Lattice perturbation theory for the null cusp string

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We reconsider the problem of discretizing the worldsheet for the gauge-fixed Green-Schwarz superstring on a null cusp background, and present a setup which fully preserves its global $U(1) \times SU(4)$ symmetry. We discuss divergences by power counting on the lattice, and study renormalizability at one loop with the example of one-point functions and one bosonic correlator of the worldsheet excitations. In order to remove UV divergences at one loop, it is necessary to introduce two extra parameters in the action, which need to be either fine-tuned at tree level or renormalized at one loop.

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I. INTRODUCTION AND DISCUSSION

In the framework of the AdS/CFT [1,2] correspondence, the expectation value of a lightlike cusped Wilson loop in $\mathcal{N}=4$ super-Yang-Mills theory is equal to the partition function of an open string propagating in $\mathrm{AdS}_5 \times S^5$ space and ending on the loop at the AdS boundary. In practice one writes

$$\langle \mathcal{W}_{\mathrm{cusp}} \rangle = \int \mathcal{D}Y \mathcal{D}\Psi e^{-S_{\mathrm{cusp}}(X_{\mathrm{cl}}+Y,\Psi)} \equiv e^{-\frac{f(g)}{8}V_2}, \qquad (1.1)$$

where S_{cusp} is obtained from the Green-Schwarz $\mathrm{AdS}_5 \times S^5$ superstring action, by parametrizing the fluctuations of the bosonic degrees of freedom $X = X_{\mathrm{cl}} + Y$ around the classical null-cusp solution X_{cl} [3,4], and by fixing the local bosonic (diffeo) and fermionic (kappa) symmetries e.g., to light-cone gauge [5]. The free energy of the open string is proportional to the worldsheet volume V_2 and we refer to the prefactor $f(g)$ as the *cusp anomaly*[1] [6,9,10]. The cusp anomaly is a function of the coupling constant $g = \frac{R^2}{4\pi\alpha'} = \frac{\sqrt{\lambda}}{4\pi}$, where R is the common radius of AdS_5 and S^5, α' is the squared string scale, while λ is the 't Hooft coupling on the gauge side of the AdS/CFT correspondence. The cusp anomaly has been calculated to next-to-next-to-leading order in a perturbative expansion in g^{-1} [8] and in dimensional regularization. Assuming integrability [11,12] and using the corresponding technology [11,13–15], the cusp anomaly can be evaluated also at finite coupling.

The Green-Schwarz $\mathrm{AdS}_5 \times S^5$ string is expected to be defined also at the nonperturbative level. A valid question is whether the nonperturbative regime of the σ model, which describes the $\mathrm{AdS}_5 \times S^5$ string at tree level in string perturbation theory, is accessible through a lattice discretization of the worldsheet (while target space remains continuous). This question is motivated by the success of the lattice as a UV nonperturbative regulator of quantum chromodynamics. This approach has been pioneered in [16–19], where a lattice-discretized version of S_{cusp} has been introduced and also used to perform Monte Carlo simulations.[2]

Once a lattice discretization of S_{cusp} and of the path integral is proposed, one still needs to understand whether the continuum limit (i.e., the limit in which the lattice spacing a vanishes) exists for physical observables, and whether the obtained continuum theory has the desired defining properties. Notice that the inverse lattice spacing a^{-1} is nothing but a uv cutoff, and the question of the existence of the continuum limit is logically equivalent to the question of cancellation of uv divergences after

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[1]In some literature, $f(g)$ is called "scaling function." From the gauge theory point of view, it governs the logarithmic behavior in the large spin anomalous dimensions of twist-two operators, and equals twice the cusp anomalous dimension of lightlike Wilson loops [6]. The same can been seen [7] at the level of the dual classical string solutions, respectively [3,4]. The normalization factor 1/8 in (1.1) also takes into account the conventions of [8].

[2]Other lattice approaches to AdS/CFT include [20–37], see also [38] and references therein.

renormalization: once a discretized action is defined as a function of a finite number of bare parameters, is it possible to cancel all uv divergences in on-shell observables with a redefinition of the bare parameters? The existence of the continuum limit at the nonperturbative level is a very complicated issue, both theoretically and numerically. However, if the lattice regularization makes sense at all, then one should recover the correct continuum theory also order by order in the perturbative expansion, i.e., in powers of g^{-1}. The goal of this paper is precisely to set the stage for such a perturbative expansion, and to discuss some peculiarities of the lattice regulator.

In Sec. II, we present a new discretization for $S_{\rm cusp}$. Contrary to the actions proposed and used in [17,18,39], the new action is invariant under the full $U(1) \times SU(4)$ group of internal symmetries. As usual in QFT, more symmetries mean less uv divergences. In Sec. III, we parametrize the fluctuations around the classical solution in analogy to what is usually done in the continuum [8] and we calculate the propagators for the lattice discretized theory.

In Sec. IV we calculate the superficial degree of divergence of the generic Feynman diagram and we show that power counting suggests that infinitely many counterterms are needed at every order in the perturbative expansion to cancel all uv divergences.

This result is not so surprising, as the Green-Schwarz action expanded around a classical background is known to be formally power-counting nonrenormalizable [40–42]. However in the continuum, when using the regularization introduced in [17,18] to which we refer as "dimensional regularization" in what follows, the cusp anomaly turns out to be finite without any counterterm, at least up to two loops [8,42]. The cancellation of divergences has been verified similarly for the two-point functions and the dispersion relation of excitations near a long spinning string in AdS_5 at one loop [43], and for a "generalized scaling function" governing the energy of a string spinning both in AdS_5 and in S^5 at two loops [44,45].[3]

In order to understand whether similar cancellations of uv divergences happen also in the lattice discretized theory, we calculate the cusp anomaly, the one-point function of the field ϕ (which parametrizes the radial direction of AdS_5), and the two-point function of x, which parametrizes the fluctuations of the string at the AdS_5 boundary. These calculations are presented in Sec. V. We will see explicitly that, in the considered lattice discretization, the situation is quite more complicated than in dimensional regularization, and it is related to the presence of power divergences. We observe the following interesting facts:

(1) The quadratic divergences cancel at one loop in the one-point function of ϕ and in the two-point function of x (while they are subtracted by hand in the cusp anomaly). At one loop, these cancellations seems quite robust in the sense that they will always happen in any reasonable discretization of the action.

(2) Linear divergences arise as well, and they generally do not cancel in all considered observables. These divergences are very specific of the lattice discretization, and arise from the particular choice of forward and backward discrete derivatives. In order to cure this problem we have introduced two extra parameters $b_\pm$ in the action that would be naturally set to 1 at the classical level. In order to remove the linear divergences at one loop, these parameters need to be either fine-tuned at tree level or renormalized at one loop.

(3) Once the linear divergences are removed by tuning or renormalization, the logarithmic divergences cancel in the cusp anomaly and in the two-point function of x (while they survive in the one-point function of ϕ in analogy to the continuum). Moreover the continuum limit of the cusp anomaly and of the dispersion relation of the worldsheet excitation with the quantum numbers of the field x are the same as the ones obtained in dimensional regularization.

The extra parameters $b_\pm$ do not seem to have any deep meaning besides the fact that they make the bare propagators particularly simple. Moreover we do not claim that the introduction and fine-tuning of these two extra parameters is enough to make all physical observables finite at all orders in perturbation theory, and this is in fact highly unlikely. Still, one would like to understand whether the number of parameters needed to achieve finiteness of physical observables via fine-tuning or renormalization is finite or not. If infinitely many parameters are necessary, then the discretized model has no predictivity, and it cannot be used as a viable nonperturbative definition of the $\mathrm{AdS}_5 \times S^5$ string in null-cusp background. A complete one-loop analysis of the divergences of n-point functions may help shed light on this issue, and we plan to carry it on in the future, with the technology developed in this paper.

One may also try to find a general mechanism that prevents linear divergences in the first place. Building on the idea that odd powers of a must be accompanied by odd powers of m, one may try to exploit a spurionic symmetry that involves the replacement $m \to -m$, the reflection of both worldsheet coordinates and an $SO(5)$ rotation, which is enjoyed by the continuous action. Such spurionic symmetry is broken by our lattice discretization. Some preliminary explorations that we do not report here indicate that it is not completely trivial to preserve this symmetry on the lattice while avoiding the doubling problem. Different options in this direction will be explored in the future.

[3]The classical worldsheet theory of the long spinning string in AdS_5 is equivalent, via an analytic continuation and a global conformal transformation, to that of the lightlike cusp solution which is of interest here, see footnote 1.

II. $U(1)\times SU(4)$ INVARIANT DISCRETIZATION

In the continuum, the $\mathrm{AdS}_5\times S^5$ superstring action in a AdS-lightcone gauge-fixing describing quantum fluctuations around the null-cusp background reads [8]

$$\begin{aligned} S^{\rm cont}_{\rm cusp} = g\int dtds\Bigg\{&\left|\partial_t x+\frac{m}{2}x\right|^2+\frac{1}{z^4}\left|\partial_s x-\frac{m}{2}x\right|^2+\left(\partial_t z^M+\frac{m}{2}z^M+\frac{i}{z^2}z_N\eta_i(\rho^{MN})^i{}_j\eta^j\right)^2\\ &+\frac{1}{z^4}\left(\partial_s z^M-\frac{m}{2}z^M\right)^2+i(\theta^i\partial_t\theta_i+\eta^i\partial_t\eta_i+\theta_i\partial_t\theta^i+\eta_i\partial_t\eta^i)-\frac{1}{z^2}(\eta^i\eta_i)^2\\ &+2i\left[\frac{1}{z^3}z^M\eta^i(\rho^M)_{ij}\left(\partial_s\theta^j-\frac{m}{2}\theta^j-\frac{i}{z}\eta^j\left(\partial_s x-\frac{m}{2}x\right)\right)\right.\\ &\left.+\frac{1}{z^3}z^M\eta_i(\rho^{M\dagger})^{ij}\left(\partial_s\theta_j-\frac{m}{2}\theta_j+\frac{i}{z}\eta_j\left(\partial_s x-\frac{m}{2}x\right)^*\right)\right]\Bigg\}, \end{aligned} \tag{2.1}$$

where

(i) x is a complex bosonic field whose real and imaginary part parametrize the fluctuations of the string (in light-cone gauge) at the boundary of AdS_5.
(ii) z^M are six real bosonic fields, i.e., $M=1,\ldots,6$; $z=\sqrt{z^M z^M}$ is the radial coordinate of the AdS_5 space, while $u^M=z^M/z$ identifies points on S_5.
(iii) The Graßmann-odd fields $\theta^i=(\theta_i)^\dagger$, $\eta^i=(\eta_i)^\dagger$, $i=1,2,3,4$ are complex anticommuting variables (no Lorentz spinor indices appear).
(iv) The matrices $(\rho^{MN})_i{}^j=(\rho^{[M}\rho^{\dagger N]})_i{}^j$ are the $SO(6)$ generators; ρ^{M}_{ij}[4] are the (traceless) off-diagonal blocks of $SO(6)$ Dirac matrices γ^M in chiral representation, see Appendix A.

The massive parameter m keeps track of the (dimensionful) light-cone momentum P_+, set to one in [8]. The action (2.1) is invariant under a $U(1)\times SU(4)$ global symmetry defined by

$$z^M\to \mathrm{Ad}(U)^{MN}z^N,\quad \theta^i\to U^i{}_j\theta^j,\quad \eta^i\to U^i{}_j\eta^j, \tag{2.2}$$

$$x\to e^{i\alpha}x,\qquad \theta^i\to e^{i\alpha/2}\theta^i,\qquad \eta^i\to e^{-i\alpha/2}\eta^j, \tag{2.3}$$

where U is an element of $SU(4)$ and its representative in the adjoint, $\mathrm{Ad}(U)$, is an element of $SO(6)$. While the original Green-Schwarz $\mathrm{AdS}_5\times S^5$ string action is invariant under diffeomorphisms and κ symmetry, these local symmetries have been fixed by the choice of light-cone gauge in Eq. (2.1). Notice that the action is not invariant under worldsheet rotations, parity ($s\to -s$), or time reversal ($t\to -t$).

In order to define the lattice-discretized theory we need to provide a discretized action, but also an explicit expression for the measure. We choose to use a flat measure for the fields, but we keep in mind that this choice is quite arbitrary as it is not invariant under reparametrization of the target $\mathrm{AdS}_5\times S_5$ target space. Given a generic observable A, expectation values in the lattice discretized theory are defined by

$$\langle A\rangle = \frac{1}{Z_{\rm cusp}}\int dxdx^* d^6z d^4\theta d^4\theta^\dagger d^4\eta d^4\eta^\dagger e^{-S_{\rm cusp}}A, \tag{2.4}$$

where $df\equiv\prod_{s,t}df(s,t)$, as usual the partition function $Z_{\rm cusp}$ is fixed by the requirement $\langle 1\rangle=1$, and $S_{\rm cusp}$ refers now to the discretised action, that we choose to be

$$\begin{aligned} S_{\rm cusp} = g\sum_{s,t}a^2\Bigg\{&\left|b_+\hat\partial_t x+\frac{m}{2}x\right|^2+\frac{1}{z^4}\left|b_-\hat\partial_s x-\frac{m}{2}x\right|^2+\left(b_+\hat\partial_t z^M+\frac{m}{2}z^M+\frac{i}{z^2}z^N\eta_i(\rho^{MN})^i{}_j\eta^j\right)^2\\ &+\frac{1}{z^4}\left(\hat\partial_s z^M\hat\partial_s z^M+\frac{m^2}{4}z^2\right)+2i(\theta^i\hat\partial_t\theta_i+\eta^i\hat\partial_t\eta_i)-\frac{1}{z^2}(\eta^i\eta_i)^2\\ &+2i\left[\frac{1}{z^3}z^M\eta^i(\rho^M)_{ij}\left(b_+\bar\partial_s\theta^j-\frac{m}{2}\theta^j-\frac{i}{z}\eta^j\left(b_-\hat\partial_s x-\frac{m}{2}x\right)\right)\right.\\ &\left.+\frac{1}{z^3}z^M\eta_i(\rho^{M\dagger})^{ij}\left(b_+\bar\partial_s\theta_j-\frac{m}{2}\theta_j+\frac{i}{z}\eta_j\left(b_-\hat\partial_s x^*-\frac{m}{2}x^*\right)\right)\right]\Bigg\}. \end{aligned} \tag{2.5}$$

[4]By convention, we will write the indices of ρ as down and those of $\rho^\dagger$ as up.

The action is written in terms of the forward and backward discrete derivatives

$$\hat{\partial}_\mu f(\sigma) \equiv \frac{f(\sigma + ae_\mu) - f(\sigma)}{a},$$
$$\bar{\partial}_\mu f(\sigma) \equiv \frac{f(\sigma) - f(\sigma - ae_\mu)}{a} \quad (2.6)$$

where e_μ is the unit vector in the direction $\mu = 0, 1$, and σ is a shorthand notation for (s, t).

Notice that the proposed discretized action (2.5) depends on four parameters: g, m, and the auxiliary parameters $b_\pm$. It is straightforward to see that the discretized action $S_{\rm cusp}$ reduces to the desired continuum action $S_{\rm cusp}^{\rm cont}$ in the naive $a \to 0$ limit, if $b_\pm \to 1$. However, as we will discuss in detail, the naive choice $b_\pm = 1$ produces undesired uv divergences at one loop. The values of $b_\pm$ need to be tuned in such a way that these uv divergences cancel. This is a sign of the fact that the lattice regulator does not manage to reproduce the cancellation of uv divergences that occurs in dimensional regularization.

An important feature of the proposed discretized action and measure is that they are invariant under the full $U(1) \times SU(4)$ internal symmetry group. This is in contrast to the discretization previously presented in [18]. The key ingredient is the use of forward and backward discrete derivatives for both the bosonic and the fermionic part of the action. This is normally avoided for fields that satisfy first-order equations of motion (usually fermions), since it breaks parity and time reversal. In our case, this is not an issue because these symmetries are already broken in the continuum action. In [18], instead, the symmetric derivative was used and, as in lattice QCD, a Wilson-like term was included to cure the resulting doubling problem, while breaking either the $U(1)$ or the $SU(4)$ symmetry.

III. PERTURBATIVE EXPANSION

On the lattice as in the continuum, the perturbative series is obtained by expanding the action around one of its minima. The $SU(4)$ symmetric point (all fields vanish in this point) is a singularity for the action because of the terms proportional to inverse powers of the radial coordinate z. As a consequence the minimum of the action must spontaneously break the internal symmetry. In the continuum an absolute minimum of the action is given by $x = x^* = 0$ and $z^M = \delta^{M6}$, and any other absolute minimum is obtained by acting with the $SU(4)$ symmetry. One can easily check that these minima are also relative minima for the discretized action. We parametrize the fluctuations around the chosen minimum which is the same way as it is done in the continuum [8]

$$z = e^\phi, \quad z^a = e^\phi \frac{y^a}{1 + \frac{1}{4}y^2},$$
$$z^6 = e^\phi \frac{1 - \frac{1}{4}y^2}{1 + \frac{1}{4}y^2}, \quad y^2 = \sum_{a=1}^{5} (y^a)^2, \quad a = 1, \ldots, 5. \quad (3.1)$$

In terms of the new variables ϕ and y^a, the path-integral measure over the z^M fields reads

$$\prod_{M=1}^{6} dz^M = e^{\sum_{s,t} \{6\phi + 5 \log(1 + \frac{y^2}{4})\}} d\phi \prod_{a=1}^{5} dy^a. \quad (3.2)$$

The contribution of the Jacobian determinant above can be conveniently included in the effective action

$$S_{\rm eff} = S_{\rm cusp} - \sum_{s,t} \left\{ 6\phi + 5 \log\left(1 + \frac{y^2}{4}\right) \right\}, \quad (3.3)$$

in terms of which expectation values of observables read

$$\langle A \rangle = \frac{1}{Z_{\rm eff}} \int dx dx^* d\phi d^5 y d^4\theta d^4\theta^\dagger d^4\eta d^4\eta^\dagger e^{-S_{\rm eff}} A. \quad (3.4)$$

Notice that the sum in the contribution to the effective action of the Jacobian determinant does not come with the corresponding a^2 factor, which means that in the naive continuum limit it diverges like a^{-2}. This should not be surprising: in the continuum this term would be proportional to $\delta^2(0)$ which yields a quadratic divergence in a hard-cutoff regularization (but it is set to zero in dimensional regularization).

The perturbative expansion, i.e., the expansion in powers of g^{-1}, is obtained by splitting the action $S_{\rm eff} = S_0 + S_{\rm int}$, where S_0 contains all quadratic terms in the fields with a coefficient proportional to g, and $S_{\rm int}$ contains all other terms. Notice that $S_{\rm int}$ also contains g-independent quadratic terms which come from the expansion of the Jacobian determinant. We focus here on the leading-order quadratic action

$$S_0 = ga^2 \sum_{s,t} \left\{ \left| b_+ \hat{\partial}_t x + \frac{m}{2} x \right|^2 + \left| b_- \hat{\partial}_s x - \frac{m}{2} x \right|^2 + b_+^2 (\hat{\partial}_t y^a)^2 + m b_+ y^a \hat{\partial}_t y^a + (\hat{\partial}_s y^a)^2 \right.$$
$$+ b_+^2 (\hat{\partial}_t \phi)^2 + m b_+ \phi \hat{\partial}_t \phi + (\hat{\partial}_s \phi)^2 + m^2 \phi^2 + 2i(\theta^i \hat{\partial}_t \theta_i + \eta^i \hat{\partial}_t \eta_i)$$
$$\left. + 2i\eta^i (\rho^6)_{ij} \left(b_+ \bar{\partial}_s \theta^j - \frac{m}{2} \theta^j \right) + 2i\eta_i (\rho^{6\dagger})^{ij} \left(b_+ \bar{\partial}_s \theta_j - \frac{m}{2} \theta_j \right) \right\}. \quad (3.5)$$

The propagators are conveniently constructed by going in momentum space. Given a function $f(s,t)$ in coordinate space, we denote by $\tilde{f}(p_0,p_1)$ the corresponding function in momentum space. On the lattice, the two are related by

$$f(s,t) = \int_{-\pi/a}^{\pi/a} \frac{d^2p}{(2\pi)^2} e^{ip_0t+ip_1s}\tilde{f}(p_0,p_1),$$
$$\tilde{f}(p_0,p_1) = \sum_{s,t} a^2 e^{-ip_0t-ip_1s} f(s,t). \tag{3.6}$$

The function $\tilde{f}(p_0,p_1)$ is periodic in both components with period $2\pi/a$, and momentum integrals are always restricted to $-\pi/a < p_k < \pi/a$ which shows explicitly that the lattice effectively enforces a hard cutoff in momentum space. As in the continuum, discrete derivatives are diagonalized in Fourier space, and read

$$\widetilde{\hat{\partial}_\mu f}(p_0,p_1) = i\hat{p}_\mu \tilde{f}(p_0,p_1),$$
$$\widetilde{\bar{\hat{\partial}}_\mu f}(p_0,p_1) = i\hat{p}_\mu^* \tilde{f}(p_0,p_1) \tag{3.7}$$

where we have defined

$$\hat{p}_\mu = e^{i\frac{ap_\mu}{2}} \frac{2}{a}\sin\frac{ap_\mu}{2}. \tag{3.8}$$

Introducing the collective bosonic and fermionic fields

$$\Phi = (\mathrm{Re}x, \mathrm{Im}x, y^1,\ldots,y^5,\phi)^t,$$
$$\Psi = (\theta_1,\ldots,\theta_4,\theta^1,\ldots,\theta^4,\eta_1,\ldots,\eta_4,\eta^1,\ldots,\eta^4), \tag{3.9}$$

the free action (3.5) can be written in momentum space as

$$S_0 = g\int_{-\pi/a}^{\pi/a} \frac{d^2p}{(2\pi)^2}\{\tilde{\Phi}^t(-p)K_{\mathrm{B}}(p)\tilde{\Phi}(p) + \tilde{\Psi}^t(-p)K_{\mathrm{F}}(p)\tilde{\Psi}(p)\}, \tag{3.10}$$

where $K_{\mathrm{B}}(p)$ is an 8×8 diagonal matrix for which the nonvanishing components given by

$$K_B^{(n,n)}(p) = \begin{cases} c_+|\hat{p}_0|^2 + c_-|\hat{p}_1|^2 + \frac{m^2}{2} & \text{if } n=1,2 \\ c_+|\hat{p}_0|^2 + |\hat{p}_1|^2 & \text{if } n=3,\ldots,7\,, \\ c_+|\hat{p}_0|^2 + |\hat{p}_1|^2 + m^2 & \text{if } n=8 \end{cases} \tag{3.11}$$

where we have defined the combinations

$$c_\pm = b_\pm^2 \mp \frac{amb_\pm}{2}, \tag{3.12}$$

and $K_{\mathrm{F}}(p)$ is an 16×16 matrix given by

$$K_F(p) = \begin{pmatrix} 0 & -\hat{p}_0^* I_{4\times4} & -\rho^6(b_+\hat{p}_1 - \frac{im}{2}) & 0 \\ -\hat{p}_0 I_{4\times4} & 0 & 0 & \rho^6(b_+\hat{p}_1 - \frac{im}{2}) \\ \rho^6(b_+\hat{p}_1^* + \frac{im}{2}) & 0 & 0 & -\hat{p}_0^* I_{4\times4} \\ 0 & -\rho^6(b_+\hat{p}_1^* + \frac{im}{2}) & -\hat{p}_0 I_{4\times4} & 0 \end{pmatrix}, \tag{3.13}$$

where we have used the identities $\rho^6 = (\rho^6)^* = -(\rho^6)^t = -\rho^{6\dagger}$ which are valid in the chosen representation (see Appendix A). The two matrices satisfy $K_B^t(p) = K_B(-p)$ and $K_F^t(p) = -K_F(-p)$.

Propagators in momentum space are defined by the entries of the inverse of these matrices up to trivial prefactors. The matrix $K_B(p)$ is diagonal and therefore easily inverted, while the matrix $K_F(p)$ is inverted by observing that

$$K_F(p)^2 = \left(|\hat{p}_0|^2 + c_+|\hat{p}_1|^2 + \frac{m^2}{4}\right) I_{16\times16}. \tag{3.14}$$

The propagators are then easily calculated:

$$\sum_\sigma a^2 e^{-ip\sigma}\langle x(\sigma)x^*(0)\rangle_0 = \frac{1}{g}\frac{1}{c_+|\hat{p}_0|^2 + c_-|\hat{p}_1|^2 + \frac{m^2}{2}}, \tag{3.15}$$

$$\sum_\sigma a^2 e^{-ip\sigma}\langle y^a(\sigma)y^b(0)\rangle_0 = \frac{1}{2g}\frac{\delta^{ab}}{c_+|\hat{p}_0|^2 + c_-|\hat{p}_1|^2}, \tag{3.16}$$

$$\sum_\sigma a^2 e^{-ip\sigma}\langle \phi(\sigma)\phi(0)\rangle_0 = \frac{1}{2g}\frac{1}{c_+|\hat{p}_0|^2 + |\hat{p}_1|^2 + m^2}, \tag{3.17}$$

$$\sum_\sigma a^2 e^{-ip\sigma}\langle \theta_i(\sigma)\theta^j(0)\rangle_0 = -\frac{1}{2g}\frac{\hat{p}_0^*\delta_i^j}{|\hat{p}_0|^2 + c_+|\hat{p}_1|^2 + \frac{m^2}{4}}, \tag{3.18}$$

$$\sum_\sigma a^2 e^{-ip\sigma}\langle \eta_i(\sigma)\eta^j(0)\rangle_0 = -\frac{1}{2g}\frac{\hat{p}_0^*\delta_i^j}{|\hat{p}_0|^2 + c_+|\hat{p}_1|^2 + \frac{m^2}{4}}, \tag{3.19}$$

$$\sum_{\sigma} a^2 e^{-ip\sigma}\langle\theta_i(\sigma)\eta_j(0)\rangle_0 = -\frac{1}{2g}\frac{\rho^6_{ij}(b_+\hat{p}_1 - \frac{im}{2})}{|\hat{p}_0|^2 + c_+|\hat{p}_1|^2 + \frac{m^2}{4}}, \tag{3.20}$$

$$\sum_{\sigma} a^2 e^{-ip\sigma}\langle\theta^i(\sigma)\eta^j(0)\rangle_0 = -\frac{1}{2g}\frac{(\rho^{6\dagger})^{ij}(b_+\hat{p}_1 - \frac{im}{2})}{|\hat{p}_0|^2 + c_+|\hat{p}_1|^2 + \frac{m^2}{4}}, \tag{3.21}$$

where σ is a shorthand notation for (s,t). All other two-point functions vanish. The denominators in the propagators reduce to a particular simple form if we choose $c_\pm = 1$, which is obtained for $b_\pm = \bar{b}_\pm$ with

$$\bar{b}_\pm = \sqrt{1 + \left(\frac{am}{4}\right)^2} \pm \frac{am}{4}. \tag{3.22}$$

As we will see in the following sections, this choice is also the correct one to reproduce continuum results for the observables we consider in this paper.

Let us turn now to the interaction vertices. The expansion of $S_{\rm eff}$ in powers of the fields x, ϕ, y, θ, and η is fairly trivial except for terms involving the forward derivative of z^M. We observe that

$$\begin{aligned}\hat{\partial}_k z^M(x) &= \frac{e^{\phi(x+ae_k)}u^M(x+ae_k) - e^{\phi(x)}u^M(x)}{a} \\ &= \frac{e^{\phi(x)+a\hat{\partial}_k\phi(x)}[u^M(x) + a\hat{\partial}_k u^M(x)] - e^{\phi(x)}u^M(x)}{a} \\ &= e^{\phi(x)}\bigg\{\hat{\partial}_k\phi(x)u^M(x) + \hat{\partial}_k u^M(x) \\ &\quad + \frac{e^{a\hat{\partial}_k\phi(x)} - 1 - a\hat{\partial}_k\phi(x)}{a}u^M(x)\bigg\}. \end{aligned} \tag{3.23}$$

The first two terms in the last expression survive in the naive $a \to 0$ limit, while the third term takes into account the violation of the Leibniz and chain rules at finite lattice spacing. By expanding the exponentials, one obtains terms that have an arbitrary number of powers of $\hat{\partial}_k\phi(x)$ multiplied by explicit powers of a. The number of derivatives and the number of factors of a are related by dimensional analysis. Analogously one finds the following formulas:

$$\hat{\partial}_k u^6(x) = \frac{-2y^c(x)\hat{\partial}_k y^c(x) - a[\hat{\partial}_k y^c(x)]^2}{2\{1+\frac{1}{4}[y^c(x) + a\hat{\partial}_k y^c(x)]^2\}\{1+\frac{1}{4}y(x)^2\}}, \tag{3.24}$$

$$\hat{\partial}_k u^b(x) = \frac{-2y^c(x)\hat{\partial}_k y^c(x) - a[\hat{\partial}_k y^c(x)]^2}{4\{1+\frac{1}{4}[y^c(x) + a\hat{\partial}_k y^c(x)]^2\}\{1+\frac{1}{4}y(x)^2\}}y^b(x). \tag{3.25}$$

Again, by expanding these expressions in y, one obtains terms an arbitrary number of powers of $\hat{\partial}_k y^c(x)$ multiplied by explicit powers of a. The number of derivatives and the number of factors of a are related by dimensional analysis.

By inspecting all terms one sees that, at each order in the perturbative expansion, the interaction Lagrangian density in x is a polynomial of the fields $\Phi(x)$, $\Psi(x)$, their first derivatives $\hat{\partial}\Phi(x)$, $\hat{\partial}\Psi(x)$, $\bar{\partial}\Psi(x)$, the lattice spacing a, and the mass m. We will not write all vertices explicitly; however, the following observations will be useful later on.

(i) Possible vertices are constrained by dimensional analysis: the boson fields have mass dimension 0, the fermion fields have mass dimension 1/2, the discrete derivatives and m have mass dimension 1, and the lattice spacing has mass dimension −1, while vertices must have dimension 2.

(ii) The considered action generates only terms that are proportional to m^0, m^1, or m^2.

(iii) Vertices exist only with 0, 2, or 4 fermion fields.

(iv) The considered action generates only terms that are proportional to a^p with $p \geq -2$. In particular terms proportional to a^{-2} are generated by the Jacobian determinant in Eq. (3.3).

IV. SUPERFICIAL DEGREE OF DIVERGENCE

The goal of this section is to show that the lattice-discretized theory is nonrenormalizable by power counting. To this end, we need to calculate the superficial degree of divergence of the generic Feynman diagram.

Feynman integrands on the lattice are periodic functions in each component of the momenta, with period $2\pi/a$. In particular they are not rational functions as in the continuum, but rational trigonometric functions of the momenta. As a consequence, the problem of establishing an appropriate power counting on the lattice is subtler than in the continuum, and it was solved completely by by Reisz [46] (see also e.g., Refs. [47,48]). Following Reisz, given a function F of the loop momenta $q_{i=1,\ldots,L}$, of the external momenta $p_{i=1,\ldots,E}$, and of the lattice spacing a, the superficial degree of divergence $\deg F$ of the function F is defined by means from its asymptotic behavior

$$F(\lambda q, p; m, a/\lambda) \overset{\lambda\to\infty}{=} C_F\lambda^{\deg F} + O(\lambda^{\deg F - 1}), \tag{4.1}$$

where $C_F \neq 0$. It is straightforward to show that $\deg(FG) = \deg F + \deg G$ and $\deg(F^{-1}) = -\deg F$. As in the continuum, each loop integral contributes with a superficial degree of divergence 2.

Denote by $\tilde{\Theta}_\alpha(p)$ the generic (bosonic or fermionic) field in momentum space. We consider here the connected n-point function in momentum space

$$\langle\tilde{\Theta}_{\alpha_1}(p_1)\cdots\tilde{\Theta}_{\alpha_E}(p_E)\rangle_c = G_\alpha(p)(2\pi)^2\sum_{\vec{n}\in\mathbb{Z}^2}\delta^2\left(\frac{2\pi}{a}\vec{n}-\sum_{i=1}^{E}p_i\right). \quad (4.2)$$

In this formula, we have used the fact that momentum conservation on the lattice takes the form of a delta comb which accounts for the $2\pi/a$ periodicity in momentum space. As in the continuum, the perturbative expansion of $G_\alpha(p)$ has a representation in terms of a sum of Feynman integrals. We introduce the amputated n-point function

$$G^{\mathrm{amp}}_{\alpha_1,\ldots,\alpha_E}(p_1,\ldots,p_E) = \sum_{\beta_1,\ldots,\beta_E} G_{\beta_1,\ldots,\beta_E}(p_1,\ldots,p_E)\prod_{e=1}^{E}[D^{-1}(p_e)]_{\alpha_e\beta_e}, \quad (4.3)$$

where $D(p)$ is the propagator matrix. $G^{\mathrm{amp}}_\alpha(p)$ has a representation in terms of a sum of Feynman integrals in which the external lines have been amputated, and we will refer to them as external legs.

Since lines that do not belong to any loop do not contribute to the superficial degree of divergence, we can restrict our analysis to diagrams that do not have such lines, i.e., one-particle irreducible diagrams. Therefore consider the generic one-particle irreducible Feynman diagram contributing to $G^{\mathrm{amp}}_\alpha(p)$, and let A be the corresponding Feynman integral. We will denote by E_B and E_F the number of external bosonic and fermionic legs respectively, and by I_B and I_F the number of internal bosonic and fermionic lines respectively. Let $l_{i=1,\ldots,I}$ be the momentum flowing in the ith internal line (with $I=I_B+I_F$), and let $p_{e=1,\ldots,E}$ be the momentum flowing in the eth external leg (with $E=E_B+E_F$). The Feynman integral has the general form

$$A=\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}\frac{d^2q_1}{(2\pi)^2}\cdots\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}\frac{d^2q_L}{(2\pi)^2}W(\hat{p},\hat{l};m,a)\prod_{i=1}^{I}D_i(\hat{l}_i;m,a), \quad (4.4)$$

where D_i is the propagator associated to the ith internal line, W is the product of all vertices, and L is the number of loops.

The internal momentum l_i can always be written as $l_i=P_i+Q_i$ where P_i is a linear combination of external momenta, and Q_i is a linear combination of loop momenta. Also, because of one-particle irreducibility, every internal line belongs to a loop, so Q_i is not identically zero. The propagators are functions of $\hat{l}_i$, whose degree of divergence is determined by looking at the asymptotic behavior

$$\hat{l}_i=e^{i\frac{a(P_i+Q_i)}{2}}\frac{2}{a}\sin\frac{a(P_i+Q_i)}{2}\xrightarrow[a\to a/\lambda]{q\to\lambda q}e^{i\frac{a(P_i+\lambda Q_i)}{2\lambda}}\frac{2\lambda}{a}\sin\frac{a(P_i+\lambda Q_i)}{2\lambda} =\lambda\hat{Q}_i+O(\lambda^0). \quad (4.5)$$

It follows easily that the degree of divergence of bosonic and fermionic propagators are the same as in the continuum, i.e.,

$$\deg D_i=\begin{cases}-2 & \text{if } i \text{ is a bosonic line}\\ -1 & \text{if } i \text{ is a fermionic line}\end{cases}. \quad (4.6)$$

The contribution to the degree of divergence of the Feynman integral of all propagators is simply

$$\deg\prod_i D_i=\sum_i\deg D_i=-2I_B-I_F. \quad (4.7)$$

Each vertex contributes to the function W with

(i) some integer power of a and m, coming from the explicit dependence on these two parameters of the interaction Lagrangian, as discussed in Sec. III;
(ii) a product of some $\hat{p}_e$ where p_e is the momentum flowing in the eth amputated external leg, coming from the discrete derivatives acting on fields in vertices which are Wick-contracted to external fields;
(iii) a product of some $\hat{l}_i$ where l_i is the momentum flowing in the ith internal line, coming from the discrete derivatives acting on fields in vertices which are Wick-contracted to fields in other vertices or possibly the same vertex.

Notice that the degree of divergence of $\hat{p}_e$ is determined by the asymptotic behavior

$$\hat{p}_e=e^{i\frac{ap_e}{2}}\frac{2}{a}\sin\frac{ap_e}{2}\xrightarrow[a\to a/\lambda]{q\to\lambda q}e^{i\frac{ap_e}{2\lambda}}\frac{2\lambda}{a}\sin\frac{ap_e}{2\lambda} =\lambda^0p_e+O(\lambda^{-1}). \quad (4.8)$$

Let P_a and P_m be the total number of a and m factors respectively, and let D_E and D_I be the total number of discrete derivative acting on external and internal lines respectively. Using Eqs. (4.5) and (4.8) one derives the asymptotic behavior

$$W(\hat{p},\hat{l};m,a)\xrightarrow[a\to a/\lambda]{q\to\lambda q}W(\lambda^0p,\lambda\hat{q};m,a/\lambda)\left[1+O\left(\frac{1}{\lambda}\right)\right] =\lambda^{D_I-P_a}W(p,\hat{q};m,a)\left[1+O\left(\frac{1}{\lambda}\right)\right], \quad (4.9)$$

which implies

$$\deg W=D_I-P_a. \quad (4.10)$$

The superficial degree of divergence of the considered Feynman integral is given by

$$\deg A = -2L + \deg W + \sum_i \deg D_i$$
$$= 2L + D_I - P_a - 2I_B - I_F. \qquad (4.11)$$

It is also interesting to calculate the mass dimension of the Feynman integral. Notice that

$$\dim D_i = \begin{cases} -2 & \text{if } i \text{ is a bosonic line} \\ -1 & \text{if } i \text{ is a fermionic line} \end{cases}, \qquad (4.12)$$

$$\dim W = P_m - P_a + D_I + D_E, \qquad (4.13)$$

which yields

$$\dim A = 2L + \dim W + \sum_i \dim D_i$$
$$= 2L + P_m - P_a + D_I + D_E - 2I_B - I_F. \qquad (4.14)$$

On the other hand, A is a term in the perturbative expansion of $G_\alpha^{\mathrm{amp}}(p)$. The mass dimension of the amputated n-point function is calculated by observing that the mass dimension of a bosonic field in Fourier space is -2, the mass dimension of a fermionic field in Fourier space is $-3/2$, and the mass dimension of the momentum-conservation delta is -2. Using Eqs. (4.2) and (4.3), one obtains

$$\dim A = \dim G^{\mathrm{amp}}$$
$$= \dim G + 2E_B + E_F = -2E_B - \frac{3}{2}E_F + 2 + 2E_B + E_F$$
$$= 2 - \frac{1}{2}E_F. \qquad (4.15)$$

Combining with Eqs. (4.11) and (4.14) we get our final formula for the degree of divergence of A:

$$\deg A = 2 - \frac{1}{2}E_F - P_m - D_E. \qquad (4.16)$$

This formula shows that the degree of divergence of one-particle irreducible diagrams cannot be larger than two. However, since the degree of divergence does not depend on the number of external bosonic legs, at any loop order the number of divergent diagrams is infinite. This implies that one needs infinitely many counterterms at any loop order to cancel the UV divergences. Without extra constraints on the counterterms one would conclude that the theory is nonrenormalizable.

Since the Feynman diagrams with $P_a = 0$ are the same ones that appear in a continuum regularization, the same conclusion holds in this case. However it is known that, in dimensional regularization, nontrivial cancellations of uv divergences happen, effectively showing that the uv counterterms are highly constrained. Even though some general argument exists for the uv finiteness of the Green-Schwarz $AdS_5 \times S^5$ string before any gauge fixing, we are not aware of a complete derivation of such constraints in the gauge-fixed theory, parametrized around the null-cusp background.

The question of whether a similar cancellation of UV divergences happens in the lattice discretization is a legitimate one. We will see with a couple of examples that unfortunately this does not work as well as in dimensional regularization: a certain amount of fine-tuning is needed in order to reproduce the continuum results.

V. SOME CALCULATIONS

A. Cusp anomaly

The partition function of the lattice-discretized theory is given by

$$Z_{\mathrm{cusp}} = \int d\Phi d\Psi e^{-S_{\mathrm{eff}}} \qquad (5.1)$$

in terms of the collective fields Φ and Ψ that are defined in Eq. (3.9) and of the effective action S_{eff} is defined in Eq. (3.3). Since the logarithm of the partition function is extensive, a complete calculation is performed by considering a finite worldsheet with area V_2. At this point the integral defining the partition function is finite and can be analytically calculated order by order in the perturbative expansion. Finally one can define the free energy density in the infinite-volume limit, i.e.,

$$\rho(g, m, a) = -\lim_{V_2\to\infty} \frac{1}{V_2} \log Z_{\mathrm{cusp}}(g, m, a, V_2). \qquad (5.2)$$

As in every statistical system, the free energy is defined up to an additive constant and only free-energy differences have physical meaning. It is also interesting to notice that rescaling the integration measure in each lattice point $d\Phi(s,t)d\Psi(s,t) \to \beta d\Phi(s,t)d\Psi(s,t)$ is equivalent to rescaling $Z_{\mathrm{cusp}} \to \beta^{\frac{V_2}{a^2}} Z_{\mathrm{cusp}}$, i.e., to redefining $\rho \to \rho - a^{-2}\log\beta$. This shows that quadratic divergences in the free energy are immaterial and can be removed by rescaling the integration measure. We propose to identify the following derivative of the free-energy density with the cusp anomalous dimension

$$f(g, m, a) = \frac{4}{m}\frac{\partial}{\partial m}\rho(g, m, a). \qquad (5.3)$$

It is straightforward to show that this derivative coincides with the standard definition in dimensional regularization, and it is also free from the normalization ambiguity.[5]

[5] Notice that in Eq. (1.1) the parameter m is set equal to 1. In the continuum, the m dependence can be reintroduced by simple dimensional analysis, yielding $Z_{\mathrm{cusp}} = e^{-\frac{f(g)}{8}m^2 V_2}$ and consequently $\rho = \frac{f(g)}{8}m^2$, which is indeed consistent with the definition (5.3).

At leading order the path integral defining the partition function reduces to a Gaussian integral, which yields

$$\rho(g,m,a) = g\frac{m^2}{2} - \frac{4}{a^2}\log(2\pi) + \frac{1}{2}\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\log\left[\frac{\det K_B(q)}{\det K_F(q)}\right] + O(g^{-1}). \tag{5.4}$$

The determinants are calculated from the explicit expressions of K_B and K_F given in Sec. III, yielding

$$\frac{\det K_B(q)}{\det K_F(q)} = \frac{(c_+|\hat{q}_0|^2 + c_-|\hat{q}_1|^2 + \frac{m^2}{2})^2(c_+|\hat{q}_0|^2 + |\hat{q}_1|^2)^5(c_+|\hat{q}_0|^2 + |\hat{q}_1|^2 + m^2)}{(|\hat{q}_0|^2 + c_+|\hat{q}_1|^2 + \frac{m^2}{4})^8}. \tag{5.5}$$

The calculation of ρ and its small-a expansion can be reduced to the following general integral:

$$\begin{aligned}\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\log a^2\left\{\sum_i(1+a\delta_i)|\hat{p}_i|^2 + M^2\right\}\\ = \frac{1}{a^2}I_{-2}^{(0,0)} + \frac{\delta_1+\delta_2}{2a} - \frac{\delta_1^2+\delta_2^2}{4} + \frac{(\delta_1-\delta_2)^2}{4\pi}\\ - \frac{M^2}{4\pi}\log(aM)^2 + M^2 I_0^{(0,0)} + O(a\log a),\end{aligned} \tag{5.6}$$

where $I_{-2}^{(0,0)} \simeq 1.166$ and $I_0^{(0,0)} \simeq 0.355$ are numerical constants. The derivation of the above asymptotic expansion and the precise definition of the constants are given in Appendix B 1. By using the above asymptotic expansion, with the convention $c_\pm = 1 + am\delta c_\pm$, after a lengthy but straightforward calculation, one gets

$$\begin{aligned}\rho(g,m,a) = g\frac{m^2}{2} - \frac{4\log(2\pi)}{a^2} + \frac{m\delta c_-}{2a} - \frac{3m^2\log 2}{8\pi} - \frac{m^2\delta c_-^2}{4}\\ + \frac{m^2\delta c_-(\delta c_- - 2\delta c_+)}{4\pi} + O(a\log a) + O(g^{-1}),\end{aligned} \tag{5.7}$$

and, correspondingly, for the cusp anomaly:

$$\begin{aligned}f(g,m,a) = 4g + \frac{\delta c_-}{2am} - \frac{3\log 2}{\pi} - 2\delta c_-^2 + \frac{2\delta c_-(\delta c_- - 2\delta c_+)}{\pi}\\ + O(a\log a) + O(g^{-1}).\end{aligned} \tag{5.8}$$

Notice that with the naive choice $b_\pm = 1$, which corresponds to $\delta c_\pm = \mp 1/2$, the cusp anomaly contains a linear divergence. On the other hand, with the special choice $b_\pm = \bar{b}_\pm$ which corresponds to $c_\pm = 1$ and $\delta c_\pm = 0$, the linear divergence is canceled, and we obtain the same result as in dimensional regularization:

$$f(g,m,0) = 4g - \frac{3\log 2}{\pi} + O(g^{-1}). \tag{5.9}$$

B. One-point functions

Let us turn to the one-point functions of the perturbative fields. Notice that $\langle x\rangle = 0$ because of the $U(1)$ symmetry, and $\langle y^a\rangle = 0$ because of the $SO(5) \subset SO(6) \simeq SU(4)$ which leaves the perturbative vacuum invariant. ϕ is the only field with a nonvanishing one-point function, which has been calculated in dimensional regularization [8,43,49]. This one-point function, as well as any n-point function of bare fields, is not expected to be uv finite. In fact it is known that $\langle\phi\rangle$ is uv divergent in dimensional regularization, and we will see that it turns out to be uv divergent also in the lattice regularization. The interest in this one-point function lies in the fact that it appears as a subdiagram in any other n-point function, and ultimately its uv divergence contributes to any physical observable. We will give an example of this mechanism in the next subsection.

There are two classes of vertices contributing to the one-point function of ϕ: single-field vertices coming from the measure

$$S_\phi = -6\sum_{s,t}\phi, \tag{5.10}$$

and three-field vertices coming from the action

$$\begin{aligned}S_{\phi\bullet\bullet} = g\sum_{s,t}a^2\Big\{-4\phi\left|b_-\hat{\partial}_s x - \frac{m}{2}x\right|^2 + c_+\hat{\partial}_t\phi\hat{\partial}_t(\phi^2) + \hat{\partial}_s\phi\hat{\partial}_s\phi^2 - 4\phi(\hat{\partial}_s\phi)^2\\ + 2c_+\hat{\partial}_t y^a\hat{\partial}_t(\phi y^a) - c_+\hat{\partial}_t\phi\hat{\partial}_t(y^2) + 2\hat{\partial}_s y^a\hat{\partial}_s(\phi y^a) - \hat{\partial}_s\phi\hat{\partial}_s(y^2) - 4\phi(\hat{\partial}_s y^a)^2\\ - 4i\phi\left[\eta^i(\rho^6)_{ij}\left(b_+\bar{\partial}_s\theta^j - \frac{m}{2}\theta^j\right) + \eta_i(\rho^{6\dagger})^{ij}\left(b_+\bar{\partial}_s\theta_j - \frac{m}{2}\theta_j\right)\right]\Big\}.\end{aligned} \tag{5.11}$$

Notice that the insertion of S_ϕ produces a tree-level diagram, while the insertion of $S_{\phi\bullet\bullet}$ produces a one-loop diagram. However, because of the mismatch in the power of g in S_ϕ and $S_{\phi\bullet\bullet}$, all these diagrams contribute to the same order in g, yielding

$$\begin{aligned}\langle\phi\rangle = &\frac{3}{gm^2a^2} + \frac{2}{gm^2}\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{c_-|\hat{q}_1|^2 + \frac{m^2}{4}}{c_+|\hat{q}_0|^2 + c_-|\hat{q}_1|^2 + \frac{m^2}{2}}\\ &- \frac{1}{2gm^2}\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{c_+|\hat{q}_0|^2 - |\hat{q}_1|^2}{c_+|\hat{q}_0|^2 + |\hat{q}_1|^2 + m^2}\\ &- \frac{5}{2gm^2}\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{c_+|\hat{q}_0|^2 - |\hat{q}_1|^2}{c_+|\hat{q}_0|^2 + |\hat{q}_1|^2}\\ &- \frac{8}{gm^2}\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{c_+|\hat{q}_1|^2 + \frac{m^2}{4}}{|\hat{q}_0|^2 + c_+|\hat{q}_1|^2 + \frac{m^2}{4}}\\ &+ O(g^{-2}). \end{aligned} \tag{5.12}$$

With the special choice $b_\pm = \bar{b}_\pm$, i.e., $c_\pm = 1$, one can use the symmetry of the integrals under $p_0 \leftrightarrow p_1$ exchange to simplify

$$\begin{aligned}\langle\phi\rangle &= -\frac{1}{g}\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{1}{|\hat{q}|^2 + \frac{m^2}{4}} + O(g^{-2})\\ &= \frac{1}{g}\left\{\frac{1}{4\pi}\log\frac{(am)^2}{4} + \frac{1}{4\pi} - I_0^{(0,0)} + O(a\log a)\right\}\\ &\quad + O(g^{-2}), \end{aligned} \tag{5.13}$$

which is logarithmically divergent, as one can explicitly see by using the asymptotic expansion given in Appendix B 2. The definition of the numerical constant $I_0^{(0,0)} \simeq 0.355$ is given in Appendix B 1. Notice that the measure, fermion-loop and x-loop contributions are separately quadratically divergent, and the cancellation of these divergences is highly nontrivial.

In the general case $c_\pm = 1 + (am)\delta c_\pm$ where $\delta c_\pm = O(a^0)$, one can again use the asymptotic expansions given in Appendix B 2, and after a lengthy calculation one gets

$$\begin{aligned}\langle\phi\rangle = &\frac{1}{g}\left\{\frac{-8\delta c_+ + \delta c_-}{\pi a} + \frac{1}{4\pi}\log\frac{(am)^2}{4}\right.\\ &\left. + \frac{1}{4\pi} - I_0^{(0,0)} + \frac{8\delta c_+^2 - \delta c_-^2}{2\pi} + O(a\log a)\right\}\\ &+ O(g^{-2}). \end{aligned} \tag{5.14}$$

Notice that the naive choice $b_\pm = 1$ corresponds to the choice $\delta c_\pm = \mp 1/2$ which yields indeed a linear divergence for $\langle\phi\rangle$:

$$\langle\phi\rangle = \frac{1}{g}\left\{\frac{9}{2\pi a} + O(\log a)\right\} + O(g^{-2}). \tag{5.15}$$

C. Two-point function

We turn now to the two-point function of the field x, which we calculate at one loop. We will use the two-point function to extract the dispersion relation of the x particle propagating on the worldsheet. In dimensional regularization and at one loop [43], both the two-point function and the dispersion relation turn out to be uv finite without any need of renormalization. We will see that this is true also at one loop in lattice perturbation theory, provided that one has chosen $c_\pm = 1$. The naive choice $b_\pm = 1$ generates uv divergences in the dispersion relation. Whether these divergences can be eliminated with a renormalization procedure is a valid question.

There are two classes of vertices contributing to the two-point function of x at one loop: three-field vertices

$$\begin{aligned}S_{xx^*\bullet} = g\sum_{s,t}a^2\bigg\{&-4\phi\left|b_-\hat{\partial}_s x - \frac{m}{2}x\right|^2\\ &+ 2\eta^i\rho^6_{ij}\eta^j\left(b_-\hat{\partial}_s x - \frac{m}{2}x\right)\\ &- 2\eta_i(\rho^{6\dagger})^{ij}\eta_j\left(b_-\hat{\partial}_s x^* - \frac{m}{2}x^*\right)\bigg\}, \end{aligned} \tag{5.16}$$

and four-field vertices

$$S_{xx^*\bullet\bullet} = 8g\sum_{s,t}a^2\phi^2\left|b_-\hat{\partial}_s x - \frac{m}{2}x\right|^2, \tag{5.17}$$

combined to give Feynman diagrams with the three different topologies illustrated in Fig. 1. Notice that the tadpole contribution will be proportional to $\langle\phi\rangle$.

On general grounds one sees that the two-point function has the following form:

$$\begin{aligned}\langle\tilde{x}(p)x^*(0)\rangle = \frac{1}{g}\bigg\{&c_+|\hat{p}_0|^2 + c_-|\hat{p}_1|^2 + \frac{m^2}{2}\\ &+ \frac{1}{g}\left(c_-|\hat{p}_1|^2 + \frac{m^2}{4}\right)\Pi_a(p) + O(g^{-2})\bigg\}^{-1}. \end{aligned} \tag{5.18}$$

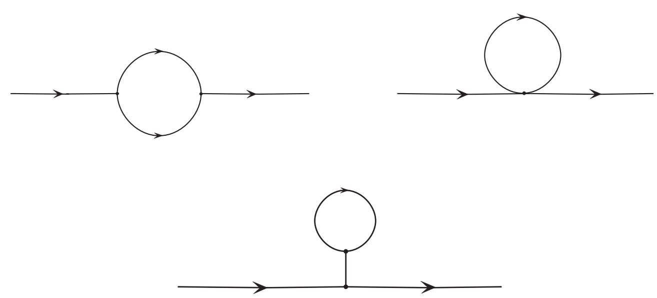

FIG. 1. Topologies of diagrams contributing to the two-point function at one loop.

The factor $(c_-|\hat{p}_1|^2 + \frac{m^2}{4})$ comes from the fact that, in all interaction vertices, x always appears in the combination $(b_-\hat{\partial}_s x - \frac{m}{2}x)$ or its complex conjugate. The function $\Pi_a(p)$ has a representation in terms of amputated Feynman diagrams and it is explicitly given by

$$\begin{aligned}\Pi_a(p) = &-4g\langle\phi\rangle + 4\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{1}{c_+|\hat{q}_0|^2 + |\hat{q}_1|^2 + m^2} \\ &- 8\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{c_-|\hat{q}_1|^2 + \frac{m^2}{4}}{c_+|\hat{q}_0|^2 + c_-|\hat{q}_1|^2 + \frac{m^2}{2}}\frac{1}{c_+|\widehat{p+q_0}|^2 + |\widehat{p+q_1}|^2 + m^2} \\ &- 8\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{\hat{q}_0}{|\hat{q}_0|^2 + c_+|\hat{q}_1|^2 + \frac{m^2}{4}}\frac{\widehat{p+q}_0^*}{|\widehat{p+q_0}|^2 + c_+|\widehat{p+q_1}|^2 + \frac{m^2}{4}}.\end{aligned} \tag{5.19}$$

All integrals in the above formula are logarithmically divergent, while the term proportional to $\langle\phi\rangle$ contains in general a linear divergence. Up to terms that vanish in the $a\to 0$ limit, one can replace $c_\pm = 1$ in the above integrals, obtaining the simpler expression

$$\begin{aligned}\Pi_a(p) = &-4g\langle\phi\rangle + 4\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{1}{|\hat{q}|^2 + m^2} - 8\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{|\hat{q}_1|^2 + \frac{m^2}{4}}{|\hat{q}|^2 + \frac{m^2}{2}}\frac{1}{|\widehat{p+q}|^2 + m^2} \\ &- 8\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{\hat{q}_0}{|\hat{q}|^2 + \frac{m^2}{4}}\frac{\widehat{p+q}_0^*}{|\widehat{p+q}|^2 + \frac{m^2}{4}} + O(a\log a).\end{aligned} \tag{5.20}$$

As in the continuum, the leading divergence of the above integrals does not depend on the external momentum; therefore, the subtracted quantity $\Delta\Pi_a(p) = \Pi_a(p) - \Pi_a(0)$ has a finite $a\to 0$ limit given by the corresponding continuum integrals, i.e.,

$$\begin{aligned}\Delta\Pi_0(p) = &-8\int_{-\infty}^{\infty}\frac{d^2q}{(2\pi)^2}\frac{q_1^2 + \frac{m^2}{4}}{q^2 + \frac{m^2}{2}}\left\{\frac{1}{(p+q)^2 + m^2} - \frac{1}{q^2 + m^2}\right\} \\ &-8\int_{-\infty}^{\infty}\frac{d^2q}{(2\pi)^2}\frac{q_0}{|\hat{q}|^2 + \frac{m^2}{4}}\left\{\frac{p_0 + q_0}{(p+q)^2 + \frac{m^2}{4}} - \frac{q_0}{q^2 + \frac{m^2}{4}}\right\} \\ &+ O(a\log a),\end{aligned} \tag{5.21}$$

while all the divergences are contained in

$$\Pi_a(0) = -4g\langle\phi\rangle - 4\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{1}{|\hat{q}|^2 + \frac{m^2}{4}} + \frac{1}{\pi} + O(a\log a), \tag{5.22}$$

where we have used the symmetry of the integrals under $p_0 \leftrightarrow p_1$ exchange to simplify them.

With the choice $c_\pm = 1$, using Eq. (5.13) one immediately sees that all divergences cancel and $\Pi_0(0) = 1/\pi$. The two-point function is finite in the continuum limit and

$$\lim_{a\to 0}\langle\tilde{x}(p)x^*(0)\rangle = \frac{1}{g}\left\{p^2 + \frac{m^2}{2} + \frac{1}{g}\left(p_1^2 + \frac{m^2}{4}\right)\Pi_0(p) + O(g^{-2})\right\}^{-1}. \tag{5.23}$$

The two-point function has poles at $p_0 = \pm iE(p_1)$ for every value of p_1, where $E(p_1)$ is the energy of a single excitation with the quantum numbers of the field x, propagating on the worldsheet with momentum p_1. In the continuum limit this is found to be

$$\begin{aligned}E(p_1)^2 &= p_1^2 + \frac{m^2}{2} + \frac{1}{g}\left(p_1^2 + \frac{m^2}{4}\right)\Pi_0\left(\sqrt{p_1^2 + \frac{m^2}{2}}, p_1\right) + O(g^{-2}) \\ &= p_1^2 + \frac{m^2}{2} - \frac{1}{gm^2}\left(p_1^2 + \frac{m^2}{4}\right)^2 + O(g^{-2}),\end{aligned} \tag{5.24}$$

where we have used the on-shell value of Π_0 (B29). The obtained dispersion relation coincides[6] with the result in [43].

However in the general case $c_\pm = 1 + (am)\delta c_\pm$ where $\delta c_\pm = O(a^0)$, $\Pi_a(0)$ and $E(p_1)$ inherit the linear divergence from $\langle\phi\rangle$. Using Eq. (5.14) one obtains

[6]To compare with [43], notice that one has to redefine the worldsheet coordinates, resulting in square masses of the fluctuations rescaled with a factor of 4.

$$\Pi_a(0) = \frac{32\delta c_+ - 4\delta c_-}{\pi a} + \frac{1 - 16\delta c_+^2 + 2\delta c_-^2}{\pi} + O(a \log a). \quad (5.25)$$

For instance, for the naive choice $b_\pm = 1$, which corresponds to $\delta c_\pm = \mp 1/2$, one obtains for the dispersion relation

$$E(p_1)^2 = p_1^2 + \frac{m^2}{2} + \frac{1}{g}\left(p_1^2 + \frac{m^2}{4}\right)\left[-\frac{18}{\pi a} + O(\log a)\right] + O(g^{-2}). \quad (5.26)$$

It is interesting to notice that, once we have set $b_\pm = 1$, the divergence in the dispersion relation cannot be eliminated by renormalizing the remaining available parameters, i.e., g and m. In other words, the choice $b_\pm = 1$ is not stable under renormalization. On the other hand, if one allows the coefficients $b_\pm$ to be renormalized along with m and g, then the divergences in the dispersion relation are eliminated e.g., by choosing

$$b_+ = 1 + \frac{1}{g_R}\frac{\frac{am_R}{8}}{2 + \frac{am_R}{2}}\left(\Pi_a(0) - \frac{1}{\pi}\right), \quad (5.27)$$

$$b_- = 1 - \frac{1}{g_R}\frac{1 + \frac{5am_R}{8}}{2 + \frac{am_R}{2}}\left(\Pi_a(0) - \frac{1}{\pi}\right), \quad (5.28)$$

$$m^2 = m_R^2\left[1 + \frac{1}{2g_R}\left(\Pi_a(0) - \frac{1}{\pi}\right)\right], \quad (5.29)$$

$$g = g_R[1 + O(g^{-1})]. \quad (5.30)$$

This choice yields a dispersion relation in the continuum limit of the same form as Eq. (5.24), except that the mass m needs to be replaced by its renormalized counterpart m_R. One could also see that the one-loop renormalization of the coupling constant can be chosen in such a way that the cusp anomaly be finite. With this discussion we do not want to imply that the chosen lattice theory is renormalizable (we do not know this). However we conclude that, if the lattice theory is renormalizable, then it is not sufficient to renormalize m and g, one also needs to introduce extra coefficients in the action and either fine-tune their tree-level values, or renormalize them.

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APPENDIX A: ρ MATRICES

In the action (2.5) the matrices ρ^M appear, which are off-diagonal blocks of the six-dimensional Dirac matrices in chiral representation

$$\gamma^M \equiv \begin{pmatrix} 0 & \rho^{M\dagger} \\ \rho^M & 0 \end{pmatrix} = \begin{pmatrix} 0 & (\rho^M)^{ij} \\ (\rho^M)_{ij} & 0 \end{pmatrix}, \quad (A1)$$

$$\rho^M_{ij} = -\rho^M_{ji}, \qquad (\rho^{M\dagger})^{il}\rho^N_{lj} + (\rho^{N\dagger})^{il}\rho^M_{lj} = 2\delta^{MN}\delta^i_j. \quad (A2)$$

The two off-diagonal blocks, carrying upper and lower indices respectively, are related by $(\rho^M)^{ij} = -(\rho^M_{ij})^* \equiv (\rho^M_{ji})^*$, so that the block with upper indices, $(\rho^{M\dagger})^{ij}$, is the conjugate transpose of the block with lower indices. A possible explicit representation is

$$\rho^1_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \rho^2_{ij} = \begin{pmatrix} 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix},$$

$$\rho^3_{ij} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\rho^4_{ij} = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{pmatrix}, \qquad \rho^5_{ij} = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix},$$

$$\rho^6_{ij} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (A3)$$

The $SO(6)$ generators are built out of the ρ matrices via

$$\rho^{MN\,i}{}_j \equiv \frac{1}{2}[(\rho^{M\dagger})^{il}\rho^N_{lj} - (\rho^{N\dagger})^{il}\rho^M_{lj}]. \quad (A4)$$

APPENDIX B: ASYMPTOTIC EXPANSIONS OF RELEVANT INTEGRALS

1. Cusp anomaly

We want to calculate the small-a expansion of the following integral:

$$F(a) = \int_{-\pi/a}^{\pi/a} \frac{d^2q}{(2\pi)^2} \log\left\{a^2\left[\sum_i \alpha_i |\hat{p}_i|^2 + M^2\right]\right\} = \frac{1}{a^2}\log(aM)^2 + \int_{-\pi/a}^{\pi/a} \frac{d^2q}{(2\pi)^2} \log\frac{\sum_i \alpha_i|\hat{p}_i|^2 + M^2}{M^2}. \tag{B1}$$

Using the Schwinger-time representation of the logarithm, i.e.,

$$\log\frac{\sum_i \alpha_i|\hat{p}_i|^2 + M^2}{M^2} = -\int_0^\infty \frac{ds}{s}\{e^{-s[a^2\sum_i \alpha_i|\hat{q}_i|^2 + (aM)^2]} - e^{-s(aM)^2}\}, \tag{B2}$$

and the change of variable $z = aq$, we obtain

$$F(a) = \frac{1}{a^2}\log(aM)^2 - \frac{1}{a^2}\int_0^\infty \frac{ds}{s} e^{-s(aM)^2}\{K(\alpha_1 s)K(\alpha_2 s) - 1\}, \tag{B3}$$

with the definition

$$K(s) = \int_{-\pi}^{\pi} \frac{dz}{2\pi} e^{-4s\sin^2\frac{z}{2}} = \frac{1}{\sqrt{4\pi s}} + O(s^{-2}). \tag{B4}$$

The function $K(s)$ is infinitely differentiable in $[0, \infty)$, and its large-s asymptotic behavior is obtained by means of a standard saddle-point analysis. We split the integral in Eq. (B3) in two regions, and we write

$$F(a) = \frac{1}{a^2}\log(aM)^2 - \frac{1}{a^2}\int_0^1 ds e^{-s(aM)^2}\frac{K(\alpha_1 s)K(\alpha_2 s) - 1}{s} - \frac{1}{a^2}\int_1^\infty \frac{ds}{s} e^{-s(aM)^2} K(\alpha_1 s)K(\alpha_2 s) + \frac{1}{a^2}\Gamma(0, (aM)^2). \tag{B5}$$

We also introduce the auxiliary function

$$G(s) = \int_s^\infty \frac{d\sigma}{\sigma} K(\alpha_1\sigma)K(\alpha_2\sigma) = \frac{1}{4\pi\sqrt{\alpha_1\alpha_2}s} + O(s^{-1}). \tag{B6}$$

Thanks to the asymptotic behavior (B4), the above integral is finite and its large-s asymptotic behaviour easily follows.

In terms of the auxiliary function, and after integration by parts, the integral in the large-s region in Eq. (B5) reads

$$\begin{aligned} -\frac{1}{a^2}\int_1^\infty \frac{ds}{s} e^{-s(aM)^2} K(\alpha_1 s)K(\alpha_2 s) &= \frac{1}{a^2}\int_1^\infty ds e^{-s(aM)^2} G'(s) \\ &= -\frac{1}{a^2}G(1) + M^2\int_1^\infty ds e^{-s(aM)^2} G(s) \\ &= -\frac{e^{-(aM)^2}}{a^2}G(1) + \frac{M^2}{4\pi\sqrt{\alpha_1\alpha_2}}\Gamma(0, (aM)^2) \\ &\quad + M^2\int_1^\infty ds e^{-s(aM)^2}\left\{G(s) - \frac{1}{4\pi\sqrt{\alpha_1\alpha_2}s}\right\}. \end{aligned} \tag{B7}$$

In the last step we have added and subtracted the leading asymptotic behavior (B6). Bringing together Eqs. (B5) and (B7), and expanding for small a, we obtain

$$F(a) = \frac{1}{a^2}I_{-2}(\alpha) - \frac{M^2}{4\pi\sqrt{\alpha_1\alpha_2}}\log(aM)^2 + M^2 I_0(\alpha) + O(a^2\log a), \tag{B8}$$

with the definitions

$$I_{-2}(\alpha) = -\gamma - \int_0^1 ds \frac{K(\alpha_1 s)K(\alpha_2 s) - 1}{s} - G(1), \tag{B9}$$

$$I_0(\alpha) = -\frac{\gamma}{4\pi\sqrt{\alpha_1\alpha_2}} + \int_0^1 ds K(\alpha_1 s)K(\alpha_2 s) + G(1) + \int_1^\infty ds\left\{G(s) - \frac{1}{4\pi\sqrt{\alpha_1\alpha_2}s}\right\}. \tag{B10}$$

By using the definition of $G(s)$ and after some straightforward algebra, one also obtains the representation

$$I_{-2}(\alpha) = -\gamma - \int_0^1 ds \frac{K(\alpha_1 s)K(\alpha_2 s) - 1}{s} - \int_1^\infty \frac{ds}{s} K(\alpha_1 s)K(\alpha_2 s), \tag{B11}$$

$$I_0(\alpha) = \frac{1-\gamma}{4\pi\sqrt{\alpha_1\alpha_2}} + \int_0^1 ds K(\alpha_1 s)K(\alpha_2 s) + \int_1^\infty ds\left\{K(\alpha_1 s)K(\alpha_2 s) - \frac{1}{4\pi\sqrt{\alpha_1\alpha_2}s}\right\}. \tag{B12}$$

We are interested in Eq. (B8) with the special choice $\alpha_i = 1 + a\delta_i$. By Taylor expanding Eq. (B8) in $a\delta_i$, we obtain

$$F(a) = \frac{1}{a^2} I_{-2}^{(0,0)} + \frac{\delta_1+\delta_2}{a} I_{-2}^{(1,0)} + \frac{\delta_1^2+\delta_2^2}{2} I_{-2}^{(2,0)} + \delta_1\delta_2 I_{-2}^{(1,1)} - \frac{M^2}{4\pi}\log(aM)^2 + M^2 I_0(1,1) + O(a\log a), \quad \text{(B13)}$$

with the definitions

$$I_{-2}^{(0,0)} = I_{-2}(1,1) = -\gamma - \int_0^1 ds \frac{[K(s)]^2-1}{s} - \int_1^\infty \frac{ds}{s}[K(s)]^2, \quad \text{(B14)}$$

$$I_{-2}^{(1,0)} = \frac{\partial I_{-2}}{\partial \alpha_1}(1,1) = -\int_0^\infty ds K'(s)K(s) = -\frac{1}{2}\int_0^\infty ds \frac{d}{ds}[K(s)]^2 = \frac{1}{2}, \quad \text{(B15)}$$

$$I_{-2}^{(1,1)} = \frac{\partial^2 I_{-2}}{\partial\alpha_1\partial\alpha_2}(1,1) = -\int_0^\infty ds s[K'(s)]^2 = -\frac{1}{2\pi}, \quad \text{(B16)}$$

$$I_{-2}^{(2,0)} = \frac{\partial^2 I_{-2}}{\partial\alpha_1^2}(1,1) = -\int_0^\infty ds s K''(s)K(s) = \int_0^\infty ds K'(s)\frac{d}{ds}[sK(s)] = \int_0^\infty ds[K'(s)]^2 + \int_0^\infty ds K'(s)K(s) = \frac{1}{2\pi} - \frac{1}{2}, \quad \text{(B17)}$$

$$I_0^{(0,0)} = I_0(1,1) = \frac{1-\gamma}{4\pi} + \int_0^1 ds[K(s)]^2 + \int_1^\infty ds\left\{[K(s)]^2 - \frac{1}{4\pi s}\right\}. \quad \text{(B18)}$$

The unknown integrals can be calculated numerically, yielding $I_{-2}^{(0,0)} \simeq 1.166$ and $I_0^{(0,0)} \simeq 0.355$.

2. One-point function

By taking the derivative with respect to M^2 of both sides of Eq. (B8), and by using the definition (B1), we obtain

$$\int_{-\pi/a}^{\pi/a} \frac{d^2q}{(2\pi)^2} \frac{1}{\sum_i \alpha_i |\hat{p}_i|^2 + M^2} = -\frac{1}{4\pi\sqrt{\alpha_1\alpha_2}}\log(aM)^2 - \frac{1}{4\pi\sqrt{\alpha_1\alpha_2}} + I_0(\alpha) + O(a^2\log a). \quad \text{(B19)}$$

Specializing to $\alpha_i = 1 + a\delta_i$ and Taylor-expanding in $a\delta_i$, we obtain

$$\int_{-\pi/a}^{\pi/a} \frac{d^2q}{(2\pi)^2} \frac{1}{\sum_i (1+a\delta_i)|\hat{p}_i|^2 + M^2} = -\frac{1}{4\pi}\log(aM)^2 - \frac{1}{4\pi} + I_0^{(0,0)} + O(a\log a). \quad \text{(B20)}$$

By applying the differential operator $\sum_i \beta_i \frac{\partial}{\partial\alpha_i}$ to both sides of Eq. (B8), and by using the definition (B1), we obtain

$$\int_{-\pi/a}^{\pi/a} \frac{d^2q}{(2\pi)^2} \frac{\sum_i \beta_i |\hat{p}_i|^2}{\sum_i \alpha_i |\hat{p}_i|^2 + M^2} = \frac{1}{a^2}\sum_i \beta_i \frac{\partial I_{-2}}{\partial\alpha_i}(\alpha) + \frac{M^2(\beta_1\alpha_2+\beta_2\alpha_1)}{8\pi(\alpha_1\alpha_2)^{3/2}}\log(aM)^2 + M^2\sum_i \beta_i \frac{\partial I_0}{\partial\alpha_i}(\alpha) + O(a^2\log a). \quad \text{(B21)}$$

Specializing to $\alpha_i = 1 + a\delta_i$ and Taylor-expanding in $a\delta_i$, we obtain

$$\int_{-\pi/a}^{\pi/a} \frac{d^2q}{(2\pi)^2} \frac{\sum_i \beta_i |\hat{p}_i|^2}{\sum_i (1+a\delta_i)|\hat{p}_i|^2 + M^2} = \frac{\beta_1+\beta_2}{a^2} I_{-2}^{(1,0)} + \frac{\beta_1\delta_1+\beta_2\delta_2}{a} I_{-2}^{(2,0)} + \frac{\beta_1\delta_2+\beta_2\delta_1}{a} I_{-2}^{(1,1)} + \frac{\beta_1\delta_1^2+\beta_2\delta_2^2}{2} I_{-2}^{(3,0)} + \frac{\beta_1\delta_2^2+\beta_2\delta_1^2+2(\beta_1+\beta_2)\delta_1\delta_2}{2} I_{-2}^{(2,1)} + \frac{M^2(\beta_1+\beta_2)}{8\pi}\log(aM)^2 + M^2(\beta_1+\beta_2) I_0^{(1,0)} + O(a\log a), \quad \text{(B22)}$$

with the following definitions:

$$I_{-2}^{(2,1)}(\alpha) = \frac{\partial^3 I_{-2}}{\partial\alpha_1^2\partial\alpha_2}(1,1) = -\int_0^\infty ds s^2 K''(s)K'(s) = -\frac{1}{2}\int_0^\infty ds s^2 \frac{d}{ds}[K'(s)]^2 = \int_0^\infty ds s[K'(s)]^2 = \frac{1}{2\pi}, \quad \text{(B23)}$$

$$I_{-2}^{(3,0)}(\alpha)=\frac{\partial^3 I_{-2}}{\partial\alpha_1^2}(1,1)=-\int_0^\infty ds s^2 K'''(s)K(s)=\int_0^\infty dsK''(s)\frac{d}{ds}[s^2K(s)]=2\int_0^\infty dssK''(s)K(s)+\int_0^\infty dss^2K''(s)K'(s)$$
$$=-2\int_0^\infty dsK'(s)\frac{d}{ds}[sK(s)]-\frac{1}{2\pi}=-2\int_0^\infty dsK'(s)K(s)-2\int_0^\infty dss[K'(s)]^2-\frac{1}{2\pi}=1-\frac{3}{2\pi}, \tag{B24}$$

$$\begin{aligned} I_0^{(1,0)} &= -\frac{1-\gamma}{8\pi}+\int_0^1 ds\, sK'(s)K(s)+\int_1^\infty ds\left\{sK'(s)K(s)+\frac{1}{8\pi s}\right\} \\ &= -\frac{1-\gamma}{8\pi}+\frac{1}{2}\int_0^1 ds\, s\frac{d}{ds}[K(s)]^2+\frac{1}{2}\int_1^\infty ds\, s\frac{d}{ds}\left\{[K(s)]^2-\frac{1}{4\pi s}\right\} \\ &= \frac{\gamma}{8\pi}-\frac{1}{2}\int_0^1 ds[K(s)]^2-\frac{1}{2}\int_1^\infty ds\left\{[K(s)]^2-\frac{1}{4\pi s}\right\} \\ &= -\frac{1}{2}I_0^{(0,0)}+\frac{1}{8\pi}, \end{aligned} \tag{B25}$$

in addition to the definitions given in the previous subsection.

3. Calculation of $\Delta\Pi_0$

The finite, continuum integral defined in the main text for the two-point function in Eq. (5.21) can be rewritten as the dimensionless integral

$$\begin{aligned} \Delta\Pi_0(p) = &-8\int\frac{d^2q}{(2\pi)^2}\left(\frac{q_1^2+1}{(q^2+2)((\tilde{p}+q)^2+4)}-\frac{1}{2}\frac{1}{q^2+4}\right) \\ &-8\int\frac{d^2q}{(2\pi)^2}\left(\frac{q_0^2+\tilde{p}_0q_0}{(q^2+1)((\tilde{p}+q)^2+1)}-\frac{1}{2}\frac{1}{q^2+1}\right)-\frac{1}{\pi} \end{aligned} \tag{B26}$$

by rescaling the momenta $\tilde{p}=\frac{m}{2}p$ and manipulating the integrals. Using standard Feynman parametrization, this can be recast as the integral

$$\Delta\Pi_0(p)=\frac{-1}{\pi}\int_0^1 dx\left(\frac{(p_0^2-p_1^2)x^2+2\tilde{p}_1^2x-(\tilde{p}^2+1)}{1+\tilde{p}^2x(1-x)}+\frac{(\tilde{p}_1^2-\tilde{p}_0^2)(1-x)^2}{4-2x+\tilde{p}^2x(1-x)}\right)-\frac{1}{\pi}. \tag{B27}$$

Reverting to $p=\frac{2}{m}\tilde{p}$ and evaluating this at the on-shell value, we obtain

$$\Delta\Pi_0\left(p;p^2=\frac{m^2}{2}\right)=\frac{-1}{m^2}\left(p_1^2+\frac{m^2}{4}\right)-\frac{1}{\pi}. \tag{B28}$$

Notice that for the choice $c_\pm=1$ where $\Pi_0(0)=\frac{1}{\pi}$, we recover the continuum limit found in [43],

$$\Pi_0\left(p;p^2=\frac{m^2}{2}\right)\Big|_{c_\pm=1}=\frac{-1}{m^2}\left(p_1^2+\frac{m^2}{4}\right). \tag{B29}$$

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