

Open-closed correspondence of K-theory and cobordism

Ralph Blumenhagen and Niccolò Cribiori

*Max-Planck-Institut für Physik (Werner-Heisenberg-Institut),
Föhringer Ring 6, 80805 München, Germany*

E-mail: blumenha@mpp.mpg.de, cribiori@mpp.mpg.de

ABSTRACT: Non-trivial K-theory groups and non-trivial cobordism groups can lead to global symmetries which are conjectured to be absent in quantum gravity. Inspired by open-closed string duality, we propose a correspondence between the two groups, which can be considered as the physical manifestation of a generalisation of the classic Conner-Floyd isomorphism. The picture is exemplified by the relations between KO-groups and Spin-cobordisms and between K-groups and Spin^c-cobordisms. We suggest that global symmetries related by such an isomorphism are generically gauged. Indeed, by combining non-torsion K-theory and cobordism groups, we recover tadpole cancellation conditions well known from type I string theory and F-theory. For torsion charges, another possibility opens up.

KEYWORDS: String and Brane Phenomenology, D-Branes, Global Symmetries

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1 Introduction

In the swampland program (see [1, 2] for reviews), the absence of global symmetries in quantum gravity (QG) plays a central role. A recent incarnation of this is the so-called cobordism conjecture [3], based on the observation that a non-vanishing cobordism group would lead to a global symmetry. Therefore, the conjecture says that any physically consistent configuration in quantum gravity must not carry any cobordism charge.¹

This is quite a generic proposal and, since it is not clear yet what the correct QG-structure is, it calls for more non-trivial tests and for relations to other maybe better understood properties of string theory. First, the conjecture is of topological nature and as such it does not rely on supersymmetry, the structure on which many established results in

¹Note that cobordism groups also play a prominent role in the computation of Dai-Freed anomalies, see e.g. [4, 5].

string theory are based. Second, in view of the fact that string theory, as we know it, is a background dependent formulation of QG, it is remarkable that due to the very definition of cobordisms even topology changing configurations are captured. Therefore, the conjecture really involves processes that one would expect in quantum gravity, though describing them (only) at the topological level. Third, the cobordism groups of purely geometric structure, like e.g. Ω^{Spin} , which are studied in the mathematical literature, are expected to be related to the closed string (gravity) sector of the theory. Indeed, strong results have recently been derived by exploiting the cobordism conjecture (for Pin^\pm -structure) in theories with 16 supercharges and a small number of compact dimensions [6–8]. This lent support for the so-called string lamppost principle.

A much better studied fundamental principle of quantum gravity is holography, as it is manifest e.g. in the AdS/CFT correspondence. In QG, there is a deep connection between gauge theories on the boundary and gravity theories in the bulk, which on the level of the string world-sheet is known as open-closed string duality. For example, a D-brane can be described by open strings ending on it, or equivalently as a coherent (boundary) state of closed string excitations. This is not just a qualitative picture, but is realised on the quantitative level, e.g. in the computation of annulus diagrams. Thus, one may ask whether there is a reflection of this duality into the cobordism conjecture.

Indeed, there is an apparent very similar open string story, namely the classification of D-branes via K-theory [9]. In this context, one talks about topologically equivalent vector bundles over manifolds wrapped by D-branes, and supersymmetry is not important either. In fact, the starting point is actually given by non-supersymmetric brane-antibrane pairs equipped with vector bundles. Moreover, vector bundles of different rank are in the same K-theory equivalence class, i.e. the open string configuration is allowed to undergo dramatic topology changes. An interesting question in this context is whether the K-theory charge need to vanish on a compact space.² In [11], a brane probe argument was given that suggested it, while in [12] the non-cancellation of the K-theory charge was related to a violation of the de Sitter swampland conjecture (see also [13]).³ If the K-theory charge is a global charge, it should better vanish in QG. Although one might object that K-theory is defined for branes carrying vector bundles on fixed background spaces, and thus gravity is actually not manifest.

It is the purpose of this work to argue that open-closed string duality suggests a correspondence between (open string) K-theory groups and appropriate (closed string) cobordism groups. In sections 2 and 3, we set the stage by briefly reviewing the salient features of the just mentioned structures and their relevance in string theory. In section 4, we first provide an heuristic physical reason why we expect K-theory and cobordism to be related and then search the mathematical literature to find a confirmation in terms of a classic theorem, called the Conner-Floyd theorem. We will see that for D-branes in type I and type II string theory, a generalisation of this theorem is needed that was formulated and proven by Hopkins-Hovey in 1992.

²It was pointed out to us by the referee that [10] implies that in fact all K-theory charges are gauged.

³In his talk at the “String Phenomenology 2019” conference, Iñaki García-Etxebarria was arguing that in the type I string a non-vanishing K-theory charge (for the non-BPS D7-brane) actually leads to a Dai-Freed anomaly.

n	0	1	2	3	4	5	6	7	8	9	10
$\widetilde{KO}(S^n)$	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	0	\mathbb{Z}	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2
D-brane	$D9$	$\widehat{D8}$	$\widehat{D7}$	$\widehat{D6}$	$D5$	$\widehat{D4}$	$\widehat{D3}$	$\widehat{D2}$	$D1$	$\widehat{D0}$	$\widehat{D(-1)}$

Table 1. K-theory classes and type I D-branes. Non-BPS D-branes are indicated with a hat.

Being now related to cobordisms, in section 5 we analyse what the cobordism conjecture implies for D-brane configurations carrying non-vanishing K-theory charge. A global symmetry in quantum gravity must either be broken or gauged. We propose that the physical significance of the Hopkins-Hovey theorem is that it provides the setting where global symmetries induced by non-vanishing cobordism/K-theory groups are in principle gauged by continuous or discrete RR forms. We show that tadpoles cancellation conditions of type I and type IIB orientifold/F-theory compactifications can be derived from a bottom-up perspective, by exploiting the proposed correspondence between cobordism and K-theory. For torsion charges, it turns out that K-theory and cobordism (most likely) decouple from one another in the tadpoles we consider. We leave a more comprehensive analysis of the effects of torsion for future studies.

This makes it evident that K-theory and cobordism are like two sides of the same coin, the first giving the D-brane contribution and the second the topological bulk contribution to tadpole cancellation conditions. From this perspective, the initial appearance of global symmetries on each side is just an artefact of ignoring the gauge field and looking at an incomplete set of charged objects.

2 D-branes and K-theory

Since the work [9], it is a well established fact that D-branes in string theory are not classified by homology but by K-theory. In type II string theory this difference is not really visible, but for the type I string it becomes apparent. Indeed, in type I it is known that there exist boundary states in CFT that break supersymmetry but are still free of any tachyonic instability, at least as long as one has only one single such non-BPS brane. Placing two branes on top of each other leads to a tachyonic mode inducing an instability of the system. As pointed out by Witten, this behaviour is captured by a non-trivial \mathbb{Z}_2 -valued K-theory class.

Concretely, the stable Dp -branes of the $d = 10$ type I string are given by the (reduced) K-theory groups $\widetilde{KO}(S^n)$ with $n = d - p - 1$. These are listed in table 1, from which the appearance of stable non-BPS branes is evident. They are related to \mathbb{Z}_2 -valued reduced K-theory classes.

One can think of these classes as describing bound states of $D9-\overline{D9}$ branes (with non-trivial vector bundles), upon wrapping them on a compact sphere S^n . The reduced K-theory group appears when one considers only initial $D9$ -brane tadpole cancelling configurations, i.e. $N_{D9} - N_{\overline{D9}} = 0$, ignoring the always present 32 $D9$ -branes in the background. This means that also potential tachyons arising from open strings stretched between these

n	0	1	2	3	4	5	6	7	8	9
$\widetilde{K}(S^n)$	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0
D-brane	D9	-	D7/D8	-	D5/D6	-	D3/D4	-	D1/D2	-

Table 2. K-theory classes and type II D-branes. There are no stable non-BPS D-branes.

32 D9-branes and the condensing $D9-\overline{D9}$ brane pairs are not captured. The latter indeed appear for the non-BPS $\widehat{D8}$ - and $\widehat{D7}$ -branes (see e.g. [14] for fate of this brane).

For the type IIB string, D-branes are classified by the unitary reduced K-theory classes $\widetilde{K}(S^n)$. Given that $\widetilde{K}(S^0) = \mathbb{Z}$ and $\widetilde{K}(S^1) = 0$, and by exploiting Bott periodicity, one finds the usual D-brane spectrum without any extra stable non-BPS brane. The story for type IIA D-branes is similar, even if a bit more subtle.⁴ It turns out [9, 15] that they are classified by $\widetilde{K}_1(S^n) = \widetilde{K}(S^{n+1})$ and the full spectrum can be recovered again thanks to Bott periodicity. The additional dimension is reminiscent of M-theory, even if a clear connection is not obvious to us.⁵ Type II D-branes and the corresponding K-theory groups are listed in table 2.

For later purposes, let us recall here a bit more about K-theory. One defines the higher reduced K-theory groups of vector bundles over a manifold X as

$$\widetilde{K}_n(X) := \widetilde{K}(\Sigma_n X), \tag{2.1}$$

where ΣX is the reduced suspension of X , which is homeomorphic to the smash product of X with S^1 ,

$$\Sigma X \cong S^1 \wedge X, \quad \Sigma_n X \cong S^n \wedge X. \tag{2.2}$$

Employing the relation $K(X) = \widetilde{K}(X \sqcup pt)$ and recalling that the 0-sphere S^0 consists of two points, one consequently has

$$K_n(pt) = \widetilde{K}_n(pt \sqcup pt) = \widetilde{K}_n(S^0) = \widetilde{K}(\Sigma_n S^0) = \widetilde{K}(S^n), \tag{2.3}$$

and similarly for the \widetilde{KO} groups. By considering not only direct sums but also tensor products of vector bundles, $K_n(pt)$ is actually a ring.

It is known that the usual BPS Dp -brane is coupled to a higher-form RR gauge field C_{p+1} , enjoying a continuous p -form gauge symmetry $C_{p+1} \rightarrow C_{p+1} + d\lambda_p$. As a consequence, there will be a conserved $(p+1)$ -form (electric) current satisfying

$$d \star F_{p+2} = \star J_{p+1}. \tag{2.4}$$

That the current is conserved follows from the fact that it is co-closed, $d \star J_{p+1} = 0$. For a static configuration, the current can be written as

$$\star J_{p+1} = \sum_i Q_i \delta^{(d-p-1)}(\Delta_{p+1,i}), \tag{2.5}$$

⁴To avoid confusion, let us emphasize that the relation to Spin^c cobordism that we propose in the following section is only meant for the type IIB case. The type IIA case is not yet understood.

⁵Recently, in [16] charge quantisation in M-theory has been related to cobordism. This is further motivated by several non-trivial consistency checks [17–19].

where the sum is over all Dp -branes carrying charge Q_i . The δ -function on the right-hand-side is the Poincare dual of the $(p+1)$ -cycle $\Delta_{p+1,i}$ wrapped by the brane. Integrating (2.4) over a compact space implies that the sum over all charges on it has to vanish.

Non-BPS $\widehat{D}p$ -branes are not coupled to higher $(p+1)$ -form RR gauge fields, but one can still assign a \mathbb{Z}_k K-theory charge to them. The framework of differential K-theory [10, 20] suggests to associate to this charge a discrete gauge symmetry, such that the total charge would vanish when integrated over a compact manifold.⁶

It is one of the most fundamental swampland conjectures that in quantum gravity there should be no global symmetries, i.e. they should either be broken or gauged. Therefore, one can consider a non-vanishing K-theory class as a topological obstruction of having such an object in a string theory background. For instance, placing a single non-BPS $\widehat{D}7$ -brane on S^2 has a $\widetilde{KO}(S^2) = \mathbb{Z}_2$ obstruction and therefore would not be allowed.

These arguments sound very similar to the recent proposal of a vanishing cobordism class in quantum gravity [3]. In fact the purpose of this work is to present a physical reason for a deep connection between the two. Before we discuss such a connection, let us review the cobordism conjecture.

3 Global symmetries and cobordism groups

The set-up is very similar to that of the last section, however there are a priori no branes involved. One considers d -dimensional string theory (i.e. $d = 10$) and places it on an n -dimensional compact space M , thus leaving $D = d - n$ non-compact dimensions. In [3], it was argued that a non-trivial cobordism group $\Omega_n^{\widetilde{QG}}$ of quantum gravity configurations with n compact dimensions implies a $(d - n - 1)$ -form global symmetry. Since there should be no global symmetries, a non-vanishing cobordism class is an obstruction to a valid quantum gravity background. Thus, if possible, such a background should be extended to remove the obstruction and arrive at vanishing cobordism classes. This can be achieved in two ways:

- **Breaking of the symmetry:** it is obtained by introducing non-trivial $(d - n - 1)$ -dimensional defects so that the global current is not conserved, i.e.

$$0 \neq d \star J_{d-n} = I_{n+1} = \sum_{\text{def},j} \delta^{(n+1)}(\Delta_{d-n-1,j}), \tag{3.1}$$

where the δ -functions are the Poincare dual of the $(d - n - 1)$ -cycles wrapped by the defects. Then, elements in the kernel of the map

$$\Omega_n^{\widetilde{QG}} \rightarrow \Omega_n^{\widetilde{QG}+\text{defects}} \tag{3.2}$$

are killed in the full theory. Here, the refined cobordism group on the right hand side is obtained from the original cobordism by mapping onto those manifolds which do not include defects.

⁶We thank the referee for pointing this out to us. We also thank Iñaki García-Etxebarria for discussions on this point.

- **Gauging of the symmetry:** it means that the cobordism class of a consistent configuration is the 0-element, i.e. $[M] = 0 \in \Omega_n^{\widetilde{QG}}$. In other words, the total charge on a compact manifold has to vanish. One says that elements in the cokernel of the map

$$\Omega_n^{\widetilde{QG}; \text{gauging}} \rightarrow \Omega_n^{\widetilde{QG}} \tag{3.3}$$

are co-killed. Here, the cobordism for the refined approximation maps into the original cobordism by forgetting the gauge field.

The first option is discussed at length in [3] and has been made further explicit in concrete non-supersymmetric string backgrounds with uncanceled NS-NS tadpoles in [21, 22]. In a few words, taking the backreaction of such configurations into account, one gets a spontaneous compactification of initially flat directions [23, 24] of finite size, whose boundary then supports the required defect. Besides, one could wonder if and how the cobordism group, which is abelian by construction, can detect non-abelian defects. For the prototype example of (p, q) 7-branes in F-theory, this puzzle has been resolved in [25] (see also [26] for a more general discussion).

The focus of the present work is on the second option, namely the gauging of the symmetry. Indeed, we will show that by exploiting the proposed correspondence between cobordism and K-theory we are led to known tadpole constraints in string theory.

3.1 Spin cobordism

The main question of course is what the full $\Omega_n^{\widetilde{QG}}$ actually is. In mathematics, one usually considers only cobordism groups of geometric structure, where the manifolds in question are often taken from a special class. Some of them have been discussed in [3] as well. For instance, one can consider cobordisms for Spin-manifolds, i.e. oriented manifolds on which one can globally define (uncharged) fermions. In this case, one says that two n -dimensional Spin-manifolds M and N are cobordant if there exists an $(n+1)$ -dimensional Spin-manifold W such that $\partial W = M \sqcup \overline{N}$, where \overline{N} is the orientation reversed manifold. The equivalence classes $[M]$ make up the Spin-cobordism group Ω_n^{Spin} , where the addition is defined via disjoint unions, i.e. $[M] + [N] = [M \sqcup N]$. Similarly to K-theory, via the cartesian product of two manifolds, $M \times N$, the Spin-cobordism theory

$$\Omega_*^{\text{Spin}} = \bigoplus_{n=0}^{\infty} \Omega_n^{\text{Spin}} \tag{3.4}$$

can be given the structure of a ring. The cobordism groups Ω_n^{Spin} and their generators $\Sigma_{n,i}$ for $0 \leq n \leq 8$ are listed in table 3.

3.2 Spin^c cobordism

Analogously, one can introduce the cobordism ring for Spin^c-manifolds, i.e. oriented manifolds on which charged fermions can be globally defined (see e.g. [27]). To make chiral

n	0	1	2	3	4	5	6	7	8
Ω_n^{Spin}	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	0	\mathbb{Z}	0	0	0	\mathbb{Z}^2
$\Sigma_{n,i}$	pt ⁺	S_p^1	$S_p^1 \times S_p^1$	0	$K3$	0	0	0	$\mathbb{B} \oplus \mathbb{H}\mathbb{P}^2$

Table 3. Cobordism classes for Spin-manifolds and their generators $\Sigma_{n,i}$. Here, by \mathbb{Z}^n we mean $\mathbb{Z} \oplus \dots \oplus \mathbb{Z}$, n times. pt⁺ is the positive oriented point, S_p^1 indicates the circle with periodic boundary conditions for the fermions, \mathbb{B} the so-called Bott space and $\mathbb{H}\mathbb{P}^2$ is the hyperbolic projective space.

indices well-defined, one needs an uplift of the second Stiefel-Whitney class from \mathbb{Z}_2 -cohomology to \mathbb{Z} -valued cohomology. In other words, a complex Spin^c-manifold Σ also carries a line-bundle \mathcal{L} so that

$$c_1(\mathcal{L}) - \frac{1}{2}c_1(\Sigma) \in H^2(\Sigma, \mathbb{Z}). \tag{3.5}$$

For Σ not Spin, a half-integer gauge flux can still make it Spin^c. Clearly, every Spin-manifold is also a Spin^c-manifold upon choosing the trivial line bundle, $\mathcal{L} = \mathcal{O}$.

The cobordism groups for Spin^c are listed in [3], where up to $n = 4$ also the generators are given. In order to extend that list to $n = 6$, let us recall how one finds the two generators of $\Omega_4^{\text{Spin}^c} = \mathbb{Z} \oplus \mathbb{Z}$. These can be found by first noticing that the two cobordism invariants in this case are the Todd class $\text{td}_4 = (c_2(\Sigma) + c_1^2(\Sigma))/12$ and $c_1^2(\Sigma)$ [28]. The question then is how to identify two complex twofolds generating the full charge lattice $\mathbb{Z} \oplus \mathbb{Z}$. This is not too difficult, as by scanning some simple projective spaces one finds

space	td ₄	c ₁ ²
$\mathcal{M}_1 = \mathbb{P}^2$	1	9
$\mathcal{M}_2 = \mathbb{P}^1 \times \mathbb{P}^1$	1	8

Therefore, $[\mathcal{G}_1] = -8[\mathcal{M}_1] + 9[\mathcal{M}_2]$ carries charge (1, 0) and $[\mathcal{G}_2] = [\mathcal{M}_1] - [\mathcal{M}_2]$ charge (0, 1) so that they can be identified as the diagonal generators of $\mathbb{Z} \oplus \mathbb{Z}$. Note that in cobordism one has $[\mathcal{G}_1] = [dP_9]$, with dP_9 being the rational elliptic surface. For $n = 6$, we show in appendix A that the two cobordism invariants listed in [28] are

$$\text{td}_6 = c_2(\Sigma) c_1(\Sigma)/24, \quad c_1^3(\Sigma)/2. \tag{3.6}$$

We notice that the projective spaces $\{\mathbb{P}^2 \times \mathbb{P}^1, (\mathbb{P}^1)^3\}$ only generate a $\mathbb{Z} \oplus 3\mathbb{Z}$ sublattice. We found instead that the following simple non-toric spaces work.

space	td ₆	c ₁ ³ /2
$\mathcal{N}_1 = \begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{matrix} 2 \\ 2 \\ 2 \end{matrix}$	1	6
$\mathcal{N}_2 = \begin{matrix} \mathbb{P}^3 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix} \begin{matrix} 3 \\ 1 \\ 1 \end{matrix}$	1	5

n	0	2	4	6
$\Omega_n^{\text{Spin}^c}$	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}^2	\mathbb{Z}^2
$\Sigma_{n,i}$	pt^+	\mathbb{P}^1	$\mathbb{P}^2 \oplus (\mathbb{P}^1)^2$	$\mathbb{P}^2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} \oplus \mathbb{P}^3 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Table 4. Cobordism classes and generators $\Sigma_{n,i}$ for Spin^c manifolds. Here, $(\mathbb{P}^k)^l = \mathbb{P}^k \times \dots \times \mathbb{P}^k$, l times and the line bundle is generically chosen as $\mathcal{L}^2 = H_\Sigma$, with H_Σ denoting the hyperplane bundle.

Therefore, $[\mathcal{H}_1] = -5[\mathcal{N}_1] + 6[\mathcal{N}_2]$ carries charge $(1, 0)$ and $[\mathcal{H}_2] = [\mathcal{N}_1] - [\mathcal{N}_2]$ charge $(0, 1)$, so that they can be identified as the diagonal generators of $\mathbb{Z} \oplus \mathbb{Z}$. A connected representative for $[\mathcal{H}_1]$ is the threefold $[dP_9 \times \mathbb{P}^1]$. Up to $n = 6$, we arrive at the cobordism classes shown in table 4.

Comparing them to the K-theory classes from the previous section, there is a striking similarity between $\widetilde{K}(S^n)$ and $\Omega_n^{\text{Spin}^c}$ as well as between $\widetilde{KO}(S^n)$ and Ω_n^{Spin} : the same classes are vanishing and also the same \mathbb{Z}_k factors appear. It is only the number of \mathbb{Z}_k -factors that differs for a couple of cases. In particular, from the tables one deduces that $\widetilde{KO}(S^n) \subseteq \Omega_n^{\text{Spin}}$ and $\widetilde{K}(S^n) \subseteq \Omega_n^{\text{Spin}^c}$. If there is such a relation between K-theory and cobordism, then one might expect it to be known in the mathematical literature. Indeed, in section 4 we will first present a physical argument for such a relation and then state a generalisation of the classic Conner-Floyd theorem, which was formulated and proven by Hopkins-Hovey and which provides the expected mathematical relation.

3.3 String cobordism

For later purposes, let us also mention the so-called string cobordism groups, Ω_n^{String} . They are relevant in the context of the heterotic string and for its S-dual type I string. One is also including in the structure the Kalb-Ramond B -field and its non-trivial Bianchi identity

$$dH = \frac{p_1(\Sigma)}{2} = -\frac{1}{16\pi^2} \text{tr}(R \wedge R), \tag{3.7}$$

where $p_1(\Sigma)$ denotes the first Pontryagin class of the Spin manifold Σ . A potential gauge field contributing like $p_1(F)$ is set to zero in (3.7). As a consequence of the Bianchi identity, the theory can only be well-defined for a trivialisation of $p_1/2$, i.e. one only includes manifolds satisfying $p_1(\Sigma)/2 = 0$ in cohomology. This is referred to as a string structure and the corresponding cobordism groups, $\Omega_n^{\text{String}}(pt)$, are called string cobordism. In table 5, we list the lower dimensional string cobordism groups together with their generators [3].

The interplay between cobordism groups and non-trivial Bianchi identities will be developed further in section 5.

4 K-theory — Cobordism isomorphism

First, let us present an intuitive physical argument for the existence of the aforementioned relation. We stress that this is an heuristic picture, and as such should not be overestimated, nevertheless it will put us on the right track.

n	0	1	2	3	4	5	6
Ω_n^{String}	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2
$\Sigma_{n,i}$	pt^+	S_p^1	$S_p^1 \times S_p^1$	S_H^3	0	0	$S_H^3 \times S_H^3$

Table 5. Cobordism classes for String-manifolds and their generators $\Sigma_{n,i}$. Here, S_H^3 means that the S^3 support a unit three-form flux H .

Recall that D-branes can be seen in two ways, as it is also explicit in their CFT description. One can either consider them as the end-points of open strings, whose quantisation describes fluctuations of the D-branes giving rise to massless gauge fields. This picture is clearly the one underlying the K-theory classification of D-branes, which is about equivalence classes of vector bundles on manifolds wrapped by the branes.

However, D-branes can also be considered as boundary states in the closed string Hilbert space, which makes it manifest that they couple to closed string excitations. At the massless level, this means that D-branes backreact on the geometry, the dilaton and also the other p -form fields present in effective string theories. Thus, D-branes can be considered as specific sourced solutions in supergravity. However, it is well known that a Dp -brane can also be considered as a solitonic magnetically charged object. As such, it carries a topological obstruction against decaying to the vacuum. In the following, we will refer to this as the (dual) bulk picture of a Dp -brane.

Due to open string - closed string duality, also called loop channel - tree channel equivalence in the CFT context, both descriptions should be equivalent. In our framework, this implies that for every obstruction to the decay of a (non-BPS) Dp -brane to the vacuum, classified e.g. in type I by $\widetilde{KO}(S^n)$ with $n = d - p - 1$, there should exist an equivalent obstruction in the dual bulk picture of the brane. The observed equivalence of K-theory and cobordism suggests that this obstruction is described by an appropriate non-vanishing cobordism class Ω_n^{QG} . Consistently, both obstructions give rise to a global $(d - n - 1)$ -form symmetry. As alluded to previously, this picture is of course a simplification and we will see later that one actually needs more than a single D-brane or O-plane to end up with a compact bulk manifold representing a given cobordism class.

Since for the construction of the $\widetilde{KO}(S^n)$ classes the 32 background $D9$ -branes were ignored, for a (non)-BPS brane in type I we expect the backreaction to only involve the closed string fields. The fermions, i.e. the gravitinos and dilatinos, are neutral so that, upon ignoring the RR fields and the dilaton, it is suggestive to consider the Spin-cobordism groups Ω_n^{Spin} , at least in first approximation.⁷ On the other hand, it was pointed out in [3] that F-theory compactifications on elliptically fibered Calabi-Yau spaces induce a Spin^c -structure on the base manifolds.⁸ This suggests a relation between the branes classified by $\widetilde{K}(S^n)$ classes and Spin^c -cobordism groups, $\Omega_n^{\text{Spin}^c}$.

⁷By ignoring the RR two-form field C_2 , at this stage we are not sensitive to potential issues with the type I Bianchi identity $dF_3 \sim \text{tr}(R \wedge R) - \text{tr}(F \wedge F)$. The latter would suggest to look at Ω_n^{String} rather than Ω_n^{Spin} .

⁸The actual relation of F-theory and Spin^c is more involved, as the relevant groups are cobordism groups of a manifold with an elliptic fibration, which are difficult to compute.

This intuitive reasoning supports the existence of a map relating cobordism to K-theory, with the following consequences:

- If there exists a \mathbb{Z}_k obstruction in some K-theory class $\widetilde{KO}(S^n)$, then also Ω_n^{Spin} should contain this obstruction.
- However, Ω_n^{Spin} could admit other equivalence classes that are not related to D-branes. Therefore, one cannot simply conclude that $\widetilde{KO}(S^n)$ is isomorphic to Ω_n^{Spin} .
- Nevertheless, as for AdS/CFT, one might expect that open-closed string duality is a one-to-one map, so that there should exist a way to make the above map an isomorphism.

In fact, in mathematics a quite advanced framework is known that substantiates the above physical intuition.

4.1 Atiyah-Bott-Shapiro orientation

The work of Atiyah-Bott-Shapiro (ABS) [29] set the ground for a relation between Spin-cobordism and K-theory. In particular, it contains the definition of ring homomorphisms

$$\alpha^c : \Omega_*^{\text{Spin}^c} \rightarrow K_*(pt), \quad \alpha : \Omega_*^{\text{Spin}} \rightarrow KO_*(pt). \tag{4.1}$$

Let us discuss these two maps in more detail.

Definition of α^c . When restricted to a fixed grade n , the homomorphism α^c is the Todd genus, i.e. for an n -dimensional manifold M we have

$$\alpha_n^c([M]) = \text{Td}(M) \in \mathbb{Z}. \tag{4.2}$$

Later, we will also use that the Todd genus is the integral over the top Todd class, i.e. $\text{Td}(M) = \int_M \text{td}_n(M)$ with $\text{td}_n(M) \in H^n(M)$. Since it actually acts on equivalence classes, for it to be well defined the Todd genus should better be a cobordism invariant. Moreover, it satisfies $\text{Td}(M \sqcup N) = \text{Td}(M) + \text{Td}(N)$ and $\text{Td}(M \times N) = \text{Td}(M) \cdot \text{Td}(N)$. Given that the Todd genus evaluates to one on any complex projective space, for the generators of $\Omega_n^{\text{Spin}^c}$ listed in table 4 one finds $\alpha_n^c([\Sigma_{n,i}]) = 1$, so that the homomorphism is surjective. The kernel of the map α_4^c is spanned by the class $[\mathbb{P}^2] - [\mathbb{P}^1 \times \mathbb{P}^1] \in \Omega_4^{\text{Spin}^c}$, while for the map α_6^c we have seen that it is spanned by

$$[\mathcal{G}_2] = [\mathcal{M}_1] - [\mathcal{M}_2] \in \Omega_6^{\text{Spin}^c}. \tag{4.3}$$

Definition of α . The story for α is similar, even if slightly more involved. Let us first look at $KO_*(pt)$. As a consequence of Bott periodicity, it can be formulated as the ring $\mathbb{Z}[s, k, b]$ modulo appropriate equivalence relations, namely [30]

$$KO_*(pt) = \mathbb{Z}[s, k, b]/(2s, s^3, kb, k^2 - 4b), \tag{4.4}$$

where s, k, b are elements of order 1, 4 and 8 respectively. An explicit definition of the map α is given in [31]

$$\alpha_n([M]) = \begin{cases} \hat{A}(M) & n = 8m, \\ \hat{A}(M)/2 & n = 8m + 4, \\ \dim H \pmod 2 & n = 8m + 1, \\ \dim H^+ \pmod 2 & n = 8m + 2, \\ 0 & \text{otherwise,} \end{cases} \quad (4.5)$$

where $\hat{A}(M)$ is the \hat{A} -genus and H (H^+) the space of (positive) harmonic spinors.

Recall from table 3 that the generator of $\Omega_1^{\text{Spin}} = \mathbb{Z}_2$ is the circle S_p^1 with periodic boundary condition for fermions. The generator of $\Omega_2^{\text{Spin}} = \mathbb{Z}_2$ is $S_p^1 \times S_p^1$. The generator of $\Omega_4^{\text{Spin}} = \mathbb{Z}$ is the Kummer surface $K3$. The generators of $\Omega_8^{\text{Spin}} = \mathbb{Z} \oplus \mathbb{Z}$ are the hyperbolic projective space $\mathbb{H}\mathbb{P}^2$ and the so-called Bott space \mathbb{B} . Therefore, the action of α on the Spin-cobordism generators is

$$\alpha_n(\Sigma_{n,i}) = 1 \text{ for } n \leq 7, \quad \alpha_8([\mathbb{B}]) = 1, \quad \alpha_8([\mathbb{H}\mathbb{P}^2]) = 0. \quad (4.6)$$

One realizes that α_n is surjective and furthermore the Kernel is non-trivial only for $n \geq 8$.

As mentioned, open-closed string duality is expected to be a one-to-one map so that there should exist a way to promote the above homomorphisms to isomorphisms. In case the map α is surjective, one can simply divide by its kernel to get an isomorphism

$$\Omega_n^{\text{Spin}}/\ker(\alpha) \cong \widetilde{KO}(S^n) \quad (4.7)$$

and similarly for α^c .⁹ Hence, we propose that this isomorphism is the mathematically precise reflection of stringy open-closed duality at the topological level.

4.2 Hopkins-Hovey isomorphism

So far, we were considering only D-branes related to the K-theory groups $K_n(pt)$ and $KO_n(pt)$. However, from our picture of open-closed duality, one would expect that the relation between K-theory and cobordism holds for type I or type II D-branes on any compact space X .

Indeed, there exists an extension of the isomorphism from the previous section to a more general case, which is known as the Hopkins-Hovey isomorphism. The latter is a generalisation of a classic theorem by Conner-Floyd [32], which in turn is a special case of the Landweber exact functor theorem [33]. To be concrete, Hopkins-Hovey proved that the maps

$$\Omega_*^{\text{Spin}}(X) \otimes_{\Omega_*^{\text{Spin}}} KO_* \rightarrow KO_*(X), \quad (4.8)$$

$$\Omega_*^{\text{Spin}^c}(X) \otimes_{\Omega_*^{\text{Spin}^c}} K_* \rightarrow K_*(X) \quad (4.9)$$

⁹The existence of this isomorphism follows trivially from Theorem C and Proposition 3.3 of [30] for $X = pt$.

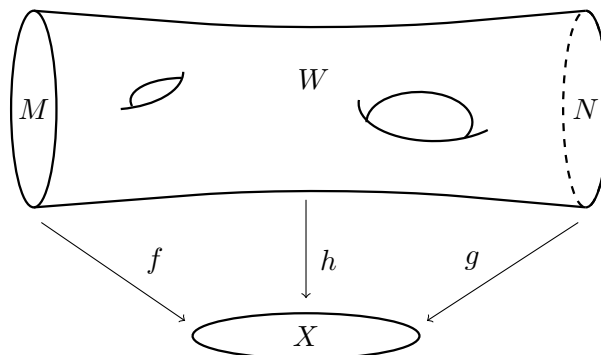


Figure 1. Cobordism (W, h) between pairs (M, f) and (N, g) .

are isomorphisms for any topological space X [34]. One says that KO (K) theory is isomorphic to the extension of scalars of Spin (Spin^c) cobordism theory through the ABS-orientation. The definition of the extension of scalars is given in appendix B.

Here, the cobordism groups $\Omega_n^G(X)$ are given by equivalence classes of pairs (M, f) of n -dimensional manifolds with G -structure together with maps $f : M \rightarrow X$. Two pairs are equivalent, $(M, f) \sim (N, g)$, if M and N are cobordant, i.e. if there exists an $(n + 1)$ -dimensional manifold W with G structure such that $\partial W = M \sqcup \bar{N}$, and if there is a map $h : W \rightarrow X$, such that $h|_M = f$ and $h|_N = g$. An illustration is given in figure 1.

Choosing $X = pt$ in the general statement of the Hopkins-Hovey theorem gives back the result from the previous section, now written as the isomorphism

$$\Omega_n^{\text{Spin}} \otimes_{\Omega_n^{\text{Spin}}} \widetilde{KO}(S^n) \cong \widetilde{KO}(S^n), \tag{4.10}$$

$$\Omega_n^{\text{Spin}^c} \otimes_{\Omega_n^{\text{Spin}^c}} \widetilde{K}(S^n) \cong \widetilde{K}(S^n), \tag{4.11}$$

where we used that $KO_n(pt) = \widetilde{KO}(S^n)$, analogously to (2.3). In this simple case, the isomorphism follows trivially from the standard result

$$R \otimes_R M \cong M, \tag{4.12}$$

which is valid for any ring R (being a module over itself) and any R -module M . The short proof is relegated to appendix C.

Without pursuing this further at the present stage, we simply conclude that as duals of $K_n(X)(KO_n(X))$ the cobordism groups $\Omega_n^{\text{Spin}^c}(X)(\Omega_n^{\text{Spin}}(X))$ should be relevant in string theory, as well.

5 Cobordism, K-theory and tadpoles

As we recalled in section 3, the cobordism conjecture states that the cobordism classes of quantum gravity must be trivial. This implies that any global symmetry detected by a non-trivial cobordism class must either be broken by defects or gauged. Since we have now the classical Hopkins-Hovey theorem available that relates certain K-theory classes to certain cobordism groups, we can wonder what the physical significance of this relation is.

What does it mean that e.g. Spin cobordism are related to KO -groups, which are relevant for the classification of D-branes in type I string theory? One can certainly also consider Spin cobordism in type II string theories, as done in [3]. Hence, what is special about type I?

To get an idea, we first observe that K-theory classifies certain defects, namely D-branes. As suggested, all global symmetries related to K-theory are gauged. Therefore, we propose that the Hopkins-Hovey isomorphism provides the relations between the D-brane defects and the cobordism groups (charges) that are coupling to the same gauge field, i.e. that can potentially appear in the same tadpole cancellation condition.

For instance, as will be discussed in more detail next, the relation between $KO_4(pt) = \mathbb{Z}$ and $\Omega_4^{\text{Spin}}(pt) = \mathbb{Z}$ indicates that the type I D5-branes and the cobordism invariant $\hat{\alpha}_4 = \hat{A}/2$ couple to the RR 6-form gauge field and thus both appear in the corresponding tadpole cancellation condition. One could also consider $\Omega_4^{\text{Spin}}(pt)$ in type IIB theory, but then it is likely not gauged but broken [3], as the Hopkins-Hovey isomorphism does not apply to this setup. In the following we substantiate our proposal by a couple of examples. We will also notice that, in the presence of torsion, one can still be able to decouple cobordism from K-theory charges originally appearing in the tadpole.

5.1 Spin-structure in type I

The non-vanishing classes $KO_n(pt) = \mathbb{Z}$ ($n = 0, 4, 8$) clearly correspond to the usual Dp -branes of type I with $p = d - n - 1$ and $d = 10$. The associated global p -form symmetry is gauged, as there are appropriate RR $(p + 1)$ -form gauge fields C_{p+1} . Thus, also the open-closed dual cobordism group $\Omega_n^{\text{Spin}}(pt)$ is expected to be gauged, so that on a compact space only configurations with vanishing charge are allowed. Let us discuss this in more detail for a first example with $n = 4$, which is central to the overall picture we intend to convey.

Gauging the global symmetry of $\Omega_4^{\text{Spin}}(pt)$. For the generator $[K3] \in \Omega_4^{\text{Spin}}$, this is the type I analogue of the example involving the heterotic string on K3 discussed in [3]. To understand the origin of the obstruction, we have to bear in mind that in computing $\Omega_n^{\text{Spin}}(pt)$ the higher form gauge field C_{d-n} was set to zero. Then, one finds a \mathbb{Z} -valued obstruction in this truncated type I theory, meaning that the generator $[K3]$ and also all other non-trivial representatives $[M]$ are not topologically consistent solutions.

As we have seen, the ABS-orientation involves the map α_n between cobordism and K-theory. Therefore, the natural candidate for the global current is

$$\star J_6 \sim \hat{\alpha}_4(K3), \tag{5.1}$$

where the four-form $\hat{\alpha}_4 \in H^4(K3)$ is such that $\int_{K3} \hat{\alpha}_4(K3) = \hat{A}(K3)/2 = \alpha_4([K3])$. Thus, the current is preserved, $d \star J_6 = 0$, and the charge of $K3$ is given by

$$\int_{K3} \star J_6 \sim \int_{K3} \hat{\alpha}_4(K3) = \alpha_4([K3]) = 1. \tag{5.2}$$

From (5.1), it is suggestive that $\hat{\alpha}_4$ should be considered as a magnetic (non-singular) current. Hence, after gauging one would obtain a magnetic Bianchi identity of the type

(up to normalisation)

$$d\tilde{F}_3 = \tilde{J}_4 \sim \hat{\alpha}_4(K3). \tag{5.3}$$

This encodes cobordism information only and thus it is not yet the end of the story, as we are now going to argue.

So far, we have considered sources of the C_2 -form and of the dual C_6 -form separately on the K-theory side (i.e. $D5$ -branes) and on the cobordism side (i.e. $K3$). We propose that the existence of the isomorphism between K-theory and cobordism suggests that upon gauging the two initially appearing global symmetries actually couple to one and the same C_2 -form (or its electric dual C_6 -form). Hence, combining them with an appropriate normalisation, we have a total Bianchi identity

$$d\tilde{F}_3 \sim \sum_i Q_i \delta^{(4)}(\Delta_{6,i}) - k \hat{\alpha}_4(M) \tag{5.4}$$

with $Q_i = \pm 1$ and where we considered general elements $[M] \in \Omega_4^{\text{Spin}}$. To warrant e.g. anomaly cancellation we know that in string theory one has $k = 24$. In fact, there is an interesting story behind the relative normalisation $k = 24$, on which we will comment in a while. Integration of (5.4) over the compact space M leads to the well-known $D5$ -brane tadpole cancellation condition,

$$\int_M \sum_i Q_i \delta^{(4)}(\Delta_{6,i}) + \dots = 24 \alpha_4(M). \tag{5.5}$$

Here the dots indicate additional contributions from magnetised $D9$ -branes, that are not yet captured by the K-theory groups we consider.

This relation makes it evident that just a geometric $[M] \neq 0$ is not a topologically consistent compactification of the type I string. Similarly, $D5$ -branes on a flat toroidal space are obstructed, too. Indeed, the left hand side of (5.5) can be considered as the $KO_4(pt)$ contribution and the right hand side as the $\Omega_4^{\text{Spin}}(pt)$ contribution. Each alone is inconsistent and therefore gives a \mathbb{Z} -valued obstruction, but by combining them one can find configurations of vanishing C_6 -form charge. Note that (5.5) means that only for multiples of 24 $D5$ -branes the charge can be cancelled.

More precisely, this gauging can be encoded in a map

$$0 = \Omega_4^{\text{Spin}+D5; U(1)_{(2)}}(pt) \longrightarrow \Omega_4^{\text{Spin}}(pt) \oplus KO_4(pt) = \mathbb{Z} \oplus \mathbb{Z}, \tag{5.6}$$

where the $U(1)_{(2)}$ factor indicates the two-form gauge symmetry. This map is defined by forgetting the C_2 -form gauge field (while keeping the $D5$ -branes). Thus, one can consider the K-theory group as the cobordism class of the defect brane, i.e. $\Omega_4^{D5}(pt) := KO_4(pt)$. We stress that in the definition of the cobordism group on the left we have to include the information about the charged objects, i.e. we have to impose the Bianchi identity (5.4) and hence a topological trivialisation of its right hand side.

Let us now comment on the charge normalization $k = 24$. We present an argument why this normalization in the tadpole condition is related to a certain string cobordism group.

Notice that the definition of the cobordism groups $\Omega_4^{\text{Spin}+D5; \text{U}(1)_{(2)}}(pt)$ is analogous to that of the more familiar so-called string-cobordism groups $\Omega_n^{\text{String}}(pt)$. Recall from section 3.3 that the latter come with a 2-form B and a trivialisation of the class $p_1/2 = 24\hat{\alpha}_4$, which in (5.4) is generalized to include also the $D5$ -branes, leading to a trivialisation of the whole class on the right hand side of the Bianchi identity. Thus, by ignoring the $D5$ -branes, for $k = 24m$ (m integer) we can define the generalized string cobordism groups

$$\Omega_n^{\text{Spin}; \text{U}(1)_{(2), k}}(pt) = \Omega_n^{\text{String}, k}(pt), \tag{5.7}$$

with a trivialization of the class $\frac{k}{48} p_1 = \frac{m}{2} p_1 \in H^4(M, \mathbb{Z})$.

Now, due to the factor k in (5.4), not the full $\mathbb{Z} \oplus \mathbb{Z}$ global symmetry on the right hand side of (5.6) is co-killed, rather a discrete $\mathbb{Z}_k \subset KO_4(pt)$ symmetry survives. This corresponds to a $D5$ -brane stack with less than k branes, so that it cannot be compensated by changing $\hat{\alpha}(M)$. The $D5$ -branes happen to be precisely the right defects which can break the global symmetry of the generalized string cobordism group,¹⁰ if $\Omega_3^{\text{String}, k}(pt) = \mathbb{Z}_k$. Hence, this reasoning has led us to a close relation between the relative normalization factor k in the tadpole constraint and a certain generalized string cobordism group. The normalization $k = 24$ realised in string theory can however not be deduced from this argument.

Notice also that the tadpole condition (5.5) is in principle valid within the whole cobordism group and thus it can be interpreted as an off-shell condition. In other words, the cobordism language allowed us to go beyond a strictly background-dependent analysis. This observation is in fact general and valid also for the other tadpole cancellations that we are going to recover in the following.

To summarise, we realise that after gauging, the (open string) K-theory groups and the (closed string) geometric cobordism groups are really two sides of the same coin. In fact, the K-theory groups are the mathematically precise definition of the cobordism groups of D-brane defects. We will present more insights below, when discussing the Spin^c -cobordisms.

Gauging discrete global symmetries. What about the \mathbb{Z}_2 -valued classes $\Omega_1^{\text{Spin}}/KO_1(pt)$ and $\Omega_2^{\text{Spin}}/KO_2(pt)$, where we start with a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ global symmetry in each case? While in type IIB theory the global symmetries related to $\Omega_1^{\text{Spin}}(pt)$ were argued to be broken with (wrapped) $O7$ -planes playing the role of the defects [3], the Hopkins-Hovey isomorphism suggests that at least the K-theory induced global symmetries are gauged.

Thus, following the same logic as in the previous example, e.g. for the case $\Omega_2^{\text{Spin}}/KO_2(pt)$ we will have a \mathbb{Z}_2 -valued charge neutrality condition

$$\int_M \sum_i Q_i \delta^{(2)}(\Delta_{8,i}) = k \alpha_2(M) \pmod{2}. \tag{5.8}$$

Here, on the left hand side are the charges of the non-BPS $\widehat{D7}$ -branes, while on the right hand side the contribution from the cobordism group. Note that the map α_2 is

¹⁰Here, we are using the fact that to break a global symmetry in $\Omega_{n-1}^{\text{Spin}; \text{U}(1)_{(n-2)}}$ we need defects (branes) of the dimension related to $KO_n(pt)$.

identical to the Arf-invariant [28]

$$\alpha_2([M]) = \text{Arf}(M), \quad (5.9)$$

which for the generator $M = S_p^1 \times S_p^1$ is equal to one.

As before, we do not know a priori what the value k of the relative normalization is. If it is even, then the right hand side of (5.8) decouples from this relation and the K-theory charge by itself has to be even. In other words, the global symmetry of $KO_2(pt) = \mathbb{Z}_2$ is gauged and the one of $\Omega_2^{\text{Spin}} = \mathbb{Z}_2$ remains unbroken at this stage and thus needs to be broken by introducing appropriate defects. In such a case, the wrapped $O7$ -planes of type IIB cannot do the job, as they are projected out in type I theory. Therefore, as also suggested for the S-dual heterotic string in [3], there must exist new so far unknown defects.

The second possibility is that the normalization k is odd. In this case, placing for instance a single non-BPS $\widehat{D7}$ -brane on the background $M = S_p^1 \times S_p^1$ would give a neutrally charged configuration and thus it would be allowed. Then, one linear combination of the \mathbb{Z}_2 -charges is gauged and the orthogonal one is broken. They both lie in the cokernel of the map

$$0 = \Omega_2^{\text{Spin}+\widehat{D7};\mathbb{Z}_2}(pt) \longrightarrow \Omega_2^{\text{Spin}} \oplus KO_2(pt) = \mathbb{Z}_2 \oplus \mathbb{Z}_2. \quad (5.10)$$

However, since just a torus $S_p^1 \times S_p^1$ is expected to be a valid background of type I string theory without placing a non-BPS $\widehat{D7}$ -brane on it, this second possibility does not seem to be realized. Thus, we conclude that k is (likely) even.

The story will repeat itself for the other \mathbb{Z}_2 K-theory classes: $KO_1(pt)$, $KO_9(pt)$ and $KO_{10}(pt)$, which are related to the non-BPS $\widehat{D8}$, $\widehat{D0}$ and $\widehat{D(-1)}$ -branes respectively. These examples make it clear that cobordism and K-theory charges paired by the Hopkins-Hovey isomorphism might still decouple from one another in the presence of torsion. Whether this will always happen is not clear yet, so that we leave a more systematic analysis of torsion charges in tadpole cancellations conditions for the future.

5.2 Spin^c-structure in F-theory: $\Omega_{2+4k}^{\text{Spin}^c}(pt)$

The non-vanishing classes $K_n(pt) = \mathbb{Z}$ (n even) correspond to the usual Dp -branes of type IIB with $p = d - n - 1$. The associated global p -form symmetry is gauged, as there are appropriate RR $(p + 1)$ -form gauge fields C_{p+1} . Thus, the story is expected to be analogous to the $KO_4(pt) = \mathbb{Z}$ example from the previous section. In particular, appropriate combinations of the global symmetries associated to the groups $K_n(pt)$ and $\Omega_n^{\text{Spin}^c}(pt)$ should be gauged.

As we have seen from the ABS-orientation, the Todd class provides the map α_n^c between cobordism and K-theory. Therefore, a natural candidate for the global current is

$$\star J_{d-n} = \alpha^c([M]) = \text{td}_n(M), \quad (5.11)$$

which indeed leads to a conserved current satisfying $d \star J_{d-n} = 0$. For all generators Σ_n from table 4, the charge is

$$\int_{\Sigma_n} \star J_{d-n} = \int_{\Sigma_n} \text{td}_n(\Sigma_n) = 1. \quad (5.12)$$

The picture that will emerge from the following analysis is that in the case where we are dealing only with backreacted $(d - n)$ -dimensional localised sources, which in these set-ups are e.g. O_{d-n-1} -planes, the current will be the Todd genus. For set-ups with different types of sources, like the $n = 6$ case below, the gauged current turns out to be a linear combination of the cobordism invariants, in particular it includes also invariants in the kernel of the ABS-orientation. Let us first consider the two cases $n = 2$ and $n = 6$ in more detail, which turn out to be related to F-theory.

Gauging the global symmetry of $\Omega_2^{\text{Spin}^c}(pt)$. Recall that for $n = 2$ the K-theory classes describe $D7$ -branes and the generator of $\Omega_2^{\text{Spin}^c}(pt)$ is $[\mathbb{P}^1]$. After gauging and combining charged K-theory and cobordism objects, we arrive at a magnetic C_0 -form tadpole cancellation condition which reads

$$\int_{\mathbb{P}^1} \sum_i Q_i \delta^{(2)}(\Delta_{8,i}) = 24 \alpha_2^c([\mathbb{P}^1]) = 12 \int_{\mathbb{P}^1} c_1(\mathbb{P}^1). \tag{5.13}$$

The normalisation factor $k = 24$ on the right hand side has been fixed such that this equation becomes precisely the well-known relation from F-theory [35] on a $K3$ surface. In this context, the $\Delta_{8,i}$ are the divisors over which the elliptic fiber becomes singular of degree Q_i and the \mathbb{P}^1 is the result of the backreaction of the (p, q) 7-branes. Hence, for the manifold \mathbb{P}^1 itself we really deal with an on-shell configuration. One might wonder whether there is a similar tadpole constraint for the NS-NS eight-form, which is the magnetic dual of the dilaton. In F-theory this constraint seems to be always satisfied among the (p, q) 7-branes so that it does not show up as a global symmetry related to a non-trivial cobordism class.

In agreement with the previous obstruction, both a single $D7$ -brane in K-theory and a pure geometric $[\mathbb{P}^1]$ in cobordism are inconsistent. However, a combination of the two ingredients can cancel the tadpole, i.e. it can lead to a configuration with vanishing C_0 -form charge. This gauging is expected to correspond to a map

$$0 = \Omega_2^{\text{Spin}^c + D7; U(1)_{(0)}}(pt) \longrightarrow \Omega_2^{\text{Spin}^c}(pt) \oplus K_2(pt) = \mathbb{Z} \oplus \mathbb{Z}, \tag{5.14}$$

where the $U(1)_{(0)}$ is the RR eight-form gauge symmetry whose total Bianchi identity and implied trivialisation has to be imposed in the definition of the full cobordism group $\Omega_2^{\text{Spin}^c + D7; U(1)_{(0)}}(pt)$. Similarly to the $K3$ example from section 5.1, a discrete $\mathbb{Z}_{24} \subset K_2(pt)$ symmetry is not co-killed. We conjecture that the relative normalisation $k = 24$ will show up in

$$\Omega_1^{\text{Spin}^c; U(1)_{(0),k}}(pt) = \mathbb{Z}_k. \tag{5.15}$$

The actual determination of such cobordism groups is left for future studies.

Moreover, on the purely mathematical level, the ABS homomorphism (4.2) seems to indicate that $[\mathbb{P}^1] \in \Omega_2^{\text{Spin}^c}$ carries the same magnetic C_0 -form charge as a single $D7$ -brane, which is the generator of $\widetilde{K}_2(S^2)$. However, we have seen that one needs $k = 24$ $D7$ -branes to get a \mathbb{P}^1 upon backreaction. This might suggest that the physically relevant

generalisation of the ABS map is rather

$$\alpha_2^c([M]) = 24 \text{Td}(M) \in 24 \mathbb{Z}, \tag{5.16}$$

which is still a group homomorphism, but ceases to be surjective. As alluded to earlier, the intuitive arguments from section 4 for the existence of a map between cobordism and K-theory are meant up to such relative normalisation factor.

Gauging the global symmetry of $\Omega_6^{\text{Spin}^c}(pt)$. For $n = 6$ we are dealing with $D3$ -branes classified by $K_6(pt) = \mathbb{Z}$ and the cobordism group $\Omega_6^{\text{Spin}^c}(pt) = \mathbb{Z} \oplus \mathbb{Z}$. Recall that the first \mathbb{Z} is the one whose charge is measured by the Todd class, the charge of the second is instead determined by $c_1^3/2$. The diagonal generators of the two were denoted as $[\mathcal{H}_1]$ and $[\mathcal{H}_2]$, where the second one generates also the kernel of α_6^c . Gauging the global three-form symmetry related to the Todd class, one expects that there exists a C_4 -form tadpole cancellation condition such that

$$\int_{\mathcal{H}_1} \sum_i Q_i \delta^{(6)}(\Delta_{4,i}) = \gamma \alpha_6^c([\mathcal{H}_1]) = \frac{\gamma}{24} \int_{\mathcal{H}_1} c_2(\mathcal{H}_1) c_1(\mathcal{H}_1), \tag{5.17}$$

where the normalisation γ needs to be determined. Recall that a connected representative for $[\mathcal{H}_1]$ is the threefold $[dP_9 \times \mathbb{P}^1]$, which can be thought of as the downstairs geometry of the orientifold $K3 \times T^2/\Omega(\sigma, I_2)$. Here, as in [36, 37], σ is a holomorphic involution of $K3$ and I_2 the reflection of the two coordinates of T^2 . Note that this orientifold only has $O3$ -planes and no $O7$ -planes.

To proceed, recall the $D3$ -brane tadpole cancellation condition [38] for F-theory compactified on a smooth elliptically fibered Calabi-Yau fourfold Y over a base B

$$\int_B \sum_i Q_i \delta^{(6)}(\Delta_{4,i}) + \dots = \frac{\chi(Y)}{24} = \int_B \left(\frac{1}{2} c_2(B) c_1(B) + 15 c_1^3(B) \right). \tag{5.18}$$

Since the second term vanishes for the generator \mathcal{H}_1 , the two conditions (5.17) and (5.18) coincide for $\gamma = 12$. Note that in the Sen limit [39] the right hand side also includes contributions from geometrized $O7$ -planes and not only $O3$ -planes.

Remarkably, the general tadpole condition (5.18) contains precisely the two cobordism invariants of $\Omega_6^{\text{Spin}^c}(pt)$, in particular $c_3(B)$ is absent. One might wonder what the role of the second charge \mathbb{Z} is. Since both \mathbb{Z} -charges appear in (5.18), it is a linear combination of the two that is gauged and couples to the C_4 -form. The current then is

$$\star J_4 \sim 12 \text{td}_6(B) + 30 \left(\frac{c_1^3(B)}{2} \right). \tag{5.19}$$

This gauging is expected to correspond to a map

$$0 = \Omega_6^{\text{Spin}^c; \text{U}(1)_{(4)}}(pt) \longrightarrow \Omega_6^{\text{Spin}^c}(pt) = \mathbb{Z} \oplus \mathbb{Z}, \tag{5.20}$$

where the $\text{U}(1)_{(4)}$ is the four-form gauge symmetry, whose Bianchi identity and the implied trivialisation are again imposed. Then, both global symmetries from $\Omega_6^{\text{Spin}^c}(pt)$ lie in the cokernel and are therefore co-killed.

We think that intuitively the appearance of two initial global symmetries is related to the fact that both pure $O3$ -planes as well as wrapped $O7$ -planes contribute to the right hand side of the tadpole equation. This means that prior to gauging, the bulk duals of pure $O3$ -planes and wrapped $O7$ -branes are not cobordant in $\Omega_6^{\text{Spin}^c}(pt)$ and give rise to two independent global symmetries.

5.3 Spin^c-structure in F-theory: $\Omega_{4k}^{\text{Spin}^c}(pt)$

We discuss now the remaining Spin^c-cobordism groups. We will be led to tadpole constraints similar to those of the previous sections, whose form is however not as clearly established.

Gauging the global symmetry of $\Omega_0^{\text{Spin}^c}(pt)$. For $n = 0$, the K-theory group classifies space-time filling $D9$ -branes so that the appearing global symmetry will be gauged via C_{10} . The related cobordism group $\Omega_0^{\text{Spin}^c}(pt) = \mathbb{Z}$ should then provide the geometric contribution to the C_{10} tadpole cancellation condition. In this case, the only natural candidate for the generator pt^+ is the space-time filling $O9$ -plane. Thus, there exist essentially two solutions of the resulting tadpole constraint, namely just the trivial 10D type IIB theory or the type I superstring.

Gauging the global symmetry of $\Omega_4^{\text{Spin}^c}(pt)$. For $n = 4$, the cobordism group $\Omega_4^{\text{Spin}^c}(pt) = \mathbb{Z} \oplus \mathbb{Z}$ suggests that the story can be similar to the $n = 6$ case above. However, there are few subtleties which we discuss below and which can lead in fact to different possibilities.

For $[K3] = 2[\mathcal{G}_1]$ one gets the same structure as for $\Omega_4^{\text{Spin}^c}(pt)$ described in section 5.1, in particular a type I background with $D9$ and $D5$ -branes. Of course, here we are assuming that one cancels the C_{10} -tadpole in a non-trivial way by an $O9$ -plane and 32 $D9$ -branes. However, the two generators of $\Omega_4^{\text{Spin}^c}(pt)$ are not Calabi-Yau so that following our logic, there should exist a C_2 tadpole condition like

$$\int_B \sum_i Q_i \delta^{(4)}(\Delta_6^i) + \dots = \int_B \left(12 \text{td}_4(B) - \frac{3}{2} c_1^2(B) \right). \quad (5.21)$$

The normalisation factors 12 and $-3/2$ have been fixed by the K3 case mentioned above and by evaluating the term $p_1 \sim \text{tr}(R \wedge R)$ (Pontryagin class) arising from the Chern-Simons term in the effective action of the $D9$ -branes and the $O9$ -plane. Note that for $c_1(B)$ not even, there will be another half-integer contribution on the left hand side from the required line bundle $\mathcal{L}^2 = c_1(B)$ supported on the $D9$ -brane wrapping the Spin^c-manifold B . Similarly to the $n = 6$ case, one could consider one \mathbb{Z} factor as the geometric counterpart of such wrapped $O9$ -branes and the other one as that of genuine $O5$ -planes.

Note that in contrast to $K3$ the generators of $\Omega_4^{\text{Spin}^c}(pt)$, namely the manifolds $B = \mathbb{P}^1 \times \mathbb{P}^1$ and $B = \mathbb{P}^2$, are off-shell configurations of type I. Could there be a different interpretation of the constraint (5.21) that is not using the type I superstring and where the manifold is a generator of $\Omega_4^{\text{Spin}^c}(pt)$ and on-shell? Recall that there exist also supersymmetric type IIB orientifolds of $K3$ with the \mathbb{Z}_2 projection $\Omega\sigma$, where σ is a holomorphic involution [36, 37]. This leads to $O5$ -planes at the fixed points of σ . Downstairs the

manifold is $[K3/2] = [dP_9]$ so that it is suggestive to get a geometrized $D5$ -brane tadpole condition of the form

$$\int_{dP_9} \sum_i Q_i \delta^{(4)}(\Delta_6^i) = k \int_{dP_9} \text{td}_4(dP_9). \tag{5.22}$$

Here, we used $\int \text{td}_4(dP_9) = 1$ and $\int c_1^2(dP_9) = 0$, leading to the identification in cobordism $[dP_9] = [\mathcal{G}_1]$. Note that for such a geometrized $O5$ -plane, the current is indeed given by the ABS isomorphism, i.e. the Todd class.

An interesting possibility is that (5.22) is analogous to the F-theory-like tadpole (5.13). Without the $O9$ -planes, there are also $NS5$ -branes present in type IIB. Such branes back-react on the geometry, so that there might exist globally consistent backreacted solutions of in general (p, q) 5-branes.¹¹ Like in F-theory, there could then exist special limits where these become the aforementioned perturbative $\Omega\sigma$ orientifolds of type IIB on $K3$. Comparing to the concrete toroidal orbifold example from [36], such a picture would suggest $k = 24$. Of course, without a concrete construction all this is fairly speculative.

Let us also mention that on-shell backgrounds for $B = \mathbb{P}^1 \times \mathbb{P}^1$ or $B = \mathbb{P}^2$ are known to appear as bases for F-theory compactifications on elliptically fibered Calabi-Yau threefolds. Usually, there is no non-trivial C_2 -form tadpole condition. However, it is known that there exist base geometries that via F-theory-heterotic duality correspond to the presence of $NS5$ -branes on the heterotic side [43–45]. With each $NS5$ -brane comes a tensor multiplet, which in the F-theory dual increases the number of 2-cycles in the base, i.e. $n_T = h^{1,1}(B) - 1$, n_T being the number of tensor multiplets.

On the heterotic side, the Hirzebruch surfaces \mathbb{F}_n correspond to smooth configurations with instanton numbers $(12+n)$ and $(12-n)$ in the two E_8 factors and no 5-branes. For all \mathbb{F}_n one finds $\int \text{td}_4 = 1$ and $\int c_1^2 = 8$ so that they are all (at least for n even) cobordant¹² to $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Consistently, the right hand side of the tadpole relation (5.21) vanishes. Thus, six-dimensional F-theory-heterotic duality suggests that there might exist a relation like

$$N_{NS5}|_{\text{het}} = n_T - 1 \stackrel{?}{=} \int_B (8 \text{td}_4(B) - c_1^2(B)) \tag{5.23}$$

for the number of $NS5$ -branes on the heterotic side.¹³

The apparent controversy of this example demonstrates that starting with incomplete cobordism structures has its limitations, but still has the potential to uncover new aspects of string theory.

Comment on gauging the global symmetry of $\Omega_8^{\text{Spin}^c}(pt)$. For $n = 8$ we know that $\Omega_8^{\text{Spin}^c}(pt) = \mathbb{Z}^4$, but we have fixed neither the four cobordism invariants nor the generators of the cobordism group. What we can say is that eventually there should be a C_6 tadpole cancellation condition reflecting a map

$$0 = \Omega_8^{\text{Spin}^c; U(1)^{(6)}}(pt) \longrightarrow \Omega_8^{\text{Spin}^c}(pt) = \mathbb{Z}^4. \tag{5.24}$$

¹¹Similar geometrized configurations have been discussed in four dimensions in [40–42].

¹²Actually, for n odd, \mathbb{F}_n is not Spin and one has an additional non-trivial line bundle on the Spin^c -manifold.

¹³Coming back to the discussion around eq. (5.22), for $B = dP_9$ we obtain $n_T = 9$ which is indeed the result for the $\Omega\sigma$ orientifold of [36].

Again, the four global symmetries of $\Omega_8^{\text{Spin}^c}(pt)$ could be bulk counterparts of wrapped $O9$, two orthogonally wrapped $O5$ and localised $O1$ -planes.

6 Conclusions

We have provided evidence that there exists a mathematical manifestation of open-closed string duality already at the purely topological level. Concretely, the relation is between K-theory and certain cobordism classes, as it is described by the generalisations of the Conner-Floyd theorem to Spin and Spin^c-manifolds.

Whereas the description of stable D -branes in terms of K-theory classes is well established, the recent proposal for the relevance of cobordism classes for consistent string backgrounds is rather on the level of a conjecture. We think that the interpretation of the Hopkins-Hovey theorem presented in this work gives more credence to the cobordism conjecture. In fact, in view of open-closed string duality one is automatically driven to the relevance of cobordism for closed string backgrounds. One could say that it encodes topological quantum gravity information on the consistent coupling of the gauge sector to gravity. By including K-theory classes, we have extended the original work of McNamara-Vafa [3] by giving a more precise mathematical meaning to the cobordism class related to defects (D-branes).

Moreover, we have exploited the consequences of this correspondence and discussed in a bottom-up approach the fate of global symmetries arising in the various setups. We suggested that the physical significance of the Hopkins-Hovey isomorphisms is that the directly involved combinations of global symmetries are gauged in the respective theories. As a consequence, orthogonal combinations of global symmetries were co-killed and hence broken. We found an intricate relation to some well-known tadpole cancellation conditions, where the bulk contributions were linear combinations of the cobordism invariants.

While it deserves further investigation, the six-dimensional example $\Omega_4^{\text{Spin}^c}(pt)$ might be considered as evidence that this approach has the potential to uncover also new aspects of string theory, even for the case of removing the global symmetries via gauging.

The examples made it evident that the intermediate appearance of extra global symmetries is a consequence of having ignored additional ingredients present in the final cobordism group Ω_n^{QG} . Hence, it is a relict of trying to narrow down the QG-structure following a bottom-up approach. Clearly, it would be very interesting to develop means to compute such refined gauge cobordism classes from first principles (see also [46]).

One could ask whether at least part of the story can be extended to more general setups. One can certainly consider D -branes in backgrounds with non-trivial H -flux, which are classified by twisted K-theory classes $K_H(X)$. Hence, one might expect a notion of H -twisted cobordism classes, as well. There have been proposals for cobordism groups relevant for the heterotic string (Ω_n^{string}) or even M-theory ($\Omega_n^{\text{pin}^+}$, but see also the recent proposal [16, 18], similar in spirit to our analysis). It would be interesting to generalize our analysis to these theories as well.

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A On the cobordisms invariants of $\Omega_6^{\text{Spin}^c}(pt)$

In [28], the cobordism invariants of $\Omega_6^{\text{Spin}^c}(pt)$ are listed as

$$c_1 \frac{\sigma}{16}, \quad \frac{c_1^3}{2} \tag{A.1}$$

with $c_1 \frac{\sigma}{16} := \frac{\sigma}{16}(\text{PD}(c_1))$, where $\text{PD}(c_1)$ is the submanifold D of the Spin^c six-manifold B which represents the Poincaré dual of $c_1(T_B)$. Below, we show that these invariants coincide with those employed in the main text.

Due to the Hirzebruch signature theorem, we can write

$$c_1 \frac{\sigma}{16} = \frac{1}{48} \int_D p_1(T_D) = \frac{1}{48} \int_B p_1(T_D) \wedge c_1(T_B), \tag{A.2}$$

where the first Pontryagin class can be expressed in terms of Chern classes as $p_1 = -2c_2 + c_1^2$. Now one invokes the normal sequence

$$0 \longrightarrow T_D \longrightarrow T_B|_D \longrightarrow N \longrightarrow 0 \tag{A.3}$$

with the normal line bundle N featuring $c_1(N) = c_1(T_B)$. Using that for the total Chern classes one has $c(T_D) = c(T_B|_D)/c(N)$, we can determine $c_1(T_D) = 0$ and $c_2(T_D) = c_2(T_B)$, i.e. we find that the hypersurface is CY. Putting everything together, we arrive at

$$c_1 \frac{\sigma}{16} = \frac{1}{48} \int_B p_1(T_D) \wedge c_1(T_B) = -\frac{1}{24} \int_B c_2(T_B) \wedge c_1(T_B) = -\text{Td}(B). \tag{A.4}$$

This shows that, consistently with the ABS orientation, the Todd class is a cobordism invariant of $\Omega_6^{\text{Spin}^c}(pt)$.

B Extension of scalars

Consider a ring R , a right R -module M and a left R -module N . Their tensor product over R is denoted as $M \otimes_R N$ and it is obtained by associating to each pair (m, n) a tensor $m \otimes_R n$. This is a ring and in fact a module over R , in particular

$$r \cdot (m \otimes n) = (r \cdot m) \otimes n, \quad (m \otimes n) \cdot r = m \otimes (n \cdot r). \tag{B.1}$$

We want to generalise the above to the case in which we have modules over different rings. Consider another ring S together with a ring homomorphism $f : R \rightarrow S$. Let M

be an R -module. Being a ring, S is a module over itself but also an R -module via the homomorphism f . Then, we can construct

$$M_S = M \otimes_R S. \tag{B.2}$$

This is called extension of scalars and it is a module over S constructed out of the R -module M . In this sense, it can be thought as a change of “ring basis” from R to S . Due to the action of f , there is a natural equivalence relation

$$(m \cdot r) \otimes s \sim m \otimes (f(r) \cdot s), \tag{B.3}$$

which allows us to define M_S as the S -module

$$M \otimes_R S = \{m \otimes s, \text{ s.t. } \forall r \in R, (m \cdot r) \otimes s = m \otimes (f(r) \cdot s)\}. \tag{B.4}$$

C Proof of the isomorphism (4.12)

In this appendix we present the proof of the standard isomorphism $R \otimes_R M \cong M$, which is valid for any ring R (which is also a module over itself) and any R -module M . We define a bilinear map $h : R \times M \rightarrow M$, such that $(r, m) \rightarrow r \cdot m$. By the universal property of the tensor product, this induces a linear map $f : R \otimes_R M \rightarrow M$, such that $r \otimes m \rightarrow r \cdot m$. The inverse linear map $g : M \rightarrow R \otimes_R M$ sends $m \rightarrow 1 \otimes m$. The isomorphism follows since $g \circ f = \mathbb{I}_M$ and $f \circ g = \mathbb{I}_{R \otimes_R M}$.

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