Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 969 (2021) 115478

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

The dynamics of non-perturbative phases via Banach bundles

Ali Shojaei-Fard

1461863596 Marzdaran Blvd., Tehran, Iran

Received 18 January 2021; received in revised form 27 May 2021; accepted 26 June 2021
Available online 29 June 2021
Editor: Clay Córdova

Abstract

Strongly coupled Dyson–Schwinger equations generate infinite power series of running coupling constants together with Feynman diagrams with increasing loop orders as coefficients. Theory of graphons for sparse graphs can address a new useful approach for the study of graph limits of sequences of partial sums corresponding to these infinite power series in the context of Feynman graphons and the cut-distance topology. Graphon models enable us to associate some new analytic graphs to non-perturbative solutions of Dyson–Schwinger equations. Homomorphism densities of Feynman graphons provide a new way of analyzing non-perturbative phase transitions. We explain the structures of topological renormalization quotient Hopf algebras of Feynman graphons which encode gauge symmetries Hopf ideals in the context of the weakly isomorphic equivalence classes corresponding to the Slavnov–Taylor / Ward–Takahashi elements. We apply Feynman graphon representations of combinatorial Dyson–Schwinger equations underlying the Connes–Kreimer renormalization Hopf algebra to construct a new class of Banach bundles for the study of the dynamics of non-perturbative phases in strongly coupled gauge field theories.

Contents

E-mail address: shojaeifa@yahoo.com.

https://doi.org/10.1016/j.nuclphysb.2021.115478

1. Introduction

On the one hand, Dyson–Schwinger equations (DSEs), as fixed point equations of Green's functions [29,50], are original tools for the study of non-perturbative aspects of gauge field theories [19,21,39,40] under strong running coupling constants [15,49]. The combinatorial versions of Green's functions are useful to formulate the solutions of DSEs in terms of power series of running coupling constants together with higher loop order Feynman diagrams as coefficients. These fundamental equations have been studied in terms of different settings derived from lattice models [16], large N limits [35,36] and the renormalization Hopf algebra [17,26,29,30,41–43,52,56,57]. On the other hand, graphons or graph functions, as symmetric bounded Lebesgue measurable functions defined on the domain $[0,1] \times [0,1]$ with images in $[0,1]$, are useful tools in infinite combinatorics for the description of graph limits of sequences of finite weighted graphs. Graphons complete the topological space of finite graphs with respect to the cut-distance topology. Homomorphism densities with respect to graphons are applied for the study of the behavior of complex networks and subgraphs in dense graphs. The interrelationship between graphons and random graphs has provided some new interesting applications of combinatorial methods to different fields of Science. The theory of graphons has been developed for dense and sparse graphs under separate settings. Stretched or rescaled graphons are generalized versions of graphons which are defined on some more general measure spaces such as the Lebesgue measure space $(\mathbb{R}_+, m)$ and L^p spaces. These analytic graphs, which have a continuum vertex set, are useful in dealing with the notion of convergence for infinite sequences of dense or sparse graphs [5,6,11,20,23,33,34]. Furthermore, recently, the Connes–Kreimer renormalization Hopf algebra has been topologically enriched via the topology of graphons to achieve some new analytic descriptions of Feynman diagrams and (non-perturbative) solutions of combinatorial DSEs in the context of graphon models. The completeness (and compactness) of the space of Feynman graphons allow us to study graph limits of sequences of Feynman diagrams with increasing loop numbers and their formal expansions in the context of infinite random graph models. This platform, which clarifies some applications of infinite combinatorics to Quantum Field Theory, has already provided some new practical tools in computing non-perturbative parameters and describing analytic evolution of strongly coupled DSEs in terms of the behavior of the partial sums [44–48].

In this research work, we apply graphon models to equip the space of Feynman diagrams of a given strongly coupled gauge field theory with the cut-distance topology. This topological setting enables us to formulate the topological renormalization Hopf algebra of Feynman diagrams of the physical theory which leads us to obtain a new completed topological Hopf algebra of Feynman graphons associated to Feynman diagrams of the physical theory. The loop number graduation is useful in dealing with DSEs in scalar theories with only one vertex-type. While the loop number graduation becomes problematic to study DSEs in gauge field theories with more than one vertex-type, this particular graduation parameter provides some advantages in our graphon setting. In other words, the loop number graduation determines the bare scaling of the pixel picture presentations of Feynman diagrams for the construction of suitable Feynman graphon models which are compatible with the combinatorial information of Feynman diagrams. At this level, the compactness of the topology of graphons enables us to control the behavior of higher loop order Feynman diagrams in the context of graph limits. The first consequence of this approach is to build a new class of graphon models for the description of non-perturbative solutions of combinatorial DSEs. The topological renormalization Hopf algebra of Feynman graphons and its quotient Hopf algebras with respect to the Hopf ideals generated by Feynman graphon representations of the Slavnov–Taylor / Ward–Takahashi elements provide a well-defined formulation for the renormalization coproduct of the complete Green's functions of Feynman graphons and solutions of DSEs. This new platform is useful to study the phase transitions of non-perturbative aspects in the context of homomorphism densities. In addition, it allows us to build a new non-perturbative generalization of the Connes–Kreimer Renormalization Group. Furthermore, we build a new Banach manifold structure on the space of all combinatorial DSEs in the physical theory which can be equipped with three fundamental Banach bundles (i.e. tangent bundle, Hopf bundle and regularization bundle). We show that mathematical tools derived from these Banach bundles can geometrically describe the dynamics of Hadron Physics in non-perturbative phases in terms of trajectories of DSEs under different running coupling constants.

1.1. Physical motivation

Generally speaking, lattice gauge field theories have provided some rigorous progresses in dealing with the dynamics of quantum systems in strong running coupling constants. The many flavor SU(3) gauge field theories have led us to some practical models for the study of physical theories beyond the Standard Model. Strongly coupled IR dynamics of these physical theories should be considered via non-perturbative settings while in the range $N_f^c \leq N_f < 16.5$, the physical theories tend to a chirally symmetric conformal fixed point [7,16,18,22,40].

Quantum Chromodynamics (QCD) is the fundamental model for the description of strong interactions of quarks and gluons under chiral symmetry breaking. QCD is explained as a matrix-valued modification of the electromagnetic theory in terms of replacing photons by gluons and electrons by quarks. Quantum fluctuations of the fields determine the force law while gluons, which carry the color charge, can interact with each other. The quantization of Chromodynamics involves the regularization and renormalization of ultraviolet divergences which generate a mass-scale where mass-dimensionless quantities become dependent on the mass-scale. The weak force is not only responsible for interactions between particles, but also it allows heavy particles to decay by emitting or absorbing some of the force carriers. The current quark-masses are the only evident scales in QCD while the nuclear weak force is the reason for any change in the quark's flavor via W bosons [49,50].

Renormalization procedure generates running coupling constants. The Landau pole is the point where the perturbative expression of the running coupling diverges. We can get the first coefficient series as an independent parameter at short distances while at relatively large distances, dependency returns due to the confinement of quarks and gluons. This means that running couplings are non-observable quantities while predictions for observables should be determined independent of the choice of the renormalization map and the regularization scheme. At a critical temperature around $T_c \approx 170$ MeV, QCD matter undergoes a deconfining phase transition into quark-gluon plasma. In perturbative QCD (i.e. the region $T \gg T_c$), we can expand different physical quantities with respect to the small enough running coupling constants. The behavior of running couplings in low energy levels (i.e. momentum transfers $Q^2 \sim 1-5$ GeV^2 while the proton's mass is approximately 1 GeV) is the original reason of the non-perturbative aspects in the context of the confinement. At this long distance level, non-perturbative solutions of DSEs are central tools to analyze the behavior of physical systems under strong running coupling constants in order to understand the hadronic structure, the quark confinement and the hadronization processes [15,19,21,39,40].

1.2. Mathematical background

The Bogoliouv–Zimmermann's forest formula in perturbative renormalization aims to remove nested loops in Feynman diagrams under a step by step recursive treatment [14]. The Kreimer's renormalization coproduct [24] encapsulates this forest formula in terms of the factorization of Feynman diagrams into their elementary components with respect to the insertion operator under a Lie algebraic setting. In fact, this factorization process reduces the computation steps of perturbative renormalization in the context of a certain Hopf algebra known as the Connes–Kreimer renormalization Hopf algebra of Feynman diagrams [4,10,14]. For a given gauge field theory Φ, $H_{\mathrm{FG}}(\Phi)$ is a graded connected commutative non-cocommutative Hopf algebra over the field $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$. Its grading structure is defined via the first Betti number of Feynman diagrams. We have $H_{\mathrm{FG}}(\Phi) = \bigoplus_{n \geq 0} H_{(n)}$ such that $H_{(0)} = \mathbb{K}$ and for each $n \geq 1$, the homogeneous component $H_{(n)}$ is the vector space generated by 1PI Feynman diagrams with the loop number n or products of 1PI Feynman diagrams with the overall loop number n [24,25,28].[1] This particular Hopf algebra can be simplified in terms of a combinatorial Hopf algebra of decorated non-planar rooted trees. This combinatorial Hopf algebra allows us to recognize a universal reformulation of the BPHZ perturbative renormalization [10,12]. The renormalization Hopf algebra and its quotient versions with respect to the Hopf ideals generated by quantum gauge symmetries provide a complete package for the description of the BPHZ perturbative renormalization in gauge field theories in terms of the Hopf–Birkhoff factorization of Feynman rules characters underlying the differential Galois theory [14,26,41,53–57]. Recent research results show the importance of the Hopf ideals of the Connes–Kreimer renormalization Hopf algebra in the Theory of Computation [43] and other generalized physical theories [38,52,53].

[1] The number of internal edges can be also applied as the graduation parameter to obtain finite dimensional homogeneous components in the renormalization Hopf algebra of Feynman diagrams. See Proposition 1.30 in [14]. In this work, the loop number graduation provides a finite type graded Hopf algebra on Feynman graphons.

On the one hand, the Connes–Kreimer Hopf algebraic setting provided a combinatorial interpretation of DSEs in terms of a particular class of recursive equations built by families of Hochschild one cocycles associated to primitive Feynman diagrams. This combinatorial setting has led us to study solutions of DSEs in the context of Hochschild Cohomology Theory [9,26,27,29,56,57], the Riemann–Hilbert correspondence [41–43], Noncommutative Geometry and Functional Analysis [44,47]. On the other hand, the theory of graphons for sparse graphs [5,6,11,33] has been applied to formulate a new analytic setting for the study of (non-perturbative) solutions of combinatorial DSEs and their renormalization program in terms of cut-distance convergent limits of sequences of graphon processes. In this direction, we determined (complete) and compact topological spaces of weakly isomorphic (rescaled and stretched) graphon classes associated to Feynman diagrams and solutions of DSEs in a (strongly coupled) gauge field theory Φ. It led us to obtain a new topological enrichment of the Connes–Kreimer renormalization Hopf algebra denoted by $H_{\mathrm{FG}}^{\mathrm{cut}}(\Phi)$ with respect to the cut-distance topology. This topological Hopf algebra is useful for the formulation of a non-perturbative generalization of the BPHZ method in dealing with the renormalization of solutions of DSEs. As a remarkable result, Feynman graphon models address a new way of studying strongly coupled DSEs in terms of the random graph processes [44–46,48].

1.3. Original achievements

At the first step, we study Feynman diagrams of a given gauge field theory in terms of pixel picture presentations and stretched or rescaled graphons to formulate a new compact topological space of these physical diagrams with respect to the cut-distance topology (i.e. Theorem 2.9). The graph limits in this new space introduce a new family of infinite Feynman diagrams (i.e. Definition 2.12). It is possible to analyze these particular graphs in terms of random graph processes built via Feynman graphon models (i.e. Corollary 2.13).

At the second step, we address an analytic description for (non-perturbative) solutions of combinatorial DSEs in the language of Feynman graphon models (i.e. Theorem 3.5). While this new picture allows us to study these equations in terms of random graph processes (i.e. Corollary 3.7), we give a new way of describing non-perturbative phase transitions via homomorphism densities of Feynman graphons associated to DSEs (i.e. Corollary 3.8).

At the third step, we consider the topological Hopf algebra of Feynman graphons (i.e. Lemma 4.1) to show some new computational advantages of Feynman graphons. We provide a new Feynman graphon approach to compute the renormalization coproduct of the complete 1PI Green's functions and solutions of DSEs in the context of some quotient versions of the Hopf algebra of Feynman graphons. These quotient Hopf algebras are derived from Hopf ideals generated by Feynman graphon representations of quantum gauge symmetries. (i.e. Lemma 4.2 and Corollaries 4.3 and 4.4).[2] Then we address the structure of a new Renormalization Group associated to the topological Hopf algebra of Feynman graphons which encodes the renormalization of DSEs (i.e. Theorem 4.10).

[2] In gauge field theories, there exist Feynman diagrams with the same loop number n and residue r which contribute to the combinatorial Green's function Γ_n^r while they might have different valent vertices. The new Hopf ideals $I_{\mathrm{ST}}^{\mathrm{graphon}}$ and $I_{\mathrm{WT}}^{\mathrm{graphon}}$ in the renormalization Hopf algebra of Feynman graphons allow us to equalize Feynman graphons corresponding to certain linear combinations of Feynman diagrams which contribute to the Slavnov–Taylor / Ward–Takahashi elements. The cut-distance topological enrichments of the resulting quotient Hopf algebras are useful to determine the renormalization coproduct of non-perturbative solutions of DSEs.

At the fourth step, we explain the structure of a new separable Banach manifold $\mathcal{S}^{\Phi,g}_{\approx}$ of weakly isomorphic equivalence classes of combinatorial DSEs in a given (strongly coupled) gauge field theory Φ with the bare coupling constant g (i.e. Theorem 5.2 and Lemma 5.5). Then we explain the constructions of the following different types of Banach bundles on $\mathcal{S}^{\Phi,g}_{\approx}$.

- Thanks to the Gâteaux differential calculus theory, the first type is the principal tangent Banach bundle

$$\left(\mathbb{G}_{\infty}(\mathbb{C}), T\mathcal{S}^{\Phi,g}_{\approx}, \mathcal{S}^{\Phi,g}_{\approx}, \pi^{\mathrm{Tan}}_{\Phi,g}\right) \tag{1.1}$$

  with respect to the complex Lie group $\mathbb{G}_{\infty}(\mathbb{C}) := \lim_{\leftarrow} \mathrm{GL}_N(\mathbb{C})$ and the natural projection $\pi^{\mathrm{Tan}}_{\Phi,g}$ (i.e. Theorem 5.8). This Banach bundle is useful for the study of the local geometric properties of trajectories or flows between DSEs where geodesics in $\mathcal{S}^{\Phi,g}_{\approx}$ address the most stable transition stages of non-perturbative phases (i.e. Remark 5.9).
- The second type is the principal Banach bundle

$$\left(\mathbb{G}_{\Phi}, H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi), \mathcal{S}^{\Phi,g}_{\approx}, \pi^{\mathrm{Hopf}}_{\Phi,g}\right), \tag{1.2}$$

  where the cut-distance topological renormalization Hopf algebra $H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi)$ of Feynman diagrams is the total space such that the complex Lie group

$$\mathbb{G}_{\Phi} := \mathrm{char}(H_{\mathrm{FG}}(\Phi), H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi)) \tag{1.3}$$

  acts on it. Cut-distance topological Hopf subalgebras generated by solutions of combinatorial DSEs are the fibers of this particular bundle (i.e. Theorem 5.12). This Banach bundle addresses a new Hopf Gauge Theory approach for the study of non-perturbative aspects of gauge field theories beyond the Standard Model.
- The third type is the principal regularization Banach bundle

$$\left(\mathbb{G}^{\mathrm{cut}}_{\Phi}(A_{\mathrm{dr}}), B_{\mathrm{DSE}} \times \mathbb{G}^{\mathrm{cut}}_{\Phi}(A_{\mathrm{dr}}), B_{\mathrm{DSE}}, \pi^{\mathrm{Triv}}_{\Phi,g}\right) \tag{1.4}$$

  with respect to the Lie group

$$\mathbb{G}^{\mathrm{cut}}_{\Phi}(A_{\mathrm{dr}}) := \mathrm{char}(H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi), A_{\mathrm{dr}}) \tag{1.5}$$

  and a given principal $\mathbb{C}^*$-bundle B_{DSE} over a convex infinitesimal neighborhood $\mathbf{\Delta}_{\mathrm{DSE}}$ of the equation DSE in the Banach space $\mathcal{S}^{\Phi,g}_{\approx}$ (i.e. Lemma 5.14 and Definition 5.16). The regularization algebra A_{dr}, as the algebra of Laurent series with finite pole part, enables us to assign meromorphic functions in the regulator. This Banach bundle is useful to give a geometric description for a non-perturbative renormalization program in terms of systems of differential equations together with singularities derived from trajectories between DSEs in $\mathcal{S}^{\Phi,g}_{\approx}$ and a particular class of connections (i.e. Theorem 5.21 and Corollary 5.22).

Each of these Banach bundles provides some new geometric data about the evolution of strongly coupled gauge field theories during the rescaling of running coupling constants.

2. Cut-distance topological space of Feynman diagrams

In this section, we are going to study the space of Feynman diagrams of a given (strongly) coupled gauge field theory in the context of the theory of graphons for sparse graphs. We associate a class of graph functions to Feynman diagrams to equip the space of Feynman diagrams with a new metric structure. This topological enrichment is a central tool for the study of graph limits of sequences of Feynman diagrams and solutions of combinatorial DSEs. In addition, it allows us to study the behavior of complicated higher loop order Feynman diagrams in terms of random graph processes.

Definition 2.1.

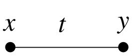
- A finite weighted graph is a graph G with a finite vertex set $V(G)$ and a finite edge set $E(G)$ together with a pair (w_V, w_E) of maps

$$w_V : V(G) \to \mathbb{R}\ ,\ \ w_E : E(G) \to \mathbb{R}\ , \tag{2.1}$$

  which associate weights to all vertices and edges of the graph. It is called an infinite weighed graph, if it has at least an infinite vertex set.
- The graph G is called oriented, if each of its edges has a direction.
- The oriented graph G with $n_G = |V(G)|$ is called a dense graph, if $|E(G)|$ is close enough to the maximum number of edges between vertices of G. It means that $1/2 < \frac{|E(G)|}{n_G(n_G-1)} \leq 1$. Otherwise it is called a sparse graph.
- For given graphs H, G, whenever $|V(G)|$ tends to infinity, the homomorphism density $t(H, G) = \frac{|\mathrm{Hom}(H,G)|}{|V(G)|^{|V(H)|}}$ determines the density of H as a subgraph in G.
- A rooted tree t is a finite connected graph which has no loop. It is oriented in such a way that every vertex of t has exactly one incoming edge except the root which has only outgoing edges. If we embed t into the plane, then it is called a planar rooted tree otherwise it is called a non-planar rooted tree. Up to the isomorphism of graphs, each rooted tree can represent an isomorphism class.
- An admissible cut of a rooted tree t is a subset c of $E(t)$ such that along any path from the root of t to each of the leaves of t, there exists at most one element of c.

Rooted trees are sparse graphs while complete graphs are dense. Graph functions or graphons, as infinite graphs with a continuum vertex set, are interesting examples of dense graphs.

Feynman diagrams are useful tools in Quantum Field Theory to encapsulate the summation over probability amplitudes corresponding to all possibilities of exchanging of virtual particles compatible with a process at a given order. The Feynman path integral method describes interactions of subatomic particles in terms of (non-)perturbative formal expansions of Feynman diagrams which contribute to Green's functions of the physical theory. Feynman diagrams simplify the description of interactions of elementary particles with respect to the parameter of time in a quantum system. Decorations (or propagators) are determined by fundamental parameters of the physical theory encoded in terms of Feynman rules. Feynman rules are bridges between integrals and diagrams. We can associate an integral factor to each edge



, (2.2)

as the integration over position parameters with respect to vertices x, y and a length parameter t. It is indeed an integral over all possible paths for the particle traveling between x and y in time t. We can interpret momentum and position parameters as Fourier transforms of each other. It is useful to translate momentum space type diagrams to their corresponding iterated integrals via Feynman rules. In general, each non-self loop associates to an integral over the corresponding momentum where sub-divergences in the original integral can be represented in terms of nested or overlapping loops.

Definition 2.2. A Feynman diagram Γ is a finite oriented decorated graph. It contains a vertex set $\Gamma^{[0]}$ for the presentation of interactions together with an edge set $\Gamma^{[1]}$ of internal and external edges. The subset $\Gamma^{[1]}_{\text{int}}$ of internal edges, which contains edges with beginning and ending vertices, presents virtual particles. The subset $\Gamma^{[1]}_{\text{ext}}$ of external edges, which contains edges with only beginning or ending vertex, presents elementary particles with assigned momenta. The graph obeys the conservation of momenta which means that the amount of momenta for input particles in each vertex is the same as the amount of momenta for output particles.

The Connes–Kreimer combinatorial Hopf algebra H_{CK} is the polynomial algebra generated by non-planar rooted trees. While the vertex number provides a graduation parameter, this polynomial algebra is equipped with a modified version of the renormalization coproduct. For each $t \in H_{\text{CK}}$, we have

$$\Delta(t) = \sum_c P_c(t) \otimes R_c(t) \ , \tag{2.3}$$

such that the sum is taken over all admissible cuts of t. In this formula, $R_c(t)$ is the subtree (generated by subtracting c from t) which contains the root of t while $P_c(t)$ is a forest of the remaining subtrees of t. The graduation parameter enables us to define an antipode recursively. Therefore H_{CK} is a free commutative non-cocommutative graded unital counital associative coassociative Hopf algebra [12,24,28].

Loops in Feynman diagrams or (1PI) primitive Feynman diagrams determine a decoration collection on rooted trees. Decorated versions of non-planar rooted trees and their formal expansions are applied to represent complicated Feynman diagrams with nested or overlapping loops [25,31]. There exists an injective Hopf algebra homomorphism from $H_{\text{FG}}(\Phi)$ to $H_{\text{CK}}(\Phi)$ as a decorated version of the Connes–Kreimer combinatorial Hopf algebra. The pair (H_{CK}, B^+) provides a simplified universal model for the Bogoliubov–Zimmermann's forest formula in perturbative renormalization [10,12,28].

Remark 2.3.

- For any Feynman diagram Γ of rank n, its tree representation t_Γ contains n vertices such that the positions of nested loops with respect to each other in Γ govern the existence of edges between vertices in t_Γ.
- The operator $B^+ : H_{\text{CK}} \to H_{\text{CK}}$ is a homogeneous linear operator which generates a new rooted tree from a forest $t_1 ... t_n$ of rooted trees with roots $r_1, ..., r_n$ by adding a new decorated vertex r together with a collection of edges $e_1, ..., e_n$ from r to $r_1, ..., r_n$. This operator can encode the insertion operator on Feynman diagrams with respect to types of vertices and external edges.

The theory of graphons in infinite combinatorics concerns the concept of convergence for sequences of finite weighted graphs. This theory is developed for the study of the spaces of dense and sparse graphs in the context of measure theoretic tools and random graph processes. It is shown that the space of graphons can topologically complete the space of finite graphs.

Definition 2.4.

- For a given σ-finite measure space (Ω, μ), a labeled stretched graphon W is a symmetric bounded measurable $\mathbb{R}$-valued function on $\Omega \times \Omega$. It is called a bigraphon, if we remove the symmetry condition.
- A relabeling W^{ρ} of W is defined in terms of applying an invertible measure preserving transformation ρ on Ω such that $W^{\rho}(x, y) := W(\rho(x), \rho(y))$.

For the closed interval $[a, b]$ equipped with the Lebesgue measure m, as the ground measure space, set $\mathcal{W}^{[a,b]}(\mathbb{R})$ as the space of labeled stretched $\mathbb{R}$-valued graphons on $[a, b] \times [a, b]$. This space is a subspace of symmetric functions in $L_{\infty}([a, b]^2)$. It is shown that each $[0, 1]$-valued graphon is equivalent to a stretched graphon on the Lebesgue measure space $\mathbb{R}_{+}$. The distance between (stretched) graphons is given by

$$d_{\mathrm{cut}}(U, V) := \inf_{\rho,\psi} \sup_{A,B\subset[a,b]} \left| \int_{A\times B} \big(U^{\rho}(x, y) - V^{\psi}(x, y)\big) dm(x) dm(y) \right|, \tag{2.4}$$

such that the infimum is over all invertible Lebesgue measure preserving transformations on $[a, b]$ and the supremum is over all Lebesgue measurable non-trivial subsets of $[a, b]$. It is shown that the space $\mathcal{W}^{[0,1]}([0, 1])$ is a compact Hausdorff topological space with respect to the pseudo-metric (2.4). The space of labeled stretched $\mathbb{R}_{+}$-valued graphons, which is topologically completed with respect to the L^1-norm, has been equipped with other norms [5,6,11,23,33,34].

In the rest of this work, we focus on stretched or rescaled versions of graphons in $\mathcal{W}^{[0,1]}([0, 1])$ or $\mathcal{W}^{[a,b]\subseteq\mathbb{R}_{+}}(\mathbb{R}_{+})$. We show that the space $\mathcal{W}^{[0,1]}([0, 1])$ is rich enough to encode graphon models of Feynman diagrams of a (strongly coupled) gauge field theory and their formal expansions which contribute to solutions of DSEs. It is important to note that by rescaling or changing the ground measure space, we can generate different graphon models for the study of graph limits of Feynman diagrams and DSEs.

Definition 2.5.

- Graphons W_1, W_2 are weakly isomorphic if there exists a graphon W and Lebesgue measure preserving transformations ρ_1, ρ_2 on $[a, b]$ such that $W_1 = W^{\rho_1}$ and $W_2 = W^{\rho_2}$ almost everywhere.
- Set $[W]_{\approx}$ as the equivalence class of graphons with respect to W up to the relabeling and the weakly isomorphic relation. It is called an unlabeled graphon. The distance (2.4) defines a metric structure on the quotient space $\mathcal{W}_{\approx}^{[a,b]\subseteq\mathbb{R}_{+}}(\mathbb{R}_{+})$ of unlabeled (stretched) graphon classes.

Definition 2.6. Let G be a finite weighted graph with the vertex set $V(G) = \{v_1, ..., v_n\}$ such that for $i = 1, ..., n$, $0 < w_V(v_i) < 1$ and $\sum_{i=1}^{n} w_V(v_i) = 1$. The adjacency matrix $A_G = (w_E(v_i v_j))_{n\times n}$ determines a bare pixel diagram P_G with n^2 rectangular elements in the unit box $[0, 1]^2$ such that the rectangular element associated to vertices v_i, v_j has the area $w_V(v_i)w_V(v_j)$. In addition, the rectangular elements associated to the 0s in A_G are decorated by white color while the rectangular elements associated to the non-zero arrays in A_G are decorated by black color.

The graph function P_G is called a pixel picture presentation of the graph G. It is a labeled [0, 1]-valued graphon while by changing the scale of the rectangular elements via any invertible Lebesgue measure preserving transformation on [0, 1], we can generate relabeled versions of this pixel picture.

Remark 2.7. If the vertex set of the graph G is not weighted, then its corresponding bare pixel picture is given by a pixel diagram in the unit square $[0, 1]^2$ such that the square elements have sides of the length $\frac{1}{n}$. The 0s in the adjacency matrix are replaced by white squares while the 1s are replaced by black squares.

Thanks to the equivalence class $[.]_{\approx}$, we equip the space of finite (weighted) graphs with the metric

$$d_{\mathrm{cut}}(G, H) := d_{\mathrm{cut}}([P_G]_{\approx}, [P_H]_{\approx}) \tag{2.5}$$

to obtain a compact Hausdorff metric space such that graphons belong to the boundary region.

Remark 2.8. The graph limit of any sequence $\{G_n\}_{n\geq 1}$ of sparse graphs, when n goes to infinity, is a graph function with almost zero density which is weakly isomorphic to the 0-graphon. It is possible to remove this issue by working on normalized graphons such as $\frac{P_{G_n}}{||P_{G_n}||_{\mathrm{cut}}}$ or rescale the ground measure space to obtain non-zero graph limits. More techniques in this direction are given in [5,6,11].

Now we are going to equip the space of Feynman diagrams of a given gauge field theory with the cut-distance topology. Decorated rooted trees, as weighted simple graphs, are useful to formulate graphon models for Feynman diagrams and their formal expansions. This leads us to introduce a new class of analytic Feynman diagrams as graph limits of sequences of Feynman diagrams.

Theorem 2.9. *The space of Feynman diagrams of a given gauge field theory is a compact Hausdorff topological space with respect to the cut-distance topology.*

Proof. For any Feynman diagram Γ with nested loops, its corresponding tree representation t_Γ is a simple finite weighted graph. We can represent the adjacency matrix A_{t_Γ} in terms of a pixel picture W_{t_Γ}. It is constructed via a uniform partition of $[0, 1]^2$ into $|t_\Gamma|^2$ boxes such that each zero array in A_{t_Γ} corresponds to the white box while each non-zero array in A_{t_Γ} corresponds to the black box.

If Γ has overlapping loops, then its tree representation $u_\Gamma = c_1 t_1 + ... + c_n t_n$ is a linear combination of decorated non-planar rooted trees. For $1 \leq i \leq n$, $W_{c_i t_i}$ is the graph function W_{t_i} defined on $I_i \times I_i$ with $m(I_i) = c_i$ such that $I_i \cap I_j = \emptyset$, $i \neq j$. A labeled stretched Feynman graphon W_{u_Γ} is determined in terms of the rescaling of the linear combination of stretched graphons $W_{c_i t_i}$. In other words, we apply affine monotone maps (as rescaling tools) to project each $W_{c_i t_i}$ to a labeled graphon $\tilde{W}_{\tilde{c}_i t_i}$ defined on a box $\tilde{I}_i \times \tilde{I}_i$ with $m(\tilde{I}_i) = \tilde{c}_i$ such that $\{\tilde{I}_i\}_{i=1}^n$ determines a partition of [0, 1]. Then we define W_{u_Γ} in terms of a normalization of the direct sum of weighted labeled graphons $\tilde{W}_{\tilde{c}_i t_i}$ namely,

$$W_{u_\Gamma} = \frac{\tilde{W}_{\tilde{c}_1 t_1} + ... + \tilde{W}_{\tilde{c}_n t_n}}{||\tilde{W}_{\tilde{c}_1 t_1} + ... + \tilde{W}_{\tilde{c}_n t_n}||_{\mathrm{cut}}} . \tag{2.6}$$

Set $[W_\Gamma]_\approx$ as the unique non-trivial class of [0, 1]-valued graphons up to the relabeling and weakly isomorphic relation corresponding to Γ. It is called an unlabeled Feynman graphon. Thanks to the metric (2.4), define a new metric on the space of Feynman diagrams given by

$$d_{\mathrm{cut}}(\Gamma_1, \Gamma_2) := d_{\mathrm{cut}}([W_{\Gamma_1}]_\approx, [W_{\Gamma_2}]_\approx) \ . \tag{2.7}$$

It determines a new separable Hausdorff totally bounded topological space of Feynman diagrams of the physical theory which is completed by Feynman graphons. □

Rooted tree representations of Feynman diagrams are sparse graphs. When the vertex number goes to infinity, the graphon models of these graphs tend to a graphon with almost zero density which is weakly isomorphic to the 0-graphon. This problem is resolved by working on the rescaled or normalized versions of graphon models. The normalized direct sum (2.6) is useful to associate non-zero graphon models to sequences of higher loop order Feynman diagrams.

Remark 2.10. Thanks to the metric (2.7), a sequence $\{\Gamma_n\}_{n\geq 1}$ of higher loop order Feynman diagrams with nested overlapping loops is convergent whenever its corresponding sequence of non-trivial stretched or rescaled Feynman graphon classes converges to a non-zero graphon class $[W]_\approx$ with respect to the cut-distance topology when n tends to infinity.

Definition 2.11.

- Set $\mathcal{S}^{\Phi}_{\mathrm{graphon},\approx}$ as the quotient space of unlabeled Feynman graphon classes corresponding to all Feynman diagrams in the physical theory Φ. It is a closed topological subspace of the space $\mathcal{W}^{[0,1]}_{\approx}([0, 1])$.
- Feynman diagrams Γ_1, Γ_2 are called weakly isomorphic, if $[W_{\Gamma_1}]_\approx = [W_{\Gamma_2}]_\approx$.

The cut-distance topological space of Feynman diagrams determines a new class of infinite Feynman diagrams which are useful for the interpretation of Green's functions of strongly coupled gauge field theories in the language of graphon models.

Definition 2.12. Thanks to Theorem 2.9, the non-zero labeled graphon W (given by Remark 2.10) can be described as the pixel picture presentation of an infinite non-planar rooted tree or forest t_∞ which is decorated by (1PI) primitive Feynman diagrams of the physical theory. Its corresponding Feynman diagram X is called the large Feynman diagram generated by the sequence $\{\Gamma_n\}_{n\geq 1}$.

Graphon models are useful to build an important collection of random graphs. We apply these random graphs to obtain some approximations of large Feynman diagrams in terms of the random graph processes.

Corollary 2.13. *Graph limits of the space of Feynman diagrams of a given gauge field theory determine random Feynman graphs.*

Proof. Let $\{\Gamma_n\}_{n\geq 1}$ be a sequence of Feynman diagrams which is cut-distance convergent to the unlabeled Feynman graphon class $[W]_\approx$. For any finite subset $S_n := \{s_1, ..., s_n\} \subset [0, 1]$, we build a new weighted graph $G(S_n, W)$ with n vertices such that the edge $s_i s_j$ has the weight $W(s_i, s_j)$. Its corresponding simple random graph $R(G(S_n, W))$ is built by including an edge $s_i s_j$ with the probability equal to its weight. When n tends to infinity, the sequence $\{R(G(S_n, W))\}_{n\geq 1}$ is convergent to an infinite random graph $R(G(S_\infty, W))$ built by choosing at random infinite countable nodes from [0, 1]. We call $R(G(S_\infty, W))$ the random Feynman graph corresponding to the graph limit of $\{\Gamma_n\}_{n\geq 1}$. $\square$

3. Dyson–Schwinger equations: from Hochschild equations to graphon representations

Interacting gauge field theories are formulated in terms of the Lagrangian formalism where Green's functions are fundamental tools for the study of the hierarchies of interactions among elementary and virtual particles. Each Green's function is given in terms of a formal expansion of iterated integrals which might contain nested sub-divergences. The renormalization process is useful to replace these ill-defined integrals by some renormalized values. Feynman diagrams are applied to encode those integrals via some combinatorial tools which enable us to encapsulate the renormalization process in terms of some combinatorial computational algorithms such as the Bogoliouv–Zimmermann's forest formula [10,14,24]. DSEs are integral equations generated by fixed point equations of Green's functions. These equations, which have a recursive nature, generate power series of running coupling constants while Feynman diagrams with higher loop orders contribute as coefficients in these series [40,50]. The strength of running couplings is the main parameter for perturbative or non-perturbative behavior of these power series. For instance, in QCD, running coupling constants at large values of the exchanged momenta are varying in the range that the convergence of perturbative expansions of Green's functions is slow or impossible [15]. In these relatively low energy levels, QCD has strong running coupling constants where the physical system should be studied under non-perturbative settings [19,21,39,49]. In higher energy levels, QCD is asymptotic free where we can generate some finite values in terms of higher order perturbation techniques and large N limits [35,36]. Thanks to the Connes–Kreimer renormalization Hopf algebra and Hochschild Cohomology Theory [9,26,56], DSEs are reformulated as Hochschild equations. This alternative setting has led us to achieve some new practical geometric and combinatorial tools for non-perturbative computations [26,27,30,41–43,57].

Thanks to the Connes–Kreimer Hopf algebraic renormalization [4,10,28,54] and Feynman graphon models built in the previous section, it is possible to describe solutions of combinatorial DSEs as the cut-distance convergent limits of sequences of finite partial sums of Feynman diagrams [44,45]. This setting addresses a new way of describing solutions of combinatorial DSEs in terms of random graph processes derived from Feynman graphon models. In addition, it provides a new analytic generalization of the BPHZ renormalization in dealing with strongly coupled DSEs. [45,46]. In this section we study DSEs on the basis of Feynman graphon models to address a new way of recognizing phase transitions in the language of homomorphism densities.

3.1. Green's functions

In ϕ^4 model with the (running) coupling constant u_0, the Lagrangian is given by

$$L(\phi) = \int d^4x \left(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} r_0 \phi^2 + \frac{1}{4!} u_0 \phi^4 \right). \tag{3.1}$$

The Feynman rules associate $\frac{1}{p^2+m^2} = \int_0^\infty dt \ \exp(-t(p^2+m^2))$ to each edge (with respect to the Schwinger parameter t), and $\int d^4x \ \exp(i \sum_j p_j x)$ to each vertex in Feynman diagrams. The first integral, which yields the propagator, and the second integral, which gives the momentum conservation for the vertices, determine a factor

$$\int \frac{d^4 p}{(2\pi)^4} \exp\big(ip(x-y) - t(p^2+m^2)\big) \tag{3.2}$$

for each edge.

Remark 3.1. The Klein–Gordon equation, $\big(\partial_\mu \partial^\mu + m^2\big)\phi = 0$, gives the equation of motion for a quantized field ϕ in the free Lagrangian density $L_0(\phi) := \int d^4x \,\big(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} r_0 \phi^2\big)$. In interacting physical theories, the interaction part of the Lagrangian leads us to formulate DSEs. Solutions of these equations determine the equations of motion [26,50].

The partition function, as the generating functional of correlation functions, is given by $Z[B] := \int \ e^{-L(\phi)+\int B\phi} \mathcal{D}\phi$. The symbol $\int \mathcal{D}\phi$ is the functional integral over a set of quantum fields in the four dimensional space-time and B is an arbitrary external field. Set $Z_0 := \int \ e^{-L_0(\phi)+\int B\phi}\mathcal{D}\phi$ to expand Z as a series in u_0 around Z_0. We have

$$Z = \int \mathcal{D}\phi \left(1 - \frac{u_0}{4!}\int_{x_1} \phi^4(x_1) + \frac{1}{2}(\frac{u_0}{4!})^2 \int_{x_1,x_2} \phi^4(x_1)\phi^4(x_2) + ...\right) e^{-L_0(\phi)+\int B\phi} . \tag{3.3}$$

The fluctuations generated by the ϕ^4 term around the Gaussian integral Z_0 deliver iterated integrals $\int^\Lambda d^4q_1...d^4q_l \prod_{i=1}^{l}(\text{propagator}(\mathrm{q_i}))$. The ultra-violet regulator Λ gives a cut-off at the upper bound of the integral [14,28]. The transition amplitudes from initial states to all finite states can be studied in the S-Matrix setting. It is possible to calculate these matrix elements in terms of a class of correlation functions known as Green's functions.

Definition 3.2. For the given action functional $S[\phi] = S_0[\phi] + u_0 S_{\text{int}}[\phi]$, the Green's function associated to interactions of N particles under the running coupling constant u_0 is given by

$$\begin{aligned} G^N(x_1, ..., x_N) &= \langle 0|T\phi(x_1)...\phi(x_N)|0\rangle = \frac{\int e^{-S[\phi]}\phi(x_1)...\phi(x_N)\mathcal{D}\phi}{\int e^{-S[\phi]}\mathcal{D}\phi} \\ &= \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \int d^4y_1...d^4y_j \langle 0|T\phi_{\text{in}}(x_1)...\phi_{\text{in}}(x_N)L_{\text{int}}(y_1)...L_{\text{int}}(y_j)|0\rangle \ , \end{aligned} \tag{3.4}$$

such that $|0\rangle$ is the vacuum ground state and ϕ_{in} is the initial state of ϕ in the infinite past.

For any 1PI Feynman diagram Γ, its residue is a new graph $\text{res}(\Gamma)$ generated by shrinking all internal edges of Γ into a new vertex $v_{\text{res}(\Gamma)}$ such that the type of $v_{\text{res}(\Gamma)}$ is determined in terms of the external edges of Γ which are connected to $v_{\text{res}(\Gamma)}$. A Green's function, which contains a formal expansion of Feynman diagrams with the residue r, is called an edge type if r is an external edge or a vertex type if r is a vertex. The collection of these two types of Green's functions has enough information to study a gauge field theory. The combinatorial versions of Green's functions are classified in terms of the residue and the loop order of Feynman diagrams which contribute to formal expansions. Thanks to Definition 3.2, for each amplitude r, we have formal expansions

$$\Gamma^r = \mathbb{I} \pm \sum_{\Gamma,\ \text{res}(\Gamma)=r} u_0^{|\Gamma|} \frac{\Gamma}{\text{Sym}(\Gamma)} . \tag{3.5}$$

The component $X^r_n = \sum_{\Gamma,\ \mathrm{res}(\Gamma)=r,\ |\Gamma|=n} \frac{\Gamma}{\mathrm{Sym}(\Gamma)}$ characterizes the formal expansion of 1PI Feynman diagrams with the loop number n which contribute to the amplitude r. For sufficiently small coupling constants, the fixed point equations of Green's functions are studied in terms of geometric series. In strongly coupled gauge field theories, these fixed points generate infinite power series of running coupling constants. These infinite expansions are studied in the context of DSEs. Fundamental identities among Feynman diagrams (i.e. Ward–Takahashi / Slavnov–Taylor identities) are useful to classify DSEs. In terms of the Ward–Takahashi identities, the photon propagator is the only propagator in QED which contributes to the running of the coupling constant. In terms of the Slavnov–Taylor identities, intermediate gauge-dependent quantities in non-abelian gauge theories create final gauge-independent results for observables [26,29,38,54].

3.2. Hochschild equations

Thanks to the Connes–Kreimer Hopf algebraic renormalization, fundamental identities among Feynman diagrams in gauge field theories are encoded in the context of Hopf ideals. The resulting quotient Hopf algebras provide a computational advantage for the renormalization of Feynman diagrams [4,26,29,38,54]. The Kreimer's renormalization coproduct, which encapsulates the Bogoliubov–Zimmermann's forest formula, is given by

$$\Delta(\Gamma) = \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma \ , \tag{3.6}$$

for each Feynman diagram Γ such that the sum is over all disjoint unions of non-trivial 1PI superficially divergent Feynman sub-diagrams of Γ [24].

Definition 3.3. For the renormalization Hopf algebra $H_{\mathrm{FG}}(\Phi)$, consider a chain complex $\{C_n\}_{n\geq 0}$ such that for each n, C_n is the set of all linear maps $T : H_{\mathrm{FG}}(\Phi) \to H_{\mathrm{FG}}(\Phi)^{\otimes n}, \Gamma \mapsto (T_1(\Gamma), ..., T_n(\Gamma))$ and $C_0 = \mathbb{K}$. Define the coboundary operator

$$\mathbf{b}T := (\mathrm{id} \otimes T)\Delta + \sum_{i=1}^{n} (-1)^i \Delta_i T + (-1)^{n+1} T \otimes \mathbb{I} \ , \tag{3.7}$$

such that Δ_i is the renormalization coproduct which acts on $T_i(\Gamma)$ [9,12].

The generators of the first rank Hochschild Cohomology $HH^1(H_{\mathrm{FG}}(\Phi))$ of this complex are important to build quantum motions. We can modify the grafting operator B^+, defined in Remark 2.3, as a linear homogeneous operator on $H_{\mathrm{FG}}(\Phi)$. For any primitive Feynman diagram γ, the operator B^+_γ is a homogeneous linear endomorphism of degree one which inserts a given Feynman diagram Γ inside γ in terms of types of vertices of γ and types of external edges of Γ. This operator allows us to reformulate the renormalization coproduct (3.6) recursively where we have,

$$\Delta B^+_\gamma = (\mathrm{id} \otimes B^+_\gamma)\Delta + B^+_\gamma \otimes \mathbb{I} \ . \tag{3.8}$$

From Definition 3.3, B^+_γ is a Hochschild one cocycle which enables us to reformulate the formal expansions (3.5) in terms of

$$\Gamma^r = \mathbb{I} \pm \sum_{k=1}^{\infty} u_0^k \sum_{\Gamma,\ \mathrm{res}(\Gamma)=r,\ |\Gamma|=k,\ |\Gamma|_{\mathrm{aug}}=1} \frac{1}{\mathrm{Sym}(\Gamma)} B_\Gamma^+(X_\Gamma)\ ,\ X_\Gamma = \prod_{e\in\Gamma^{[1]}_{\mathrm{int}}} \prod_{v\in\Gamma^{[0]}} \Gamma^v/\Gamma^e\ . \tag{3.9}$$

The powers of the couplings are determined via the loop numbers of Feynman diagrams which contribute to these expansions [27–30].

Theorem 3.4. *For a given family $\{\gamma_n\}_{n\geq 1}$ of primitive (1PI) Feynman diagrams with the corresponding Hochschild one cocycles $\{B^+_{\gamma_n}\}_{n\geq 1}$, the recursive equation*

$$X = \mathbb{I} + \sum_{n\geq 1} (\lambda g)^n \omega_n B^+_{\gamma_n}(X^{n+1}) \tag{3.10}$$

determines a class of DSEs such that the real numbers $0 < \lambda \leq 1$ govern the strength of the running coupling constants. The equation (3.10) has a unique solution $X = \sum_{n\geq 0} (\lambda g)^n X_n$ determined by the relations

$$X_n = \sum_{j=1}^{n} \omega_j B^+_{\gamma_j}\left(\sum_{k_1+...+k_{j+1}=n-j,\ k_i\geq 0} X_{k_1}...X_{k_{j+1}} \right) \in H_{\mathrm{FG}}(\Phi)\ , \tag{3.11}$$

such that $X_0 = \mathbb{I}$ is the empty graph [9,26,29].

The equation (3.10) is called a combinatorial DSE under the running coupling constant λg. Systems of these equations have also been studied under Lie algebraic, Hopf algebraic and geometric settings [17,27,29,43]. The infinite graph X is actually an object in the completion of $H_{\mathrm{FG}}(\Phi)[[\lambda g]]$ with respect to the n-adic topology given by the metric

$$d_{\mathrm{adic}}(\Gamma_1, \Gamma_2) := 2^{-\mathrm{val}(\Gamma_1 - \Gamma_2)}\ , \tag{3.12}$$

such that

$$\mathrm{val}(\Gamma) := \mathrm{Max}\left\{ n \geq 1 :\ \ \Gamma \in \bigoplus_{k\geq n} H_{(k)} \right\}\ , \tag{3.13}$$

[9,17,26,57].

For any $\Gamma = \alpha_1\Gamma_1 + ... + \alpha_s\Gamma_s$ with $\Gamma_i \in H_{(n_i)}$, set

$$n_0 := \mathrm{Min}\{n_1, ..., n_s\}. \tag{3.14}$$

For any $m \leq n_0$, $\Gamma \in \bigoplus_{k=m}^{\infty} H_{(k)}$ while $n > n_0$, $\Gamma \notin \bigoplus_{k=n}^{\infty} H_{(k)}$. Therefore $\mathrm{val}(\Gamma) = n_0$.

3.3. Graphon representations

For running couplings $\lambda g < 1$, the infinite graph X is described as a convergent geometric series. For running couplings $\lambda g \geq 1$, the sequence $\{\sum_{n=0}^{m} (\lambda g)^n X_n\}_{m\geq 1}$ of power series is divergent in the traditional approaches where X can not be an usual graph. Thanks to Feynman graphons, it is possible to find a new alternative description for X as a graph limit with respect to the cut-distance topology.

Theorem 3.5. *Feynman graphons encode non-perturbative solutions of combinatorial DSEs.*

Proof. Let DSE(λg) be a combinatorial DSE with the unique solution X_{DSE}. We set $\lambda g = 1$ to simplify the presentation. For each $m \geq 1$, set

$$Y_m := \mathbb{I} + X_1 + ... + X_m \ , \tag{3.15}$$

as the partial sums.

At the first step, thanks to Theorem 2.9, we determine the unlabeled Feynman graphon class $[W_{Y_m}]_{\approx}$ in terms of the forest representation of Y_m, for each $m \geq 1$. Each forest t_{Y_m} is a linear combination of the forest representations of X_i for $1 \leq i \leq m$. Set

$$t_{Y_m} := t_{X_1} + ... + t_{X_m} \ , \text{s.t.}, \ t_{X_i} = \alpha_1 s_1 + ... + \alpha_{p_i} s_{p_i} \ , \ 1 \leq i \leq m \ , \ |t_{Y_m}| = n_{Y_m} \ . \tag{3.16}$$

We apply a rescaling or stretching method to build a labeled Feynman graphon W_{Y_m} in terms of a normalization of the direct sum of labeled stretched Feynman graphons W_{X_i}.

At the second step, we define a new weighted version of the partial sums Y_m in terms of the n-adic metric and types of the vertices. For each vertex $v \in V(X_{\mathrm{DSE}})$, its type f_v is defined in terms of those edges in $E(X_{\mathrm{DSE}})$ which contribute to v. In other words, f_v is a set of internal and external edges. Thanks to the formula (3.11), primitive Feynman diagrams $\{\gamma_n\}_{n\geq 1}$ determine types of vertices in X_{DSE} while the grafting operators $B^+_{\gamma_n}$ determine those components of X_{DSE} which contain a particular vertex type. For any edge $v_i v_j \in E(X_{\mathrm{DSE}})$, there exist some $\gamma_i, \gamma_j \in \{\gamma_n\}_{n\geq 1}$ such that f_{v_i} is equivalent to the type of a vertex of γ_i and f_{v_j} is equivalent to the type of a vertex of γ_j. We are interested in the smallest values of i, j to define the weight of the edge $v_i v_j$ by $d_{\mathrm{adic}}(Y_{i_0}, Y_{j_0})$ such that

$$i_0 := \mathrm{Min}\{s : \ v_i \in Y_s\}, \quad j_0 := \mathrm{Min}\{t : \ v_j \in Y_t\} \ . \tag{3.17}$$

The formal expansion Y_{i_0} is the shortest partial sum of X_{DSE} which contains a vertex equivalent to v_i. In addition, set $w_i := d_{\mathrm{adic}}(X_{i_0}, \mathbb{I})$ as the weight of the vertex $v_i \in V(X_{\mathrm{DSE}})$.

At the third step, thanks to the given weighting structure, we build a new family of Feynman graphons corresponding to the partial sums Y_m. Finite expansions $a_i := \sum_{1\leq k\leq i} w_k$ (for $1 \leq i \leq m$) give us subintervals $I_i := [a_{i-1}, a_i) \subset \mathbb{R}_+$. By rescaling or stretching, for $1 \leq i \leq m$, we can consider W_{X_i} as a labeled Feynman graphon defined on $I_i \times I_i$. Therefore W_{Y_m} is a finite direct sum of Feynman graphons W_{X_i} of weights $m(I_i)$. For each $1 \leq i \leq m$, apply an affine monotone map ψ_i to project each subinterval I_i to a subinterval $\tilde{I}_i \subset [0, 1]$ to obtain a partition $\{\tilde{I}_i\}_{i=1}^m$ of $[0, 1]$. Then we have the normalized direct sum

$$W_{Y_m} := \frac{\tilde{W}_{X_1} + ... + \tilde{W}_{X_m}}{||\tilde{W}_{X_1} + ... + \tilde{W}_{X_m}||_{\mathrm{cut}}} \ , \ \tilde{W}_{X_i}(x, y) := W_{X_i}(\psi_i(x), \psi_i(y)) \ , \tag{3.18}$$

such that each $\tilde{W}_{X_i}$ is of weight $m(\tilde{I}_i)$.

At the fourth step, we build a cut-distance convergent sequence of finite random graphs with respect to the combinatorial information of the partial sums Y_m. For each $m \geq 1$, we apply a poset embedding ρ_m to project vertices of t_{Y_m} to some points in $[0, 1]$ identified by the set $V(Y_m, \rho_m)$. Then thanks to the metric (3.12), we build a sequence $\{R(Y_m)\}_{m\geq 1}$ of random graphs in such a way that for each $m \geq 1$, $R(Y_m)$ is defined by using points in $V(Y_m, \rho_m)$ such that for vertices v_i, v_j with $\rho_m^{-1}(v_i) \in X_{k_i}$ and $\rho_m^{-1}(v_j) \in X_{k_j}$, with the probability $2^{-\mathrm{val}(X_{k_i} - X_{k_j})}$, there exists an edge between v_i and v_j. This sequence is convergent to an infinite random graph $R_{X_{\mathrm{DSE}}}$ defined on the set $V(X_{\mathrm{DSE}}, \rho_\infty)$ such that ρ_∞ projects vertices of the forest $t_{X_{\mathrm{DSE}}}$ to an infinite number of nodes in $[0, 1]$. Thanks to [23,33,34], this means that the sequence $\{[W_{Y_m}]_{\approx}\}_{m\geq 1}$ is convergent with respect to the cut-distance topology. It is equivalent to say that the sequence $\{Y_m\}_{m\geq 1}$ is cut-distance convergent to $X_{\rm DSE}$.

At the fifth step, we clarify the structure of the analytic graph $W_{X_{\rm DSE}}$. When m tends to infinity, finite expansions $a_m := \sum_{1\leq k\leq m} w_k$ (for each $m \geq 1$) determine subintervals $I_m := [a_{m-1}, a_m) \subset \mathbb{R}_+$. For $m \geq 1$, we apply an affine monotone map ϕ_m to project each I_m to a subinterval $\tilde{J}_m$ of $[0, 1)$ to obtain a partition $\{\tilde{J}_m\}_{m=1}^{\infty}$ of $[0, 1)$ where by rescaling or stretching methods, we consider $\tilde{W}_{Y_m}$ as a labeled stretched Feynman graphon on $\tilde{J}_m \times \tilde{J}_m$. The compactness of the topology of graphons allows us to consider the infinite direct sum

$$W_{X_{\rm DSE}} = \frac{\tilde{W}_{X_1} + ... + \tilde{W}_{X_m} + ...}{||\tilde{W}_{X_1} + ... + \tilde{W}_{X_m} + ...||_{\rm cut}} \ , \ \tilde{W}_{X_m}(x, y) := W_{X_m}(\phi_m(x), \phi_m(y)) \ , \tag{3.19}$$

as a $[0, 1]$-valued labeled stretched Feynman graphon such that each $\tilde{W}_{Y_m}$ is of weight $m(\tilde{J}_m)$. In other words, we have $W_{X_{\rm DSE}}(x, y) = d_{\rm adic}(Y_{i_0}, Y_{j_0})$ for $x \in \tilde{J}_{i_0}, y \in \tilde{J}_{j_0}$. □

Theorem 3.5 addresses the structure of a labeled (stretched) Feynman graphon with respect to the solution $X_{\rm DSE}$ of the equation DSE. By changing affine monotone maps, we can obtain other weakly isomorphic labeled graphons corresponding to $X_{\rm DSE}$.

Definition 3.6. For a given equation DSE, define

$$[W_{\rm DSE}]_{\approx} := \left\{ W^{\rho}_{X_{\rm DSE}} : \ \rho \text{ Lebesgue measure preserving transformation on } [0,1] \right\} ,$$

as the unlabeled Feynman graphon class up to the relabeling and the weakly isomorphic relation with respect to the solution $X_{\rm DSE}$.

Corollary 3.7. *The non-perturbative solution of any equation DSE can be modeled in terms of an infinite random graph.*

Proof. Thanks to the Proof of Theorem 3.5, for each $m \geq 1$, each poset embedding ρ_m projects vertices of t_{Y_m} to $n_{Y_m} = |t_{Y_m}|$ nodes in $[0, 1]$. We build a random graph R_m which contains n_{Y_m} vertices such that for $x_i, x_j \in {\rm Im}(\rho_m)$, there exists an edge between x_i, x_j with the probability $W_{Y_m}(\rho_m^{-1}(x_i), \rho_m^{-1}(x_j))$ while there is no edge in R_m between nodes x'_i, x'_j such that x'_i or $x'_j \notin {\rm Im}(\rho_m)$. We continue this pattern to define ρ_∞ as a poset embedding which maps vertices of $t_{X_{\rm DSE}}$ to an infinite number of nodes in $[0, 1]$. These nodes are applied to build a new infinite random graph R_∞ in such a way that for $z_i, z_j \in {\rm Im}(\rho_\infty)$, there exists an edge between z_i, z_j with the probability $W_{\rm DSE}(\rho_\infty^{-1}(z_i), \rho_\infty^{-1}(z_j))$ while there is no edge in R_∞ between nodes z'_i, z'_j with z'_i or $z'_j \notin {\rm Im}(\rho_\infty)$.

Therefore the sequence $\{R_m\}_{m\geq 1}$ provides a random graph process for the description of the Feynman graphon $W_{\rm DSE}$ while R_∞ is infinite random graph model for $X_{\rm DSE}$. □

The homomorphism density given in Definition 2.1 has been extended to the level of graphons [23,33,34]. The linear space of homomorphism densities corresponding to Feynman diagrams and (partial sums of) solutions of DSEs are applied in [48] to analyze these equations under a new Gâteaux differential calculus setting. Here we consider homomorphism densities of Feynman graphons to give a new way of characterizing phase transitions of DSEs.

Corollary 3.8. *The homomorphism densities of Feynman graphons characterize phase transitions of DSEs in a given (strongly coupled) gauge field theory.*

Proof. Thanks to Theorem 3.5 and Definition 3.6, consider (strongly coupled) equations $\mathrm{DSE}_1, \mathrm{DSE}_2$ with the corresponding solutions $X_{\mathrm{DSE}_1}, X_{\mathrm{DSE}_2}$, sequences of the partial sums $\{Y_m^1\}_{m\geq 1}, \{Y_m^2\}_{m\geq 1}$ and Feynman graphon models $W_{\mathrm{DSE}_1}, W_{\mathrm{DSE}_2}$. For any $m \geq 1$, we define the induced density

$$t(Y_m^1, W_{\mathrm{DSE}_2}) = \int_{[0,1]^{|t_{Y_m^1}|}} \prod_{(i,j)\in E(t_{Y_m^1})} W_{\mathrm{DSE}_2}(x_i, x_j) \prod_{(i,j)\notin E(t_{Y_m^1})} (1 - W_{\mathrm{DSE}_2}(x_i, x_j))\, dx_1 \ldots dx_{|t_{Y_m^1}|} \ . \tag{3.20}$$

It gives the probability that the W_{DSE_2}-random graph of the order $|t_{Y_m^1}|$ is isomorphic to $t_{Y_m^1}$. This new parameter enables us to decide whether the W_{DSE_2}-random graph of the order $|t_{Y_m^1}|$ can restore the partial sum Y_m^1 or not.

Therefore for equations $\mathrm{DSE}_1, \mathrm{DSE}_2$ which belong to the same phase, there exists an order M such that for any $m \geq M$,

$$t(Y_m^1, W_{\mathrm{DSE}_2}) = t(Y_m^2, W_{\mathrm{DSE}_1}) \ . \tag{3.21}$$

If $\mathrm{DSE}_1, \mathrm{DSE}_2$ are in different phases p_1, p_2, then the value $t(Y_m^2, W_{\mathrm{DSE}_1})$ determines an approximation for the deviation of the equation DSE_2 with respect to the phase p_1 while $t(Y_m^1, W_{\mathrm{DSE}_2})$ determines an approximation for the deviation of the equation DSE_1 with respect to the phase p_2. □

4. Non-perturbative generalization of the Connes–Kreimer Renormalization Group

In this section, we introduce the renormalization Hopf algebra of Feynman graphons and modify it in terms of gauge symmetries to obtain a new graphon model for the description of the renormalization coproduct of combinatorial Green's functions and solutions of DSEs in strongly coupled gauge field theories. Then we consider the coradical filtration on Feynman graphons to build a modified version of the Connes–Kreimer Renormalization Group which works on Feynman graph limits. It governs the renormalization of (non-perturbative) solutions of DSEs in terms of a generalized version of the BPHZ renormalization program which is given in [45–47]. Therefore this Renormalization Group is capable to encode the behavior of non-perturbative phases under scale changing of running coupling constants.

Thanks to Theorem 2.9, rooted tree representations of Feynman diagrams are the key tools for the construction of graphon models. The combinatorial decorated Hopf algebra $H_{\mathrm{CK}}(\Phi)$ is graded with respect to the vertex number where vertices represent nested loops in their corresponding Feynman diagrams. In graphon models, the vertex number $|t| = n$ determines the number of subintervals $\{I_1, ..., I_n\}$ of a partition of the ground measure space $([0, 1], m)$ in the structure of a labeled graphon. When the vertex number of a tree goes to infinity (i.e. the loop number of the corresponding Feynman diagram goes to infinity), we need to deal with graph limits defined on an infinite number of subintervals $\{I_n\}_{n=1}^{\infty}$ as a partition of $[0, 1)$ or $\mathbb{R}^+$. The completeness of the topology of (stretched or rescaled) graphons enables us to recognize a new class of analytic graphs for the description of higher loop order Feynman diagrams which contribute to Green's functions in strongly coupled physical theories. We have applied this method in Theorem 3.5 to achieve a new random graph interpretation for solutions of combinatorial DSEs. This class of large Feynman diagrams belong to the boundary region of the cut-distance topological Hopf algebra of renormalization $H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi)$.

Lemma 4.1. *There exists a topological enrichment of the renormalization Hopf algebra on Feynman graphons of a given gauge field theory.*

Proof. Set $H^{\Phi}_{\mathrm{graphon}}$ as the free commutative algebra over the field $\mathbb{K}$ generated by unlabeled Feynman graphon classes corresponding to 1PI Feynman diagrams in the physical theory Φ. The loop number parameter of Feynman diagrams determines a graduation parameter on their corresponding Feynman graphons. Define $H^{\Phi,(n)}_{\mathrm{graphon}}$ as the vector space generated by classes $[W_\Gamma]_\approx$ while Γ has the loop number n or Γ is the product of 1PI Feynman diagrams with the overall loop number n.

We equip this space with the renormalization coproduct $\Delta : H^{\Phi}_{\mathrm{graphon}} \to H^{\Phi}_{\mathrm{graphon}} \otimes H^{\Phi}_{\mathrm{graphon}}$ such that for each $[W_\Gamma]_\approx$,

$$\Delta([W_\Gamma]_\approx) = \sum [W_\gamma]_\approx \otimes [W_{\Gamma/\gamma}]_\approx \ , \tag{4.1}$$

where the sum is controlled in terms of Feynman graphons associated to disjoint unions of 1PI superficially divergent Feynman subdiagrams of Γ. The tensor space $H^{\Phi}_{\mathrm{graphon}} \otimes H^{\Phi}_{\mathrm{graphon}}$ is equipped with the projective cross norm given by

$$||x||_{\mathrm{p}} := \inf \left\{ \sum_{i=1}^{n} ||W_{\gamma'_i}||_{\mathrm{cut}} ||W_{\gamma''_i}||_{\mathrm{cut}} : x = \sum_{i=1}^{n} \gamma'_i \otimes \gamma''_i \right\} , \tag{4.2}$$

with respect to the cut-norm

$$||W_{\gamma'}||_{\mathrm{cut}} := \sup_{A,B\subset[0,1]} \left| \int_{A\times B} W_{\gamma'}(x,y)dm(x)dm(y) \right| . \tag{4.3}$$

$||.||_{\mathrm{p}}$ is a norm up to the weakly isomorphic relation. Thanks to Theorem 2.9, Δ, as a linear bounded operator, is continuous. In addition, the graduation parameter enables us to define an antipode $S : H^{\Phi}_{\mathrm{graphon}} \to H^{\Phi}_{\mathrm{graphon}}$, recursively, given by

$$S([W_\Gamma]_\approx) = -[W_\Gamma]_\approx - \sum S([W_\gamma]_\approx)[W_{\Gamma/\gamma}]_\approx \ . \tag{4.4}$$

It is also a continuous operator. Therefore $H^{\Phi}_{\mathrm{graphon}}$ is a topological Hopf algebra.

Suppose a sequence $\{\Gamma_n\}_{n\geq 1}$ of Feynman diagrams is cut-distance convergent to a large Feynman diagram X. We define $\Delta([W_X]_\approx)$ as the cut-distance convergent limit of the sequence $\{\Delta([W_{\Gamma_n}]_\approx)\}_{n\geq 1}$. We also define the antipode $S([W_X]_\approx)$ as the cut-distance convergent limit of the sequence $\{S([W_{\Gamma_n}]_\approx)\}_{n\geq 1}$. Therefore, thanks to Theorem 3.5, we can topologically complete $H^{\Phi}_{\mathrm{graphon}}$ by adding unlabeled Feynman graphon classes $[W_{\mathrm{DSE}}]_\approx$ to the collection of the generators of the Hopf algebra. The resulting topological Hopf algebra is denoted by $H^{\Phi,\mathrm{cut}}_{\mathrm{graphon}}$. □

If we remove the superficial divergence condition from the coproduct (4.1), then we obtain a graphon version of the core Hopf algebra of renormalization. The renormalization Hopf algebra of Feynman graphons is like a universal Hopf algebra. It can be modified to provide a well-defined combinatorial formulation for the coproduct of the (non-perturbative) solutions of

DSEs in gauge field theories. This investigation is a direct result of the compatibility of gauge symmetries (as fundamental identities between Feynman diagrams) with the weakly isomorphic relation (which addresses an identity between Feynman graphon classes corresponding to equalized Feynman diagrams).

Lemma 4.2. *The topological renormalization Hopf algebra of Feynman graphons associated to finite Feynman diagrams in a given perturbative gauge field theory Φ such as QED provides a well-defined coproduct for the complete 1PI Green's functions of the physical theory.*

Proof. In QED, there exist two classes of Green's functions Γ^{el} , $\Gamma^{\upsilon_{\mathrm{ph\ el\ el}}}$ with respect to the electron as the elementary particle and the interaction of the photon with a pair of electrons. Thanks to the formal expansion (3.5), a Ward–Takahashi element WT_n at loop order $n > 0$ is defined by the finite formal expansion $\mathrm{WT}_n = \Gamma_n^{\upsilon_{\mathrm{ph\ el\ el}}} - \Gamma_n^{\mathrm{el}}$ [54]. Thanks to Theorem 2.9, its Feynman graphon picture W_{WT_n} is given by a finite direct sum of Feynman graphons W_Γ of weights $\frac{1}{\mathrm{Sym}(\Gamma)}$ corresponding to 1PI Feynman diagrams Γ with the loop number n and the residues $\upsilon_{\mathrm{ph\ el\ el}}$ or el.

Thanks to [55], the Ward–Takahashi elements generate the Hopf ideal I_{WT} in the Connes–Kreimer renormalization Hopf algebra $H_{\mathrm{FG}}(\mathrm{QED})$. The resulting quotient Hopf algebra $H_{\mathrm{FG}}(\mathrm{QED})/I_{\mathrm{WT}}$ identifies certain linear combinations of Feynman diagrams with different valent vertices. Define a new Hopf ideal

$$I_{\mathrm{WT}}^{\mathrm{graphon}} :=< \{[W_{\mathrm{WT}_n}]_{\approx} \ , \ n\} > \ \leq \ H_{\mathrm{graphon}}^{\mathrm{QED}} \tag{4.5}$$

generated by Feynman graphon representations of the Ward–Takahashi elements. Thanks to Lemma 4.1, we consider the topologically completed Hopf ideal $I_{\mathrm{WT}}^{\mathrm{graphon,cut}}$ with respect to the cut-distance topology. By working on the quotient topological Hopf algebra $H_{\mathrm{graphon}}^{\mathrm{QED,cut}}/I_{\mathrm{WT}}^{\mathrm{graphon,cut}}$, we obtain the renormalization coproduct of the complete Green's functions of Feynman graphons associated to QED. $\square$

Corollary 4.3. *For any amplitude r in a given strongly coupled gauge field theory Φ such as low energy QCD, the renormalization coproduct is well-defined on the complete 1PI Green's functions of Feynman graphons associated to finite Feynman diagrams.*

Proof. The non-abelian generalization of the Ward–Takahashi elements, called Slavnov–Taylor elements, are defined in terms of the Green's functions

$$\Gamma^{\mathrm{qu}}, \Gamma^{\mathrm{gh}}, \Gamma^{\mathrm{gl}}, \Gamma^{\upsilon_{\mathrm{gl\ gl\ gl}}}, \Gamma^{\upsilon_{\mathrm{gl\ gl\ gl\ gl}}}, \Gamma^{\upsilon_{\mathrm{gl\ qu\ qu}}}, \Gamma^{\upsilon_{\mathrm{gl\ gh\ gh}}}, \Gamma^{\upsilon_{\mathrm{qu\ qu}}} \tag{4.6}$$

corresponding to the elementary particles in QCD, the cubic gluon self-interaction, the quartic gluon self-interaction, the interaction of the fermion and ghost with the gluon and the self-interaction of the fermion which has mass. Thanks to the formal expansion (3.5), we have

$$\begin{aligned}
\mathrm{ST^I} &= \Gamma^{\upsilon_{\mathrm{gl\ gl\ gl}}}\Gamma^{\upsilon_{\mathrm{gl\ qu\ qu}}} - \Gamma^{\upsilon_{\mathrm{gl\ gl\ gl\ gl}}}\Gamma^{\mathrm{qu}} \ , \\
\mathrm{ST^{II}} &= \Gamma^{\upsilon_{\mathrm{gl\ gl\ gl}}}\Gamma^{\upsilon_{\mathrm{gl\ gh\ gh}}} - \Gamma^{\upsilon_{\mathrm{gl\ gl\ gl\ gl}}}\Gamma^{\mathrm{gh}} \ , \\
\mathrm{ST^{III}} &= \Gamma^{\upsilon_{\mathrm{gl\ gl\ gl}}}\Gamma^{\upsilon_{\mathrm{gl\ gl\ gl}}} - \Gamma^{\upsilon_{\mathrm{gl\ gl\ gl\ gl}}}\Gamma^{\mathrm{gl}} \ , \ [54].
\end{aligned} \tag{4.7}$$

Thanks to Theorems 2.9 and 3.5, Feynman graphon representations $W_{\mathrm{ST^I}}$, $W_{\mathrm{ST^{II}}}$, $W_{\mathrm{ST^{III}}}$ of the Slavnov–Taylor elements are given in terms of the direct sums of weighted Feynman graphons associated to 1PI Feynman diagrams which contribute to each Slavnov–Taylor element.

Thanks to [53] and Subsection 4.2 in [54], the Slavnov–Taylor elements generate the Hopf ideal I_{ST} in the Connes–Kreimer renormalization Hopf algebra $H_{\mathrm{FG}}(\mathrm{QCD})$. The resulting quotient Hopf algebra $H_{\mathrm{FG}}(\mathrm{QCD})/I_{\mathrm{ST}}$ allows us to identify certain linear combinations of Feynman diagrams with different valent vertices which contribute to the Slavnov–Taylor elements. Thanks to the Euler characteristic, in this quotient setting the loop-number grading and the vertex-type grading are equivalent. Define a new Hopf ideal

$$I_{\mathrm{ST}}^{\mathrm{graphon}} := < \{[W_{\mathrm{ST}^{\mathrm{i}}}]_{\approx} \ : \ i = \mathrm{I}, \mathrm{II}, \mathrm{III}\} > \ \leq \ H_{\mathrm{graphon}}^{\mathrm{QCD}} \tag{4.8}$$

generated by Feynman graphon representations of the Slavnov–Taylor elements. Thanks to Lemma 4.1, we consider the topologically completed Hopf ideal $I_{\mathrm{ST}}^{\mathrm{graphon,cut}}$ with respect to the cut-distance topology. We show that the resulting quotient topological Hopf algebra $H_{\mathrm{graphon}}^{\mathrm{QCD,cut}}/I_{\mathrm{ST}}^{\mathrm{graphon,cut}}$ of Feynman graphons provides the renormalization coproduct of the complete Green's functions of Feynman graphons.

On the one hand, the behavior of the Green's functions of 1PI Feynman diagrams with a finite loop number n which contribute to a fixed residue r is studied with respect to the renormalization coproduct in [4,54,57]. Proposition 4.1 in [37] completes this study in various degrees of generality. On the second hand, thanks to Theorem 2.9, each Feynman graphon $W_{\Gamma_n^r}$ is determined by the adjacency matrix of the decorated rooted tree $t_{\Gamma_n^r}$ which has n vertices. On the third hand, the one to one correspondence $\Gamma \mapsto W_\Gamma$, the coproduct (4.1) and the graduation parameter with respect to the loop number determine an isomorphism of Hopf algebras between $H_{\mathrm{graphon}}^{\mathrm{QCD}}/I_{\mathrm{ST}}^{\mathrm{graphon}}$ and $H_{\mathrm{FG}}(\mathrm{QCD})/I_{\mathrm{ST}}$.[3] Therefore we have

$$\begin{aligned}
&\sum_{\Gamma_n^r,\ |\Gamma_n^r|=n,\ \mathrm{res}(\Gamma_n^r)=r} \frac{1}{\mathrm{Sym}(t_{\Gamma_n^r})} \Delta(W_{\Gamma_n^r}) \\
&= \sum_{k=0}^{n} \sum_{\gamma^r,\ |\gamma^r|=n-k,\ \mathrm{res}(\gamma^r)=r} \frac{t_{\Gamma_n^r}|t_{\gamma^r}}{\mathrm{Sym}(t_{\gamma^r})\mathrm{Sym}(t_{\Gamma_n^r})} W_{\gamma^r} \otimes W_{\Gamma_n^r} \ ,
\end{aligned} \tag{4.9}$$

such that $t_{\Gamma_n^r}|t_{\gamma^r}$ is the number of insertion places for t_{γ^r} in $t_{\Gamma_n^r}$. $\square$

Thanks to [51], the order of the automorphism group of a rooted tree is recursively computed in terms of its branches. For a given rooted tree t, let $t_1, ..., t_n$ be the root branches with multiplicities $a_1, ..., a_n$. Then $\mathrm{Sym}(t) = \prod_{i=1}^n a_i!\ \mathrm{Sym}(t_i)^{a_i}$. The difference between the complexity of the automorphism groups of decorated rooted trees and higher loop order Feynman diagrams enables us to optimize the computations in the renormalization of Green's functions via our new graphon setting. In other words, the renormalization of Green's functions in terms of the renormalization coproduct of Feynman graphon classes associated to 1PI Green's functions given by Corollary 4.3 can reduce the computational complexity than performing the renormalization process via the traditional approach. In addition, thanks to Theorem 3.5, it is now possible to extend the equation (4.9) to the Feynman graphon representations of DSEs.

[3] The adjacency matrices associated to Feynman diagrams Γ_1, Γ_2 with different loop numbers n_1, n_2 have different sizes. This means that their corresponding bare pixel pictures are not weakly isomorphic. In addition, if Γ_1, Γ_2 are non-isomorphic Feynman diagrams with the same loop number, then their rooted tree representations $t_{\Gamma_1}, t_{\Gamma_2}$ as decorated trees are not isomorphic. Therefore, while their adjacency matrices have the same size, their corresponding bare pixel pictures can not be related to each other by any Lebesgue measure preserving transformation on [0, 1]. In other words, they are not weakly isomorphic. As the consequence, Hopf algebras $H_{\mathrm{graphon}}^{\mathrm{QCD}}/I_{\mathrm{ST}}^{\mathrm{graphon}}$ and $H_{\mathrm{FG}}(\mathrm{QCD})/I_{\mathrm{ST}}$ are isomorphic.

Corollary 4.4. *For an equation DSE (under the coupling constant $\lambda g = 1$) with respect to an amplitude r in QCD, let X_{DSE} be its solution with the corresponding sequence $\{Y_m\}_{m\geq 1}$ of the partial sums and the Feynman graphon model W_{DSE}. The behavior of the renormalization coproduct of X_{DSE} is governed by the renormalization coproduct of its partial sums.*

Proof. We know that for each $m \geq 1$, $\Delta(Y_m) = \sum_{i=1}^{m} \Delta(X_i)$. On the first hand, thanks to Theorem 3.5, the sequence $\{Y_m\}_{m\geq 1}$ is cut-distance convergent to X_{DSE}. It means that the sequence $\{[W_{Y_m}]_{\approx}\}_{m\geq 1}$ of non-zero unlabeled Feynman graphon classes is convergent to $[W_{\mathrm{DSE}}]_{\approx}$ with respect to the cut-distance topology. By applying Lemma 4.1 and Corollary 4.3, when m goes to infinity, the sequence $\{\Delta(W_{Y_m})\}_{m\geq 1}$ is convergent to $\Delta(W_{\mathrm{DSE}})$ in $H^{\mathrm{QCD,cut}}_{\mathrm{graphon}}/I^{\mathrm{graphon,cut}}_{\mathrm{ST}}$. On the second hand, for each $1 \leq i \leq m$, if we consider the Feynman graphon W_{X_i} of weight $\frac{1}{\mathrm{Sym}(t_{X_i})}$, then the Feynman graphon W_{Y_m}, as the direct sum of $W_{X_1}, ..., W_{X_m}$, is of weight $b_m := \sum_{i=1}^{m} \frac{1}{\mathrm{Sym}(t_{X_i})}$. The ordered collection $\{b_{i_1}, b_{i_2}, ...\}$ of these weights determine subintervals $J_{i_k} := [b_{i_k}, b_{i_{k+1}})$ of $\mathbb{R}_+$. By applying a suitable affine monotone map ψ_{i_k}, we can project each J_{i_k} to a subinterval $\tilde{J}_{i_k}$ in $[0, 1]$ to make a partition for $[0, 1]$. This leads us to define a new stretched or rescaled version of W_{DSE} in terms of the graph limit of the sequence $\{W_{X_1} + ... + W_{X_m}\}_{m\geq 1}$ of the finite direct sums defined on the rescaled boxes $\tilde{J}_{i_m} \times \tilde{J}_{i_m}$ such that each W_{Y_m} is of weight $m(\tilde{J}_{i_m})$.

Therefore, for each $m \geq 1$, the equation (4.9) gives a rescaled version of $\Delta(W_{Y_m})$ such that

$$\lim_{m\to\infty} \sum_{i=1}^{m} \sum_{X_i,\ |X_i|=i,\ \mathrm{res}(X_i)=r} \frac{1}{\mathrm{Sym}(t_{X_i})} \Delta([W_{X_i}]_{\approx}) \tag{4.10}$$

exists in $H^{\mathrm{QCD,cut}}_{\mathrm{graphon}}/I^{\mathrm{graphon,cut}}_{\mathrm{ST}}$. It is convergent to $[\Delta(W_{\mathrm{DSE}})]_{\approx}$ with respect to the cut-distance topology. □

Remark 4.5. Thanks to Lemma 4.2 and Corollary 4.4, the renormalization coproducts of the partial sums of any equation DSE in QED govern the coproduct of X_{DSE} in $H^{\mathrm{QED,cut}}_{\mathrm{graphon}}/I^{\mathrm{graphon,cut}}_{\mathrm{WT}}$.

Lemma 4.6. *The affine group scheme $\mathbb{G}^{\Phi}_{\mathrm{graphon}}(-) = \mathrm{Hom}(H^{\Phi}_{\mathrm{graphon}}, -)$ is determined via linear algebraic groups.*

Proof. Thanks to Lemma 4.1, each homogeneous component $H^{\Phi,(n)}_{\mathrm{graphon}}$ is generated by unlabeled graphon classes corresponding to the pixel picture presentations of decorated non-planar rooted trees t with $|t| = n$ or the pixel picture presentations of products of decorated non-planar rooted trees with the overall vertex number n. Thanks to the Cayley's formula [51], for each n, the number of non-isomorphic decorated non-planar rooted trees of the rank n is finite. Therefore $H^{\Phi,(n)}_{\mathrm{graphon}}$ is a finite dimensional vector space. Thanks to Lemma 1.20 in [14], $\mathbb{G}^{\Phi}_{\mathrm{graphon}}(-)$ is the projective limit of some linear algebraic groups dual to $H^{\Phi,(n)}_{\mathrm{graphon}}$. □

For a given gauge field theory Φ with the Hopf ideal I_{Φ} corresponding to its gauge group symmetry and any locally convex algebra A such as the regularization algebra A_{dr} of Laurent series with finite pole part, consider $\mathbb{G}^{\Phi}_{\mathrm{graphon}}(A)$ as the complex Lie group of A-valued characters of $H^{\Phi}_{\mathrm{graphon}}$. Thanks to Lemma 4.2 and Corollary 4.3, the graphon version of any Feynman rules character vanishes on $I_{\Phi}^{\text{graphon}}$. The coradical filtration is defined on Feynman diagrams. It is possible to lift this filtration onto the level of Feynman graphons.

Definition 4.7. Set $P := \text{id} - \mathbb{I} \circ \varepsilon$ such that ε is the counit of $H^{\Phi}_{\text{graphon}}$. Thanks to the formula (4.1), define the reduced coproduct $\tilde{\Delta} := P^{\otimes 2} \circ \Delta$. If $\tilde{\Delta}^n = P^{\otimes n} \circ \Delta^n$, then we set

$$H^{\Phi}_{\text{graphon},(n)} := \ker \tilde{\Delta}^{n+1} \ , \ n \geq 0 \ , \tag{4.11}$$

such that ker $\tilde{\Delta}$ determines primitive objects of the Hopf algebra of Feynman graphons. The coradical filtration given by

$$\mathbb{Q} = H^{\Phi}_{\text{graphon},(0)} \subset H^{\Phi}_{\text{graphon},(1)} \subset \ldots \subset H^{\Phi}_{\text{graphon},(n)} \subset \ldots \subset H^{\Phi}_{\text{graphon}} \ , \tag{4.12}$$

determines the graduation

$$G^{\bullet}_{\Phi,\text{graphon}} := \bigoplus_{j \geq 0} H^{\Phi}_{\text{graphon},(j)} / H^{\Phi}_{\text{graphon},(j-1)} \ , \ H^{\Phi}_{\text{graphon},(-1)} = \emptyset \ . \tag{4.13}$$

Definition 4.8. Define the 1-parameter group $\{\tilde{\theta}_t\}_{t\in\mathbb{R}}$ of automorphisms on $\mathbb{G}^{\Phi}_{\text{graphon}}(\mathbb{C})$ such that its associated infinitesimal generator $\tilde{Y} := \frac{d}{dt}|_{t=0}\tilde{\theta}_t$ is defined in terms of the filtration rank in such a way that

$$\forall \tilde{\phi} \in H^{\Phi,\vee}_{\text{graphon},(n)} : \ \tilde{Y}(\tilde{\phi}) = n\tilde{\phi} \ . \tag{4.14}$$

In other words, for each t we have

$$\tilde{\theta}_t(\tilde{\phi}) = e^{nt}\tilde{\phi} \ . \tag{4.15}$$

Remark 4.9. Each $\tilde{\theta}_t$ acts on $H^{\Phi}_{\text{graphon}}$ and its dual algebra $H^{\Phi,\vee}_{\text{graphon}}$. For each character $\tilde{\phi} \in \mathbb{G}^{\Phi}_{\text{graphon}}(A)$ and each unlabeled Feynman graphon class $[W]_{\approx}$, we have

$$\langle \tilde{\theta}_t(\tilde{\phi}), [W]_{\approx} \rangle = \langle \tilde{\phi}, \tilde{\theta}_t([W]_{\approx}) \rangle \ . \tag{4.16}$$

Theorem 4.10. *The 1-parameter group $\{\tilde{\theta}_t\}_{t\in\mathbb{R}}$ determines a Renormalization Group which acts on solutions of DSEs in a given (strongly coupled) physical theory under different running coupling constants.*

Proof. Hochschild one-cocycles associated to primitive Feynman diagrams in $H_{\text{FG}}(\Phi)$ allow us to determine the coradical filtration of Feynman diagrams which contribute to fixed point equations of Green's functions. In this setting, generators X_n of the unique solution of an equation DSE are canonically filtered in terms of powers of $L = \log S/S_0$ in the corresponding log-expansion [30]. Thanks to the renormalized Feynman rules characters in $\mathbb{G}^{\Phi}_{\text{graphon}}(A_{\text{dr}})$, this canonical filtration is lifted onto the level of Feynman graphons [46] to obtain a well-defined filtration rank for (non-perturbative) solutions of combinatorial DSEs in terms of the filtration ranks of the partial sums.

Consider an equation DSE with the corresponding sequence $\{Y_m\}_{m\geq 1}$ of the partial sums and the solution X_{DSE}. Thanks to Definition 4.7, for each $m \geq 1$, suppose r_{Y_m} is the coradical degree of Y_m which means that

$$Y_m \in G^{r_{Y_m}}_{\Phi,\text{graphon}} := \bigoplus_{j=0}^{r_{Y_m}} H^{\Phi}_{\text{graphon},(j)} / H^{\Phi}_{\text{graphon},(j-1)} \ . \tag{4.17}$$

Let $\tilde{\phi}_r \in \mathbb{G}^{\Phi}_{\text{graphon}}(A_{\text{dr}})$ be a renormalized Feynman rule character given by

$$\tilde{\phi}_r([W_{Y_m}]_{\approx}) = \sum_{j=1}^{r_{Y_m}} c^{Y_m}_j L^j, \ \ c^{Y_m}_j := c_1^{\otimes j} \tilde{\Delta}^{j-1}([W_{Y_m}]_{\approx}) \ , \tag{4.18}$$

such that $c_1^{\otimes j} : H^{\Phi}_{\text{graphon}} \times \ldots^{j \text{ times}} \ldots \times H^{\Phi}_{\text{graphon}} \to \mathbb{C}$ is a symmetric function. Since $X_{\text{DSE}} = \lim_{m\to\infty} Y_m$ with respect to the cut-distance topology (i.e. Theorem 3.5 and [45]), the sequence $\{c^{Y_m}_j\}_{m\geq 1}$ is convergent to $c^{X_{\text{DSE}}}_j$, for each j, with respect to the projective cross norm (4.2) defined in terms of the cut-distance topology. In addition, the sequence $\{\tilde{\phi}_r([W_{Y_m}]_{\approx})\}_{m\geq 1}$ is convergent to $\tilde{\phi}_r([W_{\text{DSE}}]_{\approx})$ such that we have

$$\tilde{\phi}_r([W_{\text{DSE}}]_{\approx}) = \sum_{j=1}^{r_{X_{\text{DSE}}}} c^{X_{\text{DSE}}}_j L^j \ . \tag{4.19}$$

Consider the minimal subtraction map $R_{\text{ms}} : A_{\text{dr}} \to A_{\text{dr}}$ given by

$$\sum_{n\geq -m}^{\infty} c^n z^n \in \mathbb{C}[[z, z^{-1}] \mapsto \sum_{n\geq -m}^{-1} c^n z^n \ . \tag{4.20}$$

Thanks to Atkinson Theorem [3], the Rota–Baxter algebra $(A_{\text{dr}}, R_{\text{ms}})$ determines the unique Birkhoff decomposition $(R_{\text{ms}}(A_{\text{dr}}), -(\text{id} - R_{\text{ms}})(A_{\text{dr}}))$ as a subalgebra in $A_{\text{dr}} \times A_{\text{dr}}$ [13]. It allows us to show the existence of the Birkhoff factorization for Feynman rules characters in $\mathbb{G}^{\Phi}_{\text{graphon}}(A_{\text{dr}})$. Therefore there exists a pair $(\tilde{\phi}_-, \tilde{\phi}_+)$ such that $\tilde{\phi} = \tilde{\phi}_-^{-1} * \tilde{\phi}_+$ where $\tilde{\phi}_+ \in \mathbb{G}^{\Phi}_{\text{graphon}}(\mathbb{C}[[z]])$ and $\tilde{\phi}_- \in \mathbb{G}^{\Phi}_{\text{graphon}}(\mathbb{C}[z^{-1}])$. Thanks to Lemma 4.2 and Corollary 4.3, counterterms and renormalized values generated by $(\tilde{\phi}_-, \tilde{\phi}_+)$ vanish automatically on the Hopf ideal $I^{\text{graphon}}_{\Phi}$ corresponding to the gauge group symmetry of the physical theory.

Let $\mathbf{\Delta}$ be an infinitesimal disk around $z = 0$ in the complex plane and $C = \partial\mathbf{\Delta}$ be its boundary. For a given family of regularized characters determined by a loop $\gamma_\mu : C \to \mathbb{G}^{\Phi}_{\text{graphon}}(A_{\text{dr}})$, $z \mapsto \tilde{\phi}^z$ which obeys the scaled condition $\gamma_{e^t\mu}(z) = \tilde{\theta}_{tz}(\gamma_\mu(z))$, define

$$\tilde{F}_t = \lim_{z\to 0} \tilde{\phi}^z_- \tilde{\theta}_{tz}((\tilde{\phi}^z_-)^{-1}) \ . \tag{4.21}$$

The collection $\{\tilde{F}_t\}_{t\in\mathbb{R}}$, as the 1-parameter subgroup in $\mathbb{G}^{\Phi}_{\text{graphon}}(A_{\text{dr}})$, is the Renormalization Group on the space of Feynman graphons which contribute to solutions of (regularized) DSEs and their partial sums with respect to the dimensional regularization and minimal subtraction map. The infinitesimal generator $\tilde{\beta} = \frac{d}{dt}|_{t=0}\tilde{F}_t$ governs the evolution of running coupling constants in the graphon picture of towers of DSEs. In other words, the non-perturbative beta function of the physical theory can be computed in terms of $\tilde{\beta}$. □

5. Non-perturbative dynamics via geometric platforms

Thanks to our explained graphon model approach for the description of solutions of DSEs, in this section we are going to equip the space of all DSEs of a given (strongly coupled) gauge field theory with a new separable Banach manifold structure. Then, we introduce a particular class of Banach bundles which can provide some new local and global differential geometric tools for the study of geometric flows and transition phases between DSEs. These bundles are capable to geometrically describe the real time dynamics of non-perturbative phases in terms of the behavior of combinatorial DSEs under changing the scales of running coupling constants.

5.1. Banach space of combinatorial Dyson–Schwinger equations

Feynman graphon models enable us to equip the space of DSEs of a given gauge field theory with a metric structure.

Definition 5.1. Thanks to Definition 3.6, equations $\mathrm{DSE}_1, \mathrm{DSE}_2$ with the corresponding solutions X_{DSE_1} and X_{DSE_2} are called weakly isomorphic (or weakly equivalent), if $[W_{\mathrm{DSE}_1}]_{\approx} = [W_{\mathrm{DSE}_2}]_{\approx}$. In other words, $\mathrm{DSE}_1, \mathrm{DSE}_2$ are weakly isomorphic whenever there exist Lebesgue measure preserving transformations ρ_1, ρ_2 such that their corresponding labeled Feynman graphons $W^{\rho_1}_{\mathrm{DSE}_1}, W^{\rho_2}_{\mathrm{DSE}_2}$ coincide almost everywhere.

Theorem 5.2. *Fixed point equations of Green's functions under different running coupling constants in a given (strongly coupled) gauge field theory Φ with the bare coupling constant g can be organized in a separable Banach space.*

Proof. Set $\mathcal{S}^{\Phi,g}$ as the $\mathbb{K}$-linear space generated by combinatorial DSEs of the physical theory under different running coupling constants. Thanks to Definition 5.1, we define the quotient space $\mathcal{S}^{\Phi,g}_{\approx}$ as the space of classes $[\mathrm{DSE}]_{\approx}$ up to the weakly isomorphic relation. These classes of equations can also be determined in terms of the corresponding weakly isomorphic classes of their solutions. In addition, thanks to Theorem 3.5, we have the one to one correspondence $[\mathrm{DSE}]_{\approx} \mapsto [W_{\mathrm{DSE}}]_{\approx}$ which is useful to represent each object in $\mathcal{S}^{\Phi,g}_{\approx}$ in terms of its associated unlabeled Feynman graphon class. In other words,

$$[\mathrm{DSE}]_{\approx} = [X_{\mathrm{DSE}}]_{\approx}, \quad [W_{\mathrm{DSE}}]_{\approx} = [W_{X_{\mathrm{DSE}}}]_{\approx} \ . \tag{5.1}$$

Thanks to Theorems 2.9 and 3.5, for any equation DSE with the unique solution X_{DSE} and the corresponding non-zero labeled Feynman graphon W_{DSE}, its cut-norm is defined by

$$\| \mathrm{DSE} \|_{\mathrm{cut}} := \| W_{\mathrm{DSE}} \|_{\mathrm{cut}} = \sup_{A,B\subset[0,1]} \left| \int_{A\times B} W_{\mathrm{DSE}}(x,y)dm(x)dm(y) \right| . \tag{5.2}$$

It determines a pseudo-metric on the linear space $\mathcal{S}^{\Phi,g}$ which leads us to define a metric structure on $\mathcal{S}^{\Phi,g}_{\approx}$ given by

$$d_{\mathrm{cut}}([\mathrm{DSE}_1]_{\approx}, [\mathrm{DSE}_2]_{\approx}) := d_{\mathrm{cut}}([W_{\mathrm{DSE}_1}]_{\approx}, [W_{\mathrm{DSE}_2}]_{\approx}) \ . \tag{5.3}$$

Thanks to the correspondence $\mathrm{DSE} \mapsto W_{\mathrm{DSE}}$, Theorem 3.5 and Definition 2.11, the compact Hausdorff topological space $\mathcal{S}^{\Phi}_{\mathrm{graphon},\approx}$ is rich enough to encode non-perturbative solutions of combinatorial DSEs. Therefore we complete $\mathcal{S}^{\Phi,g}_{\approx}$ via rescaled or stretched versions of Feynman graphon classes in $\mathcal{S}^{\Phi}_{\mathrm{graphon},\approx}$. In other words, by applying the proofs of Theorem 2.9 and Lemma 4.1, we cut-distance topologically complete $\mathcal{S}^{\Phi,g}_{\approx}$ as the topological vector space generated by classes $[\mathrm{DSE}]_{\approx}$.

Let $\{e_i\}_i$ be the collection of all different kinds of elementary particles of the physical theory and $\{v_j\}_j$ be the collection of all different kinds of interactions between them. Thanks to the formal expansions (3.5) and (3.9), weakly isomorphic equivalence classes of the fixed point equations of the vertex type and edge type 1PI Green's functions

$$\{\Gamma^{e_i}, \Gamma^{v_j} : e_i, v_j\}_{i,j} \tag{5.4}$$

are the generators of the Banach space $\mathcal{S}^{\Phi,g}_{\approx}$. Therefore, since the countable space of graphons corresponding to simple graphs is dense in the space of graphons [23,34], it is observed that $\mathcal{S}^{\Phi,g}_{\approx}$ is separable. □

Remark 5.3.

- To simplify the notation, we use DSE or X_{DSE} to represent any arbitrary object of the quotient space $\mathcal{S}^{\Phi,g}_{\approx}$.
- Thanks to Banach–Mazur Theorem [1], the real Banach space $\mathcal{S}^{\Phi,g}_{\approx}$ is isometrically isomorphic to a closed linear subspace of the Banach space $C[0,1]$ of all real valued continuous functions on $[0,1]$ with respect to the metric of uniform convergence.

5.2. Non-perturbative tangent Banach bundle

In this part we work with $\mathcal{S}^{\Phi,g}_{\approx}$ as a Banach manifold and consider its corresponding tangent bundle. This new bundle is useful to study the evolution of non-perturbative phases of the strongly coupled physical theory Φ in terms of the geometric information of trajectories between DSEs under different running coupling constants in $\mathcal{S}^{\Phi,g}_{\approx}$.

Lemma 5.4. *The distance between combinatorial DSEs can be determined in terms of the convergent limit of the sequence of distances between Feynman graphons of their partial sums.*

Proof. Consider equations DSE_1, DSE_2 with the solutions X_{DSE_1}, X_{DSE_2} and corresponding sequences $\{Y^{(1)}_m\}_{m\geq 1}$, $\{Y^{(2)}_m\}_{m\geq 1}$ of the partial sums.

X_{DSE_i} is the cut-distance convergent limit of the sequence $\{Y^{(i)}_m\}_{m\geq 1}$ for $i=1,2$. It means that the sequence $\{W_{Y^{(i)}_m}\}_{m\geq 1}$ of Feynman graphons is convergent to W_{DSE_i} for $i=1,2$. We show that the sequence $\{d_{\mathrm{cut}}(W_{Y^{(1)}_m}, W_{Y^{(2)}_m})\}_{m\geq 1}$ is convergent to $d_{\mathrm{cut}}(W_{\mathrm{DSE}_1}, W_{\mathrm{DSE}_2})$. Thanks to the metric (5.3), we have

$$\begin{aligned}
&\forall \epsilon > 0\ \exists N_1 :\ \forall m \geq N_1 :\ d_{\mathrm{cut}}(Y^{(1)}_m, X_{\mathrm{DSE}_1}) < \epsilon/2\,,\\
&\forall \epsilon > 0\ \exists N_2 :\ \forall m \geq N_2 :\ d_{\mathrm{cut}}(Y^{(2)}_m, X_{\mathrm{DSE}_2}) < \epsilon/2\,.
\end{aligned} \tag{5.5}$$

Set $N := \max\{N_1, N_2\}$. For all $m \geq N$, we have

$$\left| d_{\mathrm{cut}}(Y^{(1)}_m, Y^{(2)}_m) - d_{\mathrm{cut}}(X_{\mathrm{DSE}_1}, X_{\mathrm{DSE}_2}) \right| = \tag{5.6}$$

$$\left| \inf_{\rho_1,\rho_2} \sup_{A,B} \left| \int_{A\times B} (W^{\rho_1}_{Y^{(1)}_m} - W^{\rho_2}_{Y^{(2)}_m}) dm(x)dm(y) \right| - \inf_{\rho_1,\rho_2} \sup_{A,B} \left| \int_{A\times B} (W^{\rho_1}_{\mathrm{DSE}_1} - W^{\rho_2}_{\mathrm{DSE}_2}) dm(x)dm(y) \right| \right|$$

$$\leq \left| \inf_{\rho_1,\rho_2} \sup_{A,B} \int\limits_{A\times B} \left(W^{\rho_1}_{Y^{(1)}_m} - W^{\rho_2}_{Y^{(2)}_m} - W^{\rho_1}_{\mathrm{DSE}_1} + W^{\rho_2}_{\mathrm{DSE}_2} \right) dm(x)dm(y) \right| \tag{5.7}$$

$$\leq \inf_{\rho_1,\rho_2} \sup_{A,B} \left| \int\limits_{A\times B} \left(W^{\rho_1}_{Y^{(1)}_m} - W^{\rho_1}_{\mathrm{DSE}_1}\right) dm(x)dm(y) \right| \tag{5.8}$$

$$+ \inf_{\rho_1,\rho_2} \sup_{A,B} \left| \int\limits_{A\times B} \left(W^{\rho_2}_{\mathrm{DSE}_2} - W^{\rho_2}_{Y^{(2)}_m}\right) dm(x)dm(y) \right| < \epsilon/2 + \epsilon/2 \ . \quad \square$$

Lemma 5.5. *There exists a manifold structure on the space of solutions of combinatorial DSEs of a given (strongly coupled) gauge field theory Φ with the bare coupling constant g.*

Proof. Thanks to the built Banach structure on the space of weakly isomorphic equivalence classes of combinatorial DSEs (i.e. Theorem 5.2), we formulate the Gâteaux derivatives for functionals on $\mathcal{S}^{\Phi,g}_{\approx}$. A function $F : \mathcal{S}^{\Phi,g}_{\approx} \to \mathcal{S}^{\Phi,g}_{\approx}$ is Gâteaux differentiable at X_{DSE}, if there exists an operator $DF_{X_{\mathrm{DSE}}} : \mathcal{S}^{\Phi,g}_{\approx} \to \mathcal{S}^{\Phi,g}_{\approx}$ such that for all $X \in \mathcal{S}^{\Phi,g}_{\approx}$, we have

$$\lim_{t\to 0} \frac{F(X_{\mathrm{DSE}} + tX) - F(X_{\mathrm{DSE}})}{t} = DF_{X_{\mathrm{DSE}}}(X) \ . \tag{5.9}$$

In other words, all the directional derivatives exist and form a bounded linear operator. If the Gâteaux derivative of F uniformly exists on the unit ball in $\mathcal{S}^{\Phi,g}_{\approx}$, then it is called Frèchet differentiable.

This differential calculus is useful to build a manifold structure on $\mathcal{S}^{\Phi,g}_{\approx}$. For $n \geq 0$, a C^n-atlas on $\mathcal{S}^{\Phi,g}_{\approx}$ is a collection $\{(A_i, \phi_i)\}_{i\in I}$ of charts such that $A_i \subseteq \mathcal{S}^{\Phi,g}_{\approx}$, $\mathcal{S}^{\Phi,g}_{\approx} = \bigcup_i A_i$. Each ϕ_i is a bijection between A_i and an open subset of $\mathcal{S}^{\Phi,g}_{\approx}$ such that $\phi_i(A_i \cap A_j)$ is an open subset in $\mathcal{S}^{\Phi,g}_{\approx}$. Each map $\phi_j \circ \phi_i^{-1}$ is n-times Frèchet differentiable such that the n[th] derivation is a continuous function with respect to the cut-distance topology. However we can consider the atlas with a single chart which is defined globally on $\mathcal{S}^{\Phi,g}_{\approx}$ to build our promised smooth Banach manifold structure. $\square$

Remark 5.6. For a given gauge field theory Φ with the bare coupling constants $g_1, ..., g_n$, all combinatorial DSEs under different running couplings can be encoded via the Banach manifold $\mathcal{S}^{\Phi,g_1,...,g_n}_{\approx}$. This topological space can be studied as a subspace of the space of unlabeled rescaled or stretched Feynman graphon classes of the physical theory.

Thanks to Corollary 3.8, in a gauge field theory Φ with the bare coupling constant g, (non-perturbative) solutions of DSEs under a running coupling constant $\lambda_0 g$ can give us information about the status of the physical theory in a definite phase.

Definition 5.7. A real time dynamical process in the physical theory is given in terms of changing the scale of running coupling constants λg in a given equation DSE(λg) or changing the primitive Feynman diagrams $\{\gamma_n\}_{n\geq 1}$ in the combinatorial structure of the equation (i.e. Theorem 3.4) during a continuum of time. In other words, a real time dynamical process should present moving from an equation DSE_1 to another equation DSE_2 in the physical theory under changing the scales of running couplings during a continuum of time.

The Gâteaux differential calculus theory on the Banach manifold $\mathcal{S}^{\Phi,g}_{\approx}$ is the starting step to initiate the foundations of a new Gauge Theory for the study of the dynamics of non-perturbative aspects of quantum field theories beyond the Standard Model under different running coupling constants. In this regard, we are going to analyze the evolution of DSEs under different running coupling constants in the context of the geometry of $\mathcal{S}^{\Phi,g}_{\approx}$.

Theorem 5.8. *For a given strongly coupled gauge field theory Φ with the bare coupling constant g, the tangent bundle of the Banach manifold $\mathcal{S}^{\Phi,g}_{\approx}$ describes the real time dynamical processes between non-perturbative phases of the physical theory.*

Proof. For the Banach manifold $\mathcal{S}^{\Phi,g}_{\approx}$, consider its associated tangent bundle via the projection map $\pi^{\mathrm{Tan}}_{\Phi,g} : T\mathcal{S}^{\Phi,g}_{\approx} \to \mathcal{S}^{\Phi,g}_{\approx}$ such that

$$T\mathcal{S}^{\Phi,g}_{\approx} := \bigcup_{X_{\mathrm{DSE}} \in \mathcal{S}^{\Phi,g}_{\approx}} \left\{ (X_{\mathrm{DSE}}, Z) : \ Z \in T_{X_{\mathrm{DSE}}} \mathcal{S}^{\Phi,g}_{\approx} \right\} . \tag{5.10}$$

The tangent space $T_{X_{\mathrm{DSE}}} \mathcal{S}^{\Phi,g}_{\approx}$ at X_{DSE} is the vector space of all tangent vectors Z at X_{DSE}. This tangent bundle is a principal $\mathbb{G}_\infty(\mathbb{C})$-bundle such that $\mathbb{G}_\infty(\mathbb{C})$ is the projective limit of the complex linear algebraic groups $\mathrm{GL}_N(\mathbb{C})$ dual to the Hopf algebras

$$H_N = \mathbb{C}[x_{i,j}, t]_{i,j=1,\ldots,N} / (\det(x_{i,j})t - 1) \ , \ \forall N \geq 1 \ . \tag{5.11}$$

The theory of differential forms on Banach manifolds are considered in [32] where a p-form is defined as a continuous alternating p-linear map. We apply this theory to our tangent bundle and $\mathbb{G}_\infty(\mathbb{C})$-valued 0-forms on $\mathcal{S}^{\Phi,g}_{\approx}$. Therefore connections are given by $\mathfrak{g}_\infty(\mathbb{C}) = \mathrm{Lie}\ \mathbb{G}_\infty(\mathbb{C})$-valued differential 1-forms on $\mathcal{S}^{\Phi,g}_{\approx}$. The $\mathfrak{g}_\infty(\mathbb{C})$-valued 2-form $R_\omega = d\omega + \frac{1}{2}[\omega \wedge \omega]$ is the curvature form of the connection ω.

For any $\mathbb{G}_\infty(\mathbb{C})$-module E, consider a new vector bundle

$$\mathcal{E}_{\mathcal{S}^{\Phi,g}_{\approx}} = T\mathcal{S}^{\Phi,g}_{\approx} \times_{\mathbb{G}_\infty(\mathbb{C})} E \to \mathcal{S}^{\Phi,g}_{\approx} \ . \tag{5.12}$$

Any connection ω of the tangent bundle gives a $\Gamma(T^*\mathcal{S}^{\Phi,g}_{\approx} \otimes \mathcal{E}_{\mathcal{S}^{\Phi,g}_{\approx}})$-valued operator on the space of cross-sections of $\mathcal{E}_{\mathcal{S}^{\Phi,g}_{\approx}}$. We can extend this operator to a new operator d_ω of degree one on the space of $\mathcal{E}_{\mathcal{S}^{\Phi,g}_{\approx}}$-valued p-forms with the corresponding conjugate operator δ_ω. The connection ω of the tangent bundle with the curvature R_ω which satisfies the condition $\delta_\omega R_\omega = 0$ is called a generalized Yang–Mills field. These connections, which enable us to define new elements as analogous to quantities associated to frame changing, can be interpreted as carriers of interactions among combinatorial DSEs.

In this setting, a parallel transporter for generalized Yang–Mills fields is defined as a diffeomorphism between fibers. It is actually a linear map $T_\alpha : T_{X_{\mathrm{DSE}_1}} \mathcal{S}^{\Phi,g}_{\approx} \to T_{X_{\mathrm{DSE}_2}} \mathcal{S}^{\Phi,g}_{\approx}$ such that α is a trajectory from X_{DSE_1} to X_{DSE_2} in the Banach space $\mathcal{S}^{\Phi,g}_{\approx}$.

For equations $\mathrm{DSE}_1, \mathrm{DSE}_2$ with the solutions $X_{\mathrm{DSE}_1}, X_{\mathrm{DSE}_2}$ in the base space $\mathcal{S}^{\Phi,g}_{\approx}$, set

$$C_{1,2} := \left\{ \alpha_{12} : [a,b] \subset \mathbb{R} \to \mathcal{S}^{\Phi,g}_{\approx} : \ \alpha_{12}(a) = X_{\mathrm{DSE}_1}, \alpha_{12}(b) = X_{\mathrm{DSE}_2} \right\} , \tag{5.13}$$

as the collection of all smooth curves in $\mathcal{S}^{\Phi,g}_{\approx}$ which connects X_{DSE_1} to X_{DSE_2}. Define a partial order relation on $C_{1,2}$ in terms of the lengths of paths in such a way that

$$\alpha_{12} \leq \beta_{12} \Leftrightarrow l(\alpha_{12}) \leq l(\beta_{12}) \ . \tag{5.14}$$

Thanks to [8], the Banach space $\mathcal{S}_{\approx}^{\Phi,g}$, equipped with the metric

$$d_{\text{cut}}(X_{\text{DSE}_1}, X_{\text{DSE}_2}) = ||W_{\text{DSE}_1} - W_{\text{DSE}_2}||_{\text{cut}} \ , \tag{5.15}$$

is a geodesic space. It means that each pair $\text{DSE}_1, \text{DSE}_2$ of equations can be connected by a geodesic path, namely a linearly reparameterized geodesic

$$\hat{\alpha}_{12} : t \in [0,1] \mapsto (1-t)X_{\text{DSE}_1} + tX_{\text{DSE}_2} \in \mathcal{S}_{\approx}^{\Phi,g} \ . \tag{5.16}$$

It is a continuous path of length $d_{\text{cut}}(X_{\text{DSE}_1}, X_{\text{DSE}_2})$. Therefore the poset $C_{1,2}$ has a minimal object.

- For any curve $\alpha_{12} \in C_{1,2}$, the value $||d\alpha_{12}/dt|_{t=t_0}||_{\text{cut}}$ identifies the phase transition speed from DSE_1 to DSE_2 at the time t_0. Define the unit tangent operator $e_1^{\alpha_{12}}(t) = \frac{d\alpha_{12}/dt}{||d\alpha_{12}/dt||_{\text{cut}}}$.
- The unit normal operator $e_2^{\alpha_{12}}(t) = \frac{de_1^{\alpha_{12}}(t)/dt}{||de_1^{\alpha_{12}}(t)/dt||_{\text{cut}}}$ indicates the deviance of α_{12} from being a "straight" line in $\mathcal{S}_{\approx}^{\Phi,g}$.
- The curvature of the path α_{12} at the time t is given by $\kappa(t) = \frac{||de_1^{\alpha_{12}}(t)/dt||_{\text{cut}}}{||d\alpha_{12}(t)/dt||_{\text{cut}}}$. This parameter determines the amount of deviation between non-perturbative parameters generated by a trajectory of DSEs. □

Remark 5.9. Geodesics in $\mathcal{S}_{\approx}^{\Phi,g}$, which have zero curvature, are intrinsic choices of the physical theory for transitions from a non-perturbative phase to another phase.

Proof. Thanks to the theory of sub-Riemannian structures on Banach manifolds explained in Sections 2.2 and 2.3 in [2], we can define a weak sub-Riemannian structure on the Banach manifold $\mathcal{S}_{\approx}^{\Phi,g}$. The corresponding metric tensor allows us to define the energy functional on the space of continuous differentiable paths $\alpha_{12} : [0,1] \to \mathcal{S}_{\approx}^{\Phi,g}$.

Thanks to Corollary 3.8, let equations $\text{DSE}_1(\lambda_1 g), \text{DSE}_2(\lambda_2 g)$ belong to different phases of a given gauge field theory Φ under running coupling constants $\lambda_1 g, \lambda_2 g$. The minimum of the functional

$$E(\alpha_{12}) = \frac{1}{2}\int_0^1 g_{\alpha_{12}(t)}(d\alpha_{12}/dt, d\alpha_{12}/dt)dt \tag{5.17}$$

gives the geodesic $\hat{\alpha}_{12}$ which has zero curvature. If we consider the parameter t as the time evolution in the physical system, then for any $0 < t_0 < 1$, $\hat{\alpha}_{12}(t_0)$ determines an equation $\text{DSE}_{t_0}(\lambda_{t_0} g)$ in an intermediate phase under the running coupling $\lambda_{t_0} g$. Therefore the geodesic $\hat{\alpha}_{12}$ determines the shortest path of intermediate phases between initial and final phases with respect to the running couplings $\lambda_1 g$ and $\lambda_2 g$. □

5.3. Non-perturbative Hopf Banach bundle

Consider $H_{\text{FG}}^{\text{cut}}(\Phi)$ as the cut-distance topological enrichment of the Connes–Kreimer renormalization Hopf algebra of Feynman diagrams of the physical theory Φ. This topological Hopf algebra is rich enough to generate non-perturbative parameters of the renormalization of solutions of combinatorial DSEs [46]. In addition, for any equation $\text{DSE}(\lambda g)$ with the unique solution

$X_{\mathrm{DSE}} = \sum_{n\geq 0}(\lambda g)^n X_n$, components X_n (given by Theorem 3.4) determine a topological Hopf subalgebra of $H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi)$.

Lemma 5.10. *For any locally convex algebra A,*

- *The space* $\mathrm{Lin}(H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi), A)$ *of linear maps with the convolution product (defined by the renormalization coproduct) is a unital associative non-commutative algebra.*
- *The space* $\mathrm{char}(H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi), A)$ *of characters is a group such that the inversion is defined in terms of the antipode of the renormalization Hopf algebra.*
- *The space* $\partial\mathrm{char}(H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi), A)$ *of infinitesimal characters is a Lie subalgebra of the space* $(\mathrm{Lin}(H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi), A), [.,.])$ *such that the Lie bracket is the commutator with respect to the convolution product.*
- *The topology of pointwise convergence turns* $\mathrm{Lin}(H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi), A)$ *into a locally convex algebra while* $\mathrm{char}(H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi), A)$ *is a topological group with the corresponding topological Lie algebra* $\partial\mathrm{char}(H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi), A)$.

Proof. Thanks to the Kreimer's renormalization coproduct (3.6), the convolution product is defined by

$$\phi_1 * \phi_2(\Gamma) = \sum_{\gamma} \phi_1(\gamma)\phi_2(\Gamma/\gamma)\ . \tag{5.18}$$

Let $\{\Gamma_n\}_{n\geq 1}$ be a sequence of Feynman diagrams which is cut-distance convergent to a large Feynman diagram X (given in Definition 2.12). Then, thanks to Lemmas 4.1, 4.2 and Corollary 4.3, $\Delta(X)$ is given as the cut-distance convergent limit of the sequence $\{\Delta(\Gamma_n)\}_{n\geq 1}$. Therefore $\phi_1 * \phi_2(X)$ is well-defined by the cut-distance convergent limit of the sequence $\{\phi_1 * \phi_2(\Gamma_n)\}_{n\geq 1}$. In addition, $S(X)$ is given as the cut-distance convergent limit of the sequence $\{S(\Gamma_n)\}_{n\geq 1}$ where we have

$$S(\Gamma_n) = -\Gamma_n - \sum_{\gamma_n} S(\gamma_n)\Gamma_n/\gamma_n\ . \quad \square \tag{5.19}$$

The renormalization Hopf algebra is graded and connected. Therefore we can define the exponential map on the space of infinitesimal characters where we have

$$\exp : l \mapsto \sum_{n=0}^{\infty} \frac{l^{*n}}{n!}\ ,\quad l^{*n} := l * \ldots * l\ . \tag{5.20}$$

It determines a bijection map from $\partial\mathrm{char}(H_{\mathrm{FG}}(\Phi), A)$ to $\mathrm{char}(H_{\mathrm{FG}}(\Phi), A)$ with the corresponding inverse map ln.

Remark 5.11.

- The compactness of the space of Feynman graphons is enough to show that the exponential map (5.20) is well-defined on large Feynman diagrams as non-perturbative solutions of combinatorial DSEs. Therefore exp, as an analytic map, determines a bijection between spaces $\partial\mathrm{char}(H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi), A)$ and $\mathrm{char}(H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi), A)$.
- For any Feynman diagram Γ, let $Z_\Gamma : H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi) \to \mathbb{C}$ be its corresponding infinitesimal character given by $Z_\Gamma(\Gamma') = \delta_{\Gamma,\Gamma'}$ such that $\delta_{.,.}$ is the Kronecker delta on Feynman diagrams. It

is a linear map which satisfies the Leibniz rule. For any large Feynman diagram X_{DSE}, the infinitesimal character $Z_{X_{\text{DSE}}}$ is well-defined.

Theorem 5.12. *$H^{\text{cut}}_{\text{FG}}(\Phi)$ is the total space for a Hopf Banach bundle on $\mathcal{S}^{\Phi,g}_{\approx}$.*

Proof. The Lie group $\mathbb{G}_{\Phi} = \text{char}(H_{\text{FG}}(\Phi), H^{\text{cut}}_{\text{FG}}(\Phi))$ acts on $H^{\text{cut}}_{\text{FG}}(\Phi)$ in such a way that for any finite Feynman diagram Γ,

$$\mathbb{G}_{\Phi} \times H^{\text{cut}}_{\text{FG}}(\Phi) \to H^{\text{cut}}_{\text{FG}}(\Phi),\ \langle \phi, \Gamma \rangle := \phi(\Gamma)\ . \tag{5.21}$$

Since the topological Hopf algebra $H^{\text{cut}}_{\text{FG}}(\Phi)$ is totally bounded via Feynman graph limits, any character $\phi \in \mathbb{G}_{\Phi}$ is linear and bounded. Therefore for any large Feynman diagram X, as the cut-distance convergent limit of the sequence $\{\Gamma_n\}_{n\geq 1}$, the action is defined by

$$\langle \phi, X \rangle := \lim_{n\to\infty} \phi(\Gamma_n)\ . \tag{5.22}$$

Define a new principal Banach bundle $\left(\mathbb{G}_{\Phi}, H^{\text{cut}}_{\text{FG}}(\Phi), \mathcal{S}^{\Phi,g}_{\approx}, \pi^{\text{Hopf}}_{\Phi,g}\right)$ such that for any $X_{\text{DSE}} \in \mathcal{S}^{\Phi,g}_{\approx}$, the topological Hopf subalgebra $H^{\text{cut}}_{\text{DSE}}$ is the fiber $\pi^{\text{Hopf},-1}_{\Phi,g}(X_{\text{DSE}})$ of this new Banach bundle. The Lie group $\mathbb{G}_{\Phi}$ preserves these fibers. For each equation DSE in $\mathcal{S}^{\Phi,g}_{\approx}$, the Banach space $H^{\text{cut}}_{\text{DSE}}$ is called the internal space such that generators $\mathbb{I}, X_1, X_2, ...$ can determine a gauge system.

Vector fields $X_{\text{DSE}} \mapsto Z \in T_{X_{\text{DSE}}}\mathcal{S}^{\Phi,g}_{\approx}$ in the tangent bundle $T\mathcal{S}^{\Phi,g}_{\approx} \to \mathcal{S}^{\Phi,g}_{\approx}$ (i.e. Theorem 5.8) are replaced with $H^{\text{cut}}_{\text{DSE}}$-valued 0-forms on the base space $\mathcal{S}^{\Phi,g}_{\approx}$ such as

$$X_{\text{DSE}} \mapsto B^+_{\gamma}(.) \in H^{\text{cut}}_{\text{DSE}} \tag{5.23}$$

for each primitive Feynman diagram $\gamma \in H_{\text{FG}}(\Phi)$. While connections in the tangent bundle are replaced with $\mathfrak{g}_{\Phi} = \text{Lie}\ \mathbb{G}_{\Phi}$-valued 1-forms (as gauge potentials), functionals (5.23) contribute to determine generalized Yang–Mills fields in this Hopf Banach bundle. In this setting, a parallel transporter is defined as a linear map $T_{\alpha} : H^{\text{cut}}_{\text{DSE}_1} \to H^{\text{cut}}_{\text{DSE}_2}$ such that α is a trajectory from X_{DSE_1} to X_{DSE_2} in the Banach manifold $\mathcal{S}^{\Phi,g}_{\approx}$. $\square$

5.4. Non-perturbative regularization Banach bundle

For any equation DSE with the unique solution X_{DSE} in a given (strongly coupled) gauge field theory Φ, suppose $\mathbf{\Delta}_{\text{DSE}}$ is an infinitesimal convex neighborhood around X_{DSE} in $\mathcal{S}^{\Phi,g}_{\approx}$. Set $\mathbf{\Delta}^*_{\text{DSE}}$ as the corresponding punctured neighborhood which does not contain X_{DSE}. Set $C_{\text{DSE}} := \partial \mathbf{\Delta}_{\text{DSE}}$ as the boundary region. For $\mathbb{G}^{\text{cut}}_{\Phi}(A_{\text{dr}}) = \text{char}(H^{\text{cut}}_{\text{FG}}(\Phi), A_{\text{dr}})$ and $\mathcal{S}^{\Phi,g}_{\approx}$ as a topological subspace of $H^{\text{cut}}_{\text{FG}}(\Phi)$, functions $\mathbf{\Delta}^*_{\text{DSE}} \to \mathbb{G}^{\text{cut}}_{\Phi}(A_{\text{dr}})$ can generate $\mathbb{G}^{\text{cut}}_{\Phi}(A_{\text{dr}})$-valued trajectories controlled by combinatorial DSEs under different running coupling constants in a neighborhood around the equation DSE.

Lemma 5.13. *There exists a Haar measure on $\mathcal{S}^{\Phi,g}_{\approx}$.*

Proof. We equip the space $\mathcal{S}^{\Phi,g}_{\approx}$ with the symmetric difference $\bigodot$. For given equations $\text{DSE}_1, \text{DSE}_2$ with the corresponding families $\{B^+_{\gamma^i_n}\}_{n\geq 1}$ of Hochschild one cocycles for $i = 1, 2$, $\text{DSE}_1 \bigodot \text{DSE}_2$ is a new equation generated by the symmetric difference of sets $\{\gamma^1_n\}_{n\geq 1}$ and

$\{\gamma_n^2\}_{n\geq 1}$ of primitive (1PI) Feynman diagrams. This binary operation determines a compact Hausdorff topological abelian group with respect to the cut-distance topology. This enables us to equip $\mathcal{S}_{\approx}^{\Phi,g}$ with a unique (up to scalars) non-zero left-right invariant Haar measure μ_{Haar}. □

Thanks to the 1-parameter group $\{\tilde{\theta}_t\}_{t\in\mathbb{R}}$ of automorphisms on $\mathbb{G}_{\text{graphon}}^{\Phi}(\mathbb{C})$ (given by Definition 4.8 and Remark 4.9) and Remark 5.11, define a new 1-parameter group of automorphisms on $\mathbb{G}_{\Phi}^{\text{cut}}(\mathbb{C})$ which is useful for the study of the dynamics of non-perturbative phases in a neighborhood of a strongly coupled equation DSE.

Lemma 5.14. *The coradical filtration determines a new 1-parameter group of automorphisms on* $\mathbb{G}_{\Phi}^{\text{cut}}(\mathbb{C})$.

Proof. For a fixed equation DSE with the solution X_{DSE}, define a new 1-parameter group of automorphisms $\{\hat{\theta}_{Z_X}\}_{X\in\mathcal{S}_{\approx}^{\Phi,g}}$ such that Z_X is the infinitesimal character corresponding to X. For any $\tilde{\phi}\in H_{(n)}^{\text{cut},\vee}$ with the coradical filtration rank $\hat{Y}(\tilde{\phi}) = n\tilde{\phi}$, define

$$\hat{\theta}_{Z_X} \in \text{Aut}(\mathbb{G}_{\Phi}^{\text{cut}}(\mathbb{C})) : \ \hat{\theta}_{Z_X}(\tilde{\phi}) = \exp(nZ_{X-X_{\text{DSE}}}) * \tilde{\phi} \ . \tag{5.24}$$

Thanks to the exponential map (5.20) and Remark 5.11, $\hat{\theta}_{Z_X}(\tilde{\phi})$ is a well-defined character. We have

$$\begin{aligned}
&\frac{D}{DZ_X}\hat{\theta}_{Z_X}(\tilde{\phi}) = \frac{D}{DZ_X}\left(\exp(nZ_{X-X_{\text{DSE}}}) * \tilde{\phi}\right) \\
&= \frac{D}{DZ_X}\left(\exp(nZ_{X-X_{\text{DSE}}})\right) * \tilde{\phi} + \exp(nZ_{X-X_{\text{DSE}}}) * \frac{D}{DZ_X}(\tilde{\phi}) \\
&\Rightarrow \frac{D}{DZ_X}|_{X=X_{\text{DSE}}}\hat{\theta}_{Z_X}(\tilde{\phi}) = n\exp(nZ_{X-X_{\text{DSE}}})|_{X=X_{\text{DSE}}} * \tilde{\phi} \\
&= n\exp(nZ_{\mathbb{I}}) * \tilde{\phi} = n\tilde{\phi} \ .
\end{aligned} \tag{5.25}$$

Therefore

$$\frac{D}{DZ_X}|_{X=X_{\text{DSE}}}\hat{\theta}_{Z_X} = \hat{Y} \ . \quad \square \tag{5.26}$$

Definition 5.15. Thanks to Lemma 5.14, for a given equation DSE with the solution X_{DSE} and any $X\in\mathcal{S}_{\approx}^{\Phi,g}$, an operator

$$F_X : \mathbf{\Delta}_{\text{DSE}}^* \to \mathbb{G}_{\Phi}^{\text{cut}}(\mathbb{C}) \ , \tag{5.27}$$

is called a X-scaled smooth curve around DSE in the Banach space $\mathcal{S}_{\approx}^{\Phi,g}$, if

$$F_{Xe^{X_{\text{DSE}_1}}}(X_{\text{DSE}_2}) = \hat{\theta}_{Z_{X_{\text{DSE}_1}\odot X_{\text{DSE}_2}}}(F_X(X_{\text{DSE}_2})) \ , \tag{5.28}$$

such that

$$Xe^{X_{\text{DSE}_1}} = \sum_{n=0}^{\infty}\frac{XX_{\text{DSE}_1}^n}{n!} \tag{5.29}$$

is a well-defined object in the topological Hopf algebra $H_{\text{FG}}^{\text{cut}}(\Phi)$.

For the multiplicative group $\mathbb{G}_m$ associated to the Hopf algebra

$$H = \mathbb{C}[t, t^{-1}], \quad \Delta(t) = t \otimes t \ , \tag{5.30}$$

consider a principal $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$-bundle $(B_{\mathrm{DSE}}, \mathbf{\Delta}_{\mathrm{DSE}}, \pi)$ with respect to an action u of $\mathbb{G}_m$ on B_{DSE} given by rescaling the amount of running couplings in the structure of combinatorial DSEs. In other words,

$$u: \ z \times X_{\mathrm{DSE}'(\lambda g)} \mapsto X_{\mathrm{DSE}'(|z|\lambda g)} \ , \quad z \in \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^* \ . \tag{5.31}$$

Definition 5.16. We build a new trivial principal $\mathbb{G}_\Phi^{\mathrm{cut}}(A_{\mathrm{dr}})$-bundle

$$P_{\Phi,g} = \big(B_{\mathrm{DSE}} \times \mathbb{G}_\Phi^{\mathrm{cut}}(A_{\mathrm{dr}}), B_{\mathrm{DSE}}, \pi_{\Phi,g}^{\mathrm{Triv}}\big) \ , \tag{5.32}$$

such that the action of $\mathbb{G}_m$ on $P_{\Phi,g}$ is defined by

$$z \times (X_{\mathrm{DSE}'(\lambda g)}, \tilde{\phi}) \mapsto (u(z, X_{\mathrm{DSE}'(\lambda g)}), u^Y(\tilde{\phi})) \ . \tag{5.33}$$

Remove the fiber over X_{DSE} from the base space B_{DSE} to restrict the bundle $P_{\Phi,g}$ to $P_{\Phi,g}^{X_{\mathrm{DSE}}}$. Connections on this new principal bundle are given by $\mathfrak{g}_\Phi^{\mathrm{cut}}(\mathbb{C}) = \mathrm{Lie}\ \mathbb{G}_\Phi^{\mathrm{cut}}(\mathbb{C})$-valued 1-forms.

Definition 5.17. We apply cut-norm to define a partial order between combinatorial DSEs in such a way that

$$X_{\mathrm{DSE}_1} \le X_{\mathrm{DSE}_2} \Leftrightarrow ||X_{\mathrm{DSE}_1}||_{\mathrm{cut}} \le ||X_{\mathrm{DSE}_2}||_{\mathrm{cut}} \ . \tag{5.34}$$

Definition 5.18. Thanks to Lemma 5.13, for a given smooth function $\alpha : C_{\mathrm{DSE}} \to \mathfrak{g}_\Phi^{\mathrm{cut}}(\mathbb{C})$ defined on the boundary region C_{DSE} of an infinitesimal convex neighborhood $\mathbf{\Delta}_{\mathrm{DSE}}$ around a fixed equation DSE in the Banach space $\mathcal{S}_{\approx}^{\Phi,g}$, the associated time-ordered exponential is defined by

$$\mathrm{T}e^{\int_{C_{\mathrm{DSE}}} \alpha(X_{\mathrm{DSE}'}) d\mu_{\mathrm{Haar}}(X_{\mathrm{DSE}'})} := 1 + \tag{5.35}$$
$$\sum_{n \ge 1} \int_{X_{\mathrm{DSE}_1} \le \ldots \le X_{\mathrm{DSE}_n} \in C_{\mathrm{DSE}}} \alpha(X_{\mathrm{DSE}_1}) \ldots \alpha(X_{\mathrm{DSE}_n}) d\mu_{\mathrm{Haar}}(X_{\mathrm{DSE}_1}) \ldots d\mu_{\mathrm{Haar}}(X_{\mathrm{DSE}_n}) \ ,$$

such that $1 \in H_{\mathrm{FG}}^{\mathrm{cut},\vee}(\Phi)$ is the unit in the dual space corresponding to the counit in $H_{\mathrm{FG}}^{\mathrm{cut}}(\Phi)$.

Remark 5.19. The integral $\mathrm{T}e^{\int_{C_{\mathrm{DSE}}} \alpha(X_{\mathrm{DSE}'}) d\mu_{\mathrm{Haar}}(X_{\mathrm{DSE}'})}$ defines an element of $\mathbb{G}_\Phi^{\mathrm{cut}}(\mathbb{C})$. It determines the value $F(X_{\mathrm{DSE}_n})$ of the unique solution of the differential equation $DF(X_{\mathrm{DSE}'}) = F(X_{\mathrm{DSE}'})\alpha(X_{\mathrm{DSE}'})$ with the initial condition $F(X_{\mathrm{DSE}_1}) = 1$.

Definition 5.20. For the differential field $(\mathbb{C}\{z\}, \frac{d}{dz})$, we define the logarithmic derivative

$$\mathbf{D} : \mathbb{G}_\Phi^{\mathrm{cut}}(\mathbb{C}\{z\}) \to \Omega^1(\mathfrak{g}_\Phi^{\mathrm{cut}}(\mathbb{C}\{z\})) \ , \quad \tilde{\phi}_z \mapsto \tilde{\phi}_z^{-1} \frac{d}{dz} \tilde{\phi}_z \ , \tag{5.36}$$

which associates regularized characters to connections on $P_{\Phi,g}^{X_{\mathrm{DSE}}}$. A flat connection $\omega \in \Omega^1(\mathfrak{g}_\Phi^{\mathrm{cut}}(\mathbb{C}\{z\}))$ on $P_{\Phi,g}^{X_{\mathrm{DSE}}}$ is called equi-singular, if ω is $\mathbb{G}_m$-invariant, and for any solution f of the differential equation $\mathbf{D}f = \omega$ with respect to the logarithmic derivative, the restriction of f to sections $\sigma : \mathbf{\Delta}_{\mathrm{DSE}} \to B_{\mathrm{DSE}}$ has the same type of singularity.

Theorem 5.21. *Any X-scaled smooth curve $F_X : \mathbf{\Delta}^*_{\mathrm{DSE}} \to \mathbb{G}^{\mathrm{cut}}_{\Phi}(\mathbb{C})$ around the equation DSE in the Banach space $\mathcal{S}^{\Phi,g}_{\approx}$ has a unique Birkhoff factorization determined by a unique element $\hat{\beta}$ in the Lie algebra $\mathfrak{g}^{\mathrm{cut}}_{\Phi}(\mathbb{C})$.*

Proof. We apply Dimensional Regularization and Minimal Subtraction Scheme ([24,26,28]) to obtain a unique Birkhoff factorization (F_-, F_+) for F_X such that for

$$\mathcal{S}^{\Phi,g}_{\approx} \setminus C_{\mathrm{DSE}} = \Delta^+_{\mathrm{DSE}} \sqcup \Delta^-_{\mathrm{DSE}}, \; X_{\mathrm{DSE}} \in \Delta^+_{\mathrm{DSE}} , \tag{5.37}$$

we have $F_+ : \Delta^+_{\mathrm{DSE}} \to \mathbb{G}^{\mathrm{cut}}_{\Phi}(\mathbb{C})$ and $F_- : \Delta^-_{\mathrm{DSE}} \to \mathbb{G}^{\mathrm{cut}}_{\Phi}(\mathbb{C})$ as smooth functions which obey the convolution equation

$$F_X(X') = F_-(X')^{-1} * F_+(X') , \tag{5.38}$$

with respect to the renormalization coproduct.

Thanks to Lemma 5.14, define

$$G_X(X') = \hat{\theta}_{Z_{X' \odot \ln X}}(F_-(X')^{-1}) , \tag{5.39}$$

such that $\ln X = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(X-\mathbb{I})^n}{n}$. We can show that it is a X-scaled smooth map. We have

$$\begin{aligned} G_{e^{X''}X}(X') &= \hat{\theta}_{Z_{X' \odot \ln(e^{X''}X)}}(F_-(X')^{-1}) = \hat{\theta}_{Z_{X' \odot (X''+\ln X)}}(F_-(X')^{-1}) \\ &= \hat{\theta}_{Z_{X' \odot X'' + X' \odot \ln X}}(F_-(X')^{-1}) = \hat{\theta}_{Z_{X'' \odot X'}}(G_X(X')) . \end{aligned} \tag{5.40}$$

Therefore there exists a curve F_{reg} regular at $X' = X_{\mathrm{DSE}}$ such that

$$G_X(X')^{-1} * F_X(X') = \hat{\theta}_{Z_{X' \odot \ln X}}(F_{\mathrm{reg}}(X')) . \tag{5.41}$$

It allows us to apply the time-ordered exponential (i.e. Definition 5.18) on the punctured neighborhood

$$A_{\mathrm{DSE}} := < \{X'''_1, ..., X'''_n \in \mathbf{\Delta}^*_{\mathrm{DSE}} \; : \; -X' \bigodot \ln X \leq ... \leq X'''_1 \leq ... \leq X'''_n \leq ...\} > \tag{5.42}$$

around the equation DSE for the presentation of G_X. In other words, there exists an element $\hat{\beta} \in \mathfrak{g}^{\mathrm{cut}}_{\Phi}(\mathbb{C})$ such that

$$G_X(X')^{-1} = Te^{-X'^{-1} \int_{A_{\mathrm{DSE}}} \hat{\theta}_{Z_{-X'''}}(\hat{\beta}) d\mu_{\mathrm{Haar}}(X''')} . \tag{5.43}$$

Thanks to (4.21), whenever $X_{\mathrm{DSE}'} \to X_{\mathrm{DSE}}$,

$$\langle F_-(X_{\mathrm{DSE}'})^{-1} \otimes F_-(X_{\mathrm{DSE}'})^{-1}, (S \otimes \hat{\theta}_{Z_{X \odot X_{\mathrm{DSE}'}}})\Delta(\Gamma)\rangle \tag{5.44}$$

is convergent for any $\Gamma \in H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi)$. Set

$$\underline{F}_{Z_X} = \lim_{X_{\mathrm{DSE}'} \to X_{\mathrm{DSE}}} F_-(X_{\mathrm{DSE}'}) * \hat{\theta}_{Z_{X \odot X_{\mathrm{DSE}'}}}(F_-(X_{\mathrm{DSE}'})^{-1}) , \tag{5.45}$$

for any $X \in \mathcal{S}^{\Phi,g}_{\approx}$ to define the 1-parameter subgroup $\{\underline{F}_{Z_X}\}_{X \in \mathcal{S}^{\Phi,g}_{\approx}}$ as a new Renormalization Group. Thanks to Lemma 5.14, we can check that

$$\underline{F}_{Z_{X \odot X'}} = \underline{F}_{Z_X} * \underline{F}_{Z_{X'}} . \tag{5.46}$$

Thanks to (5.26), we have

$$\langle F_-(X_{\mathrm{DSE}'}) * \hat{Y}(F_-(X_{\mathrm{DSE}'})^{-1}), X' \rangle = X_{\mathrm{DSE}'}^{-1} \langle \frac{D}{DZ_X}|_{X=X_{\mathrm{DSE}}} \underline{F}_{Z_X}, X' \rangle . \tag{5.47}$$

Therefore

$$F_-(X_{\mathrm{DSE}'})^{-1} = 1 + \sum_{n \geq 1} X_{\mathrm{DSE}'}^{-n} \, \underline{d}_n , \tag{5.48}$$

such that

$$\hat{Y}(\underline{d}_{n+1}) = \underline{d}_n * \frac{D}{DZ_X}|_{X=X_{\mathrm{DSE}}} \underline{F}_{Z_X} , \ \hat{Y}(\underline{d}_1) = \frac{D}{DZ_X}|_{X=X_{\mathrm{DSE}}} \underline{F}_{Z_X} := \hat{\beta} , \tag{5.49}$$

and

$$\underline{d}_n = \int_{X_{\mathrm{DSE}_n} \leq \ldots \leq X_{\mathrm{DSE}_1} \in C_{\mathrm{DSE}}} \hat{\theta}_{Z_{-X_{\mathrm{DSE}_1}}}(\hat{\beta}) * \ldots * \hat{\theta}_{Z_{-X_{\mathrm{DSE}_n}}}(\hat{\beta}) \, d\mu_{\mathrm{Haar}}(X_{\mathrm{DSE}_1}) \ldots d\mu_{\mathrm{Haar}}(X_{\mathrm{DSE}_n}) . \tag{5.50}$$

Thanks to Definition 5.18, Remark 5.19, Equations (5.48) and (5.50), the negative component F_- can be determined by the beta function $\hat{\beta} \in \mathfrak{g}_{\Phi}^{\mathrm{cut}}(\mathbb{C})$ as the infinitesimal generator of the Renormalization Group $\{\underline{F}_{Z_X}\}_{X \in \mathcal{S}_{\approx}^{\Phi,g}}$. In other words, we have

$$F_-(X_{\mathrm{DSE}'}) = \mathrm{T} e^{-X_{\mathrm{DSE}'}^{-1} \int_{C_{\mathrm{DSE}}} \hat{\theta}_{Z_{-X'''}}(\hat{\beta}) d\mu_{\mathrm{Haar}}(X''')} , \tag{5.51}$$

for all $X_{\mathrm{DSE}'} \in \mathbf{\Delta}_{\mathrm{DSE}}^{*}$. □

Theorem 5.21 provides the basic elements for a new interpretation of non-perturbative phases in terms of systems of differential equations derived from a particular class of connections on $P_{\Phi,g}^{X_{\mathrm{DSE}}}$. Thanks to [41,43], these systems, which have irregular singularities, can be reformulated in terms of systems of Picard–Fuchs equations together with regular singularities.

Corollary 5.22. *The dynamics of non-perturbative phases generated by an equation DSE can be described via systems of singular differential equations derived from objects of the Lie algebra $\mathfrak{g}_{\Phi}^{\mathrm{cut}}(\mathbb{C})$.*

Proof. We follow the procedure explained in Chapter 1: Sections 7.1 and 7.2 in [14]. Thanks to Remark 5.19, Definition 5.20 and Theorem 5.21, there exists a bijective correspondence between flat equi-singular $\mathbb{G}_{\Phi}^{\mathrm{cut}}(\mathbb{C})$-connections on $P_{\Phi,g}^{X_{\mathrm{DSE}}}$ and objects of the Lie algebra $\mathfrak{g}_{\Phi}^{\mathrm{cut}}(\mathbb{C})$.

For an arbitrary infinitesimal character $Z_{X_{\mathrm{DSE}'}} \in \mathfrak{g}_{\Phi}^{\mathrm{cut}}(\mathbb{C})$ associated to $X_{\mathrm{DSE}'} \in C_{\mathrm{DSE}}$, $\exp(Z_{X_{\mathrm{DSE}'}})$ is the character in $\mathbb{G}_{\Phi}^{\mathrm{cut}}(\mathbb{C})$ according to the exponential map (5.20). Any smooth scaled curve generated by $C_{\mathrm{DSE}} \to \mathbb{G}_{\Phi}^{\mathrm{cut}}(\mathbb{C})$, $X_{\mathrm{DSE}'} \mapsto \exp(Z_{X_{\mathrm{DSE}'}})$ can determine a system of differential equations with irregular singularities. This system contains differential equations $\mathbf{D} \exp(Z_{X_{\mathrm{DSE}'}}) = \omega_{\mathrm{DSE}'}$ for any $X_{\mathrm{DSE}'} \in C_{\mathrm{DSE}}$ such that each $\omega_{\mathrm{DSE}'}$ is a flat equi-singular connection. □

6. Conclusion

Graphon models have been applied in Quantum Field Theory to provide some new tools for the computation of non-perturbative parameters. The space of Feynman diagrams of a given gauge field theory can be topologically completed which leads us to formulate an enrichment of the Connes–Kreimer renormalization Hopf algebra in terms of graphon representations of Feynman diagrams. This topological enrichment is useful to construct a new class of analytic Feynman diagrams as cut-distance convergent graph limits for sequences of Feynman diagrams and their formal expansions whenever their loop numbers increase. These new analytic Feynman diagrams are described in terms of random graph processes. Thanks to this investigation, this research work provided some new geometric tools for the study of the behavior of gauge field theories under strong running coupling constants. We applied homomorphism densities of Feynman graphons to characterize non-perturbative phase transitions. We formulated a new topological renormalization Hopf algebra of Feynman graphons where the graphon pictures of gauge symmetries in the language of Hopf ideals led us to obtain a well-defined formulation for the renormalization coproduct of (non-perturbative) solutions of DSEs. We explained the structure of a new non-perturbative Renormalization Group on the space of Feynman graphons which contribute to solutions of combinatorial DSEs. We constructed three types of Banach bundles on the space of DSEs of a physical theory Φ to obtain some new tools for the description of the evolution of towers of these equations. This new geometric setting led us to interpret all possible transitions between non-perturbative phases in Φ in terms of the intrinsic geometric data of trajectories in the Banach manifold $\mathcal{S}^{\Phi,g}_{\approx}$. The local and global differential geometries of this particular Banach manifold with respect to its tangent bundle can provide some new data about the real time dynamics of the physical theory described by changing DSEs in different scales of running coupling constants. Geodesics in this Banach manifold can determine the optimal transition options between non-perturbative phases. We addressed the basic elements of a new Hopf Gauge Theory model for non-perturbative Quantum Field Theory. We explained the structure of a new regularization bundle which enables us to geometrically describe the renormalization of non-perturbative solutions of combinatorial DSEs in terms of a particular class of differential systems related to equi-singular connections.

6.1. A new continuum approach

In Hadron Physics, where hadrons (such as protons and neutrons) are the composite particles constructed from quarks and gluons under strong interaction, we need to deal with the non-perturbative behavior of fixed point equations of Green's functions in the context of DSEs under different running coupling constants [19,39,40]. For a given strongly coupled gauge field theory Φ with the bare coupling constant g and Lagrangian L^{Φ}, let $\mathrm{SU}(N)$ be its gauge group such that N determines the number of flavors. The Lagrangian L^{Φ} is invariant under local transformations in $\mathrm{SU}(N)$. Therefore DSEs in the Banach manifold $\mathcal{S}^{\Phi,g}_{\approx}$, originated from the interaction part of L^{Φ}, are also invariant under $\mathrm{SU}(N)$. Thanks to graphon models, in Corollary 3.8, we have provided a new process to characterizes non-perturbative phases of physical theories in terms of the homomorphism densities of Feynman graphons associated to DSEs. The tangent bundle, Hopf bundle and regularization bundle on the Banach manifold $\mathcal{S}^{\Phi,g}_{\approx}$ given in Theorem 5.8, Theorem 5.12 and Definition 5.16 enable us to study the phase transitions (i.e. Remark 5.9) in terms of the geometric data of trajectories in $\mathcal{S}^{\Phi,g}_{\approx}$ (as the base space) with respect to different choices of bundles. Thanks to Theorem 5.21 and Corollary 5.22, trajectories in $\mathcal{S}^{\Phi,g}_{\approx}$ encode the behavior of (towers of) (renormalized) DSEs under different running coupling constants. Our study can provide a new continuum setting for the description of the dynamics of non-perturbative gauge field theories under changing the strength of running coupling constants in the structure of DSEs in terms of some differential geometric tools derived from our new Banach bundles.

6.2. Open research directions: quantum gauge symmetries via graphon models

We cut-distance topologically completed the renormalization Hopf algebra where Theorem 3.5, Lemma 4.1 are applied to formulate a new Feynman graphon version of the Slavnov–Taylor / Ward–Takahashi identities in the context of the Hopf ideals $I_{\mathrm{WT}}^{\mathrm{graphon,cut}}$, $I_{\mathrm{ST}}^{\mathrm{graphon,cut}}$ generated by a particular family of Feynman graphons. These Hopf ideals equalize Feynman graphons corresponding to certain linear combinations of Feynman diagrams which contribute to the Slavnov–Taylor / Ward–Takahashi elements. Therefore the resulting quotient topological Hopf algebras $H_{\mathrm{graphon}}^{\mathrm{QED,cut}}/I_{\mathrm{WT}}^{\mathrm{graphon,cut}}$ and $H_{\mathrm{graphon}}^{\mathrm{QCD,cut}}/I_{\mathrm{ST}}^{\mathrm{graphon,cut}}$ of Feynman graphons are capable of computing the renormalization coproducts of the complete 1PI Green's functions and the solutions of their corresponding DSEs in terms of the cut-distance convergent limits of sequences of coproducts of finite formal expansions of higher loop order Feynman diagrams (i.e. Corollaries 4.3 and 4.4).

This background initiates some new research directions about the study of DSEs in generalized gauge field theories. It is shown in Section 5 in [37] that quantum gauge symmetries in super- and non-renormalizable local gauge field theories, as generalized versions of the Slavnov–Taylor / Ward–Takahashi identities, generate Hopf ideals in the associated renormalization Hopf algebras. In this setting, the renormalization coproduct and antipode identities are generalized to describe quantum gauge symmetries in terms of Hopf ideals of the generalized renormalization Hopf algebra. In addition, the combinatorial structure of Green's functions in Quantum General Relativity coupled to QED has been studied in the context of the Hopf algebra of QGR–QED [38]. Furthermore, in [52], combinatorial DSEs at the level of ribbon graphs are considered to formulate a renormalization theory for noncommutative field theories. The Feynman graphon interpretation of the renormalization Hopf algebras of these physical theories and the graphon pictures of Hopf ideals associated to generalized quantum gauge symmetries are capable to provide some new computational tools for the study of DSEs in these generalized physical theories.

CRediT authorship contribution statement

Ali Shojaei-Fard: Conceptualization, Original Motivation, Methodology, Research, Writing / Reviewing / Editing: Original draft preparation, Writing / Reviewing / Editing: Final preparation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The author would like to thank the Referee because of fundamental comments and interesting suggestions which were helpful to improve the presentation of this work.

References

[1] E. Akin, The Metric Theory of Banach Manifolds, Lecture Notes in Mathematics, vol. 662, Springer, 1978.

[2] S. Arguillere, Sub-Riemannian geometry and geodesics in Banach manifolds, J. Geom. Anal. 30 (2020) 2897–2938.
[3] F.V. Atkinson, Some aspects of Baxter's functional equation, J. Math. Anal. Appl. 7 (1) (1963).
[4] M. Borinsky, Feynman graph generation and calculations in the Hopf algebra of Feynman graphs, Comput. Phys. Commun. 185 (12) (2014) 3317–3330.
[5] C. Borgs, J.T. Chayes, H. Cohn, N. Holden, Sparse exchangeable graphs and their limits via graphon processes, J. Mach. Learn. Res. 18 (2017) 210.
[6] C. Borgs, J.T. Chayes, H. Cohn, Y. Zhao, An L^p theory of sparse graph convergence I: limits, sparse random graph models, and power law distributions, Trans. Am. Math. Soc. 372 (2019) 3019–3062.
[7] D. Bullock, C. Frohman, J. Kania-Bartoszynska, Topological interpretations of lattice gauge field theory, Commun. Math. Phys. 198 (1998) 47–81.
[8] M.R. Bridson, A. Hafliger, Metric Spaces of Non-positive Curvature, Grundlehren der Mathematischen Wissenschaften, vol. 319, Springer, 1999.
[9] C. Bergbauer, D. Kreimer, Hopf algebras in renormalization theory: locality and Dyson–Schwinger equations from Hochschild cohomology, IRMA Lect. Math. Theor. Phys. 10 (2006) 133–164.
[10] D.J. Broadhurst, D. Kreimer, Renormalization automated by Hopf algebra, J. Symb. Comput. 27 (6) (1999) 581–600.
[11] B. Bollobas, O. Riordan, Metrics for sparse graphs, in: Surveys in Combinatorics 2009, in: LMS Lecture Notes Series, vol. 365, 2009, pp. 211–287.
[12] A. Connes, D. Kreimer, Hopf algebras, renormalization and noncommutative geometry, Commun. Math. Phys. 199 (1998) 203–242.
[13] A. Connes, D. Kreimer, Renormalization in quantum field theory and the Riemann–Hilbert problem, J. High Energy Phys. 09 (1999) 024.
[14] A. Connes, M. Marcolli, Noncommutative Geometry, Quantum Fields and Motives, Colloquium Publications, Amer. Math. Soc., vol. 55, 2008.
[15] A. Deur, S.J. Brodsky, G.F. de Teramond, The QCD running coupling, Prog. Part. Nucl. Phys. 90 (1) (2016).
[16] T. DeGrand, C. DeTar, Lattice Methods for Quantum Chromodynamics, World Scientific, 2006.
[17] L. Foissy, General Dyson–Schwinger equations and systems, Commun. Math. Phys. 327 (2014) 151–179.
[18] Z. Fodor, K. Holland, J. Kuti, et al., The running coupling of 8 flavors and 3 colors, J. High Energy Phys. 2015 (2015) 19.
[19] E. Gutierrez, A. Ahmad, A. Ayala, A. Bashir, A. Raya, The QCD phase diagram from Schwinger–Dyson equations, J. Phys. G, Nucl. Part. Phys. 41 (2014) 075002.
[20] S. Gao, P.E. Caines, Graphon control of large-scale networks of linear systems, IEEE Trans. Autom. Control 65 (10) (2020) 4090–4105.
[21] A. Hadicke, Nonperturbative approaches to determining the behavior of the gluon propagator and quark propagator in quantum chromodynamics by Schwinger–Dyson equations, Int. J. Mod. Phys. A 6 (19) (1991) 3321–3345.
[22] A. Hasenfratz, D. Schaich, A. Veernala, Nonperturbative β function of eight-flavor SU(3) gauge theory, J. High Energy Phys. 2015 (2015) 143.
[23] S. Janson, Graphons, Cut Norm and Distance, Couplings and Rearrangements, New York Journal of Mathematics, NYJM Monographs, vol. 4, State University of New York, University at Albany, Albany, NY, 2013.
[24] D. Kreimer, On the Hopf algebra structure of perturbative quantum field theories, Adv. Theor. Math. Phys. 2 (1998) 303–334.
[25] D. Kreimer, On overlapping divergences, Commun. Math. Phys. 204 (1999) 669–689.
[26] D. Kreimer, Unique factorization in perturbative QFT, Nucl. Phys., Proc. Suppl. 116 (2003) 392–396.
[27] D. Kreimer, Dyson–Schwinger equations: from Hopf algebras to number theory, in: Universality and Renormalization, Fields Inst. Commun. 50 (2007) 225–248.
[28] D. Kreimer, Algebraic structures in local QFT, Nucl. Phys., Proc. Suppl. 205–206 (2010) 122–128.
[29] D. Kreimer, Anatomy of a gauge theory, Ann. Phys. 321 (2006) 2757–2781.
[30] O. Kruger, D. Kreimer, Filtrations in Dyson–Schwinger equations: next-to j-leading log expansions systematically, Ann. Phys. 360 (2015) 293–340.
[31] T. Krajewski, R. Wulkenhaar, On Kreimer's Hopf algebra structure of Feynman graphs, Eur. Phys. J. C 7 (1999) 697–708.
[32] S. Lang, Differential and Riemannian Manifolds, 3rd edition, Graduate Texts in Mathematics, vol. 160, 1995.
[33] L. Lovasz, Very large graphs, Curr. Dev. Math. 2008 (2009) 67–128.
[34] L. Lovasz, Large Networks and Graph Limits, AMS Colloquium Publications, vol. 60, 2012.
[35] M. Marino, Lectures on non-perturbative effects in large N gauge theories, matrix models and strings, Fortschr. Phys. 62 (2014) 455–540.

[36] M. Marino, Instantons and Large N: An Introduction to Non-perturbative Methods in Quantum Field Theory, Cambridge University Press, 2015.
[37] D. Prinz, Gauge symmetries and renormalization, arXiv:2001.00104v2, 2021.
[38] D. Prinz, Algebraic structures in the coupling of gravity to gauge theories, Ann. Phys. 426 (2021) 168395.
[39] C.D. Roberts, S.M. Schmidt, Dyson–Schwinger equations: density, temperature and continuum strong QCD, Prog. Part. Nucl. Phys. 45 (2000) S1–S103.
[40] C.D. Roberts, A.G. Williams, Dyson–Schwinger equations and the application to hadronic physics, Prog. Part. Nucl. Phys. 33 (1994) 477–575.
[41] A. Shojaei-Fard, A geometric perspective on counterterms related to Dyson–Schwinger equations, Int. J. Mod. Phys. A 28 (32) (2013) 1350170.
[42] A. Shojaei-Fard, The global β-functions from solutions of Dyson–Schwinger equations, Mod. Phys. Lett. A 28 (34) (2013) 1350152.
[43] A. Shojaei-Fard, A new perspective on intermediate algorithms via the Riemann–Hilbert correspondence, Quantum Stud. Math. Found. 4 (2) (2017) 127–148.
[44] A. Shojaei-Fard, A measure theoretic perspective on the space of Feynman diagrams, Bol. Soc. Mat. Mex. (3) 24 (2) (2018) 507–533.
[45] A. Shojaei-Fard, Graphons and renormalization of large Feynman diagrams, Opusc. Math. 38 (3) (2018) 427–455.
[46] A. Shojaei-Fard, Non-perturbative β-functions via Feynman graphons, Mod. Phys. Lett. A 34 (14) (2019) 1950109.
[47] A. Shojaei-Fard, Formal aspects of non-perturbative Quantum Field Theory via an operator theoretic setting, Int. J. Geom. Methods Mod. Phys. 16 (12) (2019) 1950192.
[48] A. Shojaei-Fard, The analytic evolution of Dyson–Schwinger equations via homomorphism densities, Math. Phys. Anal. Geom. 24 (2021) 18.
[49] E.V. Shuryak, Nonperturbative phenomena and phases of QCD, in: W. Plessas, L. Mathelitsch (Eds.), Lectures on Quark Matter, in: Lecture Notes in Physics, vol. 583, 2002.
[50] F. Strocchi, An Introduction to Non-perturbative Foundations of Quantum Field Theory, International Series of Monographs on Physics, Oxford University Press, 2013.
[51] K.C. Stacey, D.A. Holton, On the automorphism group of a tree, J. Comb. Theory, Ser. B 22 (1) (1977) 10–25.
[52] A. Tanasa, D. Kreimer, Combinatorial Dyson–Schwinger equations in non-commutative field theory, J. Noncommut. Geom. 7 (1) (2013) 255–289.
[53] W.D. van Suijlekom, The structure of renormalization Hopf algebras for gauge theories I: representing Feynman graphs on BV-algebras, Commun. Math. Phys. 290 (2009) 291–319.
[54] W.D. van Suijlekom, Renormalization of gauge fields: a Hopf algebra approach, Commun. Math. Phys. 276 (2007) 773–798.
[55] W.D. van Suijlekom, The Hopf algebra of Feynman graphs in QED, Lett. Math. Phys. 77 (2006) 265–281.
[56] S. Weinzierl, Hopf algebras and Dyson-Schwinger equations, Front. Phys. 11 (2016) 111206.
[57] K. Yeats, Growth estimates for Dyson–Schwinger equations, Thesis (Ph.D.), Boston University, ProQuest LLC, Ann Arbor, MI, 2008.