pmatrix} 1 & -\frac{a_{4(u,d)}}{\alpha_{u,d}} & -\frac{a_{4(u,d)}}{\beta_{u,d}} \\ -\frac{a_{5(u,d)}}{\alpha_{u,d}} & 1 & 0 \\ -\frac{a_{5(u,d)}}{\beta_{u,d}} & 0 & 1 \end{pmatrix}, \tag{55}$$

where $a_{l(u,d)}$ $(l = 1 \div 5)$ are given in Eq. (53) and

$$\alpha_{u,d} = a_{1(u,d)} + a_{2(u,d)} - a_{3(u,d)}, \quad \beta_{u,d} = a_{1(u,d)} + a_{2(u,d)} + a_{3(u,d)}. \tag{56}$$

The quark mixing matrix is defined as, $U_{\mathrm{CKM}} = U_L^u U_L^{d\dagger}$, which has the following entries:

$$\begin{aligned} &(U_{\mathrm{CKM}})_{11} = 1 + a_{4u}a_{4d}\left(\frac{1}{\alpha_u\alpha_d} + \frac{1}{\beta_u\beta_d}\right), \\ &(U_{\mathrm{CKM}})_{22} = 1 + \frac{a_{5u}a_{5d}}{\alpha_u\alpha_d}, \quad (U_{\mathrm{CKM}})_{33} = 1 + \frac{a_{5u}a_{5d}}{\beta_u\beta_d}, \\ &(U_{\mathrm{CKM}})_{12} = -\frac{a_{4u}}{\alpha_u} - \frac{a_{5d}}{\alpha_d}, \quad (U_{\mathrm{CKM}})_{21} = -\frac{a_{5u}}{\alpha_u} - \frac{a_{4d}}{\alpha_d}, \\ &(U_{\mathrm{CKM}})_{13} = -\frac{a_{4u}}{\beta_u} - \frac{a_{5d}}{\beta_d}, \quad (U_{\mathrm{CKM}})_{31} = -\frac{a_{5u}}{\beta_u} - \frac{a_{4d}}{\beta_d}, \\ &(U_{\mathrm{CKM}})_{23} = \frac{a_{5u}a_{5d}}{\alpha_u\beta_d}, \quad (U_{\mathrm{CKM}})_{32} = \frac{a_{5u}a_{5d}}{\beta_u\alpha_d}. \end{aligned} \tag{57}$$

The model is shown to be consistent because the experimental data on quark mass and mixing angles [2] can be fitted with the quark Yukawa coupling parameters $a_{l(u,d)}$ $(l = 1 \div 5)$. Indeed, by comparing the elements of U_{CKM} in Eq. (57) with the corresponding best fit values[7] given in Refs. [2,58] as shown in Table 5, we get a prediction for the quark mixing matrix as presented in Table 6.

The result in Table 6 shows that, between the model results and the best-fit values given in Ref. [2,58], there are significant differences for three elements (23, 32, 31) while other elements are well consistent with each other. Thus, this is an acceptable result.

[7] To obtain the best-fit values of $(U_{CKM})_{ij}^{bf}$ in Tables 6 we use the Wolfenstein parameters given by Ref. [58]: $\lambda_C = 0.2250$, $\bar{\rho} = 0.148$, $\bar{\eta} = 0.348$ and $A = 0.826$.

Table 6
The best-fit results for V_{CKM} elements taken from Refs. [2,58] and the model prediction.

Observable	Experimental best-fit value	The model prediction	Percent error (%)
$(V^{exp}_{\mathrm{CKM}})_{11}$	0.974352	0.974352	0
$(V^{exp}_{\mathrm{CKM}})_{12}$	0.224998	0.224998	0
$(V^{exp}_{\mathrm{CKM}})_{13}$	$0.00138 - 0.003341i$	$0.00323 - 0.001603i$	0.229
$(V^{exp}_{\mathrm{CKM}})_{21}$	$-0.224858 - 0.00014i$	$-0.224858 - 0.00014i$	0
$(V^{exp}_{\mathrm{CKM}})_{22}$	$0.973486 - 0.00003i$	$0.973486 - 0.00003i$	0
$(V^{exp}_{\mathrm{CKM}})_{23}$	0.0420	-0.00056	98.7
$(V^{exp}_{\mathrm{CKM}})_{31}$	$0.00811 - 0.00332i$	$-0.00318 + 0.0016i$	59.4
$(V^{exp}_{\mathrm{CKM}})^{bf}_{32}$	$-0.04123 - 0.00076i$	-0.000194	99.5
$(V^{exp}_{\mathrm{CKM}})_{33}$	0.999111	0.999996	0.0886

Next, identifying quark masses in Eq. (54) with the corresponding best-fit values given in Table 6, we get

$$\begin{aligned}
&a_{1u} = 2.16 \times 10^{6}\ \mathrm{eV}, \quad a_{2u} = -8.71 \times 10^{10}\ \mathrm{eV}, \quad a_{3u} = -8.58 \times 10^{10}\ \mathrm{eV},\\
&a_{4u} = (1.4 + 1.47i)10^{8}\ \mathrm{eV},\ a_{5u} = -(1.45 + 1.47i)10^{8}\ \mathrm{eV},\\
&a_{1d} = 4.67 \times 10^{6}\ \mathrm{eV}, \quad a_{2d} = -2.14 \times 10^{9}\ \mathrm{eV}, \quad a_{3d} = -2.04 \times 10^{9}\ \mathrm{eV},\\
&a_{4d} = -(0.977 + 1.02i)10^{7}\ \mathrm{eV},\ a_{5d} = (1.01 - 1.02i)10^{7}\ \mathrm{eV}.
\end{aligned} \tag{58}$$

5. Conclusions

We have proposed a renormalizable $B - L$ SM extension based on Q_6 symmetry which accommodates fermion mass and mixing parameters with CP violation in neutrino oscillations. Both normal and inverted neutrino mass ordering as well as the smallness of the active neutrino masses are generated through type I seesaw mechanism. The obtained physical parameters are well consistent with the global fit of neutrino oscillation [1]. The model also predicts effective neutrino mass parameters of $\langle m_{ee}\rangle = 3.23 \times 10^{-3}$ eV, $m_\beta = 8.92 \times 10^{-3}$ eV for NH and $\langle m_{ee}\rangle = m_\beta = 4.96 \times 10^{-2}$ eV for IH which are all well consistent with the recent experiments.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Q_6 group and Clebsch-Gordan coefficients

Q_6 group has the 12 elements, $a^m b^k$ for $m = 0, 1, 2, 3, 4, 5$ and $k = 0, 1$, where a and b satisfy $a^6 = e$, $b^2 = a^3$ and $b^{-1}ab = a^{-1}$. All elements of Q_6 are divided into six conjugacy classes with $\underline{1}$, $\underline{1}'$, $\underline{1}''$, $\underline{1}'''$, $\underline{2}$ and $\underline{2}'$ as its six irreducible representations. The Clebsch-Gordan coefficients for all the tensor products of Q_6 are given by:

$$\begin{aligned}
&\underline{1}(x_1)\otimes\underline{1}(y_1)=\underline{1}(x_1y_1),\quad \underline{1}(x_1)\otimes\underline{1}'(y_1)=\underline{1}'(x_1y_1),\\
&\underline{1}(x_1)\otimes\underline{1}''(y_1)=\underline{1}''(x_1y_1),\quad \underline{1}(x_1)\otimes\underline{1}'''(y_1)=\underline{1}'''(x_1y_1),\\
&\underline{1}'(x_1)\otimes\underline{1}'(y_1)=\underline{1}(x_1y_1),\quad \underline{1}'(x_1)\otimes\underline{1}''(y_1)=\underline{1}'''(x_1y_1),\\
&\underline{1}'(x_1)\otimes\underline{1}'''(y_1)=\underline{1}''(x_1y_1),\quad \underline{1}''(x_1)\otimes\underline{1}''(y_1)=\underline{1}'(x_1y_1),\\
&\underline{1}''(x_1)\otimes\underline{1}'''(y_1)=\underline{1}(x_1y_1),\quad \underline{1}'''(x_1)\otimes\underline{1}'''(y_1)=\underline{1}'(x_1y_1),
\end{aligned}\tag{A.1}$$

$$\begin{aligned}
&\underline{1}\,(x_1)\otimes\underline{2}\begin{pmatrix}y_1\\y_2\end{pmatrix}=\underline{2}\begin{pmatrix}x_1y_1\\x_1y_2\end{pmatrix},\quad \underline{1}'\,(x_1)\otimes\underline{2}\begin{pmatrix}y_1\\y_2\end{pmatrix}=\underline{2}\begin{pmatrix}x_1y_1\\x_1y_2\end{pmatrix},\\
&\underline{1}''\,(x_1)\otimes\underline{2}\begin{pmatrix}y_1\\y_2\end{pmatrix}=\underline{2}'\begin{pmatrix}x_1y_1\\x_1y_2\end{pmatrix},\quad \underline{1}'''\,(x_1)\otimes\underline{2}\begin{pmatrix}y_1\\y_2\end{pmatrix}=\underline{2}'\begin{pmatrix}x_1y_1\\x_1y_2\end{pmatrix},\\
&\underline{1}\,(x_1)\otimes\underline{2}'\begin{pmatrix}y_1\\y_2\end{pmatrix}=\underline{2}'\begin{pmatrix}x_1y_1\\x_1y_2\end{pmatrix},\quad \underline{1}'\,(x_1)\otimes\underline{2}'\begin{pmatrix}y_1\\y_2\end{pmatrix}=\underline{2}'\begin{pmatrix}x_1y_1\\x_1y_2\end{pmatrix},\\
&\underline{1}''\,(x_1)\otimes\underline{2}'\begin{pmatrix}y_1\\y_2\end{pmatrix}=\underline{2}\begin{pmatrix}x_1y_1\\x_1y_2\end{pmatrix},\quad \underline{1}'''\,(x_1)\otimes\underline{2}'\begin{pmatrix}y_1\\y_2\end{pmatrix}=\underline{2}\begin{pmatrix}x_1y_1\\x_1y_2\end{pmatrix},
\end{aligned}\tag{A.2}$$

$$\begin{aligned}
&\underline{2}(x_1,x_2)\otimes\underline{2}(y_1,y_2)=\underline{1}(x_1y_2-x_2y_1)\oplus\underline{1}'(x_1y_2+x_2y_1)\oplus\underline{2}'\begin{pmatrix}x_1y_1\\x_2y_2\end{pmatrix},\\
&\underline{2}(x_1,x_2)\otimes\underline{2}'(y_1,y_2)=\underline{1}''(x_1y_1-x_2y_2)\oplus\underline{1}'''(x_1y_1+x_2y_2)\oplus\underline{2}\begin{pmatrix}x_1y_2\\x_2y_1\end{pmatrix},\\
&\underline{2}'(x_1,x_2)\otimes\underline{2}'(y_1,y_2)=\underline{1}(x_1y_2+x_2y_1)\oplus\underline{1}'(x_1y_2-x_2y_1)\oplus\underline{2}'\begin{pmatrix}x_1y_1\\x_2y_2\end{pmatrix},
\end{aligned}\tag{A.3}$$

where x_i and y_j are the multiplet components of different representations.

The rules to conjugate of all the representations of Q_6 are given by

$$\underline{1}^*(1^*)=\underline{1}(1^*),\ \underline{1}'^*(1^*)=\underline{1}'(1^*),\ \underline{1}''^*(1^*)=\underline{1}'''(1^*),\ \underline{1}'''^*(1^*)=\underline{1}''(1^*),\tag{A.4}$$

$$\underline{2}^*(1^*,2^*)=\underline{2}(-2^*,1^*),\ \underline{2}'^*(1^*,2^*)=\underline{2}'(1^*,2^*),\tag{A.5}$$

where, for example, $\underline{2}^*(1^*,2^*)$ denotes some $\underline{2}^*$ multiplet of the form $(x_1^*,x_2^*)\sim\underline{2}^*$.

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