



Fundamental energy scale of the thick brane in mimetic gravity

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Received: 23 November 2020 / Accepted: 17 October 2021 / Published online: 8 November 2021

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Abstract In this paper, thick branes generated by the mimetic scalar field with Lagrange multiplier formulation are investigated. We give three typical thick brane background solutions with different asymptotic behaviors and show that all the solutions are stable under tensor perturbations. The effective potentials of the tensor perturbations exhibit as volcano potential, Poöschl–Teller potential, and harmonic oscillator potential for the three background solutions, respectively. All the tensor zero modes (massless gravitons) of the three cases can be localized on the brane. We also calculate the corrections to the four-dimensional Newtonian potential. On a large scale, the corrections to the four-dimensional Newtonian potential can be ignored. While on a small scale, the correction from the volcano-like potential is more pronounced than the other two cases. Combining the specific corrections to the four-dimensional Newtonian potential of these three cases and the latest results of short-range gravity experiments, we get the constraint on the scale parameter as $k \gtrsim 10^{-4}\text{eV}$, and constraint on the corresponding five-dimensional fundamental scale as $M_* \gtrsim 10^5\text{TeV}$.

1 Introduction

The nature of dark matter is one of the most mysterious fields in the modern theoretical physics [1, 2]. Over the last century, the theoretical physicists have made many attempts to reveal the mask of dark matter [3–18]. One possible approach is that the general relativity should be modified at large scale [19–30]. Recently, a Weyl-symmetric extension of general relativity which is named as mimetic gravity has attracted more and more attention [31, 32].

In the mimetic gravity model, the physical metric $g_{\mu\nu}$ is determined by an auxiliary metric $\tilde{g}_{\mu\nu}$ and a scalar field ϕ

with the relation of $g_{\mu\nu} = -\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi$ [31]. In such a setup, the conformal degree of freedom is separated with a covariant way, and the extra degree of freedom can be deemed to dynamic and mimic cold dark matter. The mimetic model could be transformed into a Lagrange multiplier formulation with a potential of the mimetic scalar field. With these methods, one can obtain a viable theory confronted with the cosmic evolution. It was pointed out that this model can also drive the late-time acceleration and early-time inflation [33]. In Ref. [34], the authors pointed out that the mimetic gravity is free of ghost instability only for a positive energy density. Then, Chaichian et al. proved that the mimetic gravity based on tensor–vector theory or tensor–vector–scalar theory is free of ghost instabilities [35]. For more recent works about mimetic gravity, see Refs. [36–50].

On the other hand, since the brane world theory can address the hierarchy problem [51, 52] and the cosmological constant problem [53] successfully, it has attracted more and more attention in the last decades. In the brane world theory, our observable universe is supposed as a 3-brane embedded into a higher-dimensional bulk. The elementary particles in the standard model should be localized on the brane and gravity can propagate into the extra dimension. According to the energy distribution of the brane along the extra dimension, brane models can be divided into thin brane models and thick brane models. For a thin brane model, the energy distribution is a Dirac delta function [54–62]. For a thick brane model, the energy distributes along the extra dimension but localizes around some narrow regions [63–78]. Figure 1 shows the shapes of thin brane [52] and thick brane [79].

It is known that there is no dynamic for a thin brane. For investigating the dynamical generation of a brane and its internal structure, thick brane models were presented. A typical mechanism is that the thick brane can be generated by one or more background scalar fields coupled with gravity. It means that the features of the background scalar fields

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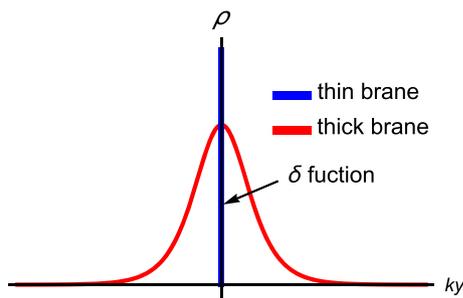


Fig. 1 The shapes of energy distributions of thin and thick branes

can influence the construction of the thick brane, namely, different scalar field can generate thick brane with different structure.

Note that, once we consider the brane world theory, a series of massive particles beyond the standard model of particle physics will be predicted, e.g., massive gravitons and massive vector particles. These massive particles will correct the forms of the four-dimensional Newtonian potential and Coulomb potential. For example, a massive graviton will contribute a correction term to the four-dimensional Newtonian potential with a form of Yukawa potential, and the corresponding total contribution depends on the mass spectrum of the gravitons.

Furthermore, many researches on the brane world theory show that the effective potentials, where the KK graviton modes trapped in, depend on the structures of brane world. It means that the different brane world structures will lead to different mass spectra of the KK gravitons, and these differences will eventually be reflected in the correction behavior to the Newtonian potential.

For the mimetic gravity, the conformal degree of freedom can be separated as a scalar field which can be considered as the cold dark matter, and the conformal formulation can also be transformed into a Lagrange multiplier formulation. It turns out that the mimetic gravity not only provides an extra degree of freedom, but also modifies the equations of the motion. Recently, in Refs. [80, 81], the authors promoted the four-dimensional mimetic gravity into the brane world theory and constructed some thick branes with richer inner structures which are excluded in the high-dimensional general relativity. Besides, such thick branes with different structures result in different characters of the gravitational resonances and the correction to the Newtonian potential. Based on their results, we can see that the properties of the mimetic gravity can relax the constraint for constructing a thick brane, it means that the thick branes with richer structures can be more easily constructed in the high-dimensional mimetic gravity. Combining the properties of the mimetic gravity, we can consider the influence of the thick brane structures on the correction to the four-dimensional Newtonian potential by constructing three thick brane models with different structures

in the higher-dimensional mimetic gravity. Then, by combining the current gravitational inverse-square law experiments [82–84], we can obtain the fundamental energy of the brane world in mimetic gravity.

The organization of this paper is as follows. In Sect. 2, we briefly introduce the mimetic theory in the brane world scenario and obtain the corresponding thick brane solutions. Then, we analyze the stability of the brane solutions under the tensor perturbations and check the localization of the massless graviton in Sect. 3. After that, we derive the corresponding correction to the four-dimensional Newtonian potential, for which the final constraints about the mimetic thick brane are given in Sect. 4. Finally, the conclusion and discussion are given in Sect. 5.

2 The model

In this section, we consider the five-dimensional mimetic gravity with the following action

$$S = \int d^5x \sqrt{-g} \left(\frac{R}{2\kappa_5^2} + \lambda \left[\partial_M \phi \partial^M \phi - U(\phi) \right] - V(\phi) \right), \tag{1}$$

where $\kappa_5^2 = 1/M_*^3$ with M_* being the five-dimensional fundamental scale and λ is a Lagrange multiplier. For simplicity, we chose the natural unit with $\kappa_5^2 = 1$.

In this paper, we set the thick brane to be static and flat. In the brane world theory, the thick brane can be generated by a five-dimensional background scalar field. On the one hand, for a static thick brane, the background field ϕ should not be a function of time. On the other hand, for a flat brane, the four-dimensional Lorentz symmetry should be satisfied on the brane. Therefore, the scalar field ϕ should only depend on the extra dimension, which means that there is no kinetic energy term of ϕ and the mimetic thick brane is free of ghost instability. In such setup, we can see that $U(\phi) = g^{MN} \partial_M \phi \partial_N \phi = e^{-2A} (\partial_z \phi)^2 > 0$.

By varying the action (1) with respect to g_{MN} , ϕ , and λ , respectively, we get the equations of motion (EoM) as

$$G_{MN} + 2\lambda \partial_M \phi \partial_N \phi - L_\phi g_{MN} = 0, \tag{2}$$

$$2\lambda \square^{(5)} \phi + 2g^{MN} \partial_M \lambda \partial_N \phi + \lambda \frac{\partial U}{\partial \phi} + \frac{\partial V}{\partial \phi} = 0, \tag{3}$$

$$g^{MN} \partial_M \phi \partial_N \phi - U = 0. \tag{4}$$

Here, $L_\phi = \lambda [g^{MN} \partial_M \phi \partial_N \phi - U]$, and the five-dimensional d'Alembert operator is defined as $\square^{(5)} = g^{MN} \nabla_M \nabla_N$. The Latin indices ($M, N = 0, 1, 2, 3, 5$) stand for the five-dimensional coordinate indices, and the Greek indices ($\mu, \nu = 0, 1, 2, 3$) represent the brane coordinate indices.

We consider the following brane world metric with four-dimensional Poincaré invariance

$$ds^2 = e^{2A(z)}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2), \tag{5}$$

where $e^{A(z)}$ or $A(z)$ is called as the warp factor. Then, Eqs. (2)–(4) can be rewritten as

$$e^{-2A}(3A'^2 + 3A'') + V(\phi) + \lambda(U - e^{-2A}\phi'^2) = 0, \tag{6}$$

$$6A'^2 + e^{2A}V(\phi) + e^{2A}\lambda(U + e^{-2A}\phi'^2) = 0, \tag{7}$$

$$\lambda\left(6A'\phi' + 2\phi'' + e^{2A}\frac{\partial U}{\partial\phi}\right) + 2\lambda'\phi' + e^{2A}\frac{\partial V}{\partial\phi} = 0, \tag{8}$$

$$e^{-2A}\phi'^2 - U = 0, \tag{9}$$

where the prime denotes the derivative with respect to the extra-dimensional coordinate z . The above equations are not independent of each other. Combining them we get three largely simplified equations for λ , $U(\phi)$, and $V(\phi)$

$$\lambda = \frac{3(A'' - A'^2)}{2\phi'^2}, \tag{10}$$

$$U(\phi) = e^{-2A}\phi'^2, \tag{11}$$

$$V(\phi) = -3e^{-2A}(A'^2 + A''). \tag{12}$$

Generally, in order to get the solutions of thick brane, we can use the super-potential approach by setting a related variable for the brane model with two independent equations and three variables, more details can refer Refs. [64, 85]. In this model, we get three independent equations, and five variables e.g, $A(z)$, ϕ , λ , $V(\phi)$, and $U(\phi)$. Inspired by the super-potential approach, we need to constraint the two variables of them. Note that all the expressions of λ , $U(\phi)$, and $V(\phi)$ depend on the warp factor $A(z)$ and the mimetic field $\phi(z)$. So, once $A(z)$ and $\phi(z)$ are given, the profiles of λ , $U(\phi)$, and $V(\phi)$ could be determined. Besides, we can see that only the variable $A(z)$ can affect the perturbed tensor equation (31). On the one hand, there are no constraints on $A(z)$ and $\phi(z)$ from the equations of motion, so they can be chosen arbitrarily in principle. However, on the other hand, a viable thick brane world model should satisfy the minimal requirement, namely, the localization of massless tensor mode (massless graviton). Therefore, not all the choices of $A(z)$ and $\phi(z)$ are achievable. Here, we will consider three typical solutions.

2.1 Volcano (VO) type thick brane

Firstly, we consider the case of the warp factor $e^{A(z)}$ as a power function of the extra-dimensional z , and the mimetic field $\phi(z)$ is a kink function. The solutions can be given as

$$e^{A(z)} = \frac{1}{\sqrt{k^2z^2 + 1}}, \quad \phi(z) = v\left(\frac{kz}{\sqrt{1 + k^2z^2}}\right)^\gamma, \tag{13}$$

where k is the scale parameter which controls the thickness of the brane, γ is a positive integer, and v is a positive parameter determining the limit of the scalar field. The corresponding expressions of λ , $U(\phi)$, and $V(\phi)$ can be expressed as

$$\lambda = -\frac{3}{2}\frac{(k^2z^2 + 1)^\gamma}{\gamma^2v^2(k^2z^2)^{\gamma-1}}, \tag{14}$$

$$U(\phi) = (kv\gamma)^2\Phi^{2-2/\gamma}(\Phi^{2/\gamma} - 1)^2, \tag{15}$$

$$V(\phi) = -3k^2(3\Phi^{2/\gamma} - 1), \tag{16}$$

where $\Phi = \phi/v$. Such brane solution will give a volcano type effective potential of the tensor perturbations.

2.2 Pöschl–Teller (PT) type thick brane

Then, we come to the hyperbolic function form of the warp factor $e^{A(z)}$ and a different kink form of the mimetic field $\phi(z)$. The expressions of warp factor and mimetic field can be given as

$$e^{A(z)} = \text{sech}(kz), \quad \phi(z) = v\tanh^\gamma(kz), \tag{17}$$

for which the other functions can be solved as

$$\lambda = -\frac{3\sinh^2(2kz)\tanh^{-2n}(kz)}{8\gamma^2v^2}, \tag{18}$$

$$U(\phi) = (kv\gamma)^2\Phi^{2-2/\gamma}(\Phi^{2/\gamma} - 1), \tag{19}$$

$$V(\phi) = \frac{3k^2(2\Phi^{2/\gamma} - 1)}{\Phi^{2/\gamma} - 1}. \tag{20}$$

2.3 Harmonic oscillator (HO) type thick brane

Finally, we choose an exponential warp factor and a kink mimetic field:

$$e^{A(z)} = e^{-k^2z^2}, \quad \phi(z) = v\left(\frac{kz}{\sqrt{1 + k^2z^2}}\right)^\gamma. \tag{21}$$

The specific expressions of λ , $U(\phi)$, and $V(\phi)$ are solved as follows

$$\lambda = -\frac{3(2k^2z^2 + 1)(k^2z^2 + 1)^{\gamma+2}}{\gamma^2v^2(k^2z^2)^{\gamma-2}}, \tag{22}$$

$$U(\phi) = (kv\gamma)^2\Phi^{2-2/\gamma}\left(1 - \Phi^{2/\gamma}\right)^3 e^{\frac{2\Phi^{2/\gamma}}{1-\Phi^{2/\gamma}}}, \tag{23}$$

$$V(\phi) = \frac{6k^2(3\Phi^{2/\gamma} - 1)}{\Phi^{2/\gamma} - 1} e^{\frac{2}{\Phi^{2/\gamma}-1}}. \tag{24}$$

The shapes of these three kinds of warp factors and the two kinds of mimetic scalar fields are shown in Fig. 2. Figure 2a and b show that if γ is an odd integer, the mimetic scalar field would be a single-kink (the black dashed lines) for $\gamma = 1$, and it will become a double-kink (the red lines) with $\gamma \geq 3$. Besides, if γ is an even integer, the scalar field

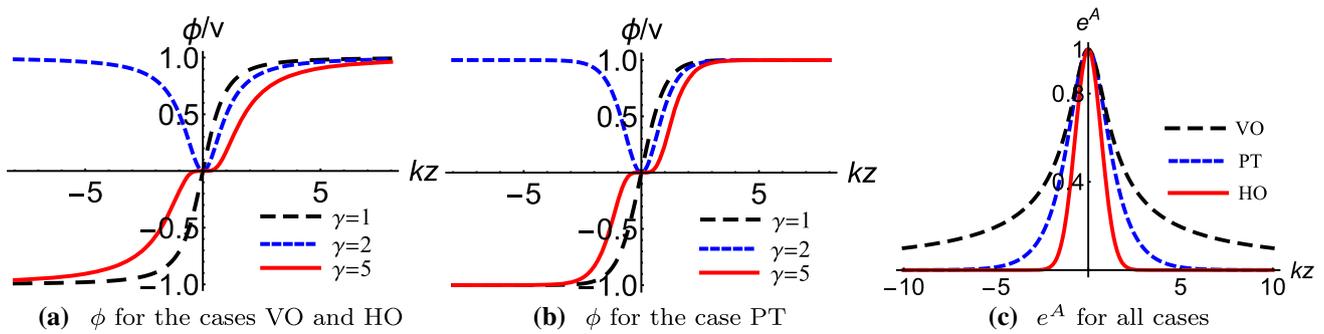


Fig. 2 The shapes of the mimetic field $\phi(z)$, and the warp factors e^A for all cases

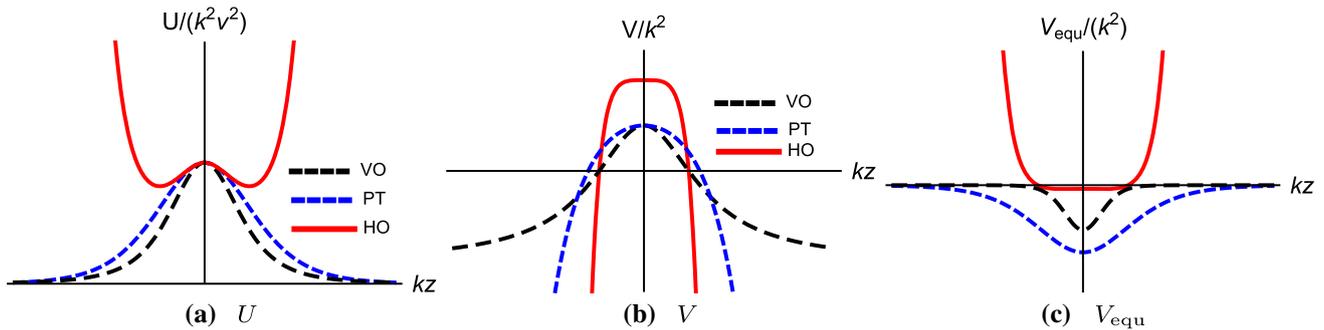


Fig. 3 The shapes of the scalar potentials $U(\phi)$ and $V(\phi)$ from the action, and the equivalent potential V_{equ} for the canonical version of the mimetic field

will be not a kink configuration (the blue dashed line) anymore. For a general thick brane model, the background scalar field should be a kink configuration. While, in mimetic thick brane model, due to the Lagrange multiplier which can cause excess degrees of freedom, the non-kink scalar field can also generate a thick brane. Although these solutions have the same limit of $e^A|_{z \rightarrow \infty} \rightarrow 0$, they differ with the asymptotic behaviors at infinity of the extra dimension, with the attenuation intensity $\text{HO} > \text{PT} > \text{VO}$ (see Fig. 2c).

Figure 3 shows that for the three cases of the thick brane solutions, the potentials $U(\phi)$ and $V(\phi)$ are everywhere well-defined and extend along the extra dimension with different behaviors. From Fig. 3a, we can see that the potentials U for the VO and PT cases extend along the extra dimension but localize around $z = 0$. For the HO case, the potential U distributes along the extra dimension as a harmonic-oscillator-like potential. For the potential V of the VO case, Fig. 3b shows that it tends to a constant at $z \rightarrow \pm\infty$ (the black dashed line). While, for the PT and HO cases, we have $V \rightarrow -\infty$ at $z \rightarrow \pm\infty$, which seems indicate an instability of the scalar profile. However, it should be noted that the scalar field is not canonical. By combining the process from Eq. (24) to Eq. (29) in Ref. [86], we can rewrite the EoM of the scalar field into a canonical form as

$$\hat{\phi}'' + 3A'\hat{\phi}' = \frac{\partial V_{\text{equ}}}{\partial \hat{\phi}}, \tag{25}$$

where

$$\hat{\phi} = \int e^{3(\hat{A}-A)} \phi' dz, \tag{26}$$

$$\hat{A} = \int \frac{A'A'' - 3A'^3 + A'''}{3(A'^2 - A'')} dz, \tag{27}$$

$$V_{\text{equ}} = \int e^{6(\hat{A}-A)} \left(\frac{U'}{2} + \frac{V'Ue^{2A}}{3(A'' - A'^2)} \right) dz. \tag{28}$$

Figure 3c shows the shapes of the equivalent potential V_{equ} for the three cases. From Fig. 3c, we can see that there are no singular boundary behaviors at $z \rightarrow \pm\infty$ for the three cases, which means that the profile of the background scalar field is stable for all the cases.

3 linear perturbations and localization

In this section, we consider the linear perturbations of the metric and their localization. It is well known that the linear perturbations of a background metric can be decomposed into three parts: the transverse-traceless tensor modes, the scalar modes, and the transverse vector modes (the so-called scalar–vector–tensor (SVT) decomposition), for which the three kinds of modes decouple with each other [87].

According to the Bardeen formalism for metric fluctuations [88], the extra dimension part of the metric fluctuations

can be expressed as a scalar mode. In the brane world theory, the linear field equations of the tensor mode, vector mode, and scalar mode of the perturbation of the background space-time are decoupled. The localized tensor zero mode of the perturbation describes the four-dimensional massless graviton which reduces the four-dimensional Newtonian potential. While the localized scalar zero mode of the perturbation will result in the fifth force. It has been shown in Ref. [89] that for the brane world model in mimetic gravity, the scalar zero mode can not be localized on the brane and has no contribution to the four-dimensional effective potential. Therefore, there is no additional fifth force. Thus we can only consider the tensor zero mode in this paper, and focus on the influence of the structure for thick brane on the correction to the Newtonian potential. By combining the short-range experiments, we obtain the constraint of the mimetic gravity. For the tensor perturbations, it can be set as follows

$$ds^2 = e^{2A(z)} [(\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu + dz^2]. \tag{29}$$

Here, $h_{\mu\nu} = h_{\mu\nu}(x^\mu, z)$ depends on all the coordinates. Combining the specific perturbed metric (29) and the transverse-traceless (TT) condition, i.e., $\partial^\mu h_{\mu\nu} = \eta^{\mu\nu} h_{\mu\nu} = 0$, we can simplify the perturbed Ricci tensor as

$$\delta R_{\mu\nu} = -\frac{1}{2} (\square^{(4)} + \partial_z^2 + 2A'' + 6A'^2 + 3A'\partial_z) h_{\mu\nu}, \tag{30}$$

$$\delta R_{55} = 0, \tag{31}$$

where the four-dimensional d'Alembertian is defined as $\square^{(4)} \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu$. Besides, the tensor perturbations of the equations of motion (2) can be expressed as the following form

$$\delta R_{MN} = \frac{2}{3} \delta g_{MN} (\lambda U + V). \tag{32}$$

Then, by combining Eqs. (10), (11), (12), and the perturbed metric (29), the right hand side of Eq. (32) can be simplified as

$$\frac{2}{3} \delta g_{\mu\nu} (\lambda U + V) = -(A'' + 3A'^2) h_{\mu\nu}, \tag{33}$$

$$\frac{2}{3} \delta g_{55} (\lambda U + V) = 0. \tag{34}$$

Therefore, the perturbed tensor equation can be obtained as

$$-\frac{1}{2} \square^{(4)} h_{\mu\nu} - \frac{1}{2} h''_{\mu\nu} - \frac{3}{2} A' h'_{\mu\nu} = 0. \tag{35}$$

Next, we make a KK decomposition with the following form as

$$\tilde{h}_{\mu\nu} = \sum \epsilon_{\mu\nu}^{(n)}(x) \tilde{\Psi}_n(z), \tag{36}$$

where the polarization tensor $\epsilon_{\mu\nu}$ also satisfies the TT condition $\partial^\mu \epsilon_{\mu\nu} = \epsilon_{\mu}{}^\mu = 0$. Bringing the KK decomposition into the perturbed tensor equation (35), we can get a

four-dimensional massive Klein–Gordon equation for the polarization tensor $\epsilon_{\mu\nu}(x)$ and an equation for the extra-dimensional part $\tilde{\Psi}_n(z)$:

$$(\square^{(4)} - m_n^2) \epsilon_{\mu\nu}^{(n)}(x) = 0, \tag{37}$$

$$-\tilde{\Psi}_n''(z) - 3A'\tilde{\Psi}_n'(z) = m_n^2 \tilde{\Psi}_n(z). \tag{38}$$

Furthermore, by redefining the extra dimensional part as $\tilde{\Psi}_n(z) = e^{-\frac{3}{2}A} \Psi_n(z)$, we obtain a Schrödinger-like equation for the new function $\Psi_n(z)$ of the extra-dimensional part:

$$-\partial_z^2 \Psi_n(z) + V_l(z) \Psi_n(z) = m_n^2 \Psi_n(z), \tag{39}$$

with the effective potential $V_l(z)$ given by

$$V_l(z) = \frac{3}{2} A'' + \frac{9}{4} A'^2. \tag{40}$$

We should note that with this effective potential (40), the above Schrödinger-like equation can be rewritten as $\mathcal{H}\Psi_n(z) = m_n^2 \Psi_n(z)$, where the Hamiltonian operator is given as $\mathcal{H} = Q^+ Q$, with $Q = -\partial_z + \frac{3}{2} \partial_z A$. Since the eigenvalue of the Hamiltonian operator \mathcal{H} is positive definite, there do not exist negative m_n^2 modes, namely, there are no tachyonic tensor modes.

The abstract expression of the effective potential $V_l(z)$ shows that it only depends on the warp factor $A(z)$, namely, different asymptotic behaviors of the warp factor $A(z)$ can result in different properties of the effective potentials. We will discuss the properties of the effective potentials with the warp factors $A(z)$ given in Sect. 2.

Shapes of three kinds of effective potentials $V_l(z)$ are shown in Fig. 4. From Fig. 4a, we can see that the effective potential $V_l(z)$ of case VO is a volcano-like potential with a single potential well, and the potential well becomes narrower and deeper with the parameter k increases. The asymptotic behavior of the volcano-like potential shows that there are no other bound states except the zero mode, and the mass spectrum of the massive excited states is continuous from $m_n > 0$. Figure 4b describes the shape of effective potential $V_l(z)$ for case PT. It shows that the second warp factor $A(z)$ leads to a PT potential behavior as $V_l(z)|_{z \rightarrow \pm\infty} = 9k^2/4$. The scale parameter k determines the width and the depth of the potential well. For this PT effective potential, there are two bound states, i.e., the zero mode with $m_0 = 0$ and the first excited state with $m_1 = \sqrt{2}k$, and the mass spectrum is also continuous from $m_n \geq \sqrt{2}k$. For case HO, Fig. 4c shows that the effective potential $V_l(z)$ has the behavior of a harmonic-oscillator potential with the parameter k controls the the mass spectrum. For the harmonic-oscillator potential, all of the states are bound states with mass $m_n = \sqrt{6}nk$, the index n means n -th eigenstate. These different potentials lead to different mass spectra of massive gravitons, which can result in different corrections to the Newtonian potential.

At the end of this section, we consider the zero mode of the tensor perturbations by setting $m_n^2 = 0$ in Eq. (39). It is easy to get the solution of the zero mode:

$$\Psi_0(z) \propto e^{\frac{3}{2}A(z)}. \tag{41}$$

One can verify that the zero modes for the above three different types of warp factors are square-integrable and hence all the zero modes are localized around the brane. Thus, the four-dimensional Newtonian potential can be realized on the brane.

4 The correction to Newtonian potential

In the above section, we have considered the tensor perturbations and obtained the Schrödinger-like equation (39), we also demonstrated that the zero mode can be localized on the brane for all the cases to recover the four-dimensional gravity. In this section, we consider the massive KK modes of the gravitons which can cause the correction to the Newtonian potential in four-dimensional theory.

In the thick brane scenario, the energy density of the brane has a distribution along the extra dimension. Therefore, for simplicity, Refs. [63, 90–93] considered the gravitational potential between two point-like sources of mass M_1 and M_2 located at the origin of the extra dimension, i.e., $z = 0$. We can express the gravitational potential between two masses on the brane as

$$\begin{aligned} V(r) &= -\frac{M_1 M_2}{M_{pl}^2} \frac{1}{r} - \frac{M_1 M_2}{M_*^3} \sum_{J_n \neq 0} \frac{e^{-mr}}{r} |\Psi_n(0)|^2 \\ &= -\frac{M_1 M_2}{M_{pl}^2} \frac{1}{r} \left(1 + \frac{M_{pl}^2}{M_*^3} \sum_{J_n \neq 0} \Delta u(r) \right), \end{aligned} \tag{42}$$

where M_{pl} and M_* are the effective four-dimensional Planck scale and the five-dimensional fundamental scale, respectively. \sum_n stands for summation or integration (or both) with respect to n , depending on the respective discrete or continuous character of the massive KK modes. Besides, we set $\Delta u(r) = e^{-mr} |\Psi_n(0)|^2$.

We can focus on the curvature term of the action (1) from which we will derive the effective four-dimensional scale M_{pl} :

$$M_*^3 \int d^5x \sqrt{-g} R \supset M_{pl}^2 \int d^4x \sqrt{-g^{(4)}(x^\mu)} R^{(4)}(x^\mu). \tag{43}$$

Therefore, the relation between the effective Planck scale M_{pl} and the fundamental scale M_* is given by

$$M_{pl}^2 = M_*^3 \int_{-\infty}^{+\infty} dz e^{3A}. \tag{44}$$

So, the gravitational potential between two masses on the brane can be simplified as

$$\begin{aligned} V(r) &= -\frac{M_1 M_2}{M_{pl}^2} \left(\frac{1}{r} + \Delta V(r) \right) \\ &= -\frac{M_1 M_2}{M_{pl}^2} \frac{1}{r} \left(1 + \Delta U(r) \right), \end{aligned} \tag{45}$$

$$\Delta U(r) = \left(\int_{-\infty}^{+\infty} dz e^{3A} \right) \sum_{J_n \neq 0} \Delta u(r), \tag{46}$$

where $\Delta V(r) = \frac{1}{r} \Delta U(r)$ is the correction term to the Newtonian potential, $\Delta U(r)$ the relative correction term and $\Delta u(r)$ the correction factor. Next, we calculate the corrections to the Newtonian potential for the three cases.

Case VO: We substitute the warp factor (13) into the Schrödinger-like equation (39), and get the reduced Schrödinger-like equation

$$-\partial_z^2 \Psi_n + \frac{3k^2(5k^2 z^2 - 2)}{4(k^2 z^2 + 1)^2} \Psi_n = m_n^2 \Psi_n. \tag{47}$$

To solve the above equation, we consider the behavior of the effective potential at infinity of the extra dimension:

$$V(z) \sim \frac{15}{4z^2}. \tag{48}$$

The approximate solution is given by a linear combination of Bessel functions as

$$\Psi_n(z) = \sqrt{z} \left(C_1 J_2(m_n z) + C_2 Y_2(m_n z) \right), \tag{49}$$

where C_1 and C_2 are arbitrary constants, $J_2(m_n z)$ and $Y_2(m_n z)$ are the first and second Bessel functions, respectively. In Ref. [63], the authors calculated the expression of $\Psi_n(0)$:

$$\Psi_n(0) \sim \left(\frac{m_n}{k} \right)^{1/2}. \tag{50}$$

So, the correction factor $\Delta u(r)$ to the Newtonian potential with a massive graviton m_n is

$$\Delta u(r) = e^{-m_n r} |\Psi_n(0)|^2 = \frac{m_n e^{-m_n r}}{k}. \tag{51}$$

From Fig. 4a, we can see that the spectrum of the massive gravitons is continuous. Therefore, the relative correction term to the Newtonian potential resulted by all the massive gravitons is

$$\Delta U(r) = \left(\int_{-\infty}^{+\infty} dz e^{3A} \right) \int_0^\infty dm \Delta u(r) = \frac{2}{k^2 r^2}. \tag{52}$$

The correction to the Newtonian potential for case VO is the same form as that of the Randall–Sundrum brane model $\Delta V(r) = \frac{1}{r} \Delta U(r) \sim 1/k^2 r^3$.

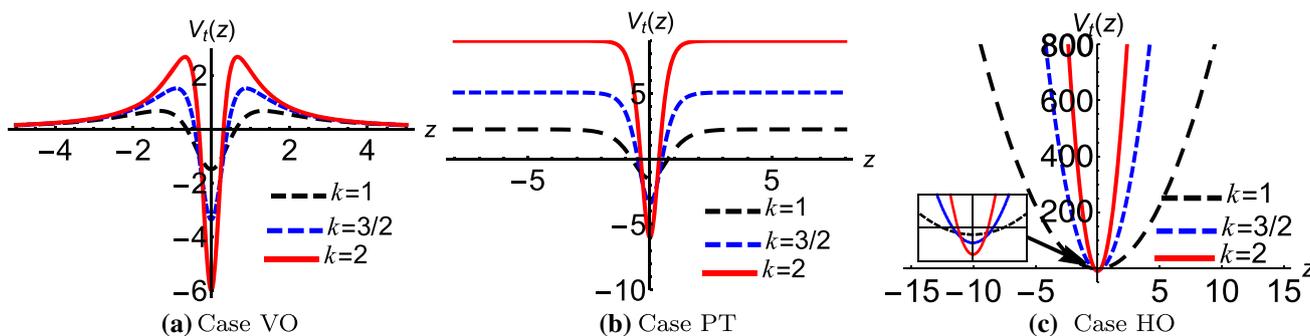


Fig. 4 The effective potentials $V_t(z)$ of the tensor perturbations for all cases

Case PT: For this case with the warp factor (17), the corresponding effective potential turns into

$$V_t(z) = \frac{9}{4}k^2 - \frac{15}{4}k^2 \text{sech}^2(kz), \tag{53}$$

and the Schrödinger-like equation can be expressed as

$$\left(-\partial_z^2 - \frac{15}{4}k^2 \text{sech}^2(kz)\right)\Psi_n = E_n \Psi_n, \tag{54}$$

where $E_n = m_n^2 - \frac{9}{4}k^2$. It can be shown that there are two bound states in this potential. The first one is the ground state $\Psi_0(z)$ with $E_0 = -\frac{9}{4}k^2$, and it is in fact the zero mode since the mass is zero: $m_0 = 0$. The second one is the first excited state $\Psi_1(z)$ with $E_1 = -k^2/4$, which represents a massive graviton with mass $m_1 = \sqrt{2}k$. The two bound states wave functions are

$$\Psi_0(z) = C_0 \text{sech}^{3/2}(kz), \tag{55}$$

$$\Psi_1(z) = C_1 \sinh(kz) \text{sech}^{3/2}(kz). \tag{56}$$

Here C_0 and C_1 are the normalization constants. We should note that $\Psi_1(0) = 0$ which means that the first excited state does not contribute to the correction of the Newtonian potential.

The continuous spectrum starts at $E_n = 0$, corresponding to $m_n^2 \geq 9k^2/4$. These excited states asymptotically turn into plane waves, and represent delocalized KK massive gravitons. Their explicit expressions can be given in terms of the associated Legendre functions of the first kind:

$$\Psi_n(z) = \sum_{\pm} C_{\pm}^n P_{3/2}^{\pm\sigma} \tanh(kz), \tag{57}$$

where C_{\pm}^n are m_n -dependent parameters and

$$\sigma = \sqrt{\frac{9}{4}k^2 - m_n^2}. \tag{58}$$

In order to calculate the correction from the continuous modes, we need to compute the normalization constants C_{\pm}^n . According to Refs. [93–95], we can reduce the constants C_{\pm}^n

as

$$C_+^n = C_-^n = \frac{1}{\sqrt{2\pi}} |\Gamma(1 + \sigma)|, \tag{59}$$

where $\Gamma(1 + \sigma)$ is the gamma function. So, $\Psi_n(0)$ can be expressed as the following form

$$\Psi_n(0) = \frac{\Gamma(1 - \sigma)}{\Gamma(-\frac{1}{4} - \frac{\sigma}{2})\Gamma(\frac{7}{4} - \frac{\sigma}{2})}, \tag{60}$$

and for a massive graviton $\Delta u(r)$ is

$$\begin{aligned} \Delta u(r) &= e^{-m_n r} |\Psi_n(0)|^2 \\ &= e^{-m_n r} \left| \frac{\Gamma(1 - \sigma)}{\Gamma(-\frac{1}{4} - \frac{\sigma}{2})\Gamma(\frac{7}{4} - \frac{\sigma}{2})} \right|^2. \end{aligned} \tag{61}$$

Although we have obtained the parsed expression of the Newtonian potential for a massive graviton, it is cumbersome for the final result to integrate (61) directly. So, our approach is to use $|\Psi_n(0)|^2$ as a fitting function with its approximate behavior, i.e.,

$$|\Psi_n(0)|^2 \approx a_1 \tanh(a_2(m - m_0)) + a_3, \tag{62}$$

where $a_1 = \frac{26903}{100000}$, $a_2 = \frac{13}{40}$, $a_3 = \frac{4937}{100000}$, and $m_0 = 3k/2$. The relative correction to the Newtonian potential for all the massive gravitons is

$$\begin{aligned} \Delta U(r) &= \int_{-\infty}^{+\infty} dz e^{3A} \int_{\frac{3}{2}k}^{\infty} dm \Delta u(r) \\ &= b_1 \frac{\pi e^{-\frac{3}{2}kr}}{kr} \left[-b_2 kr \psi^{(0)}(b_1 kr) - b_3 \right. \\ &\quad \left. + b_2 kr \psi^{(0)}\left(b_1 kr + \frac{1}{2}\right) \right]. \end{aligned} \tag{63}$$

where $b_1 = \frac{10}{13}$, $b_2 = \frac{26903}{100000}$, $b_3 = \frac{35711}{250000}$, and $\psi^{(0)}(x)$ is the logarithmic derivative of the Gamma function: $\psi^{(0)}(x) \equiv \frac{d}{dx} \ln \Gamma(x)$. Note that the term in the square bracket of (63) is almost a constant, we can get an approximate expression

of the relative correction term of the Newtonian potential:

$$\Delta U(r) \propto \frac{e^{-\frac{3}{2}kr}}{kr}. \tag{64}$$

Case HO: The corresponding Schrödinger equation (39) for case HO can be expressed as the following form

$$-\partial_z^2 \Psi_n + (9k^4 z^2 - 3k^2) \Psi_n = m_n^2 \Psi_n. \tag{65}$$

The normalized solution is given by

$$\Psi_n = \sqrt{\frac{3}{\pi}} \sqrt{\frac{1}{n!}} 2^{-\frac{n}{2}} e^{-\frac{3}{2}k^2 z^2} H_n(\sqrt{3}kz), \tag{66}$$

where n is a positive integer and H_n is Hermite polynomial. The corresponding mass spectrum is $m_n = k\sqrt{6n}$, which means that the mass gap decreases with the mass. The expression of wave function Ψ_n at $z = 0$ can be rewritten as

$$\Psi_n(0) = \frac{\sqrt[4]{3\pi} 2^{n/2}}{\sqrt{n!} \Gamma(\frac{1-n}{2})}. \tag{67}$$

Then, $\Delta u(r)$ for a massive graviton is

$$\begin{aligned} \Delta u(r) &= e^{-m_n r} |\Psi_n(0)|^2 \\ &= e^{-k\sqrt{6n}r} \frac{\sqrt{3\pi} 2^n}{n! \Gamma^2(\frac{1-n}{2})}. \end{aligned} \tag{68}$$

Note that $\Delta u(r) = 0$ for an odd n . Therefore, the odd modes of the massive gravitons do not contribute to the correction of the Newtonian potential. In order to calculate the correction to the Newtonian potential for all the massive gravitons, we should do some tedious but simple steps. The $\Delta u(r)$ for this case can be rewritten as

$$\begin{aligned} \Delta u(r) &= e^{-m_n r} |\Psi_n(0)|^2 = e^{-2k\sqrt{3a}r} \frac{\sqrt{3\pi} 2^{2a}}{(2a)! \Gamma^2(\frac{1-2a}{2})^2} \\ &\approx e^{-2k\sqrt{3a}r} \frac{\sqrt{3}}{\pi \sqrt{a}}, \quad \left(a = n/2 = \frac{m_n^2}{12k^2} \right), \end{aligned} \tag{69}$$

However, it is cumbersome to sum Δu directly. From Fig. 5, we can see that $\sum_a \Delta u$ can be fitted by $\int \Delta u da$. For simplicity, we replace sum with integration and obtain the following approximate result

$$\begin{aligned} \Delta U(r) &= \sum_{a=1}^{\infty} \Delta u(r) \approx \sum_{a=1}^{\infty} e^{-2k\sqrt{3a}r} \frac{\sqrt{3}}{\pi \sqrt{a}} \\ &\approx \int_1^{\infty} e^{-2k\sqrt{3a}r} \frac{\sqrt{3}}{\pi \sqrt{a}} da = \frac{e^{-2\sqrt{3}kr}}{\pi kr}. \end{aligned} \tag{70}$$

Figure 6 shows the correction factor Δu of the massive gravitons (plot as $\Delta u^{1/3}$) and the relative correction terms ΔU contributed by all the massive gravitons for the three cases, respectively. From Fig. 6a, we can see that the correction factor Δu is smaller for larger graviton mass, and the

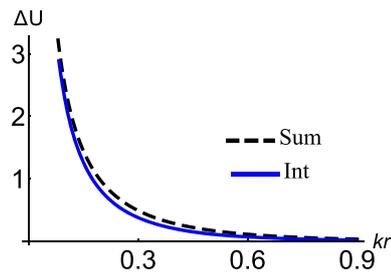


Fig. 5 Plot of the correction term $\Delta U(kr)$ for case HO. The dashed black line is the result of $\sum_{a=1}^{10000} \Delta u$, and the blue line the result of $\int_{a=1}^{10000} \Delta u$

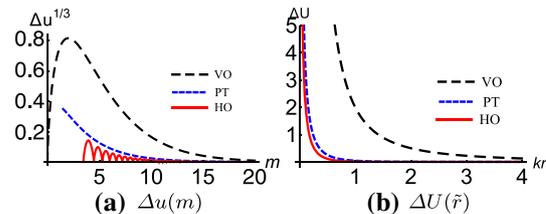


Fig. 6 The correction factor Δu of the massive gravitons (plot as $\Delta u^{1/3}$) and the relative correction ΔU for the three cases. The parameter kr in (a) is set as $kr = 1$

attenuation trends of the three cases are slightly different. On the one hand, the different forms of $|\Psi_n(0)|^2$, the square of the massive graviton mode on the brane, are given by

$$|\Psi_n(0)|^2 \sim \begin{cases} m_n/k, & \text{case VO;} \\ a_1 \tanh(a_2(m_n - m_0)) + a_3, & \text{case PT;} \\ \frac{\sqrt{3\pi} 2^n}{n! \Gamma^2(\frac{1-n}{2})}, \quad n = \frac{m_n^2}{6k^2}, & \text{case HO.} \end{cases} \tag{71}$$

On the other hand, for different models, the mass range of the massive gravitons which dominate the correction to the gravitational potential are different, i.e., $(0 \sim \infty)$, $[3k/2 \sim \infty)$, $[2\sqrt{3}k \sim \infty)$ for the three cases, respectively. Note that for case VO and case PT the mass spectra are continuous, while the mass spectrum of case HO is discrete. These differences lead to different forms of the relative correction terms ΔU , which are shown in Fig. 6b. From Fig. 6b, we can see that on small scales $\Delta U > 1$, which means that the correction term dominates the Newtonian potential, and on large scales the relative correction term decays to zero rapidly. That is to say, the effect of all massive gravitons on the Newtonian potential can be ignored on large scales. We note that the correction term ΔU of case VO is more remarkable than the others.

So far, we have obtained the expressions of the three corrections to the Newtonian potential. Then, we can obtain the constraints on the parameters of our models by combining the latest tests of the gravitational inverse-square law [82–84]. In these experiments, the authors considered the following

Table 1 Constraints of the scale parameter k and the fundamental M_* for the three cases at $r = 210 \mu\text{m}$

Model	ΔU	\tilde{r}	$k_{\text{min}}(\text{eV})$	M_*	$M_*^{\text{min}}(\text{TeV})$
Case VO	$1/k^2 r^2$	14.1	6.7×10^{-3}	$\frac{k}{2} M_{pl}^2$	7.9×10^5
Case PT	$\frac{e^{-\frac{3}{2}kr}}{kr}$	1.6	7.4×10^{-4}	$\frac{2k}{\pi} M_{pl}^2$	4.2×10^5
Case HO	$\frac{e^{-2\sqrt{3}kr}}{\pi kr}$	1.0	4.8×10^{-4}	$\sqrt{3/\pi} k M_{pl}^2$	4.1×10^5

Table 2 Constraints of the scale parameter k and the fundamental M_* for the three cases at $r = 52 \mu\text{m}$

Model	ΔU	\tilde{r}	$k_{\text{min}}(\text{eV})$	M_*	$M_*^{\text{min}}(\text{TeV})$
Case VO	$1/k^2 r^2$	4.5	8.6×10^{-3}	$\frac{k}{2} M_{pl}^2$	8.6×10^5
Case PT	$\frac{e^{-\frac{3}{2}kr}}{kr}$	0.7	1.4×10^{-3}	$\frac{2k}{\pi} M_{pl}^2$	5.1×10^5
Case HO	$\frac{e^{-2\sqrt{3}kr}}{\pi kr}$	0.5	1.0×10^{-3}	$\sqrt{3/\pi} k M_{pl}^2$	5.3×10^5

four-dimensional gravitational potential

$$V(r) = V_N(r) [1 + \alpha \exp(-r/\lambda)], \tag{72}$$

where the parameter r is the separation between two masses, λ and α are the length scale and strength of the Yukawa type correction. As shown in Refs. [83,84], the corresponding values of α, λ, r are

$$\{\alpha, r, \lambda\} = \{1, 210 \mu\text{m}, 48 \mu\text{m}\}, \tag{73}$$

$$\{\alpha, r, \lambda\} = \{0.45, 52 \mu\text{m}, 38.6 \mu\text{m}\}, \tag{74}$$

and the magnitudes of the corresponding Yukawa correction term are

$$\alpha e^{-r/\lambda} \sim \begin{cases} 0.01, & r = 210 \mu\text{m}, \\ 0.1, & r = 52 \mu\text{m}. \end{cases} \tag{75}$$

Note that, the magnitude of the correction term of the gravitational potential should be independent of its form. Therefore, it is natural to set the same magnitudes for the correction terms of our models when we choose the same separations as the separations in Refs. [83,84]. Obviously, the Yukawa type correction $\alpha \exp(-r/\lambda)$ can be considered as a form of ΔU in our models. To get the constraints on the parameters, we can set that the upper limit of the correction term ΔU is the same with the magnitude of $\alpha e^{-r/\lambda}$, which means that $\Delta U < 0.01$ for $r = 210 \mu\text{m}$ or $\Delta U \leq 0.1$ for $r = 52 \mu\text{m}$.

By using the above assumptions, we get the critical points with $\Delta U(\tilde{r}) = 0.01$ with the separation $r = 210 \mu\text{m}$ [83] and $\Delta U(\tilde{r}) = 0.1$ for $r = 52 \mu\text{m}$ [84], where $\tilde{r} = kr$. After calculation, we get the critical values of \tilde{r} for the three cases. Note that the relative correction term $\Delta U(\tilde{r})$ decreases monotonically with \tilde{r} , in other words, to make sure ΔU is less than the critical points for the test experiments, the scale parameter k should satisfy the relation $k > \tilde{r}/r$. Besides, we can get the constraints on the five-dimensional fundamental scale M_* based on Eq. (44).

We give the constraints on the parameter k and the fundamental scale M_* of our three models in Tables 1 and 2. Comparing these results, we can see that the constraints of parameters k and M_* based on the experimental data in Ref. [84] are stronger than the constrains by Ref. [83]. Here we should note that the specific correction to the Newtonian potential depends on the structure of brane, namely the warp factor $A(z)$ of the thick brane model. Combining Eqs. (44), (71), and Fig. 6, we can see that the warp factor not only affects the mass spectrum of massive gravitons and the expression of $|\Psi_n(0)|^2$, but also affects the relation between M_{pl} and M_* . In other words, the specific limits of k and M_* are also model-dependent. Therefore, comparing our three thick brane models, the limits of k and M_* of case VO are stricter than other two cases for both the two experimental data. Then, we can get the conclusion that the critical value of the scale parameter k is at least 10^{-4}eV , and the five-dimensional fundamental scale M_* should be at least 10^5TeV .

5 Conclusion

In this paper, we considered the thick brane model generated by a mimetic scalar field with the Lagrange multiplier formulation. With the existence of excess degrees of freedom, we constructed three background solutions. Although, these three solutions have the same limit of $e^A|_{z \rightarrow \infty} \rightarrow 0$, the asymptotic behaviors of these three cases are different with $\text{HO} > \text{PT} > \text{VO}$. These different asymptotic behaviors cause different effective potentials of tensor perturbations, which lead to different corrections to the Newtonian potential by the massive KK gravitons.

We got the specific expressions of the effective potentials for the three cases. They are volcano-like potential, PT potential, and harmonic oscillator potential. We showed that all the solutions are stable under the tensor perturbations and the zero modes of tensor perturbations can be localized on the

branes. Therefore, the four-dimensional Newtonian potential can be recovered.

We also calculated the corrections to the Newtonian potential for the three cases. For case VO, the relative correction term $\Delta U \propto 1/(kr)^2$ is the same as the RS model [52]. Although, the relative correction terms ΔU of case PT and case HO have the same form with $\Delta U \approx \exp(-\beta kr)/kr$, the specific values $\beta = -3/2$ and $\beta = -2\sqrt{3}$ will lead to big difference on a small scale. For the corrections to the Newtonian potential of these three cases, the results show that the four-dimensional Newtonian potential can be recovered on large scales. On a small scale, the three cases have different behaviors, the correction to the Newtonian potential of case VO is more pronounced than the other two cases. Combining the specific correction terms to the Newtonian potential for these three models and the latest experiments of the gravitational inverse-square law [83, 84], we obtained the constraints that the scale parameter k is at least 10^{-4} eV, and the corresponding five-dimensional fundamental scale M_* should be at least 10^5 TeV.

Acknowledgements This work was supported by the National Natural Science Foundation of China (Grants nos. 11875151 and 11947025), the Fundamental Research Funds for the Central Universities (Grants no. lzujbky-2019-ct06 and lzujbky-2021-pd08), and the 111 Project under Grant no. B20063. T.T. Sui and Y.P. Zhang were supported by the scholarship granted by the Chinese Scholarship Council (CSC).

Data Availability Statement The manuscript has associated data in a data repository. [Authors' comment: Our data is presented in the form of formulas, pictures and tables, which are enough to explain our results. There is no need to list specific data.]

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Funded by SCOAP³.

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