

Massive Nambu–Goldstone fermions and bosons for non-relativistic superconformal symmetry: Jackiw–Pi vortices in a trap

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We discuss a supersymmetric extension of a non-relativistic Chern–Simons matter theory, known as the supersymmetric Jackiw–Pi model, in a harmonic trap. We show that the non-relativistic version of the superconformal symmetry, called the super-Schrödinger symmetry, is not spoiled by an external field including the harmonic potential. It survives as a modified symmetry whose generators have explicit time dependences determined by the strength of the trap, the rotation velocity of the system, and the fermion number chemical potential. We construct $1/3$ Bogomol’nyi–Prasad–Sommerfield (BPS) states of trapped Jackiw–Pi vortices preserving part of the modified superconformal symmetry and discuss fluctuations around static BPS configurations. In addition to the bosonic massive Nambu–Goldstone modes, we find that there exist massive Nambu–Goldstone fermions associated with broken generators of the modified super-Schrödinger symmetry. Furthermore, we find that eigenmodes form supermultiplets of a modified supersymmetry preserved by the static BPS backgrounds. As a consequence of the modified supersymmetry, infinite towers of explicit spectra can be found for eigenmodes corresponding to bosonic and fermionic lowest Landau levels.
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1. Introduction

Non-trivial external background fields are useful tools to study various aspects of field theories. When a generic background field is turned on in a physical system, it may break a symmetry of the system and drastically change the structure of the model. However, it has been shown that if an external field can be viewed as a chemical potential term associated with a conserved charge, a version of the Nambu–Goldstone (NG) theorem can still be applied even when a symmetry appears explicitly broken by the external field. The crucial difference from the standard NG theorem is that the corresponding NG mode in this case has a non-vanishing mass precisely determined by the symmetry algebra. Such *massive Nambu–Goldstone bosons* have been discussed in Refs. [1–4], and the scattering amplitudes of massive NG modes were recently studied in Ref. [5].

In Refs. [6–10], various properties of the massive NG modes associated with the non-relativistic conformal symmetry, called the Schrödinger symmetry [11,12], have been revealed in the $(2+1)$ -dimensional non-linear Schrödinger system in a harmonic trap. One of the most important observations is that the Schrödinger symmetry survives even in the presence of external background fields including the harmonic potential. More precisely, a modified Schrödinger symmetry generated by time-dependent operators remains in such a background. In general, when a symmetry generated by an operator with an explicit time dependence is spontaneously broken, the associated NG

modes have a non-vanishing mass determined by the commutation relation between the corresponding broken generator and the Hamiltonian. As in the case of the Lorentz and Galilean symmetry, a time-dependent symmetry can be used to study the dynamical properties of the system. For example, in the non-linear Schrödinger system in a harmonic trap, time-dependent solutions can be generated from static ones by applying the time-dependent modified Schrödinger symmetry.

In this paper we discuss the supersymmetric Jackiw–Pi model and study vortices and massive NG modes in a non-trivial background. The Jackiw–Pi model is a field-theoretic framework describing anyons in terms of the non-linear Schrödinger system coupled with a Chern–Simons gauge field [13]. As with the standard non-linear Schrödinger model, the Jackiw–Pi model has a modified (time-dependent) Schrödinger symmetry in various backgrounds. Non-topological vortex solutions, called Jackiw–Pi vortices [14,15], have been discussed in such backgrounds [16–22], and in particular time-dependent solutions were constructed by making use of maps between the models with and without the external fields.

The Jackiw–Pi model without a background field has a supersymmetric extension which possesses a non-relativistic superconformal symmetry, called the super-Schrödinger symmetry [23–25]. In this paper we show that external background fields corresponding to the harmonic potential, the spatial rotation, and the flavor and fermion number chemical potentials do not spoil the superconformal symmetry as well as the Schrödinger symmetry. In the presence of such external fields, the whole super-Schrödinger symmetry becomes a time-dependent symmetry of the type which has been discussed in the context of the supersymmetric harmonic oscillator in quantum mechanics [26,27].

We also discuss Jackiw–Pi vortices in the non-trivial background fields and construct their $1/3$ Bogomol’nyi–Prasad–Sommerfield (BPS) states, which are invariant under part of the time-dependent supersymmetry. The moduli matrix formalism, which has been used to describe the moduli space of non-Abelian vortices [28–33], can also be applied to write down a formal solution of the $1/3$ BPS equation in this system. For each choice of a holomorphic matrix $H_0(z)$ we can obtain a BPS configuration of trapped Jackiw–Pi vortices by solving the Gauss law equation. Generic $1/3$ BPS solutions turn out to be Q-soliton-like configurations, that is, they are time-dependent stationary configurations stabilized by conserved charges. They are new time-dependent solutions which are different from the known solutions obtained by using the maps between the models with and without the external fields [16–21].

The BPS solutions become static configurations if $H_0(z)$ takes one of special forms corresponding to the fixed points of the spatial and flavor rotation. We discuss fluctuations around them and show that bosonic and fermionic eigenmodes form supermultiplets of the unbroken time-dependent supersymmetry. There are two types of supermultiplets: one is a generic supermultiplet composed of a pair of bosonic and fermionic modes; the other is a short supermultiplet consisting only of a bosonic component. In particular, we show that in addition to bosonic massive NG modes associated with spontaneously broken generators of the modified Schrödinger symmetry, there exist *massive Goldstinos* corresponding to spontaneously broken modified supercharges. They consistently form supermultiplets, as expected from the super-Schrödinger algebra. In addition to those massive NG modes, we exactly derive eigenvalue spectra of infinite towers of short and long supermultiplets corresponding to the bosonic and fermionic lowest Landau levels, respectively.

The organization of the paper is as follows. In Sect. 2, we briefly review the super-Schrödinger symmetry in the supersymmetric Jackiw–Pi model and show that there exists a modified super-Schrödinger symmetry even in the presence of generalized chemical potential terms including the harmonic potential. In Sect. 3, we discuss $1/3$ BPS solutions of trapped non-Abelian Jackiw–Pi

vortices which preserve part of the modified superconformal symmetry. By applying the moduli matrix formalism, we write down formal solutions and show that static configurations correspond to fixed points of the rotation and flavor symmetry. In Sect. 4, we investigate fluctuations around static BPS backgrounds and elucidate the structure of supermultiplets of eigenmodes, including bosonic and fermionic massive NG modes. Section 5 is devoted to a summary and discussions. In Appendix A, the generalized chemical potential, modified symmetry, and massive NG mode are reviewed, and an example is shown in the free Schrödinger system in Appendix B.

2. Supersymmetric Jackiw–Pi model in a harmonic trap

2.1. SUSY Jackiw–Pi model and super-Schrödinger symmetry

The supersymmetric (SUSY) Jackiw–Pi model consists of a gauge field A_μ and pairs of bosonic matter fields ϕ_I and fermionic matter fields ψ_I . For simplicity, we consider the case of a $U(N)$ gauge field A_μ with N_F matter pairs (ϕ_I, ψ_I) ($I = 1, \dots, N_F$) in the (N, N_F) representation of the $U(N)$ gauge group and the $SU(N_F)$ flavor symmetry. It would be straightforward to extend the following discussion to more general settings. By using $N \times N_F$ matrix notation for the matter fields,

$$\phi \equiv (\phi_1, \phi_2, \dots, \phi_{N_F}), \quad \psi \equiv (\psi_1, \psi_2, \dots, \psi_{N_F}), \quad (2.1)$$

the action of the supersymmetric Jackiw–Pi model can be written as

$$S = \int dt d^2x \text{Tr} \left[\phi^\dagger \hat{\Delta}_0 \phi + \frac{1}{m} \psi^\dagger \hat{\Delta}_0 \psi - \frac{\pi}{km} M^2 - \frac{\pi}{km^2} \psi^\dagger Y \psi \right] + k S_{\text{CS}}, \quad (2.2)$$

where the trace is taken over the flavor indices. Just for notational convenience, we have introduced the $N_F \times N_F$ matrix M and the $N \times N$ matrix Y defined by

$$M \equiv \phi^\dagger \phi + \frac{1}{m} \psi^\dagger \psi, \quad Y \equiv \phi \phi^\dagger - \frac{1}{m} \psi \psi^\dagger + \frac{k}{\pi} i F_{z\bar{z}}. \quad (2.3)$$

The symbol $\hat{\Delta}_0$ denotes the differential operator which gives the standard non-relativistic kinetic term

$$\hat{\Delta}_0 \equiv i \mathcal{D}_t + \frac{1}{m} (\mathcal{D}_z \mathcal{D}_{\bar{z}} + \mathcal{D}_{\bar{z}} \mathcal{D}_z), \quad (2.4)$$

where $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$ are the complex coordinates on the two-dimensional plane. The covariant derivative and the field strength are defined by $\mathcal{D}_\mu \phi \equiv (\partial_\mu + i A_\mu) \phi$, $\mathcal{D}_\mu \phi^\dagger \equiv \partial_\mu \phi^\dagger - i \phi^\dagger A_\mu$ and $F_{\mu\nu} \equiv -i[\mathcal{D}_\mu, \mathcal{D}_\nu]$, etc.

The parameter k is the Chern–Simons level and S_{CS} is the Chern–Simons term normalized as

$$S_{\text{CS}} \equiv \frac{1}{4\pi} \int \text{tr} \left[A \wedge dA + \frac{2i}{3} A \wedge A \wedge A \right]. \quad (2.5)$$

By rescaling the gauge field as $A_\mu \rightarrow A_\mu/k$, we can see that in the infinite level limit $k \rightarrow \infty$, this model reduces to the free theory whose equations of motion are given by the Schrödinger equation. In addition to the standard Schrödinger symmetry (see Appendix B for the details of the Schrödinger symmetry), the action is invariant under the non-relativistic version of superconformal symmetry, namely the super-Schrödinger symmetry. We can show that this system has the same symmetry as the free supersymmetric Schrödinger system even for finite k .

Table 1. Generators of the super-Schrödinger symmetry. The generators above and below the dashed line are bosonic and fermionic operators, respectively.

• H : time translation	• P_i : translation
• J : rotation	• D : dilatation
• B^i : Galilean symmetry	• C : special Schrödinger symmetry
• $\mathcal{N}$: central charge: phase rotation	• $\mathcal{N}_f$: fermion number symmetry

• Q, q, S : supersymmetry	

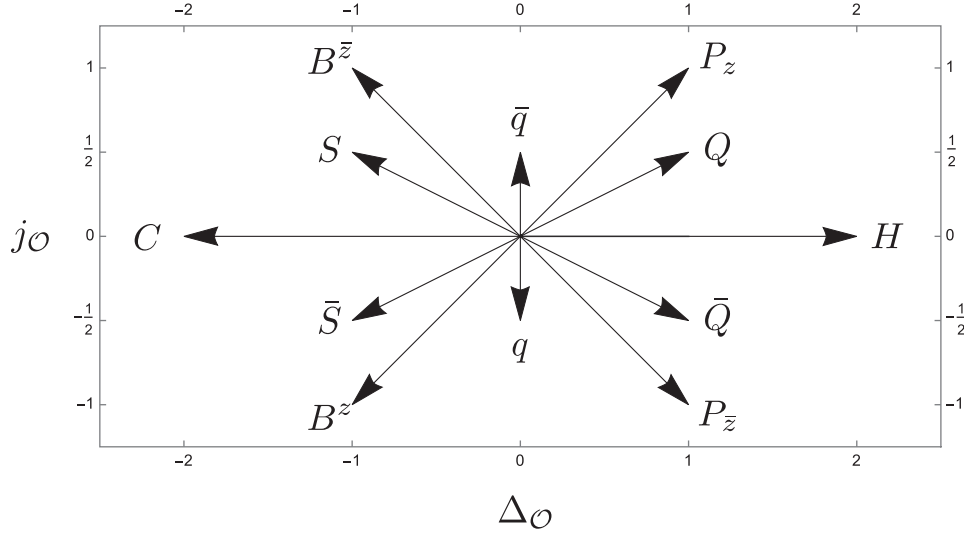


Fig. 1. $(\Delta_{\mathcal{O}}, j_{\mathcal{O}})$ of the generators.

2.1.1. Super-Schrödinger algebra

The generators of the super-Schrödinger symmetry are summarized in Table 1. The non-vanishing bosonic part of their commutation relation is given by

$$[H, B^z] = -iP_z, \quad [H, C] = iD, \quad [P_z, C] = -iB^z, \quad [P_z, B^{\bar{z}}] = -2im\mathcal{N}, \quad (2.6)$$

$$[J, \mathcal{O}] = -j_{\mathcal{O}}\mathcal{O}, \quad [D, \mathcal{O}] = -i\Delta_{\mathcal{O}}\mathcal{O}, \quad [\mathcal{N}_f, \mathcal{O}] = -q_f^{\mathcal{O}}\mathcal{O}, \quad (2.7)$$

where $B^z = B_1 + iB_2$, $B^{\bar{z}} = B_1 - iB_2$, $P_z = (P_1 - iP_2)/2$, and $P_{\bar{z}} = (P_1 + iP_2)/2$. The symbol $\mathcal{O}$ denotes any eigenoperator of $(J, D, \mathcal{N}_f)$. The eigenvalues of (J, D) are summarized in Fig. 1, and the fermion numbers are $q_{\mathcal{O}} = 1$ for (Q, q, S) and $q_{\mathcal{O}} = 0$ for the bosonic operators. The non-vanishing part of the commutation relation containing the fermionic generators is given by

$$\{Q, \bar{Q}\} = 2H, \quad \{Q, \bar{q}\} = P_z, \quad \{Q, \bar{S}\} = D + iJ - \frac{3}{2}i\mathcal{N}_f, \quad (2.8)$$

$$\{q, \bar{q}\} = m\mathcal{N}, \quad \{q, \bar{S}\} = -B^z, \quad \{S, \bar{S}\} = 2C, \quad (2.9)$$

$$[H, S] = iQ, \quad [C, Q] = -iS, \quad [P_z, S] = [B^z, Q] = 2iq. \quad (2.10)$$

2.1.2. Bosonic part of super-Schrödinger symmetry

Let ξ^μ be the non-relativistic version of the conformal Killing vector,

$$\xi^t = \varepsilon_H + 2\varepsilon_D t - \varepsilon_C t^2, \quad \xi^z = -2(\varepsilon_P + \varepsilon_{\bar{B}} t) + (\varepsilon_D - \varepsilon_C t + i\varepsilon_J)z, \quad \xi^{\bar{z}} = \bar{\xi}^{\bar{z}}, \quad (2.11)$$

where ε_O are transformation parameters. Then the bosonic part of the super-Schrödinger transformations takes the form

$$\delta\phi = [\xi^\mu \mathcal{D}_\mu + \lambda + i\alpha]\phi, \quad \delta\psi = [\xi^\mu \mathcal{D}_\mu + \lambda + i\alpha + i\varepsilon_f]\psi, \quad (2.12)$$

$$\delta\phi^\dagger = [\xi^\mu \mathcal{D}_\mu + \lambda - i\alpha]\phi^\dagger, \quad \delta\psi^\dagger = [\xi^\mu \mathcal{D}_\mu + \lambda - i\alpha - i\varepsilon_f]\psi^\dagger, \quad (2.13)$$

$$\delta A_\mu = \xi^\nu F_{\nu\mu}, \quad (2.14)$$

where the real functions λ and α are given by

$$\lambda = \varepsilon_D - \varepsilon_C t, \quad \alpha = \varepsilon_N + m z \varepsilon_B + m \bar{z} \varepsilon_{\bar{B}} + \frac{m}{2} |z|^2 \varepsilon_C. \quad (2.15)$$

2.1.3. Fermionic part of super-Schrödinger symmetry

To see the invariance of the action under supersymmetry, let us first note that the following transformation does not change the action:

$$\delta\phi = \frac{1}{m} (m\zeta_q - 2i\zeta_Q \mathcal{D}_z) \psi, \quad \delta\psi = -(m\bar{\zeta}_q - 2i\bar{\zeta}_Q \mathcal{D}_{\bar{z}}) \phi, \quad (2.16)$$

$$\delta\phi^\dagger = -\frac{1}{m} (m\bar{\zeta}_q + 2i\bar{\zeta}_Q \mathcal{D}_{\bar{z}}) \psi^\dagger, \quad \delta\psi^\dagger = -(m\zeta_q + 2i\zeta_Q \mathcal{D}_z) \phi^\dagger, \quad (2.17)$$

$$\delta A_z = -\frac{2\pi}{km} \bar{\zeta}_Q \phi \psi^\dagger, \quad \delta A_{\bar{z}} = \frac{2\pi}{km} \zeta_Q \psi \phi^\dagger, \quad (2.18)$$

$$\delta A_t = \frac{\pi}{km^2} (m\zeta_q + 2i\zeta_Q \mathcal{D}_z) \psi \phi^\dagger + (h.c.), \quad (2.19)$$

where ζ_q and ζ_Q are fermionic SUSY transformation parameters corresponding to the supercharges q and Q , respectively. Actually, there exists one more supersymmetry generated by the supercharge S whose transformation law can be obtained from Eqs. (2.16)–(2.19) by promoting the transformation parameters ζ_q and ζ_Q into the following functions depending on the coordinates $(t, z, \bar{z})$:

$$\zeta_q = \varepsilon_q - \bar{z} \varepsilon_S, \quad \zeta_Q = \varepsilon_Q + t \varepsilon_S, \quad (2.20)$$

where $(\varepsilon_q, \varepsilon_Q, \varepsilon_S)$ are transformation parameters corresponding to the supercharges (q, Q, S) :

$$\delta = \varepsilon_q q + \varepsilon_Q Q + \varepsilon_S S + (h.c.). \quad (2.21)$$

2.2. Harmonic trap and modified super-Schrödinger symmetry

Now let us put the SUSY Jackiw–Pi system in a harmonic trap. The harmonic potential term can be introduced by adding the Noether charge C (corresponding to the special Schrödinger transformation) to the Hamiltonian. As shown in Appendix A, a Hamiltonian with such *generalized* chemical potential terms possesses a *modified* symmetry even though the original symmetry appears explicitly broken.

In the present case, the generalized chemical potential terms can be turned on by introducing the following external gauge field A_μ^{ex} as $A_\mu \rightarrow A_\mu + A_\mu^{\text{ex}}$:

$$A_\mu^{\text{ex}} dx^\mu = \frac{i}{2} m \tilde{\omega} (\bar{z} dz - z d\bar{z}) + \left[\frac{m}{2} (\omega^2 - \tilde{\omega}^2) |z|^2 - \mu_f \hat{\mathcal{N}}_f - \mu_a \hat{\mathcal{N}}_a \right] dt, \quad (2.22)$$

where $\hat{\mathcal{N}}_f$ is the fermion number operator,

$$\hat{\mathcal{N}}_f \psi = \psi, \quad \hat{\mathcal{N}}_f \phi = 0, \quad (2.23)$$

and $\hat{\mathcal{N}}_a$ ($a = 1, \dots, N_F$) are the flavor number operators:

$$\hat{\mathcal{N}}_a \phi_b = \delta_{ab} \phi_b, \quad \hat{\mathcal{N}}_a \psi_b = \delta_{ab} \psi_b. \quad (2.24)$$

The parameters $(\omega, \tilde{\omega}, \mu_f, \mu_a)$ correspond to the following *generalized* chemical potentials:

- ω : the strength of the harmonic trap
- $\tilde{\omega}$: the angular velocity of the rotation
- μ_f : the fermion number chemical potential
- μ_a : the flavor symmetry chemical potential.

In the presence of the external gauge fields, the differential operator $\hat{\Delta}_0$ in the kinetic terms is replaced by the differential operator $\hat{\Delta}$ obtained by replacing the covariant derivatives with those with the external field

$$\begin{aligned} \hat{\Delta} &\equiv i\tilde{\mathcal{D}}_t + \frac{1}{m} \left(\tilde{\mathcal{D}}_z \tilde{\mathcal{D}}_{\bar{z}} + \tilde{\mathcal{D}}_{\bar{z}} \tilde{\mathcal{D}}_z \right) \\ &= \hat{\Delta}_0 - \tilde{\omega} (z\mathcal{D}_z - \bar{z}\mathcal{D}_{\bar{z}}) - \frac{m\omega^2}{2} |z|^2 + \mu_f \hat{\mathcal{N}}_f + \sum_{a=1}^{N_F} \mu_a \hat{\mathcal{N}}_a, \end{aligned} \quad (2.25)$$

where $\tilde{\mathcal{D}}_\mu$ denotes the covariant derivative including the external field,

$$\tilde{\mathcal{D}}_\mu \phi \equiv (\partial_\mu + iA_\mu + iA_\mu^{\text{ex}}) \phi, \text{ etc.} \quad (2.26)$$

Since the differential operator $\hat{\Delta}$ does not commute with some generators of the super-Schrödinger transformation, it appears that the part of the super-Schrödinger symmetry including the supersymmetry is broken in the Jackiw–Pi system in the harmonic trap. Although the original super-Schrödinger transformation is no longer a symmetry of the action, there exists a modified super-Schrödinger symmetry even in the presence of the external field. By using the general method explained in Appendix A, we can find the following modified super-Schrödinger symmetry (see Eqs. (B.19)–(B.23) in Appendix B for the explicit forms of the symmetry transformations).

2.2.1. Bosonic part of modified super-Schrödinger symmetry

To write down the bosonic part of the modified super-Schrödinger symmetry it is convenient to introduce $\eta_I(t)$ defined by the following differential equations:

$$i\partial_t(\eta_{\bar{B}} \mp i\omega\eta_P) = (\tilde{\omega} \pm \omega)(\eta_{\bar{B}} \mp i\omega\eta_P), \quad i\partial_t(\eta_C + 2i\omega\eta_D) = 2\omega(\eta_C + 2i\omega\eta_D), \quad (2.27)$$

$$\partial_t\eta_J = -\tilde{\omega}\partial_t\eta_H = -2\tilde{\omega}\eta_D, \quad \partial_t\eta_N = \partial_t\eta_f = 0. \quad (2.28)$$

By using these functions, “the non-relativistic conformal Killing vector ξ^μ for the modified Schrödinger symmetry” can be written as

$$\xi^t = \eta_H, \quad \xi^z = -2\eta_P + (\eta_D + i\eta_J)z, \quad \xi^{\bar{z}} = -2\eta_{\bar{P}} + (\eta_D - i\eta_J)\bar{z}. \quad (2.29)$$

Then the bosonic part of the modified super-Schrödinger transformations takes the form

$$\delta\phi = [\xi^\mu \mathcal{D}_\mu + \lambda + i\alpha]\phi, \quad \delta\psi = [\xi^\mu \mathcal{D}_\mu + \lambda + i\alpha + i\eta_f]\psi, \quad (2.30)$$

$$\delta\phi^\dagger = [\xi^\mu \mathcal{D}_\mu + \lambda - i\alpha]\phi^\dagger, \quad \delta\psi^\dagger = [\xi^\mu \mathcal{D}_\mu + \lambda - i\alpha - i\eta_f]\psi^\dagger, \quad (2.31)$$

$$\delta A_\mu = \xi^\nu F_{\nu\mu}, \quad (2.32)$$

with $\lambda = \eta_D$ and

$$\alpha = \eta_{\mathcal{N}} + m(\eta_B - i\tilde{\omega}\eta_{\bar{P}})z + m(\eta_{\bar{B}} + i\tilde{\omega}\eta_P)\bar{z} + m\left[\frac{1}{2}\eta_C + \tilde{\omega}\eta_J - \frac{1}{2}(\omega^2 - \tilde{\omega}^2)\eta_H\right]|z|^2. \quad (2.33)$$

These are essentially the same as the modified Schrödinger symmetry in the free system (see Appendix B for the explicit forms of the symmetry transformations). By appropriately identifying the integration constants of the differential equations in Eqs. (2.27) and (2.28) with the transformation parameters $\varepsilon_{\mathcal{O}}$, we can confirm that this transformation reduces to the standard Schrödinger symmetry when the chemical potential terms are turned off ($\omega = \tilde{\omega} = 0$).

It is worth noting that the $SU(N_F)$ flavor symmetry is also not broken but modified as

$$\delta\phi = i\phi \mathfrak{T}(t), \quad \delta\psi = i\psi \mathfrak{T}(t), \quad (2.34)$$

where $\mathfrak{T}(t)$ denotes a time-dependent generator of $SU(N_F)$ such that

$$i\partial_t \mathfrak{T}(t) = [\mathcal{M}, \mathfrak{T}(t)], \quad \mathfrak{T}(0) \in \mathfrak{su}(N_F), \quad \mathcal{M} \equiv \text{diag}(\mu_1, \dots, \mu_{N_F}). \quad (2.35)$$

2.2.2. Fermionic part of modified super-Schrödinger symmetry

The fermionic part of the modified super-Schrödinger transformation takes the same form as the unmodified one in Eqs. (2.16)–(2.19) if the covariant derivative is promoted as $\mathcal{D}_\mu \rightarrow \tilde{\mathcal{D}}_\mu$ and ζ_q and ζ_Q are replaced with the functions satisfying the differential equation

$$\begin{pmatrix} i\partial_t - \mu_f & i\bar{z}(\tilde{\omega}^2 - \omega^2) \\ i\partial_{\bar{z}} & i\partial_t - \mu_f - 2\tilde{\omega} \end{pmatrix} \begin{pmatrix} \zeta_q \\ \zeta_Q \end{pmatrix} = 0, \quad \partial_z \zeta_Q = \partial_{\bar{z}} \zeta_Q = \partial_z \zeta_q = 0. \quad (2.36)$$

The general solution takes the form

$$\zeta_q = e^{-i\mu_f t} \varepsilon_q - e^{-i(\mu_f + 2\tilde{\omega})t} \bar{z} f'(t), \quad \zeta_Q = e^{-i(\mu_f + 2\tilde{\omega})t} f(t), \quad (2.37)$$

where the function $f(t)$ is given by

$$f(t) = \frac{1}{2} \left[\left(\varepsilon_Q + \frac{i\varepsilon_S}{\omega} \right) e^{i(\tilde{\omega} - \omega)t} + \left(\varepsilon_Q - \frac{i\varepsilon_S}{\omega} \right) e^{i(\tilde{\omega} + \omega)t} \right], \quad (2.38)$$

and $f'(t)$ is the time derivative of $f(t)$. The integration constants ($\varepsilon_q, \varepsilon_Q, \varepsilon_S$), which can be interpreted as the transformation parameters of the modified symmetry, are chosen so that ζ_q and ζ_Q reduce to the original forms in Eq. (2.20) in the limit $\omega, \tilde{\omega}, \mu_f \rightarrow 0$.

By applying these transformations to the action with the generalized chemical potential terms, we can explicitly check that the action is invariant under these modified symmetries.¹

3. 1/3 BPS equation and Jackiw–Pi vortices

3.1. 1/3 BPS condition

Since the SUSY Jackiw–Pi system has the modified super-Schrödinger symmetry even in the harmonic trap, it is possible to consider BPS states of the Jackiw–Pi vortices [14], which preserve part of the modified super-Schrödinger symmetry. A BPS condition can be obtained by requiring $\delta\psi = 0$ for each choice of the transformation parameters $(\varepsilon_q, \varepsilon_Q, \varepsilon_S)$. We can obtain a BPS equation with no explicit time dependence by setting

$$\varepsilon_q = 0, \quad \varepsilon_S = -i\omega\epsilon_Q. \quad (3.1)$$

By using the differential operators ∇_z and $\nabla_{\bar{z}}$ defined by

$$\nabla_z \equiv \mathcal{D}_z - \frac{1}{2}m\omega\bar{z}, \quad \nabla_{\bar{z}} \equiv \mathcal{D}_{\bar{z}} + \frac{1}{2}m\omega z \quad (3.2)$$

we can write the BPS equation corresponding to Eq. (3.1) as

$$\nabla_{\bar{z}}\phi = 0. \quad (3.3)$$

Any solution of this BPS solution satisfies the full set of equations of motion if the following first-order differential equations are also satisfied:

$$iF_{z\bar{z}} + \frac{\pi}{k}\phi\phi^\dagger = 0, \quad \nabla_t\phi = 0, \quad (3.4)$$

where we have defined

$$\nabla_t \equiv \mathcal{D}_t + \frac{\pi i}{km}\phi\phi^\dagger + i(\omega - \tilde{\omega})(z\mathcal{D}_z - \bar{z}\mathcal{D}_{\bar{z}}) + i\sum_{a=1}^{N_F}(\omega - \mu_a)\hat{\mathcal{N}}_a. \quad (3.5)$$

In the following, we consider field configurations satisfying the set of equations in Eqs. (3.3) and (3.4) with asymptotic behaviors²

$$\phi \rightarrow 0, \quad F_{\mu\nu} \rightarrow 0. \quad (3.6)$$

¹ Although it is possible to directly show the invariance of the action by using the explicit form of the transformation, we can bypass the complicated computation by performing the coordinate transformation

$$x^\mu = (t, z, \bar{z}) \rightarrow X^M = (T, Z, \bar{Z}) \text{ with } T = \frac{1}{\omega} \tan \omega(t - t_0), \quad Z = \sqrt{T'(t)} e^{i\tilde{\omega}t} z, \quad \bar{Z} = \sqrt{T'(t)} e^{-i\tilde{\omega}t} \bar{z}$$

and the field redefinition $(A_\mu, \phi, \psi) \rightarrow (A_M, \Phi, \Psi)$ given by $A_\mu dx^\mu = A_M dX^M$ and

$$\phi(t, z, \bar{z}) = \sqrt{T'(t)} e^{i\hat{\alpha}} \Phi(T, Z, \bar{Z}), \quad \psi(t, z, \bar{z}) = \sqrt{T'(t)} e^{i\hat{\alpha}} \Psi(T, Z, \bar{Z}),$$

with $\hat{\alpha} = (\mu_f \hat{\mathcal{N}}_f + \mu_a \hat{\mathcal{N}}_a)t - \frac{m\omega^2}{2}|z|^2 T$. For an arbitrary value of $t_0 \in \mathbb{R}$, the action expressed in terms of the fields (A_M, Φ, Ψ) and the coordinates $(T, Z, \bar{Z})$ takes the original form in Eq. (2.2), and the modified super-Schrödinger symmetry reduces to the original one. Then we can check the invariance of the Lagrangian (up to total derivative) in each coordinate patch by choosing the parameter $t_0 \in \mathbb{R}$ in such a way that the singularities of the coordinate transformation at $t = t_0 + \frac{\pi}{\omega}(n + 1/2)$ ($n \in \mathbb{Z}$) are not contained in the patch. Note that the super-Schrödinger symmetry itself is non-singular everywhere.

² Since the generalized chemical potential terms introduced to deform the action can be regarded as the external gauge field $A_\mu \rightarrow A_\mu + A_\mu^{\text{ex}}$, the solution of the deformed equation of motion A_μ^{sol} can be mapped to

3.2. General BPS solution

To write down the general solution of Eqs. (3.3) and (3.4) it is convenient to introduce an arbitrary $N \times N_F$ holomorphic matrix $H_0(t, z)$, called the moduli matrix [28,29]. By using $H_0(t, z)$, we can formally solve Eqs. (3.3) and (3.4) as

$$\phi = S^{-1} H_0(t, z) \text{diag}(e^{i\mu_1 t}, e^{i\mu_2 t}, \dots, e^{i\mu_{N_F} t}), \quad (3.7)$$

$$A_{\bar{z}} = -iS^{-1} \partial_{\bar{z}} S + \frac{i}{2} m \omega z, \quad (3.8)$$

$$A_t = -\frac{\pi}{km} \phi \phi^\dagger + i(\omega - \tilde{\omega})(z A_z - \bar{z} A_{\bar{z}}) - \omega, \quad (3.9)$$

where the $N \times N$ matrix $S(t, z, \bar{z})$ is an element of the complexified gauge group $U(N)^\mathbb{C} \cong GL(N, \mathbb{C})$ satisfying

$$\partial_{\bar{z}}(\partial_z \Omega \Omega^{-1}) = m\omega - \frac{\pi}{k} H_0 H_0^\dagger \Omega^{-1}, \quad \Omega \equiv S S^\dagger. \quad (3.10)$$

This equation ensures that the Gauss law $iF_{z\bar{z}} + \frac{\pi}{k} \phi \phi^\dagger = 0$ is satisfied. The BPS equation $\nabla_{\bar{z}} \phi = 0$, which can be rewritten as

$$\partial_{\bar{z}} H_0(t, z) = 0, \quad (3.11)$$

is automatically satisfied for an arbitrary choice of the holomorphic matrix $H_0(t, z)$. The remaining equation $\nabla_t \phi = 0$ determines the time dependence of the solution as

$$\left[\partial_t + i(\omega - \tilde{\omega})(z \partial_z - \bar{z} \partial_{\bar{z}}) \right] S = \left[\partial_t + i(\omega - \tilde{\omega})(z \partial_z - \bar{z} \partial_{\bar{z}}) \right] H_0 = 0. \quad (3.12)$$

It follows from this equation that S and H_0 have no explicit t dependence if they are written in terms of the coordinates z_* and $\bar{z}_*$ defined as

$$z_* \equiv e^{i(\tilde{\omega} - \omega)t} z, \quad \bar{z}_* \equiv e^{-i(\tilde{\omega} - \omega)t} \bar{z}. \quad (3.13)$$

This implies that the whole system is rotating in the z -plane with angular velocity $\tilde{\omega} - \omega$. By solving Eq. (3.10) for Ω , physical quantities such as energy density profiles can be explicitly obtained for an arbitrarily chosen $H_0(z_*)$.

Note that for $N = N_F = 1$, Eq. (3.10) can be rewritten into the vortex equation classified as follows [34]:

$m\omega = 0, k < 0$	Jackiw–Pi	[14]
$m\omega > 0, k < 0$	Ambjørn–Olesen	[35,36]
$m\omega > 0, k > 0$	Taubes	[37]
$m\omega > 0, k = \infty$	Bradlow	[38]
$m\omega < 0, k < 0$	Popov	[39]

a solution of the original equation of motion $A_\mu^o = A_\mu^{\text{sol}} + A_\mu^{\text{ex}}$. Nevertheless, the original system with boundary condition $A_\mu^o \rightarrow 0$ and the deformed system with the boundary condition in Eq. (3.6) are inequivalent since Eq. (3.6) corresponds to the boundary condition $A_\mu^o \rightarrow A_\mu^{\text{ex}}$.

Although Eq. (3.10) has an identical form to the vortex equation, the boundary conditions are different and consequently the vortices in our setup have some distinctive physical properties.

The stability of the solution is guaranteed by the conserved charges associated with the spatial rotation and the internal phase rotation. We can show that for given values of the Noether charges, the energy of the system is bounded from below as

$$E \geq (\omega - \tilde{\omega})J + \sum_{a=1}^{N_F} (\omega - \mu_a)\mathcal{N}_a, \quad (3.14)$$

where J and $\mathcal{N}_a$ are the angular momentum and the flavor symmetry Noether charges,

$$J = \int d^2x \phi_a^\dagger (z\mathcal{D}_z - \bar{z}\mathcal{D}_{\bar{z}})\phi_a, \quad \mathcal{N}_a = \int d^2x \phi_a^\dagger \phi_a \text{ (no sum over } a). \quad (3.15)$$

The BPS solution saturates this lower bound for the energy. This is an example of Q solitons, that is, solitons which are stabilized by Noether charges.

It is worth noting that we can obtain more general solutions of the equations of motion (breathing solutions, etc.) by applying the modified Schrödinger transformation to the BPS configurations discussed in this section. Such solutions preserve different combination of the supercharges and satisfy a certain time-dependent BPS equations.

3.3. Static BPS solution

Although a generic BPS configuration is a stationary solution which depends on time t , the solution in Eqs. (3.7)–(3.9) becomes static if H_0 is chosen so that the resulting scalar field ϕ is invariant under the rotation and the flavor transformations, that is,

$$\hat{J}\phi = \hat{\mathcal{N}}_a\phi = 0 + \{\text{infinitesimal gauge transformation}\}. \quad (3.16)$$

By appropriately fixing the gauge, the static solution with, e.g., $\mu_a \hat{\mathcal{N}}_a\phi = \boldsymbol{\mu}\phi \equiv \text{diag}(\mu_1, \dots, \mu_N)\phi$, can be written as

$$\phi = (e^{-\frac{1}{2}\boldsymbol{\sigma}zL} | \mathbf{0}_{N \times (N_F - N)}), \quad (3.17)$$

$$A_{\bar{z}} = -\frac{i}{2}\partial_{\bar{z}}(\boldsymbol{\sigma} - m\omega|z|^2), \quad (3.18)$$

$$A_t = -\frac{\pi}{km}\phi\phi^\dagger + i(\omega - \tilde{\omega})(zA_z - \bar{z}A_{\bar{z}} + i\mathbf{L}) - (\omega - \boldsymbol{\mu}), \quad (3.19)$$

where $\mathbf{L} = \text{diag}(l_1, l_2, \dots, l_N)$ is an $N \times N$ diagonal matrix with $l_i \in \mathbb{Z}_{\geq 0}$. The matrix $\boldsymbol{\sigma} = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_N)$ denotes a set of real profile functions satisfying

$$\partial_z \partial_{\bar{z}} \sigma_i = m\omega - \frac{\pi}{k}|z|^{2l_i} e^{-\sigma_i}, \quad (z\partial_z - \bar{z}\partial_{\bar{z}})\sigma_i = 0, \quad (3.20)$$

with asymptotic behavior $\sigma_i \rightarrow m\omega|z|^2$. We can show that the subleading part of σ_i takes the form

$$\sigma_i \rightarrow m\omega|z|^2 - \rho_i \log|z|^2. \quad (3.21)$$

From this asymptotic behavior, it follows that the real parameters ρ_i correspond to the magnetic fluxes and the flavor charges

$$\frac{1}{2\pi} \int d^2x iF_{z\bar{z}} = -\frac{1}{2} \text{diag}(\rho_1, \dots, \rho_N), \quad \mathcal{N}_a = \frac{k}{2\pi} \rho_a. \quad (3.22)$$

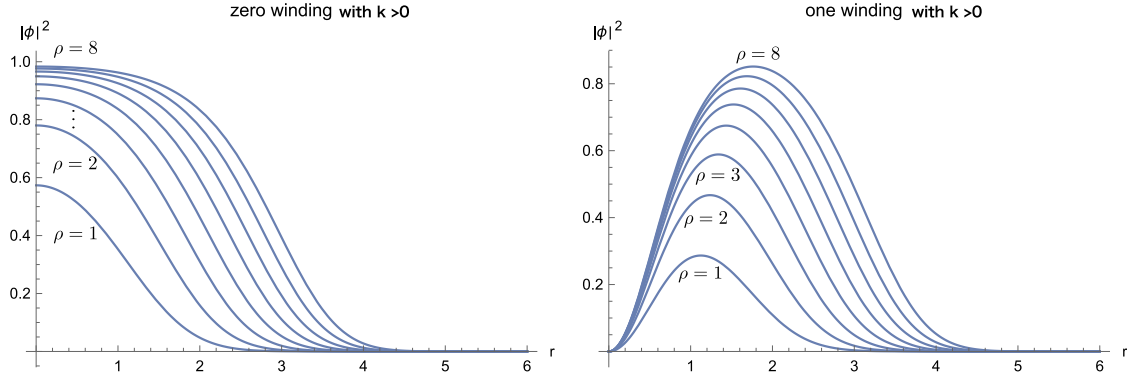


Fig. 2. Profiles of particle number density $|\phi|^2$ for $k = \pi$, $N = N_F = 1$, $m\omega = 1$.

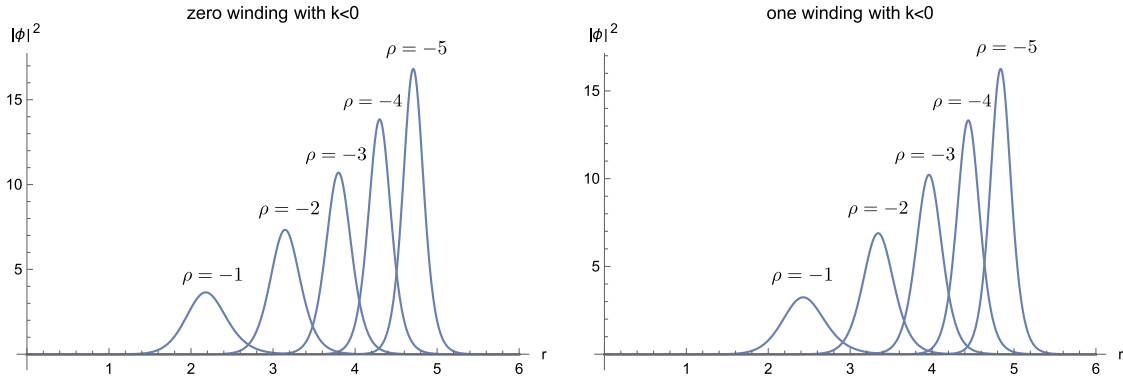


Fig. 3. Profiles of particle number density $|\phi|^2$ for $k = -\pi$, $N = N_F = 1$, $m\omega = 1$.

See Figs. 2 and 3 for the profiles of the charge density.

As in Ref. [20], time-dependent solutions can be obtained from these static solutions by applying the modified Schrödinger symmetry. Although such solutions do not satisfy the BPS equation in Eq. (3.3), they preserve a certain time-dependent linear combination of the supercharges.

4. Spectrum of fluctuation modes in BPS background

In this section we consider fluctuations of the fields $(\delta A_\mu, \delta\phi, \delta\psi)$ around a BPS background (A_μ, ϕ) and show that, in addition to the so-called massive NG modes in the bosonic fluctuations, there exist fermionic massive NG modes associated with the broken supercharges of the modified super-Schrödinger symmetry.

4.1. Linearized equations for fluctuations

When we discuss fluctuations of the bosonic fields, it is convenient to remove the gauge zero modes by imposing the gauge-fixing condition on the fluctuations as

$$\delta A_t = -\frac{\pi}{km}(\delta\phi\phi^\dagger + \phi\delta\phi^\dagger) - i(\omega - \tilde{\omega})(z\delta A_z - \bar{z}\delta A_{\bar{z}}). \quad (4.1)$$

This gauge-fixing condition does not completely remove unphysical gauge zero modes since there remain gauge degrees of freedom generated by $\Lambda \in \mathfrak{u}(N)$ such that

$$\mathcal{D}_t \Lambda + i(\omega - \tilde{\omega})(z\mathcal{D}_z \Lambda - \bar{z}\mathcal{D}_{\bar{z}} \Lambda) + \frac{\pi i}{km}[\phi\phi^\dagger, \Lambda] = 0. \quad (4.2)$$

This residual gauge degrees of freedom can be fixed by imposing the additional gauge-fixing condition as

$$i(\mathcal{D}_z \delta A_{\bar{z}} + \mathcal{D}_{\bar{z}} \delta A_z) + \frac{\pi}{k} (\delta \phi \phi^\dagger - \phi \delta \phi^\dagger) = 0. \quad (4.3)$$

The Gauss law equation $iF_{z\bar{z}} + \frac{\pi}{k} \phi \phi^\dagger = 0$ reduces to the linearized Gauss law for the fluctuation fields:

$$i(\mathcal{D}_z \delta A_{\bar{z}} - \mathcal{D}_{\bar{z}} \delta A_z) + \frac{\pi}{k} (\delta \phi \phi^\dagger + \phi \delta \phi^\dagger) = 0. \quad (4.4)$$

If the gauge-fixing condition in Eq. (4.1) and the linearized Gauss law in Eq. (4.4) are satisfied, the linearized equations for the fluctuation fields can be written as

$$\left(i\nabla_t + \frac{2}{m} \nabla \tilde{\nabla} \right) \begin{pmatrix} \delta A_{\bar{z}} \\ \delta \phi \end{pmatrix} = 0, \quad \left(i\nabla_t + \frac{2}{m} \tilde{\nabla} \nabla \right) \delta \psi = 0, \quad (4.5)$$

where the differential operators are given by³

$$\nabla_t \equiv \mathcal{D}_t + \frac{\pi i}{km} \phi \phi^\dagger + i(\omega - \tilde{\omega}) \left(\hat{J} - \frac{3}{2} \hat{\mathcal{N}}_f \right) + i \sum_{a=1}^{N_F} (\omega - \mu_a) \hat{\mathcal{N}}_a - i(\mu_f + \omega + \tilde{\omega}) \hat{\mathcal{N}}_f \quad (4.6)$$

and

$$\nabla = \begin{pmatrix} \frac{\pi i}{k} \hat{\phi}^\dagger \\ \nabla_z \end{pmatrix}, \quad \tilde{\nabla} = \begin{pmatrix} i\hat{\phi} & \nabla_{\bar{z}} \end{pmatrix}. \quad (4.7)$$

The operators $\hat{\phi}$ and $\hat{\phi}^\dagger$ denote the right multiplications of ϕ and $\phi^\dagger$, e.g. $\hat{\phi} \cdot \delta A_{\bar{z}} = \delta A_{\bar{z}} \phi_a$, and the differential operators ∇_z and $\nabla_{\bar{z}}$ are defined by

$$\nabla_z \equiv \mathcal{D}_z - \frac{1}{2} m \omega \bar{z}, \quad \nabla_{\bar{z}} \equiv \mathcal{D}_{\bar{z}} + \frac{1}{2} m \omega z. \quad (4.8)$$

4.2. Eigenmode expansion and supermultiplets

Here we consider fluctuations around the static BPS background in Eqs. (3.17)–(3.19). Let us consider the eigenmode expansion of the fluctuations

$$\begin{pmatrix} \delta A_{\bar{z}} \\ \delta \phi \end{pmatrix} = \sum_n \varphi_n(t) \begin{pmatrix} u_{g,n} \\ u_{s,n} \end{pmatrix}, \quad \delta \psi = \sum_n \chi_n(t) u_{f,n}, \quad (4.9)$$

where $(u_{g,n}, u_{s,n})$ and $u_{f,n}$ are bosonic and fermionic mode functions satisfying the eigenmode equations

$$\left(i\nabla(\epsilon_{b,n}) + \frac{2}{m} \nabla \tilde{\nabla} \right) \begin{pmatrix} u_{g,n} \\ u_{s,n} \end{pmatrix} = 0, \quad \left(i\nabla(\epsilon_{f,n}) + \frac{2}{m} \tilde{\nabla} \nabla \right) u_{f,n} = 0, \quad (4.10)$$

³ The generator $\hat{J}$ denotes the angular momentum operator including the spin part:

$$\hat{J} \equiv z \mathcal{D}_z - \bar{z} \mathcal{D}_{\bar{z}} + \hat{S}.$$

where $\nabla(\epsilon)$ is the operator which can be obtained from ∇_t in Eq. (4.6) by replacing the time derivative $i\partial_t$ with an eigenvalue ϵ . Then the linearized equations reduce to the following equations for the bosonic and fermionic degrees of freedom $\varphi_n(t)$ and $\chi_n(t)$:

$$i\partial_t \varphi_n(t) = \epsilon_{b,n} \varphi_n(t), \quad i\partial_t \chi_n(t) = \epsilon_{f,n} \chi_n(t). \quad (4.11)$$

Note that since $\nabla(\epsilon)$ commutes with $\nabla\tilde{\nabla}$ and $\tilde{\nabla}\nabla$,

$$\left[\nabla(\epsilon_{b,n}), \nabla\tilde{\nabla} \right] \begin{pmatrix} u_{g,n} \\ u_{s,n} \end{pmatrix} = 0, \quad \left[\nabla(\epsilon_{f,n}), \tilde{\nabla}\nabla \right] u_{f,n} = 0, \quad (4.12)$$

the solution to the bosonic (fermionic) linearized equation can be decomposed into simultaneous eigenmodes of $\nabla(\epsilon)$ and $\nabla\tilde{\nabla}$ ($\tilde{\nabla}\nabla$).

Since the BPS background configuration preserves a linear combination of three complex supercharges (q, Q, S) , eigenmodes of the bosonic and fermionic fluctuations are paired so that they form supermultiplets of the unbroken supersymmetry. Let (u_g, u_s) be a bosonic eigenmode with eigenvalue ϵ_b . The partner fermionic eigenmode u_f can be obtained as

$$u_f = \tilde{\nabla} \begin{pmatrix} u_g \\ u_s \end{pmatrix}, \quad \epsilon_f = \epsilon_b - \mu_f - \omega - \tilde{\omega}. \quad (4.13)$$

On the other hand, any fermionic eigenmode u_f with eigenvalue ϵ_f can be mapped to its partner bosonic eigenmode as

$$\begin{pmatrix} u_g \\ u_s \end{pmatrix} = \nabla u_f, \quad \epsilon_b = \epsilon_f + \mu_f + \omega + \tilde{\omega}. \quad (4.14)$$

Since (u_g, u_s) and u_f are eigenmodes of $\nabla\tilde{\nabla}$ and $\tilde{\nabla}\nabla$ respectively, the sequential mappings (*boson* $\rightarrow$ *fermion* $\rightarrow$ *boson*) and (*fermion* $\rightarrow$ *boson* $\rightarrow$ *fermion*) do not give new eigenmodes,

$$\begin{pmatrix} u_g \\ u_s \end{pmatrix} \rightarrow \nabla\tilde{\nabla} \begin{pmatrix} u_g \\ u_s \end{pmatrix} \propto \begin{pmatrix} u_g \\ u_s \end{pmatrix}, \quad u_f \rightarrow \tilde{\nabla}\nabla u_f \propto u_f. \quad (4.15)$$

Therefore, a generic supermultiplet consists of a pair of bosonic and fermionic eigenmodes.

It is worth noting that, unlike the case of ordinary supersymmetry, bosonic and fermionic eigenmodes in a supermultiplet have different eigenfrequencies, $\epsilon_b - \epsilon_f = \mu_f + \omega + \tilde{\omega}$. This is due to the unbroken supersymmetry being part of the modified supersymmetry which explicitly depends on the time t . One can check that the unbroken supersymmetry becomes independent of t when $\epsilon_b - \epsilon_f = \mu_f + \omega + \tilde{\omega} = 0$.

4.2.1. Short supermultiplets

Although a generic supermultiplet is made up of a pair of bosonic and fermionic eigenmodes, there also exist short supermultiplets, each of which consists of only a single bosonic eigenmode. Such a short multiplet can be found by solving the linearized BPS equation $\nabla_{\bar{z}}\delta\phi + i\delta A_{\bar{z}}\phi = 0$, i.e.

$$\tilde{\nabla} \begin{pmatrix} u_g \\ u_s \end{pmatrix} = 0. \quad (4.16)$$

For a bosonic eigenmode satisfying this equation, the *boson* $\rightarrow$ *fermion* mapping in Eq. (4.13) vanishes. Furthermore, there is no fermionic eigenmode such that ∇u_f is a solution of the linearized

BPS equation since the operator $\tilde{\nabla}\nabla = \nabla_z\nabla_{\bar{z}} - m\omega$ has no zero mode.⁴ Therefore, this type of short multiplet consists of only a single bosonic mode.

4.2.2. Linearized Gauss law and gauge-fixing condition

Since the bosonic component of a long supermultiplet is an element of $\text{im}\nabla$, the linearized Gauss law in Eq. (4.4) and the gauge-fixing condition in Eq. (4.3) are automatically satisfied:

$$i\mathcal{D}_z u_g + \frac{\pi}{k} u_s \phi^\dagger = i\mathcal{D}_z \left(\frac{\pi i}{k} u_f \phi^\dagger \right) + \frac{\pi}{k} \nabla_z u_f \phi^\dagger = 0. \quad (4.17)$$

On the other hand, for any solution of the linearized BPS equation in Eq. (4.16) (element of $\text{Ker}\tilde{\nabla}$), we can always find a short multiplet satisfying the constraints in Eqs. (4.4) and (4.3) by using the symmetry of the eigenmode equation,

$$\begin{pmatrix} u_g \\ u_s \end{pmatrix} \rightarrow \begin{pmatrix} u_g - \mathcal{D}_{\bar{z}}\Lambda \\ u_s + i\Lambda\phi \end{pmatrix}, \quad (4.18)$$

where $\Lambda \in \mathfrak{gl}(N)$ is an $N \times N$ matrix satisfying $\nabla(\epsilon_b)\Lambda = 0$ and

$$i \left[\mathcal{D}_z \mathcal{D}_{\bar{z}} \Lambda - \frac{\pi}{k} \Lambda \phi \phi^\dagger \right] = i\mathcal{D}_z u_g + \frac{\pi}{k} u_s \phi^\dagger. \quad (4.19)$$

In this way, we can find physical short multiplets satisfying the linearized Gauss law equation and the gauge-fixing condition.

4.3. Bosonic and fermionic massive Nambu–Goldstone modes

Since the static BPS configuration in Eqs. (3.17)–(3.19) breaks part of the super-Schrödinger symmetry, there exist NG modes in the fluctuations of the fields.

4.3.1. Bosonic massive Nambu–Goldstone modes

In the presence of the external fields, the super-Schrödinger symmetry is modified in such a way that the generators explicitly depend on the time t . Consequently, the corresponding NG modes become massive. The bosonic NG modes satisfying the constraints of Eqs. (4.4) and (4.3) take the form

$$P_z - i\omega B^{\bar{z}} \rightarrow \begin{pmatrix} u_g \\ u_s \end{pmatrix} = \begin{pmatrix} -iF_{z\bar{z}} - m\omega \\ -i\nabla_z \phi \end{pmatrix}, \quad \epsilon_b = \tilde{\omega} - \omega, \quad (4.20)$$

$$P_z + i\omega B^{\bar{z}} \rightarrow \begin{pmatrix} u_g \\ u_s \end{pmatrix} = \begin{pmatrix} -iF_{z\bar{z}} \\ -i\nabla_z \phi \end{pmatrix}, \quad \epsilon_b = \tilde{\omega} + \omega. \quad (4.21)$$

$$D + 2i\omega C \rightarrow \begin{pmatrix} u_g \\ u_s \end{pmatrix} = \begin{pmatrix} zF_{z\bar{z}} \\ (z\mathcal{D}_z + \bar{z}\mathcal{D}_{\bar{z}} + 1)\phi \end{pmatrix}, \quad \epsilon_b = 2\omega, \quad (4.22)$$

where the first NG mode generated by $P_z - i\omega B^{\bar{z}}$ is in a short multiplet and we have used the symmetry in Eq. (4.18) so that it satisfies the constraints in Eqs. (4.4) and (4.3). These three complex

⁴ $\tilde{\nabla}\nabla = \nabla_z\nabla_{\bar{z}} - m\omega$ is a negative definite operator, since for any function f ,

$$\int d^2x \bar{f} (\nabla_z\nabla_{\bar{z}} - m\omega) f = - \int d^2x \left[\left| \left(\partial_{\bar{z}} + \frac{m\omega}{2} z \right) f \right|^2 + m\omega |f|^2 \right] < 0.$$

modes (and their complex conjugate) correspond to the broken modified symmetry generated by six real operators (translation, Galilean, dilatation, and special conformal symmetry). There also exist massive NG modes corresponding to the broken modified flavor symmetry in Eq. (2.34).

4.3.2. Fermionic massive Nambu–Goldstone modes

Since the BPS configuration breaks part of the supersymmetry, there also exist fermionic NG modes. As in the bosonic case, the modified supersymmetry transformations explicitly depend on time and hence the corresponding fermionic NG modes are massive. There are two such fermionic massive NG modes corresponding to the broken fermionic generators q and $Q + i\omega S$,

$$q \rightarrow u_f = \phi, \quad \epsilon_f = -\mu_f, \quad (4.23)$$

$$Q + i\omega S \rightarrow u_f = z\phi, \quad \epsilon_f = -\mu_f - \tilde{\omega} + \omega. \quad (4.24)$$

We can check that the NG modes generated by $(P_z + i\omega B^{\bar{z}}, q)$ and $(D + 2i\omega C, Q + i\omega S)$ are the pairs of supermultiplets related by the boson–fermion mapping discussed above.

4.4. Infinite towers of eigenmodes in static BPS background

As we have seen in the previous section, the bosonic and fermionic massive NG modes have eigenfrequencies given by the chemical potentials with the integer coefficients determined by the charges of the corresponding generators. Here we show that there are infinite towers of eigenmodes with such eigenvalues.

Let us first consider the case of short multiplets. Since the BPS equation $\nabla_{\bar{z}}\phi$ is satisfied by Eqs. (3.7)–(3.9) for an arbitrary matrix $H_0(t, z)$, the linearized BPS equation can be solved by using the linearized version of Eqs. (3.7)–(3.9), which takes the form

$$\begin{pmatrix} u_g \\ u_s \end{pmatrix} = \begin{pmatrix} 0 \\ e^{-\frac{1}{2}\sigma} \delta H_0(z) \end{pmatrix} + \begin{pmatrix} -\mathcal{D}_{\bar{z}}\Lambda \\ i\Lambda\phi \end{pmatrix}, \quad (4.25)$$

where Λ is the $N \times N$ matrix determined by the constraint⁵

$$i \left[\mathcal{D}_z \mathcal{D}_{\bar{z}} \Lambda - \frac{\pi}{k} \Lambda \phi \phi^\dagger \right] = \frac{\pi}{k} e^{-\frac{1}{2}\sigma} \delta H_0(z) \phi^\dagger. \quad (4.26)$$

This solution of the linearized BPS equation satisfies the eigenmode equation with eigenvalue ϵ if the matrix $\delta H_0(z)$ is chosen so that $\Lambda(\epsilon) \delta H_0 = 0$. This condition is satisfied when δH_0 has one non-zero component given by a monomial of z . For example, if δH_0 has z^l ($l \in \mathbb{Z}_{\geq 0}$) in its (j, J) -component,

$$(\delta H_0)_{iI} = z^l \delta_{ij} \delta_{IJ}, \quad (4.27)$$

the eigenfrequency of the corresponding short multiplet is given by

$$\epsilon_b = (l - l_j)(\omega - \tilde{\omega}) - \mu_J + \mu_j. \quad (4.28)$$

The NG mode generated by $P_z - i\omega B^{\bar{z}}$ corresponds to a linear combination of the modes with $(l, J) = (l_j - 1, j)$. The NG modes corresponding to the broken (modified) flavor symmetry are also contained in these towers of eigenmodes.

⁵ The matrix Λ can also be written as $\Lambda = ie^{-\frac{1}{2}\sigma} \delta S$ by using the solution δS of the linearized version of Eq. (3.10) which determines the matrix S .

In addition to the short multiplets, we can also find exact spectra of a class of ordinary long supermultiplets. Such supermultiplets can be obtained from the fermionic eigenmode corresponding to the lowest Landau level. For example, u_f is an eigenmode with frequency

$$\epsilon_f = (l - l_j)(\omega - \tilde{\omega}) - \mu_J + \mu_j - \mu_f \quad (4.29)$$

if u_f has a non-zero monomial in the (j, J) -component

$$(u_f)_{iI} = e^{-\frac{1}{2}\sigma_j} z^l \delta_{ij} \delta_{IJ}. \quad (4.30)$$

The fermionic massive NG modes in Eqs. (4.23) and (4.24) correspond to the linear combinations of the eigenmodes with $(l, J) = (l_j, j)$ and $(l, J) = (l_j + 1, j)$, respectively. The corresponding bosonic mode can be obtained by applying the map in Eq. (4.14),

$$(u_g)_{iI} = \frac{\pi i}{k} e^{-\frac{1}{2}(\sigma_j + \sigma_J)} z^l \bar{z}^{l_J} \delta_{ij} \delta_{IJ}, \quad (u_s)_{iI} = e^{-\frac{1}{2}\sigma_j} z^{l-1} \left(l - \frac{1}{2} r \partial_r \sigma_j \right) \delta_{ij} \delta_{IJ}, \quad (4.31)$$

where $r = |z|$. As we have seen above, this bosonic mode has eigenfrequency related to that of the fermionic mode as in Eq. (4.13),

$$\epsilon_b = (l - l_j)(\omega - \tilde{\omega}) - \mu_J + \mu_j - \omega - \tilde{\omega}. \quad (4.32)$$

The towers of the eigenmodes in Eqs. (4.27) and (4.30) can be interpreted as the lowest Landau levels in the bosonic and fermionic sectors, respectively. As was done in the non-linear Schrödinger system [40], it would be interesting to discuss the low-energy dynamics of such degrees of freedom with physically distinctive properties.

5. Summary and discussion

In this paper we have discussed the supersymmetric Jackiw–Pi model in the harmonic trap. The super-Schrödinger symmetry of the original SUSY Jackiw–Pi model is modified in the presence of the external background fields which correspond to the generalized chemical potential terms including the harmonic potential. We have seen that the 1/3 BPS states of Jackiw–Pi vortices, which preserve part of the modified supersymmetry, are stationary configurations rotating around the origin. They become static when the moduli matrix is at the fixed points of the spatial rotation and the flavor symmetry. We have investigated fluctuations around the static BPS backgrounds and revealed the structure of supermultiplets of eigenmodes. In addition to the bosonic massive NG modes, we identified the fermionic massive NG modes associated with the broken modified superconformal symmetry. We have also found the eigenmode spectra of the infinite towers of supermultiplets corresponding to the bosonic and fermionic lowest Landau levels.

While we have discussed one of the simplest examples of (modified) non-relativistic supersymmetry in the Jackiw–Pi model, it is known that there exist Chern–Simons matter systems with extended non-relativistic supersymmetries [41–43]. It would be interesting to investigate bosonic and fermionic massive NG modes in the extended models such as the non-relativistic Aharony–Bergman–Jafferis–Maldacena model. Another direction to be explored is to clarify the relation between the quantum states of the Jackiw–Pi vortices in the harmonic potential and the spectrum of the chiral primary operators [44,45] from the viewpoint of the non-relativistic version of the state–operator mapping [46]. If we set $\mu_f + \omega + \tilde{\omega} = 0$ in our model, the explicit time dependence of the supersymmetry preserved by the 1/3 BPS states disappears and hence we can compactify the time direction without

breaking the supersymmetry. Such a situation is quite similar to the Ω -background [47,48], and it would be possible to compute certain types of superconformal indices by using the supersymmetric localization method [49]. As in the case of the vortex partition functions in two-dimensional $\mathcal{N} = (2, 2)$ theories [50,51], the moduli matrix method, which was used to describe the BPS vortex solution, would play a crucial role in the localization computation and hence it is an important future work to investigate the structure of the space of the BPS solutions from the viewpoint of the moduli matrix formalism and its relation to the Atiyah–Drinfeld–Hitchin–Manin-like construction discussed in Ref. [22].

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Appendix A. Generalized chemical potential and massive Nambu–Goldstone mode

In this section we summarize the basic properties of the generalized chemical potential, modified symmetry, and the massive NG mode.

Consider a system with a symmetry generated by conserved charges Q^a obeying the commutation relation

$$[Q^a, Q^b] = if^{ab}{}_c Q^c. \quad (\text{A.1})$$

If some of these generators have explicit time dependence, they do not commute with the Hamiltonian. It follows from the Noether theorem and the Heisenberg equations of motion

$$\frac{dQ^a}{dt} = i[H, Q^a] + \frac{\partial Q^a}{\partial t} = 0 \quad (\text{A.2})$$

that the explicit time dependence of the conserved charges can be written as

$$Q^a = (e^{\mathcal{H}t})^a{}_b \underline{Q}^b, \quad (\text{A.3})$$

where $\mathcal{H}^a{}_b$ is the matrix defined by

$$[H, Q^a] = i\mathcal{H}^a{}_b Q^b \quad (\text{A.4})$$

and $\underline{Q}^a$ are operators which do not have explicit time dependence and satisfy the same commutation relation as Q^a ,

$$\frac{\partial \underline{Q}^a}{\partial t} = 0, \quad [\underline{Q}^a, \underline{Q}^b] = if^{ab}{}_c \underline{Q}^c. \quad (\text{A.5})$$

Note that $\underline{Q}^a$ are identical to the symmetry generators Q^a if the Q^a do not have explicit time dependence. Using $\underline{Q}^a$, we can add “chemical-potential-like terms” to the Hamiltonian without introducing explicit time dependence,

$$\tilde{H} = H - \mu_a \underline{Q}^a, \quad (\text{A.6})$$

where μ_a are parameters corresponding to the chemical potentials. In the presence of these terms, which we call generalized chemical potential terms, the original symmetry of the system is explicitly broken. Nevertheless, we can find the same number of conserved charges as follows. Consider a linear combination of $\underline{Q}^a$ with time-dependent coefficients

$$\tilde{Q}^a = \mathcal{G}^a_b(t) \underline{Q}^b. \quad (\text{A.7})$$

The Heisenberg equation of motion implies

$$\frac{d\tilde{Q}^a}{dt} = i[\tilde{H}, \tilde{Q}^a] + \frac{\partial \tilde{Q}^a}{\partial t} = \left[-\mathcal{G}^a_b \left(\mathcal{H}^b_c - \mu_d f^{db}_c \right) + \frac{\partial}{\partial t} \mathcal{G}^a_b \right] \underline{Q}^c. \quad (\text{A.8})$$

This equation implies that $\tilde{Q}^a$ are conserved charges if the coefficients $\mathcal{G}^a_b(t)$ are chosen so that they satisfy

$$\frac{\partial}{\partial t} \mathcal{G}^a_b = \mathcal{G}^a_c \left(\mathcal{H}^c_b - \mu_d f^{dc}_b \right). \quad (\text{A.9})$$

For constant chemical potentials μ_a , the conserved charges $\tilde{Q}^a$ are given by

$$\tilde{Q}^a = \mathcal{G}^a_b(t) \underline{Q}^b = (e^{\tilde{\mathcal{H}}t})^a_b \underline{Q}^b, \quad \tilde{\mathcal{H}}^a_b \equiv \mathcal{H}^a_b - \mu_c f^{ca}_b. \quad (\text{A.10})$$

We call the symmetry generated by $\tilde{Q}^a$ “modified symmetry.” It is worth noting that even for time-dependent μ_a , we can construct $\tilde{Q}^a$ by using the solution of Eq. (A.9).

Appendix A.1. Massive Nambu–Goldstone mode

Here we briefly review the massive NG mode, which appears when a symmetry with explicit time dependence is spontaneously broken.

Let us assume that the matrix $\tilde{\mathcal{H}}$ is diagonalizable. Then, corresponding to the eigenvector of $\tilde{\mathcal{H}}$, there exists a linear combination of the modified conserved charge in Eq. (A.10) such that

$$\tilde{Q}^\alpha = e^{-i\alpha t} \underline{Q}^\alpha, \quad (\text{A.11})$$

where α is an eigenvalue of $\tilde{\mathcal{H}}$ and $\underline{Q}^\alpha$ is a certain linear combination of the operators $\underline{Q}^a$. The Heisenberg equation for $\tilde{Q}^\alpha$ implies that

$$\tilde{Q}^\alpha(t) = e^{-i\alpha t} e^{i\tilde{\mathcal{H}}t} \underline{Q}^\alpha(0) e^{-i\tilde{\mathcal{H}}t}, \quad (\text{A.12})$$

where $\underline{Q}^\alpha(0)$ denotes the operator $\underline{Q}^\alpha$ at $t = 0$ (Schrödinger picture). Consider a matrix element $\tilde{Q}^\alpha(t)$ between two energy eigenstates $|E_1\rangle$ and $|E_2\rangle$, given by

$$\langle E_2 | \tilde{Q}^\alpha(t) | E_1 \rangle = e^{i(E_2 - E_1 - \alpha)t} \langle E_2 | \underline{Q}^\alpha(0) | E_1 \rangle. \quad (\text{A.13})$$

Since the matrix element of the conserved charge is independent of time, the right-hand side cannot have time dependence. Therefore, if a matrix element of $\tilde{Q}^\alpha$ is nonzero,

$$\langle E_2 | \tilde{Q}^\alpha | E_1 \rangle \neq 0, \quad (\text{A.14})$$

then there exists a gap between the two energy eigenstates,

$$\Delta E = E_2 - E_1 = \alpha. \quad (\text{A.15})$$

The massive mode corresponding to this gap is the massive NG mode, whose mass is exactly given by an eigenvalue of $\tilde{\mathcal{H}}^a_b = \mathcal{H}^a_b - \mu_c f^{ca}_b$. See Ref. [3] for a more precise argument for the massive NG theorem.

Appendix B. Schrödinger symmetry in free Schrödinger system

In this section we briefly review the Schrödinger symmetry, the generalized chemical potential, and the modified symmetry in the free Schrödinger system as an example. The action of the free Schrödinger system in (2+1) dimensions takes the form

$$S = \int dt d^2x \bar{\phi} \left[i\partial_t + \frac{1}{2m} \partial_i^2 \right] \phi. \quad (\text{B.1})$$

This is a non-relativistic system, in which the canonical commutation relation is given by

$$[\phi(x), \bar{\phi}(x')] = \delta^2(x - x'). \quad (\text{B.2})$$

This system is invariant under the Schrödinger symmetry:

time translation	$\delta_H \phi = \partial_t \phi,$
dilatation	$\delta_D \phi = (2t\partial_t + x^i \partial_i + 1)\phi,$
rotation	$\delta_J \phi = (x_1 \partial_2 - x_2 \partial_1)\phi,$
special Schrödinger symmetry	$\delta_C \phi = \left[t^2 \partial_t + t(x_i \partial_i + 1) - i\frac{m}{2} x_i^2 \right] \phi,$
Galilean boost	$\delta_{B_i} \phi = (t\partial_i - imx_i)\phi,$
translation	$\delta_{P_i} \phi = -\partial_i \phi,$
phase rotation	$\delta_{\mathcal{N}} \phi = -i\phi.$

The corresponding conserved charges are given by

Hamiltonian	$H = \int d^2x \bar{\phi} \left[-\frac{1}{2m} \partial_i^2 \right] \phi,$
dilatation charge	$D = \int d^2x \bar{\phi} \left[-\frac{t}{m} \partial_i^2 + i(x_i \partial_i + 1) \right] \phi,$
angular momentum	$J = \int d^2x \bar{\phi} \left[i(x_1 \partial_2 - x_2 \partial_1) \right] \phi,$
special Schrödinger charge	$C = \int d^2x \bar{\phi} \left[-\frac{t^2}{2m} \partial_i^2 + it(x_i \partial_i + 1) + \frac{m}{2} x_i^2 \right] \phi,$
Galilean boost charge	$B_i = \int d^2x \bar{\phi} \left[it\partial_i + mx_i \right] \phi,$
momentum	$P_i = \int d^2x \bar{\phi} \left[-i\partial_i \right] \phi,$
particle number	$\mathcal{N} = \int d^2x \bar{\phi} \phi.$

Table B1. Generator spins and conformal weights.

	H	P_z	$P_{\bar{z}}$	D	J	$\mathcal{N}$	B_z	$B_{\bar{z}}$	C
$\Delta_{\mathcal{O}}$	2	1	1	0	0	0	-1	-1	-2
$j_{\mathcal{O}}$	0	-1	1	0	0	0	1	-1	0

These charges satisfy the Schrödinger algebra, whose non-trivial part takes the form

$$[H, B_i] = -iP_i, \quad [C, P_i] = iB_i, \quad [B_i, P_j] = im\delta_{ij}\mathcal{N}, \quad [H, C] = iD, \quad (\text{B.3})$$

$$[J, \mathcal{O}] = -j_{\mathcal{O}}\mathcal{O}, \quad [D, \mathcal{O}] = -i\Delta_{\mathcal{O}}\mathcal{O}, \quad (\text{B.4})$$

where $\mathcal{O}$ denotes an arbitrary eigenoperator of the Cartan part $(J, D, \mathcal{N})$ with eigenvalues $(j_{\mathcal{O}}, \Delta_{\mathcal{O}})$. The spins $j_{\mathcal{O}}$ and conformal weights $\Delta_{\mathcal{O}}$ of the generators are given by Table B1, where $P_z = (P_1 - iP_2)/2$ and $B_z = B_1 + iB_2$. The conserved charges given above can be written as linear combinations of the time-independent operators defined in Eq. (A.3):

$$\begin{aligned}
H &= \underline{H} & \underline{H} &= \int d^2x \bar{\phi} \left[-\frac{1}{2m} \partial_i^2 \right] \phi, \\
D &= \underline{D} + 2t\underline{H} & \underline{D} &= \int d^2x \bar{\phi} [i(x_i \partial_i + 1)] \phi, \\
J &= \underline{J} & \underline{J} &= \int d^2x \bar{\phi} [-i(x_1 \partial_2 - x_2 \partial_1)] \phi, \\
C &= \underline{C} + t\underline{D} + t^2\underline{H} & \underline{C} &= \int d^2x \bar{\phi} \left[\frac{m}{2} x_i^2 \right] \phi, \\
B_i &= \underline{B}_i - t\underline{P}_i & \underline{B}_i &= \int d^2x \bar{\phi} [mx_i] \phi, \\
P_i &= \underline{P}_i & \underline{P}_i &= \int d^2x \bar{\phi} [-i\partial_i] \phi, \\
\mathcal{N} &= \underline{\mathcal{N}} & \underline{\mathcal{N}} &= \int d^2x \bar{\phi} \phi.
\end{aligned}$$

Note that the time dependence is at most quadratic since the matrix $\mathcal{H}$ defined in Eq. (A.4) satisfies $\mathcal{H}^3 = 0$ and hence $e^{\mathcal{H}t} = 1 + \mathcal{H}t + (\mathcal{H}t)^2/2$ in the case of the Schrödinger symmetry.

Now let us deform the Hamiltonian by introducing the generalized chemical potential terms,

$$\tilde{H} = H - \mu_a \underline{Q}^a. \quad (\text{B.5})$$

For the most generic chemical potential, the deformed Hamiltonian terms takes the form

$$\begin{aligned}
\tilde{H} &= \int d^2x \bar{\phi} \left[-\frac{1}{2m} \partial_i^2 - i(\mu_D x_i + \mu_J \epsilon_{ij} x_j - \mu_{P_i}) \partial_i \right. \\
&\quad \left. - \left(i\mu_D + \frac{m}{2} \mu_C x_i^2 + m\mu_{B_i} x_i + \mu \right) \right] \phi. \quad (\text{B.6})
\end{aligned}$$

These chemical potential terms can also be written as an external gauge field as

$$\tilde{H} = \int d^2x \bar{\phi} \left[-\frac{1}{2m} \tilde{\mathcal{D}}_i^2 + A_i^{\text{ex}} \right] \phi, \quad (\text{B.7})$$

where $\tilde{D}_\mu \equiv \partial_\mu + iA_\mu^{\text{ex}}$ with

$$A_\mu^{\text{ex}} dx^\mu = \frac{i}{2} m \tilde{\omega} \left[\overline{(z - z_0)} dz - (z - z_0) d\bar{z} \right] + \left[\frac{m}{2} (\omega^2 - \tilde{\omega}^2) |z - z_0|^2 \right] dt + d\alpha. \quad (\text{B.8})$$

Here, $z = x_1 + ix_2$ is the complex coordinate and the function α is given by

$$\alpha = \frac{m\mu_D}{2} |z - z'|^2 - \mu t. \quad (\text{B.9})$$

The chemical potentials μ_a and the parameters $(\omega, \tilde{\omega}, \mu, z_0, z')$ are related by

$$\mu_J = -\tilde{\omega}, \quad \mu_C = -|\omega + i\mu_D|^2, \quad \mu_N = \mu - \frac{m}{2} \left[(\omega^2 - \tilde{\omega}^2) |z'|^2 + |\tilde{\omega} z_0 + i\mu_D z'|^2 \right], \quad (\text{B.10})$$

$$\mu_{B_1} + i\mu_{B_2} = (\omega^2 - \tilde{\omega}^2) z' + (\tilde{\omega} - i\mu_D)(\tilde{\omega} z_0 + i\mu_D z'), \quad \mu_{P_1} + i\mu_{P_2} = -i(\tilde{\omega} z' + i\mu_D z'). \quad (\text{B.11})$$

Equations (B.7) and (B.8) show that the generalized chemical potential terms for the Schrödinger symmetry can be essentially regarded as the harmonic potential and constant magnetic field characterized by the frequencies ω and $\tilde{\omega}$, respectively.

For simplicity, let us focus on the case with $\mu_D = \mu_{B_i} = \mu_{P_i} = 0$ ($z_0 = z' = 0$). In the presence of $\mu_C = -\omega$, $\mu_J = -\tilde{\omega}$, and $\mu_N = \mu$, the modified conserved charges in Eq. (A.10) are given by

$$\tilde{H} = \underline{H} + \omega^2 \underline{C} + \tilde{\omega} \underline{J} - \mu \underline{N}, \quad (\text{B.12})$$

$$\tilde{N} = \underline{N}, \quad (\text{B.13})$$

$$\tilde{J} = \underline{J}, \quad (\text{B.14})$$

$$\tilde{Q}_\pm = e^{i(\omega \pm \tilde{\omega})t} \left[\omega (\underline{B}_1 \pm i\underline{B}_2) + i(\underline{P}_1 \pm i\underline{P}_2) \right], \quad (\text{B.15})$$

$$\tilde{Q}_0 = e^{2i\omega t} \left[\omega^2 \underline{C} - i\omega \underline{D} - \underline{H} \right]. \quad (\text{B.16})$$

These conserved charges are chosen so that the matrix $\tilde{H}$ takes the diagonal form, that is,

$$[\tilde{H}, \tilde{J}] = 0, \quad [\tilde{H}, \tilde{N}] = 0, \quad [\tilde{H}, \tilde{Q}_\pm] = -\omega_\pm \tilde{Q}_\pm, \quad [\tilde{H}, \tilde{Q}_0] = -2\omega \tilde{Q}_0, \quad (\text{B.17})$$

with $\omega_\pm = \omega \pm \tilde{\omega}$. These equations implies that they satisfy the conservation law

$$\frac{d\tilde{Q}^a}{dt} = i[\tilde{H}, \tilde{Q}^a] + \frac{\partial \tilde{Q}^a}{\partial t} = 0. \quad (\text{B.18})$$

Note that $\tilde{Q}_\pm$ and $\tilde{Q}_0$ are complex quantities and hence $\tilde{Q}_\pm^\dagger$ and $\tilde{Q}_0^\dagger$ are also conserved.

The explicit forms of the transformations corresponding to the modified conserved charges are given by

$$\delta_{\tilde{H}} \phi = \partial_t \phi, \quad (\text{B.19})$$

$$\delta_{\tilde{N}} \phi = i\phi, \quad (\text{B.20})$$

$$\delta_{\tilde{J}} \phi = i[z\partial_z - \bar{z}\partial_{\bar{z}}] \phi, \quad (\text{B.21})$$

$$\delta_{\pm} \phi = e^{i\omega_\pm t} [\partial_1 \pm i\partial_2 + m\omega(x_1 \pm ix_2)] \phi, \quad (\text{B.22})$$

$$\delta_0 \phi = e^{2i\omega t} [\partial_t + i(m\omega^2 |z|^2 - \mu) + i\omega(z\partial_z + \bar{z}\partial_{\bar{z}} + 1) - i\tilde{\omega}(z\partial_z - \bar{z}\partial_{\bar{z}})] \phi. \quad (\text{B.23})$$

We can show that the deformed system, which is described by the action

$$S = \int dt d^2x \left[i\bar{\phi}\tilde{D}_t\phi + \frac{1}{2m}\bar{\phi}\tilde{D}_i^2\phi \right], \quad (\text{B.24})$$

is invariant under the modified transformations given above. The last two equations in Eq. (B.17) imply that $\tilde{Q}_\pm^\dagger$ and $\tilde{Q}_0^\dagger$ are essentially the creation operators of massive quanta,

$$[\tilde{H}, \tilde{Q}_\pm^\dagger] = \omega_\pm \tilde{Q}_\pm^\dagger, \quad [\tilde{H}, \tilde{Q}_0^\dagger] = 2\omega \tilde{Q}_0^\dagger. \quad (\text{B.25})$$

Therefore, if a state is not invariant under the modified symmetry generated by $\tilde{Q}_\pm^\dagger$ and $\tilde{Q}_0^\dagger$, there exist massive NG modes with masses $\omega_\pm$ and 2ω .

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