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Quiver matrix model of ADHM type and BPS state counting in diverse dimensions

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We review the problem of Bogomol'nyi–Prasad–Sommerfield (BPS) state counting described by the generalized quiver matrix model of Atiyah–Drinfield–Hitchin–Manin type. In four dimensions the generating function of the counting gives the Nekrasov partition function, and we obtain a generalization in higher dimensions. By the localization theorem, the partition function is given by the sum of contributions from the fixed points of the torus action, which are labeled by partitions, plane partitions and solid partitions. The measure or the Boltzmann weight of the path integral can take the form of the plethystic exponential. Remarkably, after integration the partition function or the vacuum expectation value is again expressed in plethystic form. We regard it as a characteristic property of the BPS state counting problem, which is closely related to the integrability.

Subject Index A10, B27

1. Introduction

It is well known the instantons (anti-self-dual connections) in four-dimensional gauge theory allow Atiyah–Drinfield–Hitchin–Manin (ADHM) construction [1,2]. From the viewpoint of string theory the ADHM description can be obtained by considering the $D4$–$D0$ system in type IIA string theory, where the matrices, which are basic dynamical variables in ADHM construction, come from the open strings connecting D-branes[1] [3–6]. The low-energy effective theory on the world volume of D-branes is the dimensional reduction of ten-dimensional super Yang–Mills theory. The original anti-self-duality of the gauge field is translated to the Bogomol'nyi–Prasad–Sommerfield (BPS) condition for the world volume theory on $D4$-branes, while the ADHM equations are obtained as the BPS condition on $D0$-branes. Since the world volume of $D0$-branes has no spacial direction, the theory is reduced to supersymmetric quantum mechanics (in fact a matrix model), which we call the ADHM matrix model.

The ADHM description of the four-dimensional instantons (or the BPS solitons in five-dimensional theory from the viewpoint of M-theory) also plays a significant role in the computation of the instanton partition function of Nekrasov [7–9], which provides a microscopic derivation of the Seiberg–Witten prepotential of four-dimensional $\mathcal{N} = 2$ Yang–Mills theory. By introducing a sufficiently large number of torus actions on the ADHM moduli space, we can employ the Atiyah–Bott-type localization formula to compute the path integral. The fixed points of the topic action are isolated and labeled by a tuple of partitions (or Young diagrams). Then the partition function is expressed as a summation over the contribution from each fixed point, which is in turn given by the equivariant character of the tangent space at the fixed point as a module of the torus action.

[1] In this article we only consider $U(n)$ gauge theory.

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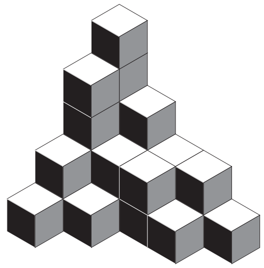

$$\pi = \begin{pmatrix} 5 & 3 & 2 & 2 & 1 \\ 4 & 2 & 2 & 1 & \\ 2 & 1 & & & \\ 1 & & & & \end{pmatrix}, \qquad |\pi| = 26.$$

Fig. 1. Plane partition as a three-dimensional Young diagram.

In the following we will discuss some intriguing aspects in generalizing this story to higher dimensions by replacing $D4$-branes with Dp-branes ($p = 2d = 6, 8$), where the fixed points are labeled by higher-dimensional generalizations of the partition, called a plane partition ($d = 3$) or a solid partition ($d = 4$). The BPS condition on $D6$- and $D8$-branes can be identified with the higher-dimensional instanton equation in six and eight dimensions, respectively [10], while the BPS condition on $D0$-branes gives what we call a quiver matrix model of ADHM type. In the same manner as the four-dimensional case, the moduli space $\mathcal{M}_{n,k}$ of the quiver matrix model is topologically labeled by the number n of Dp-branes and the number k of $D0$-branes. We call k the instanton number in analogy with the four-dimensional case. To obtain the partition function of $U(n)$ gauge theory on the Dp-brane, we will fix n and take a summation of k over non-negative integers.

1.1. *Fixed points of the torus action and* $(d-1)$*-partitions*

Let t_i collectively denote equivariant parameters of the torus action on the moduli space $\mathcal{M}_{n,k}$ of matrix equations of ADHM type. In general, they consist of the equivariant parameters of the torus action on the (flat) space-time coordinate $(z_1, \ldots, z_d) \in \mathbb{C}^d$ (the Ω background parameters of Nekrasov), the Cartan subgroup of the gauge symmetry $G_C = U(n)$ (the Coulomb moduli parameters) and of the flavor symmetry G_F (mass parameters). We can identify the equivariant K group of a point with the ring of Laurent polynomials in the equivariant parameters $K_T(\mathrm{pt}) = \mathbb{C}[t_i^{\pm 1}]$. Hence, the equivariant character at the fixed points takes a value in $K_T(\mathrm{pt})$.

Recall that a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ is a non-increasing sequence of positive integers such that $\lambda_{\ell+1} = 0$ for finite ℓ. It is useful to represent λ in terms of the Young diagram. We denote $|\lambda| = \sum_{i=1}^{\infty} \lambda_i$, which is the total number of boxes (cells) in the corresponding Young diagram. We can consider a higher-dimensional generalization or a $(d-1)$-partition $\pi = \{\pi_{i_1,\ldots,i_{d-1}}\}$, $(i_1, \ldots, i_{d-1}) \in \mathbb{Z}_{>0}^{d-1}$, which is an array of positive integers with the (obvious) higher-dimensional generalization of the non-increasing condition; for example, $\pi_{i,j} \geq \pi_{i+1,j}$ and $\pi_{i,j} \geq \pi_{i,j+1}$ when $d = 3$, and $\pi_{i_1,\ldots,i_{d-1}} \neq 0$ for only a finite set of $(i_1, \ldots, i_{d-1})$ (see Fig. 1). When $d = 3$ and $d = 4$, this is usually called a plane and a solid partition, respectively. We define $|\pi| = \sum_{(i_1,\ldots,i_{d-1})} \pi_{i_1,\ldots,i_{d-1}}$, which means the volume (the number of boxes, cubes, ...) of the $(d-1)$-partition π. It turns out that the fixed points of the toric action on $\mathcal{M}_{n,k}$ are isolated and in one-to-one correspondence with the set of n-tuples of $(d-1)$-partitions $\vec{\pi} = (\pi^\alpha)_{\alpha=1}^n$, and that $|\vec{\pi}| := \sum_{\alpha=1}^n |\pi^\alpha|$ is identified with the instanton number k. Thus, in the sector of instanton number k, we can reduce the quiver matrix model to a statistical model with the configuration space $\Pi_k^n := \{\vec{\pi} \mid |\vec{\pi}| = k\}$, where the equivariant character at each fixed point $\vec{\pi}$ gives the Boltzmann weight of the model.

1.2. Partition function and plethystic exponential

It is interesting that the Boltzmann weight derived from the ADHM matrix model takes the form of the plethystic exponential (see Sect. 2 for a definition) P.E.$[\chi_{\bar{\pi}}(t_i)]$. We define the topological partition function by a weighted sum over the total configuration space $\cup_{k\geq 0}\Pi_k^n$ where, introducing the box-counting parameter $\mathfrak{q}$, we multiply the volume (the number of boxes, cubes, ...) of $(d-1)$-partitions $|\pi|$ as the additional Boltzman weight:

$$Z_{\text{top}}(t_i;\mathfrak{q}) := \left\langle \text{P.E.}[\chi_{\bar{\pi}}(t_i)]\right\rangle = \sum_{\pi} \mathfrak{q}^{|\pi|}\text{P.E.}[\chi_{\bar{\pi}}(t_i)]. \tag{1.1}$$

That is, if we identify the instanton number k as the particle number, the topological partition function corresponds to the grand canonical ensemble in statistical mechanics. The phenomena on which we will focus in this article is that in the computation of the topological partition function, the expectation value of the plethystic exponential is again expressed by the plethystic exponential:

$$\left\langle \text{P.E.}[\chi_{\pi}(t_i)]\right\rangle = \text{P.E.}[F(t_i;\mathfrak{q})]. \tag{1.2}$$

Since the plethystic exponential can be regarded as the character of the symmetric algebra $S^\bullet V$ of a G-module V, this is an example of "super"-integrability that the expectation value of the character gives another character, which we encounter typically in the matrix model and plays an important role for extending the realm of symmetric functions [11,12].

When $d=3$, with the computation of the topological partition function $Z_{\text{top}}(t_i;\mathfrak{q})$ we may associate equivariant (or K-theory) vertices [13], which are generalizations of the refined topological vertex [14,15]. In fact, in an appropriate limit of the Ω background parameters q_i, the equivariant vertex reduces to the refined topological vertex. Since the refined topological vertex is characterized as the intertwining operator of the quantum toroidal algebra of $\mathfrak{gl}(1)$ [16], it is tempting to expect some quantum algebras behind the "super"-integrability of Eq. (1.2). Furthermore, since physically the partition function in Eq. (1.1) is nothing but the generation function of the numbers of BPS states, this seems to be along the same line of BPS / vertex operator algebra correspondence, the correspondence of the algebra of BPS states with the chiral algebra of some two-dimensional CFT. It may be interesting to look at a cohomological Hall algebra associated with the quiver of ADHM type [17].

The paper is organized as follows. In the next section we introduce the plethystic exponential. We can regard it as a character of the symmetric algebra and hence it plays a significant role in this article. After presenting ADHM-type matrix model equations coming from the BPS condition of the D-brane system in Sect. 3, we discuss a matrix model formulation or the measure for eigenvalues of matrices in Sect 4. The measure is given in terms of the plethystic exponential, and hence is naturally expressed by power sum functions of eigenvalues. In Sect. 5 we compute the equivariant character of the tangent space at the fixed points. Finally, we present the plethystic forms of the partition function in Sect. 6. In each section after Sect. 4 we first review the well-established case of $d=2$ (the original ADHM equation) and then try to generalize it to higher dimensions. From the viewpoint of mathematics, one of the crucial points is that though the fixed points of the torus action are still isolated and labeled by a higher-dimensional generalization of the partition, the tangent space at each fixed point is not smooth any more and it is defined only virtually.

I would like to dedicate this article to the memory of Prof. Tohru Eguchi, who passed away last year. My collaboration with him started when both of us participated in the inaugural project at the

Newton Institute in the summer of 1992. Our interest was in the interplay of a topological string as a two-dimensional topological quantum field theory and integrable systems such as $w_{1+\infty}$ algebra, the Toda lattice hierarchy [18–20], which became one of the main themes in my research afterwards. After almost a decade I had a second chance of collaboration on the five-dimensional lift of Seiberg–Witten theory, the Nekrasov partition function, and topological strings [21–23], which are closely related to the subject reviewed in the present paper. I am very grateful to Eguchi-san for these fruitful and inspiring collaborations. Though I was not his student, I learned how to enjoy research through collaboration with Eguchi-san.

2. Plethystic exponential

For a function $F(t_1, t_2, \ldots, t_\ell)$ we define the plethystic exponential by

$$\text{P.E.}[F(t_1, t_2, \ldots, t_\ell)] = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} F(t_1^k, t_2^k, \ldots, t_\ell^k)\right). \tag{2.1}$$

Let us assume that $F(t_1, t_2, \ldots, t_\ell)$ can be expanded as follows:

$$F(t_1, t_2, \ldots, t_\ell) = \sum_{n_1, \ldots, n_\ell \in \mathbb{Z}} a_{n_1 \cdots n_\ell} t_1^{n_1} \cdots t_\ell^{n_\ell} \tag{2.2}$$

with $a_{0\cdots 0} = 0$. Then we see that

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k} F(t_1^k, t_2^k, \ldots, t_\ell^k) &= \sum_{n_1, \ldots, n_\ell \in \mathbb{Z}} a_{n_1 \cdots n_\ell} \sum_{k=1}^{\infty} \frac{1}{k} t_1^{kn_1} \cdots t_\ell^{kn_\ell} \\ &= - \sum_{n_1, \ldots, n_\ell \in \mathbb{Z}} a_{n_1 \cdots n_\ell} \log(1 - t_1^{n_1} \cdots t_\ell^{n_\ell}). \end{aligned} \tag{2.3}$$

Thus, the plethystic exponential factorizes as an infinite product:

$$\text{P.E.}[F(t_1, t_2, \ldots, t_\ell)] = \prod_{n_1, \ldots, n_\ell \in \mathbb{Z}} (1 - t_1^{n_1} \cdots t_\ell^{n_\ell})^{-a_{n_1 \cdots n_\ell}}. \tag{2.4}$$

In fact, when $F(t_1, t_2, \ldots, t_\ell)$ is a character of a G-module V, with t_i parametrizing the Cartan subgroup of G,

$$F(t_1, t_2, \ldots, t_\ell) = \mathrm{Tr}_V \, g, \tag{2.5}$$

the plethystic exponential computes the character of the symmetric algebra $S^k V$:

$$\sum_{k=1}^{\infty} s^k \, \mathrm{Tr}_{S^k V} g^k = \text{P.E.}[s \cdot F(t_1, t_2, \ldots, t_\ell)]. \tag{2.6}$$

The MacMahon function is a typical example of the plethystic exponential:

$$M(t) := \prod_{n=1}^{\infty} (1 - t^n)^{-n} = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k[t^k]^2}\right), \tag{2.7}$$

where we have introduced the notation

$$[x] := x^{\frac{1}{2}} - x^{-\frac{1}{2}} = -[x^{-1}]. \tag{2.8}$$

Note that

$$F(t)=\frac{1}{[t]^2}=t\frac{\partial}{\partial t}\left(\frac{1}{1-t}\right)=\sum_{n=1}^{\infty}nt^n. \tag{2.9}$$

Another example which is also ubiquitous in our computation is

$$(x;q)_\infty=\prod_{n=0}^{\infty}(1-xq^n)=\text{P.E.}\left[-\frac{x}{1-q}\right]=\text{P.E.}\left[\frac{x/\sqrt{q}}{[q]}\right]. \tag{2.10}$$

It is curious that the generating function of the counting of solid partitions does not seem to allow a plethystic expression. In fact, the conjecture of MacMahon, which assumes a plethystic form, fails.

3. ADHM-type equation as BPS condition

To write down the matrix equations of ADHM type, we introduce two vector spaces N and K with $\dim_{\mathbb{C}} N = n$ and $\dim_{\mathbb{C}} K = k$. They are associated with Dp-branes ($p = 2d = 4, 6, 8$) and $D0$-branes, respectively, and the dimensions give the numbers of these branes. An ADHM-type equation is supposed to describe the BPS bound states of $D0$-branes (instantons) with the background Dp-branes. In all the cases the equation of motion is invariant under the gauge symmetry $U(k)$ acting on the vector space K. Note that since the matrix equations of ADHM type describe the theory on $D0$-branes the gauge symmetry is $U(k)$, while $U(n)$ symmetry on Dp-branes is regarded as the flavor symmetry. In the following we list the equations of the quiver matrix model. There are two types of open string with boundaries on $D0$-branes: one is a $k\times k$ matrix in $\mathrm{Hom}_{\mathbb{C}}(K,K)$, where both ends are attached to $D0$-branes, and the other is a $k\times n$ matrix in $\mathrm{Hom}_{\mathbb{C}}(N,K)$ together with the conjugate, which describes open strings stretching between $D0$- and Dp-branes.

- $d=2, X=\mathbb{C}^2$ ($D0$–$D4$ system, the original ADHM equation) [24]:

$$\mu_{\mathbb{C}}=[B_1,B_2]+IJ=0, \tag{3.1}$$

$$\mu_{\mathbb{R}}(\zeta)=\left[B_1,B_1^\dagger\right]+\left[B_2,B_2^\dagger\right]+II^\dagger-J^\dagger J-\zeta\cdot E_{k\times k}=0\quad(\zeta>0), \tag{3.2}$$

  where $B_{1,2}\in\mathrm{Hom}_{\mathbb{C}}(K,K)$ and $I,J^\dagger\in\mathrm{Hom}_{\mathbb{C}}(N,K)$.
- $d=3, X=\mathbb{C}^3$ ($D0$–$D6$ system) [25–27]:

$$\mu_{\mathbb{C}}=\left[B_i,B_j\right]+\frac{1}{2}\epsilon_{ijk}\left[B_k^\dagger,Y\right]=0, \tag{3.3}$$

$$\mu_{\mathbb{R}}(\zeta)=\sum_{i=1}^{3}\left[B_i,B_i^\dagger\right]+\left[Y,Y^\dagger\right]+II^\dagger-\zeta\cdot E_{k\times k}=0\quad(\zeta>0), \tag{3.4}$$

$$\mu_B=Y\cdot I=0, \tag{3.5}$$

  where $B_{1,2,3},Y\in\mathrm{Hom}_{\mathbb{C}}(K,K)$ and $I\in\mathrm{Hom}_{\mathbb{C}}(N,K)$.
- $d=4, X=\mathbb{C}^4$ ($D0$–$D8$ system) [28,29]:

$$\mu_{\mathbb{C}}=[B_a,B_b]+\frac{1}{2}\Omega_{abcd}\left[B_c^\dagger,B_d^\dagger\right]=0, \tag{3.6}$$

$$\mu_{\mathbb{R}}(\zeta)=\sum_{i=1}^{4}\left[B_i,B_i^\dagger\right]+II^\dagger-\zeta\cdot E_{k\times k}=0\quad(\zeta>0), \tag{3.7}$$

where $B_{1,2,3,4} \in \mathrm{Hom}_{\mathbb{C}}(K,K)$, $I \in \mathrm{Hom}_{\mathbb{C}}(N,K)$, and Ω_{abcd} is a component of the Calabi–Yau four-form, with $\Omega \wedge \overline{\Omega} = \mathrm{vol}_8$.

The origin of Y in the case of $d = 3$ is rather subtle, but the equations can be obtained by a dimensional reduction of those for $d = 4$ by putting $Y = B_4$. Or we can regard it as a consequence of "tachyon condensation" [29]. The additional condition $\mu_B = 0$, which only appears for $d = 3$, means that $D0$-branes cannot escape along the normal direction to $D6$-branes. A similar condition appears in the BPS condition for the spiked instanton [30–33]. Thus, it might be more natural to consider a $D8$–$D6$–$D0$ system as a generalization of the spiked instanton. It has been argued that a constant B field (a background flux) is required for the existence of bound states of $D0$–$D6$ and $D0$–$D8$ systems [34,35] (see also Ref. [33] for a related discussion); ee implicitly assume that such a flux is turned on, if necessary.

In each case we can discard the D term condition $\mu_{\mathbb{R}}(\zeta) = 0$ (or the real component of the hyperKähler moment map) with $\zeta > 0$ in favor of the following stability condition:

If a subspace $K' \subset K$ satisfies $I(N) \subset K'$ and $B_a(K') \subset K'$, then $K' = K$,

with the gauge symmetry being complexified to $GL(k,\mathbb{C})$. We can show that the F-term condition $\mu_{\mathbb{C}} = 0$ implies that the B_a are commuting, $[B_a, B_b] = 0$. In the case of $d = 4$ it follows from $\mathrm{Tr}\,(\mu_{\mathbb{C}})^2 = 0$. In other cases we use the stability condition to show the vanishing of J or Y. Then, the stability condition implies that the vector space K is spanned by the action of B_a on the subspace ("vacuum") $I(N)$:

$$K = \mathbb{C}[B_a] \cdot I(N). \tag{3.8}$$

We will use this property when we compute the equivariant character of the tangent space in Sect. 5.

The formal complex dimensions of the moduli space are computed by subtracting the gauge degrees of freedom and constraints from the total number of components of the matrices:

- $d = 2$: $2k^2 + 2nk - k^2 - k^2 = 2nk$,
- $d = 3$: $3k^2 + k^2 + nk - k^2 - 3k^3 - nk = 0$,
- $d = 4$: $4k^2 + nk - k^2 - 3k^3 = nk$.

Note that if we did not introduce Y in $d = 3$, the computation would be

$$3k^2 + nk - k^2 - 3k^3 = (n-k)k,$$

and we cannot have a good moduli space. Since the dimensions are not necessarily even for $d = 4$, the moduli space cannot be hyperKähler. In fact, for $d > 2$ the moduli space is not smooth and the tangent space only has a virtual meaning.

When $n = 1$, which corresponds to Abelian gauge theory on Dp-branes, we expect the moduli space to be mathematically equivalent to the Hilbert scheme of k points on $\mathbb{C}^d$. It is known that when $d > 2$ it is qualitatively different from the case of $\mathbb{C}^2$ [36]. It is desirable to clarify the meaning of the generalized ADHM conditions from the viewpoint of the Hilbert scheme of k points on $\mathbb{C}^d$.

4. Matrix model description

One can construct a cohomological matrix model by imposing ADHM-type BPS conditions as a gauge-fixing condition of a cohomological matrix model, which is achieved in Becchi–Rouet–Stora–Tyutin (BRST) manner. In the case of $d = 2$, ADHM constraints are obtained as hyperKähler moment maps and this leads to integration over the Higgs branch of supersymmetric quantum mechanics [37,38]. Equivariant localization of topological (BRST) symmetry allows us to compute the partition function as a residue integral over the eigenvalues (diagonal elements) of the matrix. It turns out that the poles of the residue integral are labeled by partitions, and after the residue integral we obtain a summation over the partitions.

4.1. $d = 2$ (from localization to Macdonald polynomials)

Let $\{x_i\}_{i=1}^k$ be the Cartan variables of $GL(k)$ or the eigenvalues of $k \times k$ matrices. The equivariant integration over the instanton moduli space $\mathcal{M}_{n,k}$ is reduced to a contour integral,

$$Z_k = \frac{1}{k!} \oint \prod_{i=1}^{k} \frac{dx_i}{2\pi\sqrt{-1}x_i} z_k(x_i, u_\alpha, q_1, q_2), \tag{4.1}$$

where we have divided the integral by the order of the Weyl group (we will order the eigenvalues) and $\{u_\alpha = e^{a_\alpha}\}_{\alpha=1}^n$ is the Cartan variables for $GL(n)$ symmetry coming from n $D4$-branes. The parameters $q_i = e^{\epsilon_i}$ are Ω-background parameters or the equivariant parameters of the torus action $(z_1, z_2) \to (q_1 z_1, q_2 z_2)$ on $\mathbb{C}^2$. The full partition function is

$$Z^{4\mathrm{D}}(u_\alpha, q_i; \mathfrak{q}) = 1 + \sum_{k=1}^{\infty} \mathfrak{q}^k Z_k, \tag{4.2}$$

and we will see, by introducing the power sum function $p_n(x)$ of the eigenvalues, that the integrand $z_k(x_i, u_\alpha, q_1, q_2)$ allows a plethystic expression. Note that we may identify $\log z_k(x_i, u_\alpha, q_1, q_2)$ as an effective action of the matrix model. The contributions to $z_k(x_i, u_\alpha, t_1, t_2)$ are evaluated as follows:[2]

- Jacobian (Vandelmonde determinant) from the change of variables to diagonal variables:

$$\prod_{i \neq j} \left(1 - \frac{x_i}{x_j}\right). \tag{4.3}$$

  This factor is also regarded as the contribution of the $GL(k)$ gauge symmetry of ADHM constraints.
- Contribution of ADHM constraints:

$$\prod_{i,j} \left(1 - q_1 q_2 \frac{x_i}{x_j}\right) = (1 - q_1 q_2)^k \prod_{i \neq j} \left(1 - q_1 q_2 \frac{x_i}{x_j}\right). \tag{4.4}$$

- Contribution of matrix variables $B_{1,2}, I, J$:

$$\prod_{i,j} \left(1 - q_a \frac{x_i}{x_j}\right)^{-1} = (1 - q_a)^{-k} \prod_{i \neq j} \left(1 - q_a \frac{x_i}{x_j}\right) \quad \text{from } B_a \text{, and} \tag{4.5}$$

$$\prod_{i=1}^{k} \prod_{\alpha=1}^{n} \left(1 - \frac{x_i}{u_\alpha}\right)^{-1}, \qquad \prod_{i=1}^{k} \prod_{\alpha=1}^{n} \left(1 - q_1 q_2 \frac{u_\alpha}{x_i}\right)^{-1} \quad \text{from } I \text{ and } J. \tag{4.6}$$

[2] These contributions are in one-to-one correspondence with the terms in the equivariant character to be given in the next section.

Let us rescale the variable $u_\alpha \to \sqrt{q_1 q_2} u_\alpha$ to make the last two contributions symmetric:

$$\prod_{i=1}^{k}\prod_{\alpha=1}^{n}\left(1-\sqrt{q_1q_2}\frac{x_i}{u_\alpha}\right)^{-1}, \qquad \prod_{i=1}^{k}\prod_{\alpha=1}^{n}\left(1-\sqrt{q_1q_2}\frac{u_\alpha}{x_i}\right)^{-1}. \tag{4.7}$$

In terms of the function

$$S(z) := \frac{(1-z)(1-q_1q_2z)}{(1-q_1z)(1-q_2z)}, \tag{4.8}$$

we can write the integrand as follows:

$$z_k(x_i, a_\alpha, q_1, q_2) = \left(\frac{1-q_1q_2}{(1-q_1)(1-q_2)}\right)^k \frac{\displaystyle\prod_{i\neq j} S\left(\frac{x_i}{x_j}\right)}{\displaystyle\prod_{i=1}^{k}\prod_{\alpha=1}^{n}\left(1-\sqrt{q_1q_2}\frac{x_i}{u_\alpha}\right)\left(1-\sqrt{q_1q_2}\frac{u_\alpha}{x_i}\right)}. \tag{4.9}$$

Now, in terms of the power sum variables $p_m = \sum_{i=1}^{k} x_i^m$, we can rewrite the measure $z_k(x_i, u_\alpha, q_1, q_2)$ for the contour integral in a plethystic form:

$$\begin{aligned}
\log(z_k(x_i,u_\alpha,q_1,q_2)) &= \sum_{m=1}^{\infty}\frac{1}{m}(q_1^m+q_2^m-q_1^mq_2^m)\sum_{i,j=1}^{k}\left(\frac{x_i}{x_j}\right)^m - \sum_{m=1}^{\infty}\frac{1}{m}\sum_{i\neq j}\left(\frac{x_i}{x_j}\right)^m \\
&\quad + \sum_{m=1}^{\infty}\frac{(\sqrt{q_1q_2})^m}{m}\sum_{i=1}^{k}\sum_{\alpha=1}^{n}\left[\left(\frac{x_i}{u_\alpha}\right)^m + \left(\frac{u_\alpha}{x_i}\right)^m\right] \\
&= k\sum_{m=1}^{\infty}\frac{1}{m} - \sum_{m=1}^{\infty}\frac{1}{m}(1-q_1^m)(1-q_2^m)p_m p_{-m} \\
&\quad + \sum_{m=1}^{\infty}\frac{(\sqrt{q_1q_2})^m}{m}\sum_{\alpha=1}^{n}(p_m u_\alpha^{-m} + p_{-m}u_\alpha^m).
\end{aligned} \tag{4.10}$$

Using the holonomy variables

$$U_m := \sum_{\alpha=1}^{n} u_\alpha^m \tag{4.11}$$

of $U(n)$ gauge fields, we have

$$\begin{aligned}
&\log(z_k(x_i,u,q_1,q_2)) = \log\Lambda^k \\
&\quad + \sum_{m=1}^{\infty}\frac{1}{m}\Big[-(1-q_1^m)(1-q_2^m)p_m p_{-m} + (\sqrt{q_1q_2})^m(p_m U_{-m} + p_{-m}U_m)\Big],
\end{aligned} \tag{4.12}$$

where we have introduced $\log\Lambda := \sum_{m=1}^{\infty}\frac{1}{m}$. By the change of variables

$$\alpha_m := \frac{(\sqrt{q_1q_2})^m}{1-q_1^m}U_m - (1-q_2^m)p_m, \tag{4.13}$$

we can eliminate linear terms in p_m to obtain

$$\log(z_k(x_i,u,q_1,q_2)) = \log\Lambda^k + \sum_{m=1}^{\infty}\frac{1}{m}\left[-\frac{(1-q_1^m)}{(1-q_2^{-m})}\alpha_m\alpha_{-m} + \frac{1}{(1-q_1^{-m})(1-q_2^{-m})}\right]. \quad (4.14)$$

Thus, we have

$$Z_k = \frac{1}{k!}\frac{\Lambda^k}{\prod_{i,j=1}^{\infty}(1-q_1^i q_2^j)}\oint\prod_{i=1}^{k}\frac{dx_i}{2\pi\sqrt{-1}x_i}\exp\left(-\sum_{m=1}^{\infty}\frac{1}{m}\frac{(1-q_1^m)}{(1-q_2^{-m})}\alpha_m\alpha_{-m}\right). \quad (4.15)$$

We may eliminate Λ by the renormalization of the instanton expansion parameter $\mathfrak{q}$. The universal factor $\prod_{i,j=1}^{\infty}(1-q_1^i q_2^j)^{-1}$ shoud be identified with the perturbative factor.

In the Abelian case the contour integral in Eq. (4.15) is related to the inner product for Macdonald polynomials [39]. To see this, we should note that the poles of the contour integral are labeled by partitions λ with $|\lambda| = k$, and the positions of the poles are given by

$$x_i = u\cdot q_1^{a-\frac{1}{2}}q_2^{b-\frac{1}{2}}, \qquad (a,b)\in\lambda, \quad (4.16)$$

where $u = U_1$, and we have $U_m = u^m$ for the Abelian case. Hence, the power sum takes the following values at the poles:

$$p_1^{(\lambda)} = u\sum_{a=1}^{\ell(\lambda)}\sum_{b=1}^{\lambda_a} q_1^{a-\frac{1}{2}}q_2^{b-\frac{1}{2}}, \quad (4.17)$$

$$\alpha_1^{(\lambda)} = u\sqrt{q_1q_2}\sum_{i=1}^{\infty}q_1^{i-1}q_2^{\lambda_i}. \quad (4.18)$$

Thus, we recover the topological locus:

$$\xi_i = uq_1^{i-\frac{1}{2}}q_2^{\lambda_i+\frac{1}{2}}. \quad (4.19)$$

This also explains the implication of the change of variables in Eq. (4.13). In summary, after the contour integration we have

$$Z_k = \frac{1}{k!}\frac{\Lambda^k}{\prod_{i,j=1}^{\infty}(1-q_1^i q_2^j)}\sum_{|\lambda|=k}\exp\left(-\sum_{m=1}^{\infty}\frac{1}{m}\frac{(1-q_1^m)}{(1-q_2^{-m})}\alpha_m^{(\lambda)}\alpha_{-m}^{(\lambda)}\right). \quad (4.20)$$

Note that the measure factor coincides with the (q,t)-deformed Vandermonde determinant

$$\begin{aligned}\Delta_{q,t}(\xi)^2 &:= \exp\left(\sum_{k=1}^{\infty}\frac{1}{k}\frac{1-q^k}{1-t^k}(N-\alpha_k\alpha_{-k})\right)\\ &= \prod_{n=1}^{\infty}\prod_{1\le a\neq b\le N}\frac{1-t^n\xi_a/\xi_b}{1-qt^n\xi_a/\xi_b},\end{aligned} \quad (4.21)$$

with $(q,t) = (q_1, q_2^{-1})$. This is employed to define the inner product on the space of symmetric polynomials that leads to Macdonald polynomials [40].

The integrand of the residue integral can be expressed in terms of the plethystic exponential, and taking the logarithm we may recognize the “effective” action for the eigenvalues, which is in turn expressed by the power sum. Then, the integral can be related to the inner product for the Macdonald polynomials. This also means that the effective action is bilinear in the power sums (the free boson operators). To construct a refined topological vertex we have to insert a vertex operator. It is curious that the insertion induces the interaction term in the effective action.

4.2. $d = 3$

From the ADHM-type conditions, we can similarly obtain a contour integral representation of the partition function with instanton number k:

$$Z_k = \frac{1}{k!} \oint \prod_{i=1}^{k} \frac{dx_i}{2\pi\sqrt{-1}x_i} z_k(x_i, u_\alpha, q_a), \tag{4.22}$$

where

$$z_k(x_i, u_\alpha, q_a) = \frac{\prod_{i=1}^{k}\prod_{\alpha=1}^{n}\left(1 - q_1q_2q_3\frac{u_\alpha}{x_i}\right)\prod_{i\neq j}\left(1-\frac{x_i}{x_j}\right)\prod_{1\le a<b\le 3}\prod_{i,j}\left(1-q_aq_b\frac{x_i}{x_j}\right)}{\prod_{i=1}^{k}\prod_{\alpha=1}^{n}\left(1-\frac{x_i}{u_\alpha}\right)\prod_{a=1,2,3}\prod_{i,j}\left(1-q_a\frac{x_i}{x_j}\right)\prod_{i,j}\left(1-q_1q_2q_3\frac{x_i}{x_j}\right)}. \tag{4.23}$$

It is curious to see the role of the pole at $x_j = q_1q_2q_3x_i$ in the contour integral. An analogous computation to the case of $d = 2$ leads to the following plethystic form of the measure:

$$\log(z_k(x_i, u, q_a)) = \log \Lambda^k + \sum_{m=1}^{\infty} \frac{1}{m}\big[- (1-q_1^m)(1-q_2^m)(1-q_3^m)p_m p_{-m} + (\sqrt{q_1q_2q_3})^m (p_m U_{-m} - p_{-m}U_m)\big]. \tag{4.24}$$

The crucial change here is the relative sign in the linear terms, which prevents us making a complete square by a change of variable, as in Eq. (4.13). The flip of the relative sign causes an asymmetry in exchanging the positive modes and the negative modes. As discussed in the next section, this seems to be related to the fact that, in contrast to the case of $d = 2$, we do not have a hyperKähler (holomorphic symplectic) structure any more when $d = 3$.

4.3. $d = 4$

We obtain

$$z_k(x_i, u_\alpha, q_a) = \frac{\prod_{i\neq j}\left(1-\frac{x_i}{x_j}\right)\prod_{1\le a<b\le 3}\prod_{i,j}\left(1-q_aq_b\frac{x_i}{x_j}\right)}{\prod_{i=1}^{k}\prod_{\alpha=1}^{n}\left(1-\frac{x_i}{u_\alpha}\right)\prod_{a=1}^{3}\prod_{i,j}\left(1-q_a\frac{x_i}{x_j}\right)\prod_{i,j}\left(1-q_4^{-1}\frac{x_i}{x_j}\right)}. \tag{4.25}$$

As argued in the next section, we have to choose a "chiral-half" of the full Euler character, which depends on the ordering of the set $\{(ab) \mid 1 \le a \neq b \le 4\}$. Here we choose $\{(12), (13), (23); (14), (24), (34)\}$ by taking z_4 as a "preferred" direction. As argued in Ref. [33], due to the choice of the ordering, we should be careful with the order of the contour integral.

Using the Calabi–Yau condition $q_1q_2q_3q_4 = 1$, we can obtain a plethystic form of $z_k(x_i, u_\alpha, q_a)$ as follows:

$$\log(z_k(x_i, u_\alpha, q_a)) = \log \Lambda^k + \sum_{m=1}^{\infty} \frac{1}{m}\big[q_1^m + q_2^m + q_3^m + q_4^{-m}\big] p_m p_{-m}$$

$$- \sum_{m=1}^{\infty} \frac{1}{m}\big[1 + q_1^m q_2^m + q_1^m q_3^m + q_2^m q_3^m\big] p_m p_{-m} + \sum_{m=1}^{\infty} \frac{1}{m} p_m U_{-m}$$

$$= \log \Lambda^k + \sum_{m=1}^{\infty} \frac{1}{m} p_m U_{-m} - \sum_{m=1}^{\infty} \frac{1}{m}(1-q_1^m)(1-q_2^m)(1-q_3^m) p_m p_{-m}. \tag{4.26}$$

Remarkably, this is quite close to Eq. (4.24). Since this is a "chiral-half" of the full Euler character, only the negative modes U_{-m} appear.

5. Equivariant character of (virtual) tangent space at fixed points

The fixed points of the toric action of T^d on $\mathbb{C}^d$ and the Cartan subalgebra of the gauge symmetry G_C are labeled by an n-tuple of $(d-1)$ partitions. In terms of the equivariant parameters $u_\alpha := e^{a_\alpha}$ of the Cartan subgroup of $U(n)_C$, the character of the vector space N (the Chan–Paton bundle for the background Dp branes) is[3]

$$N = \sum_{\alpha=1}^{n} u_\alpha. \tag{5.1}$$

Then, from the structure of the vector space K in Eq. (3.8), its equivariant character at the fixed point $\{\pi_\alpha\}_{\alpha=1}^n$ is

$$K_\pi = \sum_{\alpha=1}^{n} u_\alpha \left(\sum_{(i,j,k)\in\pi^\alpha} q_1^{1-i} q_2^{1-j} q_3^{1-k} \right), \tag{5.2}$$

where for illustration we write the formula for $d = 3$, but generalization to other cases, where π stands for partition ($d = 2$) and solid partition ($d = 4$), should be clear. With these basic ingredients we can compute the (Euler) characters of the deformation complex for the ADHM-type equation in each dimension.

5.1. $d = 2$

The fixed points are labeled by an n-tuple of partitions (colored Young diagrams) λ_α and

$$\chi_{4D}(u_\alpha, q_i) = N^*K + q_1q_2K^*N - (1-q_1)(1-q_2)K^*K, \tag{5.3}$$

where the positive contributions N^*K and $t_1t_2K^*N$ come from I and J, $(q_1 + q_2)K^*K$ from $B_{1,2}$, while the negative ones $-q_1q_2K^*K$ from the F-term constraint and $-K^*K$ from the gauge symmetry. The difference in the numbers of positive coefficients (+1) and negative coefficients (−1) is $2nk$, which is exactly the (complex) dimensions of the tangent space. After cancellations only positive terms survive, and when $n = 1$ it has a nice combinatorial formula [24]:

$$\chi_{4D}(q_i) = \sum_{s\in\lambda} \left(q_1^{-\ell(s)} q_2^{a(s)+1} + q_1^{\ell(s)+1} q_2^{-a(s)} \right), \tag{5.4}$$

[3] With an abuse of notation we use the same notation for the character.

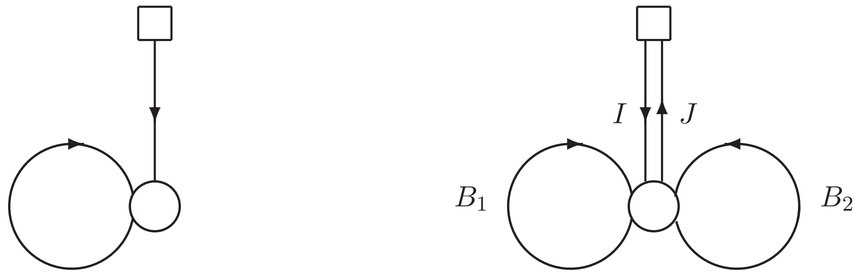

Fig. 2. ADHM quiver (right) as the double of the Jordan (framed $\widehat{A_0}$) quiver (left).

where $a(s)$ and $\ell(s)$ are the arm and the leg length of the box s in the Young diagram λ. Note that in the Abelian case the dependence on u_α disappears. In the non-Abelian ($n > 1$) case the fixed points are labeled by an n-tuple of Young diagrams $\vec{\lambda} = (\lambda^\alpha)$, and we need the arm and leg length of the box $s = (i,j) \in \lambda$ with respect to a second Young diagram μ:

$$a_\mu(i,j) := \nu_i - j, \qquad \ell_\mu(i,j) := \nu_j^\vee - i. \tag{5.5}$$

Then, an explicit formula for the equivariant character is

$$\begin{aligned} \chi_{4D}(u_\alpha, q_i) &= \sum_{\alpha,\beta=1}^{n} N_{\alpha\beta}, \\ N_{\alpha\beta}(u_\alpha, q_i) &= \frac{u_\beta}{u_\alpha}\left(\sum_{s\in\lambda^\alpha} q_1^{\ell_{\lambda^\beta}(s)} q_2^{a_{\lambda^\alpha}(s)+1} + \sum_{t\in\lambda^\beta} q_1^{\ell_{\lambda^\alpha}(s)+1} q_2^{-a_{\lambda^\beta}(s)}\right). \end{aligned} \tag{5.6}$$

Introducing the polarization[4]

$$P_2(u_\alpha, q_i) = N^*K + (q_1 - 1)K^*K \tag{5.7}$$

and the notation $\hbar := q_1 q_2$, we can express the character as follows:

$$P_2 + \hbar P_2^* = N^*K - (1 - q_1)K^*K + q_1q_2(K^*N - (1 - q_1^{-1})K^*K) = \chi_{4D}. \tag{5.8}$$

Note that $\hbar$ is the scaling factor of the symplectic form $\omega = dz_1 \wedge dz_2$. This decomposition of the equivariant character χ_{4D} reflects the fact that the moduli space of the ADHM matrix model is an example of Nakajima quiver varieties, which is defined as a hyperKähler quotient. The relevant quiver is called the Jordan quiver, which consists of a single vertex with a single loop (Fig. 2). More precisely, it is the framed Jordan quiver with a framing of $\mathbb{C}^n$. When $n = 1$ or in $U(1)$ gauge theory, the associated quiver variety is nothing but the Hilbert scheme $\mathrm{Hilb}_k\mathbb{C}^2$ of k-points on $\mathbb{C}^2$, where k is physically the number of $D0$-branes or the instanton number of the anti-self-dual connection. From this viewpoint the moduli space has the structure of a cotangent bundle and the polarization P_2 represents the contribution of the base space described by the Jordan quiver, where we subtract K^*K coming from the gauge symmetry. Then the second term corresponds to the fiber of the cotangent bundle, and the multiplication of the weight $\hbar$ is necessary.

[4] By definition, a polarization of a symplectic manifold X is an equivariant K theory class $P = T^{1/2}X \in K_T(X)$, such that the tangent space is represented as $TX = P + \hbar P^*$.

The equivariant character in Eq. (5.3) is also derived from the equivariant Chern character of the universal bundle $\mathcal{E}$ [41,42]. The virtue of this derivation is that it is applicable for more general gauge groups of type *SO* and *Sp* [43]. To construct the universal bundle $\mathcal{E}$, let m^I be local coordinates on the moduli space of instantons. The tangent space of the moduli space is spanned by solutions to the linearized equations with a gauge-fixing condition. Let $\{\psi^I_\mu(x,m)\}$ denote a basis of the tangent space at $m \in \mathcal{M}_{\rm inst}$. For a family of instantons $A_\mu(x,m)$ parametrized by m, we have

$$\frac{\partial A_\mu(x,m)}{\partial m_I} = h_{IJ}\psi^J_\mu + D_\mu \alpha_I. \tag{5.9}$$

Since the derivative of $A_\mu(x,m)$ does not necessarily satisfy the gauge-fixing condition we need a compensating gauge transformation $D_\mu\alpha_I$. With an appropriate choice of the gauge-fixing condition, for example $(D^*)^\mu\psi^I_\mu(x,m) = 0$, we can find a unique α_I. Combining A_μ with the parameter of the compensating gauge transformation α_I, we can define a one-form $\mathcal{A}(x,m) = A_\mu dx^\mu + \alpha_I dm^I$ which can be regarded as a connection of the universal bundle $\mathcal{E}$ on $\mathbb{R}^4 \times \mathcal{M}_{\rm inst}$ whose fiber is the fundamental representation $\mathbb{C}^n$ of $U(n)$. In the following we fix a complex structure of the space-time $\mathbb{R}^4$ and identify $\mathbb{R}^4 \simeq \mathbb{C}^2$. Then the spinor bundle $S^+ \oplus S^-$ on $\mathbb{R}^4$ is naturally identified with the space of $(0,k)$-forms $\Lambda^{(0,0)} \oplus \Lambda^{(0,1)} \oplus \Lambda^{(0,2)}$ on $\mathbb{C}^2$.[5] With this identification the Dirac operator is translated to the $\bar\partial$ operator.

The equivariant Chern character of the universal bundle $\mathcal{E}$ is computed as the Euler character of the complex

$$0 \longrightarrow K\otimes\Lambda^{(0,0)} \xrightarrow{\tau_z} K\otimes\Lambda^{(0,1)} \oplus N\otimes\Lambda^{(0,2)} \xrightarrow{\sigma_z} K\otimes\Lambda^{(0,2)} \longrightarrow 0, \tag{5.10}$$

where

$$\tau_z = \begin{pmatrix} B_1 - z_1 \\ B_2 - z_2 \\ J \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} -(B_2 - z_2) & B_1 - z_1 & I \end{pmatrix}, \tag{5.11}$$

and the ADHM condition guarantees that Eq. (5.10) is a complex: $\sigma_z \circ \tau_z = 0$. One can also check that $\mathrm{Ker}\,\sigma_z = \mathrm{Coker}\,\tau_z = 0$ [24]. Taking the alternating sum, we obtain

$$\mathrm{Ch}_q(\mathcal{E})(u_\alpha; q_i) = N(u_\alpha) - (1-q_1)(1-q_2)K(u_\alpha; q_i). \tag{5.12}$$

Now, the equivariant version of the index theorem tells us that the equivariant index of the Dirac operator coupled with the adjoint bundle $\mathcal{E}\otimes\mathcal{E}^*$ is[6]

$$\mathrm{Ind}_q\, \bar\partial_{\mathcal{E}\otimes\mathcal{E}^*} = \int_{\mathbb{C}^2} \mathrm{Ch}_q(\mathcal{E}\otimes\mathcal{E}^*)\mathrm{Td}_q(\mathbb{C}^2), \tag{5.13}$$

where the equivariant version of the Todd class is

$$\mathrm{Td}_q(\mathbb{C}^2) = \frac{x_1 x_2}{(1-e^{x_1})(1-e^{x_2})}, \tag{5.14}$$

[5] In general, the spinors on a Calabi–Yau manifold are equivalent to $(0,k)$-forms, where the chirality of the spinor corresponds to the parity of k.

[6] The Dirac operator on a complex manifold is related to the $\bar\partial$ operator by the twist of the square root of the determinant of the tangent bundle, which is trivial for the Calabi–Yau case.

where

$$x_i = \epsilon_i + \delta(z_i)\frac{dz_i \wedge d\bar{z}_i}{2\pi\sqrt{-1}} \tag{5.15}$$

are the equivariant Chern roots of the tangent bundle to $\mathbb{R}^4 \simeq \mathbb{C}^2$, given by equivariantly closed two-forms. We should use the Chern class of $\mathcal{E} \otimes \mathcal{E}^*$, because we consider the adjoint bundle whose fiber is the adjoint representation of $U(n)$. It shoud be easy to generalize the computation to the bi-fundamental representation. The integration over the space-time $\mathbb{C}^2 \simeq \mathbb{R}^4$ corresponds to the push-forward for the projection $\pi : \mathbb{R}^4 \times \mathcal{M}_{\text{inst}} \longrightarrow \mathcal{M}_{\text{inst}}$ and can be evaluated by the localization by the torus action $(z_1, z_2) \to (q_1 z_1, q_2 z_2)$, whose unique fixed point is the origin $z_1 = z_2 = 0$. The Hamiltonian of the torus action is $\epsilon_1|z_1|^2 + \epsilon_2|z_2|^2$, and the localization theorem for the equivariant closed forms gives

$$\int_{\mathbb{C}^2} \mathrm{Ch}_q(\mathcal{E} \otimes \mathcal{E}^*)\mathrm{Td}_q(\mathbb{C}^2) = \frac{\mathrm{Ch}_q(\mathcal{E} \otimes \mathcal{E})\mathrm{Td}_q(\mathbb{C}^2)|_{(0,0)}}{\epsilon_1\epsilon_2} = -\frac{N^*N}{(1-q_1)(1-q_2)} + \chi_{4D}, \tag{5.16}$$

where the first term, which survives even $k = 0$, is regarded as a perturbative part.

5.2. $d = 3$

The fixed points are labeled by an n-tuple of plane partitions and

$$\chi_{6D}(u_\alpha, q_i) = N^*K - q_1q_2q_3K^*N - (1-q_1)(1-q_2)(1-q_3)K^*K, \tag{5.17}$$

where N^*K, $(q_1+q_2+q_3)K^*K$, and $q_1q_2q_3K^*K$ come from dynamical matrix variables $I, B_{1,2,3}$, and Y, while $-(q_1q_2+q_2q_3+q_3q_1)K^*K$ comes from the F-term constraints and $-K^*K$ from the gauge symmetry. Finally, $-q_1q_2q_3K^*N$ comes from the constraint $\mu_B = 0$. When we impose the Calabi–Yau condition $q_1q_2q_3 = 1$, the character is anti-self-dual, $\chi_{6D} + \chi_{6D}^* = 0$, which is a consequence of the Serre duality. By the anti-self-duality, the measure on the space of plane partitions becomes uniform (up to sign); in fact it is $(-1)^{nk}$. Hence the partition function reduces to the MacMahon function.

Now, the analogue of the polarization in Eq. (5.7) in $d = 3$ is

$$P_3(u_\alpha, q_i) := N^*K + (q_1+q_2+q_3-1)K^*K, \tag{5.18}$$

and we set $\hbar = q_1q_2q_3$; then, we have

$$\chi_{6d} = P_3 - \hbar P_3^*. \tag{5.19}$$

Note that the relative sign between P and P^* should be negative for odd d. Consequently, the interpretation of the decomposition in Eq. (5.19) is rather different from the case $d = 2$. That is, Eq. (5.19) reflects what is called symmetric obstruction theory in mathematics, where the first term corresponds to the deformation space of matrix variables coming from the framed quiver with a single vertex and three loops with the subtraction of gauge symmetry, while the second term is the contributions from the obstruction space, which are represented by anti-ghosts for constraints and the secondary ghost for the gauge symmetry. The symmetric obstruction theory gives a moduli space of virtual dimension zero.

5.3. $d = 4$

The fixed points are labeled by an n-tuple of solid partitions and

$$\chi_{8D}(u_\alpha, q_i) = N^*K - (1-q_1)(1-q_2)(1-q_3)K^*K = P_4, \tag{5.20}$$

where N^*K and $(q_1 + q_2 + q_3 + q_1q_2q_3)K^*K$ come from I and $B_{1,2,3,4}$, while $-(q_1q_2 + q_2q_3 + q_3q_1)K^*K$ from the F-term constraints and $-K^*K$ from the gauge symmetry. Note that what we called the polarization in lower-dimensional cases is obtained as the character of the deformation complex of the ADHM-type condition. In other words, Eq. (5.20) is a "chiral-half" of the full Euler character $\chi_E(\mathcal{E}) = \sum_{i=0}^{4}(-1)^i \mathrm{Ext}^i(\mathcal{E}, \mathcal{E})$:

$$\chi_E(u_\alpha, q_i) = N^*K + K^*N - (1-q_1)(1-q_2)(1-q_3)(1-q_4)K^*K = \chi_{8D} + \chi_{8D}^*,$$

where we have used $\hbar = q_1q_2q_3q_4 = 1$. Contrary to the odd-dimensional case, the full character is self-dual, $\chi_E^* = \chi_E$. To define a "chiral-half" of the full Euler character, we use the real structure of $\mathrm{Ext}^2(\mathcal{E}, \mathcal{F})$, which is allowed by the Serre duality of $\mathrm{Ext}^i(\mathcal{E}, \mathcal{E})$. This seems consistent with the idea that $d = 4$ theory is a holomorphic version of the Donaldson theory [10]. By taking a "chiral-half" of the full Euler character we consider a square root of the tangent space and hence there is an ambiguity of the choice of sign. We can fix it locally, but the global consistency is a non-trivial issue. Mathematically this is the problem of the orientability of the moduli space.

6. Topological partition function

As emphasized in Sect. 1, all the partition functions in the following can be expressed as plethystic exponentials.

6.1. $d = 2$ (Abelian $\mathcal{N} = 2^*$ theory)

Using the formula in Eq. (5.4) the partition function of $U(1)$ theory with adjoint matter is

$$Z^{4D}_{U(1),\mathrm{adj}}(q_a, \mu; \mathfrak{q}) = \sum_\lambda \mathfrak{q}^{|\lambda|} \prod_{s\in\lambda} \frac{1 - \mu q_1^{-\ell(s)} q_2^{a(s)+1}}{1 - q_1^{-\ell(s)} q_2^{a(s)+1}} \frac{1 - \mu q_1^{\ell(s)+1} q_2^{-a(s)}}{1 - q_1^{\ell(s)+1} q_2^{-a(s)}}, \tag{6.1}$$

where the parameter $\mu := e^{-m}$ is the equivariant (mass) parameter for the $U(1)$ flavor symmetry of the adjoint matter. Thus, physically $Z^{4D}_{U(1),\mathrm{adj}}$ is the Nekrasov partition function of $\mathcal{N} = 2^*$ theory. We can show that it has the following plethystic form [39,44–46]:

$$\begin{aligned} Z^{4D}_{U(1),\mathrm{adj}}(q_a, \mu; \mathfrak{q}) &= \mathrm{P.E.}\left[F(q_a, \mu, \mathfrak{q})\right], \\ F(q_a, \mu; \mathfrak{q}) &:= \frac{-\sqrt{\mu\mathfrak{q}}[\mu q_1][\mu q_2]}{[q_1][q_2][\mu\mathfrak{q}]} = \frac{\mathfrak{q}}{1-\mu\mathfrak{q}} \frac{(1-\mu q_1)(1-\mu q_2)}{(1-q_1)(1-q_2)}. \end{aligned} \tag{6.2}$$

There are two natural limits for μ: the decoupling limit $\mu \to 0$ and the massless ($\mathcal{N} = 4$) limit $\mu \to 1$. In the latter case the measure on the space of partitions is uniform and we obtain the generating function of the counting of partitions:

$$Z^{4D}_{U(1),\mathcal{N}=4}(\mathfrak{q}) = \mathrm{P.E.}\left[\frac{\mathfrak{q}}{1-\mathfrak{q}}\right] = \frac{1}{(\mathfrak{q};\mathfrak{q})_\infty}. \tag{6.3}$$

On the other hand, in the former limit the measure becomes the (refined) Plancherel measure and corresponds to the pure $U(1)$ theory, which is geometrically engineered by the conifold geometry:

$$Z^{4D}_{U(1),\mathrm{adj}}(q_a; \mathfrak{q}) = \mathrm{P.E.}\left[\frac{\mathfrak{q}/\sqrt{q_1q_2}}{[q_1][q_2]}\right]. \tag{6.4}$$

When $q_1 = q_2^{-1} = t$ it gives a generalized McMahon function,

$$Z^{4D}_{U(1),\mathrm{adj}}(t;\mathfrak{q}) = \mathrm{P.E.}\left[\frac{-\mathfrak{q}}{[t]^2}\right] = \prod_{n=1}^{\infty}(1+\mathfrak{q}t^n)^{-n}, \tag{6.5}$$

where $\mathfrak{q}$ plays the role of the Kähler parameter of the conifold. Thus, this example gives a kind of interpolation between the counting of partitions and plane partitions.

Equation (6.2) can be deduced as follows.[7] First we note the "removable" boxes of a non-empty partition have vanishing leg and arm length $\ell(s) = a(s) = 0$, because if they have non-empty leg or arm, we cannot remove them from the diagram. From the measure factor in Eq. (6.1) we see that the measure on the non-empty partition has zeros at $\mu q_1 = 1$ and $\mu q_2 = 1$. Note that these zeros are preserved under $(q_1, q_2, \mu) \to (q_1^k, q_2^k, \mu^k)$. Thus, we conclude that F has the factor $[\mu q_1] \cdot [\mu q_2]$. Moreover, when $\mu = 1$ the measure is independent of q_1 and q_2. Hence, we arrive at

$$F(q_a, \mu; \mathfrak{q}) \sim \frac{[\mu q_1] \cdot [\mu q_2]}{[q_1] \cdot [q_2]}. \tag{6.6}$$

Now let us specialize $q_2 = q_1^{-1}$ and take the limit $q_1 \to 0$. Then

$$Z^{4D}_{U(1),\mathrm{adj}}(q_a, \mu; \mathfrak{q}) = \sum_{\lambda} \mathfrak{q}^{|\lambda|} \prod_{s\in\lambda} \frac{q_1^{h(s)} - \mu}{q_1^{h(s)} - 1} \frac{1 - \mu q_1^{h(s)}}{1 - q_1^{h(s)}} \to \sum_{\lambda} (\mu\mathfrak{q})^{|\lambda|}, \tag{6.7}$$

where $h(s) = \ell(s) + a(s) + 1$ is the hook length. On the other hand,

$$\frac{[\mu q_1] \cdot [\mu q_2]}{[q_1] \cdot [q_2]} = \frac{[\mu q_1] \cdot [\mu^{-1} q_1]}{[q_1]^2} \to 1 \tag{6.8}$$

in this limit. Hence, we find that

$$F(q_a, \mu; \mathfrak{q}) \sim \frac{\mu\mathfrak{q}}{1 - \mu\mathfrak{q}} \frac{[\mu q_1] \cdot [\mu q_2]}{[q_1] \cdot [q_2]} = \frac{-\sqrt{\mu\mathfrak{q}}[\mu q_1][\mu q_2]}{[\mu\mathfrak{q}][q_1][q_2]}. \tag{6.9}$$

We can also prove Eq. (6.2) by assuming the invariance of the topological string amplitudes under the change of the preferred direction of the refined topological vertex [46]. When we deduce the refined topological vertex from the equivariant vertex, the preferred direction is related to the ways the limit $|q_i| \to \infty$ can keep the Calabi–Yau combination of q_i constant (see the next subsection). This reminds us of the fact that the perturbative string theory can be obtained by taking appropriate limits of M theory.

The $\mathcal{N} = 2^*$ theory can be regarded as $\widehat{A_0}$ quiver gauge theory. As we have seen in the last section, the quiver for the ADHM equation is the double of the framed $\widehat{A_0}$ quiver, and when the framing is $U(1)$ the Nakajima variety is nothing but the Hilbert scheme of a point on $\mathbb{C}^2$. This coincidence seems to be the origin of the symmetric property of the topological partition function derived above.

[7] Strictly speaking, we assume that the partition function has a plethystic form.

6.2. $d = 3$

For the Abelian case $n = 1$, by the localization theorem the partition function is given by the summation over the plane partitions:[8]

$$Z^{6D}_{U(1)}(q_a;\mathfrak{q}) = \sum_{\pi} (-\mathfrak{q})^{|\pi|}\hat{\mathbf{a}}(\chi_\pi) = \left\langle \widehat{\text{P.E.}}[\chi_{6D}] \right\rangle, \tag{6.10}$$

where $\hat{\mathbf{a}}$ is defined by

$$\hat{\mathbf{a}}\left(\sum_i m_i w_i\right) = \prod_i [w_i]^{m_i}, \qquad m_i \in \mathbb{Z}, w_i \in T^\vee. \tag{6.11}$$

Here, m_i is the multiplicity of the character (weight) w_i of the torus T that acts on the moduli space. Note that $\hat{\mathbf{a}}$ gives the character of symmetrized symmetric products:

$$\hat{\mathbf{a}}(-\text{character of } V) = \text{character of } \hat{S}^\bullet V, \tag{6.12}$$

where $\hat{S}^\bullet V = (\det V)^{\frac{1}{2}} \cdot S^\bullet V$.

It turns out that the partition function allows a plethystic expression [25,36]:

$$Z^{6D}_{U(1)}(q_a;\mathfrak{q}) = \text{P.E.}\left[F_1(q_a,\mathfrak{q})\right], \tag{6.13}$$

where

$$F_1(q_a;\mathfrak{q}) = \frac{[q_1q_2][q_2q_3][q_3q_1]}{[q_1][q_2][q_3][\sqrt{\hbar}\mathfrak{q}][\sqrt{\hbar}/\mathfrak{q}]}, \tag{6.14}$$

with $\hbar := q_1q_2q_3$. It is tempting to identify the parameter $\hbar$ with the mass parameter μ in the previous example. In fact, both are related to the weight of the line bundle over $X = \mathbb{C}^2$ and $X = \mathbb{C}^3$, where the total space is six and ten dimensions, respectively. However, an important difference here is that the decoupling limit is not well defined, while the Calabi–Yau limit $\hbar \to 1$ is still well defined. It seems this is related to the fact that the tangent space is smooth in the $d = 2$ case, but it is singular (the tangent space only has a meaning as a virtual bundle) for $d > 2$. In the Calabi–Yau limit the partition function reduces to the MacMahon function,

$$F(q_a;\mathfrak{q}) = \frac{[q_1^{-1}][q_2^{-1}][q_3^{-1}]}{[q_1][q_2][q_3][\mathfrak{q}][\mathfrak{q}^{-1}]} = \frac{1}{[\mathfrak{q}]^2}. \tag{6.15}$$

Another interesting limit is the "refined topological vertex limit,"where we take $q_1, q_3 \to 0$ with $|q_1| \ll |q_3|$ and $q_2 \to \infty$, keeping $\hbar$ constant. In such a limit, q_3 corresponds to a preferred direction of the refined topological vertex. From the relation

$$\frac{[\hbar t]}{[t]} \to \begin{cases} \hbar^{\frac{1}{2}} & t \to \infty, \\ \hbar^{-\frac{1}{2}} & t \to 0, \end{cases} \tag{6.16}$$

we find

$$F(q_a;\mathfrak{q}) = \frac{-[q_1\hbar][q_2\hbar][q_3\hbar]}{[q_1][q_2][q_3][\sqrt{\hbar}\mathfrak{q}][\sqrt{\hbar}/\mathfrak{q}]} \to \frac{-\hbar^{-\frac{1}{2}}}{[\sqrt{\hbar}\mathfrak{q}][\sqrt{\hbar}/\mathfrak{q}]} = \frac{-1/\sqrt{q_4q_5}}{[q_4][q_5]}, \tag{6.17}$$

[8] In the six-dimensional case the natural counting parameter is $(-1)^n\mathfrak{q}$, because with this choice the partition function of $U(n)$ theory reduces to the nth power of the MacMahon function [27,47] in the Calabi–Yau limit $\hbar \to 1$.

which can be identified with the refined conifold amplitude with the Kähler parameter -1.

When $n > 1$ (the non-Abelian case) the fixed points are labeled by an n-tuple of plane partitions (colored partitions) $\vec{\pi} = (\pi^\alpha)_{\alpha=1}^n$ and the topological partition function is

$$Z^{6D}_{U(n)}(u_\alpha, q_a; \mathfrak{q}) = \sum_{\vec{\pi}} ((-1)^n \mathfrak{q})^{|\vec{\pi}|} \prod_{\alpha,\beta=1}^{n} \hat{\mathbf{a}}(V_{\alpha\beta}), \tag{6.18}$$

where

$$V_{\alpha\beta}(u_\alpha, q_a) = \frac{u_\alpha}{u_\beta} \left(\sum_{(i,j,k)\in\pi^\beta} q_1^{1-i} q_2^{1-j} q_3^{1-k} - \sum_{(r,s,t)\in\pi^\alpha} q_1^r q_2^s q_3^t \right.$$
$$\left. -(1-q_1)(1-q_2)(1-q_3) \sum_{\substack{(r,s,t)\in\pi^\alpha \\ (i,j,k)\in\pi^\beta}} q_1^{r-i} q_2^{s-j} q_3^{t-k} \right). \tag{6.19}$$

One of the significant properties of $Z^{6D}_{U(n)}(u_\alpha, q_a; \mathfrak{q})$ so defined is that it is completely independent of the equivariant parameters u^α for the framing torus, or the Coulomb branch moduli, which physically means the distance of the $D6$-branes. For lower instanton numbers this crucial property in proved in Ref. [47] by checking the vanishing of residues at the possible poles of $Z^{6D}_{U(n)}(q_a; \mathfrak{q})$ as a rational function in the equivariant parameters u^α. Quite recently, it was proved for arbitrary k by examining the contour integral representation of the partition function discussed in the last section [49]. Once we know that $Z^{6D}_{U(n)}(q_a; \mathfrak{q})$ is independent of u^α, we can evaluate the partition function in a judicious scaling of u^α, for example $u^\alpha = L^\alpha$ with $L \to \infty$ [47]. Then we can see that, for $\alpha < \beta$ [49],

$$\lim_{L\to\infty} \hat{\mathbf{a}}(V_{\alpha\beta})\hat{\mathbf{a}}(V_{\beta\alpha})|_{u_\alpha = L^\alpha} = (-\hbar^{\frac{1}{2}})^{|\pi^\beta| - |\pi^\alpha|}, \tag{6.20}$$

which implies

$$\begin{aligned}
\lim_{L\to\infty} Z^{6D}_{U(n)}(L^\alpha, q_a; \mathfrak{q}) &= \sum_{\vec{\pi}} ((-1)^n \mathfrak{q})^{|\vec{\pi}|} \prod_{\alpha=1}^{n} \hat{\mathbf{a}}(V_{\alpha\alpha}) \prod_{1\le\alpha<\beta\le n} (-\hbar^{\frac{1}{2}})^{|\pi^\beta| - |\pi^\alpha|} \\
&= \sum_{\vec{\pi}} \prod_{\alpha=1}^{n} ((-1)^n \mathfrak{q})^{|\pi^\alpha|} \hat{\mathbf{a}}(V_{\alpha\alpha}) (-\hbar^{\frac{1}{2}})^{(-n-1+2\alpha)|\pi^\alpha|} \\
&= \prod_{\alpha=1}^{n} Z^{6D}_{U(1)}(q_a; \mathfrak{q}\hbar^{\alpha - \frac{n+1}{2}}).
\end{aligned} \tag{6.21}$$

Hence, we have a factorization of the $U(n)$ partition function in Eq. (6.18) into a product of n $U(1)$ partition functions with shifted instanton number counting parameters [29,48]. This factorization surely relies on the independence of $Z^{6D}_{U(n)}(q_a; \mathfrak{q})$ of the Coulomb moduli, and is related to the orbifold action of $\mathbb{Z}_n$ on the transverse direction to the $D6$-branes.

In fact, we can use the following identity for generic variables z_1, z_2 to derive the $U(n)$ partition function in Eq. (6.13) from Eq. (6.21):

$$\sum_{\ell=1}^{n} \frac{1}{[z_1^{n+1-\ell} z_2^{1-\ell}][z_1^{\ell-n} z_2^{\ell}]} = \frac{1}{n} \sum_{\ell=0}^{n-1} \frac{1}{[\omega^\ell z_1][\omega^{-\ell} z_2]} = \frac{[(z_1 z_2)^n]}{[z_1 z_2][z_1^n][z_2^n]}, \tag{6.22}$$

where ω is an nth root of unity, $\omega^n = 1$. The second equality of Eq. (6.22) is just a simple consequence of

$$\frac{1}{n}\sum_{\ell=0}^{n-1}\omega^{k\ell} = \delta_{0,k(\mathrm{mod}\ n)}. \tag{6.23}$$

But, as pointed out in Ref. [29], it is amusing to note that the first equality of Eq. (6.22) allows a geometric interpretation, though it can also be checked by induction on n. To see the geometric meaning let us consider the asymptotically locally Euclidean (ALE) resolution $\widetilde{S_n} \longrightarrow \mathbb{C}^2/\mathbb{Z}_n$ of the $\mathbb{Z}_n$-orbifold of $\mathbb{C}^2$, where $\mathbb{Z}_n$ acts by $(z_1, z_2) \to (\omega z_1, \omega^{-1} z_2)$. We can compare the equivariant index of the Dirac operator before and after the resolution. Recall that the equivariant index of the Dirac operator on $\mathbb{C}^2$ is simply

$$\mathrm{Ind}_q D = \frac{1}{[z_1][z_2]}. \tag{6.24}$$

Then, we can recognize the middle of Eq. (6.22) as the orbifold version of the equivariant index. On the other hand, on the ALE space $\widetilde{S_n}$ there appear n fixed points over the origin, which is the original fixed point of the torus action. For example, when $n = 2$, the ALE space $\widetilde{S_2}$ is nothing but the Eguchi–Hanson space [50], which is isomorphic to the cotangent bundle of $\mathbb{CP}^1$. Thus we find two fixed points: the north and the south poles of $\mathbb{CP}^1$. The weights at these fixed points are exactly those that appear on the left-hand side of Eq. (6.22). Hence, it is the blow-up version of the index. Mathematically, the equality of the two versions follows from the fact the the fiber of the resolution is compact [29].

Since we already know that the $U(1)$ partition function has the plethystic form in Eq. (6.13), we can compute a plethystic form of the $U(n)$ partition function in Eq. (6.13) as follows:

$$Z^{6D}_{U(n)}(q_a; \mathfrak{q}) = \mathrm{P.E.}\left[F_n(q_a, \mathfrak{q})\right], \tag{6.25}$$

where

$$\begin{aligned} F_n(q_a; \mathfrak{q}) &:= \sum_{\alpha=1}^{n} F_1(q_a; \mathfrak{q}\hbar^{\alpha-\frac{n+1}{2}}) \\ &= \frac{[q_1q_2][q_2q_3][q_3q_1]}{[q_1][q_2][q_3]}\sum_{\alpha=1}^{n}\frac{1}{[\mathfrak{q}\hbar^{\alpha-\frac{n}{2}}][\mathfrak{q}^{-1}\hbar^{1-\alpha+\frac{n}{2}}]}. \end{aligned} \tag{6.26}$$

By applying Eq. (6.22) with $z_1 = \hbar^{\frac{1}{2}}\mathfrak{q}^{-\frac{1}{n}}$, $z_2 = \hbar^{\frac{1}{2}}\mathfrak{q}^{\frac{1}{n}}$, we finally obtain

$$F_n(q_a; \mathfrak{q}) = \frac{[q_1q_2][q_2q_3][q_3q_1][\hbar^n]}{[q_1][q_2][q_3][\hbar][\hbar^{\frac{n}{2}}\mathfrak{q}][\hbar^{\frac{n}{2}}\mathfrak{q}^{-1}]}. \tag{6.27}$$

In Ref. [13] the equivariant (K-theory or M-theory) vertex is defined by

$$V(\lambda, \mu, \nu) = \sum_{\pi \to (\lambda,\mu,\nu)} (-\mathfrak{q})^{|\pi|}\hat{\mathbf{a}}(\chi_\pi), \tag{6.28}$$

where the summation is taken for the plane partitions with a fixed asymptotic condition (λ, μ, ν). Note that $|\pi|$ and χ_π have to be regularized by taking the edge contributions into account. It was recently shown that if one of three partitions (λ, μ, ν) is empty, $V(\lambda, \mu, \nu)$ has a plethystic expression [51]. It will be very interesting to see if this property holds for the full vertex.

6.3. $d = 4$

The following plethystic form of the partition function is conjectured in Refs. [28,29]:

$$Z^{8D}_{U(n)}(q_a, \nu_\alpha, \mu_\alpha; \mathfrak{q}) = \text{P.E.}\left[F(q_a, \frac{\prod \nu_\alpha}{\prod \mu_\alpha}, -\mathfrak{q})\right],$$

where

$$F(q_a, s; \mathfrak{q}) := \frac{[q_1q_2][q_2q_3][q_3q_1][s]}{[q_1][q_2][q_3][q_4][\sqrt{s}\mathfrak{q}][\mathfrak{q}/\sqrt{s}]}.$$

Note that $[q_4] = -[q_1q_2q_3]$ due to the Calabi–Yau condition $q_1q_2q_3q_4 = 1$. $\nu_\alpha = e^{ia_\alpha}$ are the Coulomb branch parameters (associated with the position of the $D8$-branes) for the gauge symmetry $U(n)_C$, and $\mu_\alpha = e^{-m_\alpha}$ is the mass parameter (associated with the position of the $\overline{D8}$-branes) for the flavor symmetry $U(n)_F$. It is remarkable that the final result only depends on the ratio $\prod \nu_\alpha/\prod \mu_\alpha$, which is comparable to the fact that the partition function is independent of the Coulomb moduli u_α in six dimensions.

For $U(1)$ theory we can take $\nu = 1$ by choosing the position of a single brane as the origin. With $\mu = e^{-m}$ we find

$$F(q_a, \mu; \mathfrak{q}) = \frac{-[q_1q_2][q_2q_3][q_3q_1][\mu]}{[q_1][q_2][q_3][q_4][\sqrt{\mu}\mathfrak{q}][\mathfrak{q}/\sqrt{\mu}]} = \frac{[q_1q_2][q_2q_3][q_3q_1][\mu]}{[q_1][q_2][q_3][q_4][\sqrt{\mu}\mathfrak{q}][\sqrt{\mu}/\mathfrak{q}]}.$$

It seems that we cannot produce the uniform measure on the space of solid partitions by tuning parameters. This is consistent with the fact that there is no known plethystic formula for the generating function of the counting of solid partitions. For example, the massless limit $\mu \to 1$ gives the trivial result $F = 0$.

Let us put $q_a = e^{-R\epsilon_a}$ and $\mu = e^{-Rm}$, and take the limit $R \to 0$; then

$$\begin{aligned} F(t_a, \mu; \mathfrak{q}) &\to \exp\left(\frac{m(\epsilon_1+\epsilon_2)(\epsilon_2+\epsilon_3)(\epsilon_3+\epsilon_1)}{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}\sum_{n=1}^{\infty}\frac{1}{n}\frac{\mathfrak{q}^n}{(1-\mathfrak{q}^n)}\right) \\ &= M(\mathfrak{q})^{\frac{m(\epsilon_1+\epsilon_2)(\epsilon_2+\epsilon_3)(\epsilon_3+\epsilon_1)}{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}}. \end{aligned} \tag{6.29}$$

It is the MacMahon function that appears in this limit. The generating function of the counting of the solid partition never appears.

It is interesting that a reduction to six dimensions is achieved by tuning the mass parameters (the positions of the $\overline{D8}$-branes) which triggers a tachyon condensation of the $D8$–$\overline{D8}$ system to $D6$-branes [29]. The condition is $\nu_\alpha = q_4\mu_\alpha$, which gives $s = q_4^n$, and we obtain

$$F(q_a; \mathfrak{q}) := \frac{[q_1q_2][q_2q_3][q_3q_1][\hbar^n]}{[q_1][q_2][q_3][\hbar][\hbar^{\frac{1}{2}}\mathfrak{q}][\mathfrak{q}\hbar^{-\frac{1}{2}}]},$$

where $\hbar = q_1q_2q_3 = q_4^{-1}$. Up to sign, this agrees with Eq. (6.27).

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