Validity check of the KATIE parton level event generator in the k_t-factorization and collinear frameworks

R. Kord Valeshabadi, M. Modarres,* and S. Rezaie
Department of Physics, University of Tehran, 1439955961 Tehran, Iran

(Received 4 August 2021; accepted 11 August 2021; published 16 September 2021)

In the present paper, we check and study the validity of the KATIE parton level event generator by calculating the inclusive electron-proton (ep) dijet and the proton-proton (pp) Drell-Yan electron-pair production's differential cross sections in the k_t- and collinear factorization frameworks. The Martin-Ryskin-Watt (MRW) unintegrated parton distribution functions (UPDFs) are used as the input UPDFs. The results are compared with those of ZEUS ep inclusive dijet and ATLAS p-p Drell-Yan electron-pair productions, experimental data. The KATIE parton level event generator can directly calculate the cross sections in the k_t-factorization framework. It was noticed by van Hameren, the author of KATIE, that the lab to the Breit transformation in this generator was not correctly implemented, so the produced output did not cover the ep ZEUS experimental data in which the mentioned transformation is applied. The author of the code fixed the bug by implementing the correct transformation in the KATIE parton event generator. Then, we could appropriately produce the ep inclusive dijet differential cross section, in comparison with those of ZEUS data. It is also shown that the Martin-Ryskin-Watt at the next-to-leading order level, with the angular ordering constraint, can successfully predict the ATLAS p-p Drell-Yan data. Finally, as it is expected, we conclude that the k_t factorization is an appropriate tool for the small longitudinal parton momenta and high center of mass energies, with respect to the collinear one.

DOI: 10.1103/PhysRevD.104.054019

I. INTRODUCTION

Monte Carlo event generators are essential tools for experimentalists and theoreticians who attempt to simulate hadronic collisions. These generators are mostly based on the assumption of collinear factorization, in which the partons are assumed to behave collinearly in the hadrons. However, generally, this assumption is not correct and partons should be allowed to have transverse momenta (k_t). Hence the k_t factorization [1,2] comes into play, in which the transverse momenta of partons can be considered as an extra variable into the hard interaction calculations. This framework allows us to calculate the differential cross sections for different processes by using unintegrated parton distribution functions (UPDFs).

One of the parton level event generators that can calculate the cross sections for arbitrary final state particles, in the above framework, is the KATIE [3]. The program of this generator can produce the Les Houches Event File (LHEF) [4], where thereafter, this LHEF file can be passed to CASCADE3 [5] event generator to perform parton showering models. The KATIE, due to its simplicity, has this capability to be put into use for calculating different differential cross sections with the various UPDFs. So, it is a suitable choice to perform phenomenological studies of these distributions, as well. Therefore, in order to obtain reliable results, one task is to test this generator for different processes and compare the results to the experimental data. Hence, checking the validity of this parton level event generator at this early stage is crucially important for future studies of the k_t-factorization framework.

The k_t-factorization framework considers a more realistic picture of partons inside the proton with respect to collinear factorization, and hence it is expected to give a better description of the experimental data, especially in high energies (much greater than few TeV). Although, due to complications which arise when the transverse momentum of parton come into play, obtaining a fully transverse momentum dependent evolution equation is a challenging task. For example, the Catani-Ciafaloni-Fiorani-Marchesini evolution equation [6–9], which is based on the angular ordering of soft gluon emissions, is not defined for all quark flavors. However, recently, another approach, which is based on the parton branching method [10,11], allows one to obtain the UPDFs naturally by solving the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations

*Corresponding author.
mmodares@ut.ac.ir

[12–14]. With this method one can also obtain the UPDFs for both quarks and gluon and it is shown to have successful predictions [15,16].

Another approach for defining UPDFs is the Martin-Ryskin-Watt (MRW) approach [17,18] which assumes the parton evolves according to the DGLAP evolution equations [12–14] until the last parton emission, and then it resums the no-emission parton probability to the hard factorization scale via the Sudakov form factor. This method is investigated in detail [19–25] and is shown to be successful in describing the data of different processes at the LHC, Tevatron, etc., [26–34].

Recently, the KATIE generator has extended its application to the electron(positron)-proton (ep) collision [35]. Therefore, for the first time, we calculated the inclusive dijet production differential cross sections in the k_t-factorization framework and compared our results to those of the ZEUS Collaboration data [36,37]. It was observed that our results [36] cannot describe the data in a satisfying way, and hence we found that the lab to Breit transformation was not appropriately implemented in this parton level generator.

In the present work, we intend to study the prediction of the MRW formalism at the leading (LO) and next-to-leading order (NLO) levels for two different processes, i.e., the electron(positron)-proton (ep) inclusive dijet and proton-proton (p-p) Drell-Yan differential cross sections which can be compared with the ZEUS [37] and ATLAS [38] Collaborations data, respectively. The KATIE parton level event generator is used in order to check the validity of this generator by correcting the Breit transformation and also investigating the MRW method.

So, the paper is organized as follows: In Sec. II the theoretical framework is presented which includes the KATIE parton level event generator and the MRW formalism. The results and discussions are given in Sec. III and Sec. IV is devoted to the conclusions.

II. THE THEORETICAL FRAMEWORK

In this section, we intend to first explain the KATIE parton level event generator, and then discuss the UPDFs which are used to obtain the differential cross sections for the ep inclusive dijet and the p-p Drell-Yan processes to be compared with those of the ZEUS and ATLAS Collaboration data, respectively.

A. The KATIE parton level event generator

This parton level event generator is mainly composed of four parts, i.e., the input file, the optimization stage, the event generation, and the histograms creation. In the input file of this histogram, one should write information about the subprocesses, the factorization and renormalization scales, the experimental cuts, the off shellness or on shellness of partons, the PDFs, the UPDFs, the order of non-QCD couplings, and the energies of incoming particles. Furthermore, it is also possible to calculate the multiparton scattering, instead of the single one. It is also worth mentioning that one can use the desired UPDFs by providing grid files in columns of $\ln(x)$, $\ln(k_t^2)$, $\ln(\mu^2)$, and $f_a(x, k_t^2, \mu^2)$ for each parton flavor. Additionally, it is possible to directly use UPDFs grid files of TMDLIB [39].

After providing an input file, an optimization of all subprocesses and event generation should be performed, which are not of interest for the end user. Finally, one can obtain differential cross sections by a FORTRAN file with the name "'create_eventfile.f90." In this file, the histograms of interests are developed and the program can read the recorded events in a file which is called the raw file and create the distributions of interests. Additionally, in this part, one can produce the LHEF file, which can itself, with the help of CASCADE, [40] make showering and hadronization.

Therefore, this generator is a beneficial phenomenological tool to investigate the k_t-factorization framework and different UPDFs models. In the next subsection, we give an overview of the MRW UPDF models at the LO and NLO levels which will be used through this report.

B. The MRW UPDFs

The MRW UPDFs at the leading order level (LO-MRW UPDFs) are developed by Martin, *et al.* [17,18], and are based on the DGLAP evolution equations. In this framework, by choosing the factorization scale of the DGLAP evolution equation to be the transverse momentum of the parton, a parton evolves collinearly in the proton, until it reaches the last evolution step. In this step, the parton, which has the transverse momentum k_t, emits a parton and evolves to the factorization scale μ without any further real emissions. Therefore, the LO-MRW UPDFs are defined as follows:

$$f_a(x, k_t^2, \mu^2) = T_a(k_t^2, \mu^2)\frac{\alpha_s^{\mathrm{LO}}(k_t^2)}{2\pi} \times \sum_{b=q,g}\int_x^1 \left[P_{ab}^{\mathrm{LO}}(z) f_b^{\mathrm{LO}}\left(\frac{x}{z}, k_t^2\right)\right]\mathrm{d}z, \quad (1)$$

where $f_b^{\mathrm{LO}}(\frac{x}{z}, k_t^2)$ in the above equation is the momentum weighted parton density at the LO, and can be either $\frac{x}{z}q^{\mathrm{LO}}(\frac{x}{z}, k_t^2)$ or $\frac{x}{z}g^{\mathrm{LO}}(\frac{x}{z}, k_t^2)$ for quark (antiquark) or gluon, respectively. Additionally, $T_a(k_t^2, \mu^2)$ is the Sudakov form factor and resums no real emissions probability from the scale k_t^2 to μ^2 and which is

$$T_a(k_t^2 \leq \mu^2, \mu^2) = \exp\left(-\int_{k_t^2}^{\mu^2}\frac{\mathrm{d}\kappa_t^2}{\kappa_t^2}\frac{\alpha_s^{\mathrm{LO}}(\kappa_t^2)}{2\pi}\sum_{b=q,g}\int_0^1 \xi P_{ba}^{\mathrm{LO}}(\xi)\mathrm{d}\xi\right). \quad (2)$$

It should also be noted that the Sudakov form factor is defined for the $k_t^2 \leq \mu^2$ and for the $k_t > \mu$, $T_a \to 1$.

One should note that the LO-MRW UPDFs, in Eqs. (1) and (2), which can be defined for all quark flavors $f_q(x, k_t^2, \mu^2)$ and gluon $f_g(x, k_t^2, \mu^2)$ are divergent at $z \to 1$ and $\xi \to 1$ for the diagonal terms, i.e., $P_{qq}^{\mathrm{LO}}(z)$, $P_{gg}^{\mathrm{LO}}(z)$, $P_{qq}^{\mathrm{LO}}(\xi)$, $P_{gg}^{\mathrm{LO}}(\xi)$. To avoid such divergences, which happen as a result of soft gluon emissions, one can either use the angular ordering of the soft gluon emissions or the strong ordering of partons, along the evolution ladder, to put a cutoff on z and ξ. The authors of this model adopted the angular ordering of the soft gluon emissions, hence we use the same cutoff here, i.e.,

$$z_{\max} = \frac{\mu}{(k_t + \mu)}, \qquad \xi_{\max} = \frac{\mu}{(\kappa_t + \mu)}. \tag{3}$$

The $z_{\max}$ and $\xi_{\max}$ are the maximum allowed values of z and ξ. Therefore, one should put a Heaviside step function for the diagonal terms of Eqs. (1) and (2), i.e., $\Theta(z_{\mathrm{Max}} - z)$ and $\Theta(\xi_{\mathrm{Max}} - \xi)$, respectively.

It should be noted that the LO-MRW formalism is limited to the $k_t \geq \mu_0$, and hence for defining such a distribution in the limit $k_t < \mu_0$, the density of the partons are assumed to be constant at fixed x and μ^2 in Refs. [17,18], and satisfy the normalization condition. Therefore, one obtains the density of partons in the limit $k_t < \mu_0$, as

$$\frac{1}{k_t^2} f_a(x, k_t^2 < \mu_0^2, \mu^2) = \frac{1}{\mu_0^2} a(x, \mu_0^2) T_a(\mu_0^2, \mu^2), \tag{4}$$

where, in the above equation, we fix $\mu_0 = 1$ GeV.

The LO-MRW formalism is also extended to the next-to-leading order level (NLO-MRW) in which, instead of the scale k_t^2, the virtuality of the parton along the evolution ladder is used, i.e., $k^2 = \frac{k_t^2}{(1-z)}$. Additionally, the PDFs, the strong ordering coupling constant, and the splitting functions are at the NLO level. However, it is shown in [17] with the use of hte NLO strong ordering coupling constant along with the NLO-PDFs that one can obtain results close to those used in the fully NLO case. For simplicity, the first form is used, where the PDFs and strong ordering coupling constant are at the NLO level, but the splitting functions are at the LO level. Therefore one can write the NLO-MRW UPDFs as

$$f_a(x, k_t^2, \mu^2) = \sum_{b=q,g} \int_x^1 \frac{\alpha_s^{\mathrm{NLO}}(k^2)}{2\pi} T_a(k^2, \mu^2) \times \left[P_{ab}^{\mathrm{LO}}(z) f_b^{\mathrm{NLO}}\left(\frac{x}{z}, k^2\right)\right] \Theta(\mu^2 - k^2) \mathrm{d}z, \tag{5}$$

where, in the above equation, we implicitly insert, once again, a Heaviside step function to avoid the soft gluon divergence. There is also an additional cutoff $\Theta(\mu^2 - k^2)$ in the above equation, which limits the virtuality in the region of $k^2 < \mu^2$. Because $k^2 = \frac{k_t^2}{(1-z)}$, in the limit of large fractional momenta this scale can exceed the factorization scale and such a cutoff can prevent it. Furthermore, this cutoff limits the transverse momentum of the parton to the region less than factorization scale, in contrast to the LO case, where the partons can freely have transverse momentum larger than the factorization scale. Using this cutoff, however, has a huge negative effect in the smaller center of mass energies, i.e., when the large fractional momenta can play significant role. Because of this small center of mass energy, z and k^2 become larger and the strong ordering cutoff $\Theta(\mu^2 - k^2)$ can suppress quarks and gluon distributions. It is also obvious that due to the dependency of the factorization scale and the Sudakov form factor on z, one should move them to the argument of z integral. Finally, the Sudakov form factor in this model is

$$T_a(k^2, \mu^2) = \exp\left(- \int_{k^2}^{\mu^2} \frac{\mathrm{d}\kappa^2}{\kappa^2} \frac{\alpha_s^{\mathrm{NLO}}(\kappa^2)}{2\pi} \sum_{b=q,g} \int_0^1 \xi P_{ba}^{\mathrm{LO}}(\xi) \mathrm{d}\xi \right). \tag{6}$$

III. RESULTS AND DISCUSSION

In this section, we intend to calculate the differential cross sections in the k_t-factorization frameworks with the LO and NLO-MRW UPDFs by utilizing the KATIE parton level event generator. In this calculation, we calculate and provide the grid files of our UPDFs for each quark flavor and gluon, using the MMHT2014lo68cl and MMHT2014nlo68cl PDFs [41,42] of the LHAPDF6 library [43], as an input PDFs for our model in Eqs. (1) and (2). It should also be noted that the UPDFs are divided by k_t^2, due to the definition of differential cross section in the KATIE generator.

A. KATIE results for the ZEUS inclusive dijet production data

The ZEUS Collaboration data [37] are collected from the collisions of the protons with energy 920 GeV and electrons or positrons with energy 27.5 GeV. The photon virtuality Q^2 is between 125 GeV $< Q^2 <$ 20000 GeV. The minimum transverse energy of the two jets in the Breit frame should be larger than $E_{T,B} > 8$ GeV and the invariant mass of the two jets is required to be greater than $M_{jj} > 20$ GeV. The inelasticity is between $0.2 < y < 0.6$. Additionally, at least two jets are required to have the pseudorapidity in the range $-1 < \eta^{\mathrm{lab}} < 2.5$, in the lab frame.

We can simply perform such a calculation with the help of the KATIE parton level generator [3]. For calculation of the dijet production, we considered the subprocesses $\gamma^* + q \to q + g$ and $\gamma^* + g \to q + \bar{q}$ with the LO-MRW and NLO-MRW UPDFs. We also set the factorization and the renormalization scales $\mu_F = \mu_R = Q$, with $n_f = 5$.

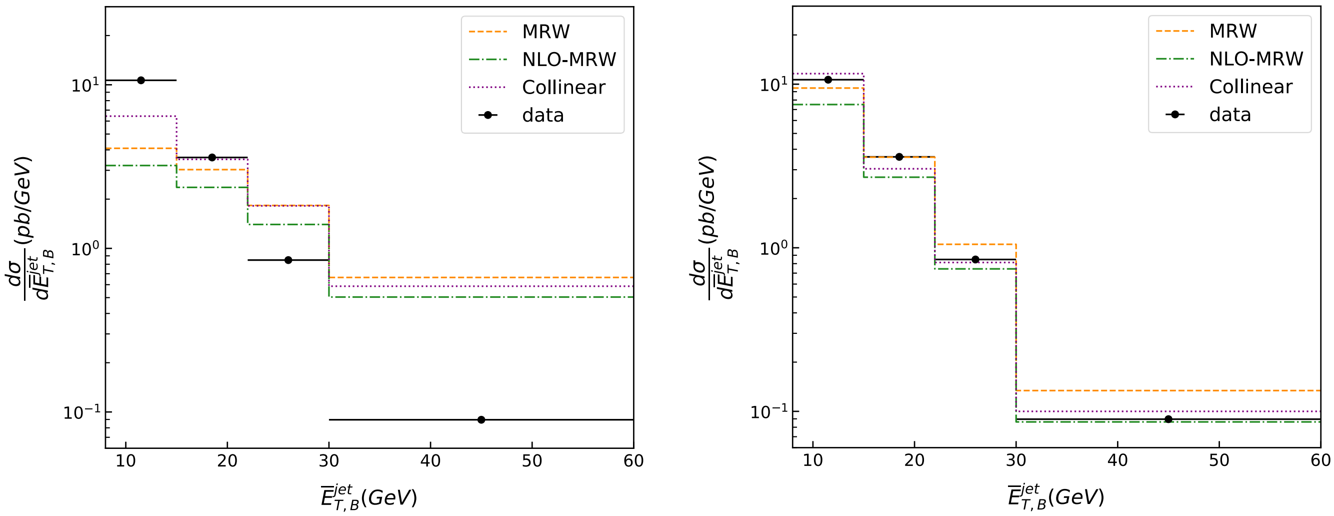

FIG. 1. The comparison of LO-MRW, NLO-MRW UPDFs k_t, and collinear factorizations ep inclusive dijet differential cross section with those of ZEUS Collaboration data [37]. The differential cross section is produced with the wrong (corrected) lab to the Breit transformation, left panel (right panel), in the KATIE.

In the left panel of Fig. 1, the differential cross section with respect to the mean transverse jet energy in the Breit frame is obtained and compared to the experimental data. As can be seen in this figure, the result of the k_t factorization with the LO-MRW and NLO-MRW UPDFs are strangely off the data. Because in the previous works it was shown that the MRW UPDFs can produce the corresponding data reasonably [26–34], to confirm what makes our calculation get worse, we evaluated the differential cross section within the collinear factorization framework using the MMHT2014nlo68cl PDFs. The prediction of the collinear factorization, as depicted in the left panel of Fig. 1, convinces us that there should be something wrong with this parton level event generator [3].

In order to diagnose the reason for such results, we decided to visualize the three dimensional plot with respect to the momenta p_x, p_y, and p_z of 100 events generated by KATIE in the Briet frame; see Fig. 2. In this figure, the three dimensional momenta of the initial quark, virtual photon, final quark, and gluon of the subprocess, i.e., $\gamma^* + q \to q + g$, in the Breit frame within the collinear factorization are shown. In the collinear factorization framework, the virtual photon should collide head to head along the z direction with the initial parton, according to the

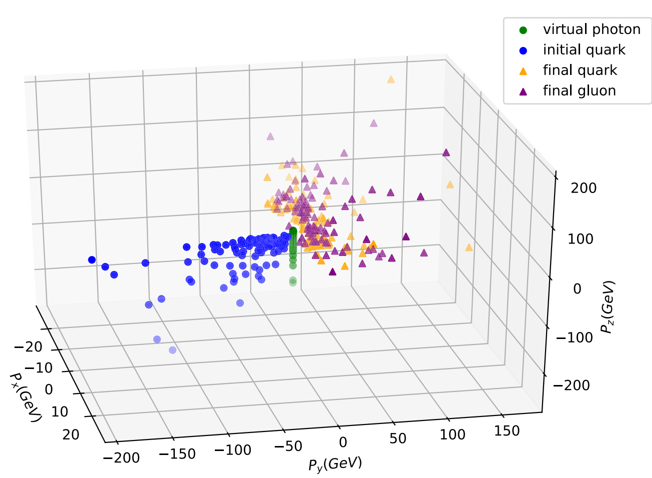

FIG. 2. The three dimensional plot, with respect to the momenta p_x, p_y, and p_z, of 100 events in the collinear factorization framework, for the subprocess $\gamma^* + q \to q + g$, generated by the actual KATIE parton level generator, before the Breit frame correction is made.

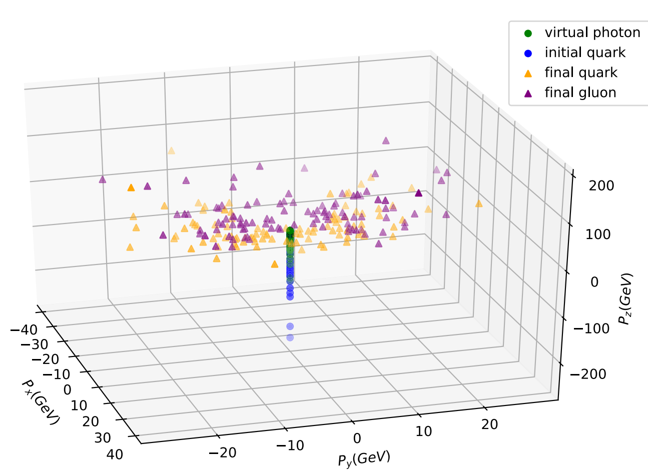

FIG. 3. The three dimensional plot, with respect to the momenta p_x, p_y, and p_z, of 100 events in the collinear factorization framework for the subprocess $\gamma^* + q \to q + g$ generated by the KATIE parton level generator after correcting the lab to Breit transformation.

Breit frame transformation, which does not happen as expected. According to the definition, in the Breit frame, the virtual photon has no energy component and only has momentum along the $-z$ direction, $q = (0, 0, 0, -Q)$, and collinearly scatters head to head with the quark in the $+z$ direction [37] which has momentum xP (P is the proton momentum). Therefore, it is obvious that there is problem due to the wrong choice of lab to Breit frame transformation and it should be fixed [44].

This lab to Breit transformation can be simply fixed, for example, according to Ref. [45] (note that the minimum requirement is $\mathbf{q} + 2x\mathbf{P} = 0$). If we make such corrections in the implementation of the KATIE, one can see that events are now behaving according to our expectation; see Fig. 3. Now, the virtual photon has a head to head collision with the initial quark along the z direction.

Now we are in a position to perform our calculation once again with the LO-MRW and NLO-MRW UPDFs, in addition to the collinear factorization framework with the PDFs, and compare them to the experimental data; see the right panel of Fig. 1. Here, one can find that the collinear factorization can predict the data well, while the results of LO-MRW UPDFs in the large mean transverse energy of final state jets overestimate the data, which is mostly due to the large UPDFs at large parton transverse momenta; see Ref. [15] for details. Additionally, the NLO-MRW UPDFs prediction underestimated the data due to this fact that the choice of k^2 as a scale in this limit of

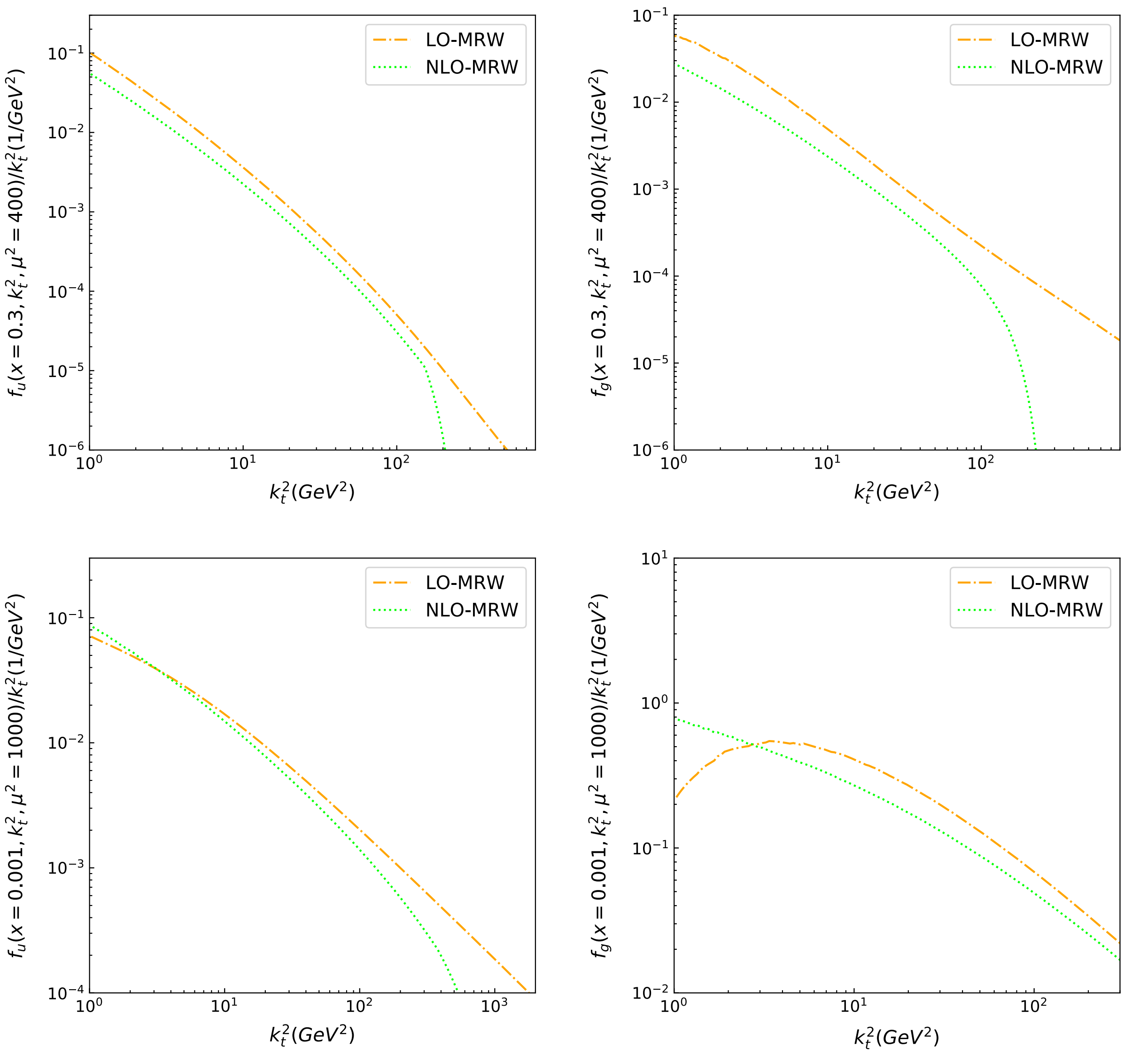

FIG. 4. The left-upper (right-upper) panel shows the comparison of up quark (gluon) LO-MRW and NLO-MRW UPDFs$/k_t^2$ at $x = 0.3$ and $\mu^2 = 400$. The left-lower (right-lower) panel shows the comparison of up quark (gluon) LO-MRW and NLO-MRW UPDFs$/k_t^2$ at $x = 0.001$ and $\mu^2 = 1000$.

energy leads to the much larger scale than k_t^2 in the LO-MRW formalism. As a result of this fact and also the additional cutoff $\Theta(\mu^2 - k^2)$ in this method, one could expect worse estimation of cross section for the NLO-MRW UPDFs at this range of energy; see the top panels of Fig. 4 to gain an insight about these two different UPDFs for the up quark and the gluon at $x = 0.3$ and $\mu^2 = 400$. However, one should note that the k_t factorization works much more better at small $x \leq 0.01$ and very high center of mass energy ≥1 TeV, which could be very useful for saving computer time [31] with respect to the next-to-next-to-leading order (NNLO) collinear calculation in future LHeC [46].

B. KATIE results for the ATLAS p-p Drell-Yan electron-pair production data

In this section we investigate the normalized differential cross section with respect to the transverse momentum of the Drell-Yan electron-pair production in the k_t-factorization framework in comparison with the ATLAS collaboration data [38]. The data is related to the collision of proton-proton with center of mass energy 13 TeV. The electron (positron) transverse momenta are larger than $p_t^{e^-(e^+)} > 27$ GeV with the invariant mass of the lepton pair $66\ \mathrm{GeV} < M^{e^-e^+} < 116\ \mathrm{GeV}$. Additionally, the absolute value of the pseudorapidity of each electron (positron) is $|\eta| < 2.47$, excluding $1.37 < |\eta| < 1.52$. For calculating the cross section we consider $q + \bar{q} \to e^- + e^+$ and $q + g \to e^- + e^+ + q$ subprocesses with the factorization and renormalization scale $\mu_R = \mu_F = \sqrt{M^{e^-e^+} + p_T^{e^-e^+}}$, with $n_f = 5$.

In Fig. 5, we perform such a comparison for the UPDFs of LO-MRW and NLO-MRW. It should be noted that the total cross section obtained with the LO-MRW and NLO-MRW UPDFs are about 687.20 pb, 683.79 pb, respectively, while the experimental cross section is measured to be about 738.3 pb. Here, we only mention the central values of the cross sections because we did not calculate the uncertainty; see Table I. The NNLO and LO total cross sections are also tabulated. As can be seen, the total predicted cross sections using both the LO-MRW and NLO-MRW are close to the experimental measurement. However, one can find that in contrast to the ZEUS energy range, where the prediction of the NLO-MRW UPDFs undershoots the data, we obtain the results that satisfactorily cover the data in most of the regions. While, regarding Fig. 5, the prediction of the LO-MRW still overshoots the data. For comparison, in the bottom panels of Fig. 4, we also plot $f_u(x = 0.001, k_t^2, \mu^2 = 1000)/k_t^2$ and $f_g(x = 0.001, k_t^2, \mu^2 = 1000)/k_t^2$ using the LO and NLO-MRW UPDFs models. It should be noted that the suppression of NLO-MRW UPDFs, which happens at the energy range of the ZEUS experiment, does not occur for the LHC experiment. Because, at small z, the virtuality k^2 does not become much larger than the factorization scale, and hence the additional cutoff $\Theta(\mu^2 - k^2)$ does not have much effect to make the result to undershoot the data.

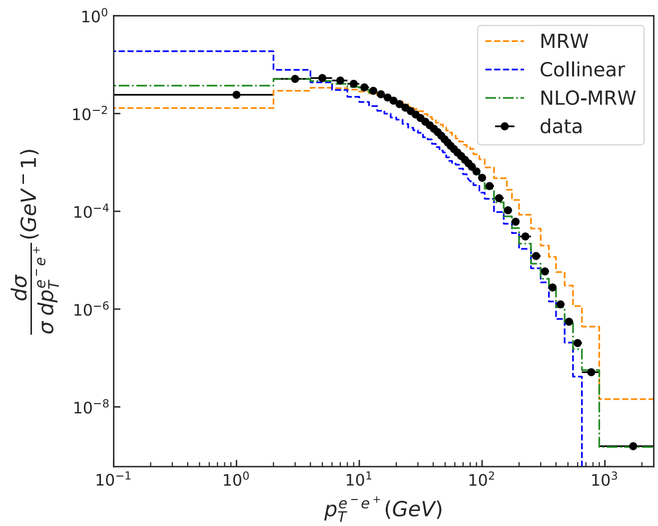

FIG. 5. The comparison of LO-MRW and NLO-MRW UPDFs k_t-factorization differential cross sections for the p-p Drell-Yan electron-pair production with the ATLAS Collaboration data [38] versus the transverse momentum of electron-positron, $p_T^{e^-e^+}$.

To make a proper comparison, the collinear calculation with KATIE for the above two processes is also plotted in Fig. 5 which gives worse results with respect to the k_t-factorization formalisms, i.e., LO and NLO-MRW. One should note that the rise of the differential cross section at small $p_T^{e^-e^+}$ is due to an incomplete soft gluon resummation technique which should be used to make QCD predictions at low $p_T^{e^-e^+}$ finite [47] and it does not happen in the case of the k_t-factorization calculations [47] (in order to obtain finite results in the case of the collinear calculation we imposed $p_T^{e^-e^+} \geq 0.4$ GeV). The application of NNLO collinear pQCD to the above differential cross section covers fully the data [38].

Finally, in contrast to our dijet calculations, one can find that the application of the k_t factorization in the KATIE parton level event generator produces the differential cross

TABLE I. The central prediction values of the LO-MRW, NLO-MRW UPDFs k_t- and collinear factorizations of total ep inclusive dijet cross sections in the fiducial volume in the electron decay channel, using KATIE, as well as the prediction/experiment of each model. The NNLO model is from Ref. [38].

Model	$\sigma^{\mathrm{prediction}}$	$\sigma^{\mathrm{experiment}}$	$\sigma^{\mathrm{prediction}}/\sigma^{\mathrm{experiment}}$
LO-MRW	683.79 pb	738.3 pb	0.926
NLO-MRW	687.20 pb	738.3 pb	0.930
Collinear-KATIE	586.03 pb	738.3 pb	0.793
NNLO [38]	703.00 pb	738.3 pb	0.952

section that is consistent with the Drell-Yan experimental data.

IV. CONCLUSION

In this work, we investigated the KATIE parton level event generator for calculating the differential cross section of ep inclusive dijet and p-p Drell-Yan productions within the k_t-factorization framework in two energy ranges of the ZEUS and ATLAS Collaboration data, respectively. In order to perform such calculations, we used the LO-MRW and NLO-MRW UPDFs with angular ordering constraint. For the ep dijet differential cross section, it was noticed by van Hameren [3,44], the author of KATIE, that the lab to Breit transformation was not correctly performed, in this parton level event generator, so the author (van Hameren, see [3,44]) fixed the bug in this code. After that, we calculated the differential cross section and obtained acceptable results. Then surely, we could obtain satisfactory outputs when the p-p Drell-Yan electron-pair production differential cross section in the above framework was calculated and compared with those of ATLAS Collaboration data. We understood that the NLO-MRW, although it underestimates the ZEUS Collaboration data, can in fact predict the data of the ATLAS accurately. However, we observed that the LO-MRW overshot the data in both the ZEUS and ATLAS Collaboration data at large transverse momenta. It was found that the k_t factorization is very useful tool for the small $x \leq 0.01$ and very high center of mass energy $\geq$1 TeV and could save much computer time [31] with respect to NNLO collinear calculation, especially in the future LHeC [46] and the present p-p LHC.

[1] Yu. L. Dokshitzer, V. A. Khoze, A. H. Mueller, and S. I. Troyan, *Basics of Perturbative QCD*, (Editions Frontiers, Singapore, 1991).
[2] J. Collins, *Foundations of Perturbative QCD* (Cambridge University Press, Cambridge, England, 2013).
[3] A. van Hameren, Comput. Phys. Commun. **224**, 371 (2018).
[4] J. Alwall *et al.*, Comput. Phys. Commun. **176**, 300 (2007).
[5] S. Baranov *et al.*, arXiv:2101.10221.
[6] M. Ciafaloni, Nucl. Phys. **B296**, 49 (1988).
[7] S. Catani, F. Fiorani, and G. Marchesini, Nucl. Phys. **B336**, 18 (1990).
[8] S. Catani, F. Fiorani, and G. Marchesini, Phys. Lett. B **234**, 339 (1990).
[9] G. Marchesini, Nucl. Phys. **B445**, 49 (1995).
[10] F. Hautmann, H. Jung, A. Lelek, V. Radescu, and R. Žlebčík, J. High Energy Phys. 01 (2018) 070.
[11] F. Hautmann, H. Jung, A. Lelek, V. Radescu, and R. Žlebčík, Phys. Lett. B **772**, 446 (2017).
[12] Y. L. Dokshitzer, Sov. Phys. JETP **46**, 641 (1977).
[13] V. N. Gribov and L. N. Lipatov, Yad. Fiz. **15**, 781 (1972).
[14] G. Altarelli and G. Parisi, Nucl. Phys. **B126**, 298 (1977).
[15] F. Hautmann, L. Keersmaekers, A. Lelek, and A. van Kampen, Nucl. Phys. **B949**, 114795 (2019).
[16] A. B. Martinez *et al.*, Phys. Rev. D **100**, 074027 (2019).
[17] A. D. Martin, M. G. Ryskin, and G. Watt, Eur. Phys. J. C **66**, 163 (2010).
[18] G. Watt, A. D. Martin, and M. G. Ryskin, Eur. Phys. J. C **31**, 73 (2003).
[19] M. Modarres and H. Hosseinkhani, Nucl. Phys. **A815**, 40 (2009).
[20] M. Modarres and H. Hosseinkhani, Few-Body Syst. **47**, 237 (2010).
[21] H. Hosseinkhani and M. Modarres, Phys. Lett. B **694**, 355 (2011).
[22] H. Hosseinkhani and M. Modarres, Phys. Lett. B **708**, 75 (2012).
[23] M. Modarres, H. Hosseinkhani, and N. Olanj, Nucl. Phys. **A902**, 21 (2013).
[24] M. Modarres, H. Hosseinkhani, and N. Olanj, Phys. Rev. D **89**, 034015 (2014).
[25] M. Modarres, H. Hosseinkhani, and N. Olanj, Int. J. Mod. Phys. A **32**, 1750121 (2017).
[26] M. Modarres, H. Hosseinkhani, N. Olanj, and M. Masouminia, Eur. Phys. J. C **75**, 556 (2015).
[27] M. Modarres, M. Masouminia, H. Hosseinkhani, and N. Olanj, Nucl. Phys. **A945**, 168 (2016).
[28] M. Modarres, M. Masouminia, R. A. Nik, H. Hosseinkhani, and N. Olanj, Nucl. Phys. **B922**, 94 (2017).
[29] M. Modarres, M. Masouminia, R. A. Nik, H. Hosseinkhani, and N. Olanj, Nucl. Phys. **B926**, 406 (2018).
[30] R. A. Nik, M. Modarres, and M. R. Masouminia, Phys. Rev. D **97**, 096012 (2018).
[31] M. Modarres, R. A. Nik, R. K. Valeshbadi, H. Hosseinkhani, and N. Olanj, J. Phys. G **46**, 105005 (2019).
[32] A. V. Lipatov and M. A. Malyshev, Phys. Rev. D **94**, 034020 (2016).
[33] S. P. Baranov, A. V. Lipatov, and N. P. Zotov, Phys. Rev. D **89**, 094025 (2014).
[34] A. V. Lipatov and N. P. Zotov, Phys. Rev. D **90**, 094005 (2014).
[35] A. van Hameren, *Proc. Sci.*, DIS2019 (2019) 139.
[36] R. K. Valeshabadi, M. Modarres, S. Rezaie, and R. Aminzadeh Nik, J. Phys. G **48**, 085009 (2021).
[37] H. Abramowicz, I. Abt, L. Adamczyk *et al.* (The ZEUS Collaboration), Eur. Phys. J. C **70**, 965 (2010).
[38] G. Aad *et al.* (ATLAS Collaboration), Eur. Phys. J. C **80**, 616 (2020).
[39] J. F. Hautmann *et al.*, Eur. Phys. J. C **74**, 3220 (2014).
[40] S. Baranov, A. B. Martinez, L. I. E. Banos, F. Guzman, F. Hautmann, H. Jung, A. Lelek, J. Lidrych,

A. Lipatov, M. Malyshev *et al.*, Eur. Phys. J. C **81**, 425 (2021).
[41] L. A. Harland-Lang, A. D. Martin, P. Motylinski, and R. S. Thorne, Eur. Phys. J. C **75**, 435 (2015).
[42] L. A. Harland-Lang, A. D. Martin, P. Motylinski, and R. S. Thorne, Eur. Phys. J. C **76**, 10 (2016).
[43] A. Buckley, J. Ferrando, S. Lloyd, K. Nordström, B. Page, M. Rüfenacht, M. Schönherr, and G. Watt, Eur. Phys. J. C **75**, 132 (2015).
[44] A. van Hameren (private communication); see https://bitbucket.org/hameren/katie/commits/.
[45] R. Devenish and A. Cooper-Sarkar, *Deep Inelastic Scattering* (Oxford University Press, 2004).
[46] J. L. A. Fernandez *et al.* (LHeC Study Group), J. Phys. G **39**, 075001 (2012).
[47] A. V. Lipatov, M. A. Malyshev, and N. P. Zotov, J. High Energy Phys. 12 (2011) 117.