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M5 branes on $\mathbb{R}^{1,1}\times$Taub-NUT

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Abstract: We study M5 branes on $\mathbb{R}^{1,1}\times$Taub-NUT that we view as a singular fibration. Reducing the M5 branes along the fiber gives 5d SYM. Due to the singularity, this 5d theory has a gauge anomaly and it is not supersymmetric. To cure these problems we add a supersymmetric gauged chiral WZW theory on the 2d submanifold where the circle fiber vanishes. In addition we add a mass term for the five scalar fields located on this submanifold. With all this, we obtain a fully supersymmetric and gauge invariant theory.

Keywords: Field Theories in Higher Dimensions, M-Theory, Supersymmetric Gauge Theory

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Contents

1 Introduction

Bosonization is the equivalence between a 2d theory of a Dirac fermion and a theory of a scalar field whose non-Abelian generalization is a WZW theory [9]. The partition function factorizes, which can be better understood by coupling the chiral part to a background gauge field, resulting in a chirally gauged WZW theory [8]. An analogous situation happens in 6d for the selfdual three-fom of the M5 brane. If we compute the partition function of a nonchiral three-form then it factorizes. This factorization can be better understood if one couples the chiral part to a background C-field that is a three-form gauge potential in 11d supergravity [11]. A somewhat different kind of chirally gauged WZW theory appears as the boundary theory of M2 branes [14].

If we put the M5 brane on $\mathbb{R}^{1,1}\times$Taub-NUT and view the Taub-NUT space as a circle fibration over $\mathbb{R}^3$, then the radius of that circle vanishes at the origin, where we have the submanifold $\mathbb{R}^{1,1}$. This has a brane interpretation of a D4 brane intersecting with a D6 brane. Near the intersection there are open strings stretching between the two branes. More generally, on the intersection brane, which is $\mathbb{R}^{1,1}$, we have chiral fermions in the bifundamental of $\mathrm{U}(N) \times \mathrm{U}(Q)$ where in the brane picture there are Q coincident D6 intersecting with N coincident D4 branes. These chiral fermions have a gauge anomaly that cancels the corresponding gauge anomaly of the 5d SYM that lives on the D4 brane [10]. By bosonizing we get a chirally gauged WZW on $\mathbb{R}^{1,1}$ and again that theory has a gauge anomaly that cancels the corresponding gauge anomaly of the D4 brane [5]. The gauge anomaly of the D4 brane comes from a gravi-photon term in eq. (2.1). In this paper we will extend this analyzis to the supersymmetric case. We find that the combined system of 5d SYM on the D4 branes and supersymmetric chirally gauged supersymmetric WZW theories plus additional mass terms for the five scalar fields on $\mathbb{R}^{1,1}$ results in a supersymmetric and gauge invariant theory.

1.1 Outline

In section 2 we present the five-dimensional SYM theory obtained by reducing the Abelian M5 brane on a circle bundle. The relation with the Abelian M5 brane can be found in [3] and is not repeated here. In this reference one term that is proportional to the derivative of the radius r of the circle fibration is missing in the fermionic terms in the action, a mistake that has also propagated to [12]. This term is crucial in order to combine the terms in the action into Weyl covariant derivatives that act on the matter fields. It is also needed in order for the action to be supersymmetric. In order to show that the action is supersymmetric, we need to use eqs. (3.2), (3.3) in this paper (which are new results) in addition to the Killing spinor equation (2.2) that was also presented in [3].

In section 3 we obtain eqs. (3.2) and (3.3) that impose conditions on the geometry. We then present two classes of solutions to these conditions. One class is conformally flat spacetimes of a certain type and the other class is of the form $\mathbb{R}^{1,1}\times$ multi-Taub-NUT. We do not expect these to exhaust all possible geometries so we are leaving a classification problem of geometries as a future problem. In this paper we are interested in singular fibrations where the graviphoton has a nonvanishing magnetic charge at the singular locus. For conformally flat spaces the graviphoton vanishes. This leaves us with $\mathbb{R}^{1,1}\times$ multi-Taub-NUT. Much of section 3 is a review of old results, but the conditions (3.2) and (3.3) are new, and the message we want to convey here is that these old results can be derived from just using (3.2) and (3.3), a strategy that we think may be applied to also find some other five-manifolds on which these 5d SYM theories may live. If the reader is happy with accepting multi-Taub-NUT as the starting point, then section 3 can be skipped.

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In section 4 we show that the 5d SYM theory on $\mathbb{R}^{1,1}\times$ multi-Taub-NUT has a gauge anomaly and a supersymmetry anomaly at the singular locus of the fibration. The gauge anomaly has been discussed previously in [5] where it was argued that this gauge anomaly shall be canceled by adding a chiral gauged WZW theory at the singular locus. The supersymmetry anomaly is a new result.

In section 5 we present a supersymmetric extension of the chiral gauged WZW theory on the singular locus. The singular locus in this case is $\mathbb{R}^{1,1}$ or a number of copies thereof, in the case of multi-Taub-NUT. This theory is supersymmetric if we do not make a supersymmetry variation of the gauge potential that we may view as a background gauge field from the perspective of WZW theory, coming from the 5d SYM theory. What we find is that if we make a supersymmetry variation of the gauge field induced from the 5d SYM theory, then the WZW theory receives a supersymmetry variation that exactly cancels against a corresponding term in the supersymmetry variation of the 5d SYM action.

In section 6 we discuss the WZW current and perform a consistency check using equations of motion. We also note that a mass term for the five scalar fields at the singular locus has to be added in order to fully cancel the supersymmetry anomaly of the 5d SYM.

In section 7 we have a discussion where put our results in the context of what has been done previously in the literature. We also discuss some issues in the previous literature that we have found.

There are a couple of appendices with further details. One particularly important new result in these appendices is the Weyl projection (A.1) that we need in order to preserve supersymmetry on a generic circle fibration under the dimensional reduction from the M5 brane.

2 The 5d super Yang-Mills

We put the Abelian M5 brane on a circle bundle with the metric

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu + r^2\left(d\psi + \kappa_\mu dx^\mu\right)^2$$

Here x^μ are coordinates on the five-dimensional base manifold, $\psi \sim \psi + 2\pi$ is the fiber coordinate parametrizing the circle fiber with circumference $2\pi r$. The radius r can depend on the base-manifold coordinates. The gravi-photon is denoted κ_μ and its curvature is denoted $W_{\mu\nu} = \partial_\mu\kappa_\nu - \partial_\nu\kappa_\mu$. We perform dimensional reduction along the circle fiber. That results in an Abelian 5d SYM. This has a non-Abelian generalization whose action is given by [3]

$$S = \int d^5x\sqrt{-G}\frac{1}{4\pi^2 r}\mathcal{L}$$

where the Lagrangian density is

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}^2 + \frac{r}{4}\varepsilon^{\mu\nu\lambda\kappa\tau}\omega(A)_{\mu\nu\lambda}W_{\kappa\tau} - \frac{1}{2}(D_\mu\phi^A)^2 - \frac{m^2}{2}(\phi^A)^2\\ &+\frac{i}{2}\bar{\psi}\Gamma^\mu D_\mu\psi - \frac{i}{4r}\bar{\psi}\Gamma^\mu\psi\partial_\mu r + \frac{ir}{16}\bar{\psi}\Gamma^{\mu\nu}\Gamma^{\widehat{\psi}}\psi W_{\mu\nu} - \frac{1}{2}\bar{\psi}\Gamma^A\Gamma^{\widehat{\psi}}[\psi,\phi^A] + \frac{1}{4}[\phi^A,\phi^B]^2\end{aligned}$$

where a trace over the Lie algebra generators is understood and not written out explicitly for notational simplicity. The field content is a gauge field A_μ, five scalar fields ϕ^A and a fermionic field ψ. We use here an 11d notation, where the gamma matrices are 11d. Likewise the fermionic field is an 11d Majorana spinor that is reduced to 6d where it is Weyl projected, and subsequently dimensionally reduced to 5d. The details are summarized in the appendix D. The mass squared is given by the following rather complicated expression

$$m^2 = \frac{R}{5} - \frac{r^2}{20} W_{\mu\nu}^2 + \frac{3}{5}\frac{\nabla^2 r}{r} - \left(\frac{\nabla_\mu r}{r}\right)^2$$

and

$$\omega(A)_{\mu\nu\lambda} = A_\mu \partial_\nu A_\lambda - \frac{2i}{3} A_\mu A_\nu A_\lambda$$

is the Chern-Simons three-form, where it is understood that the indices μ, ν, λ shall be antisymmetrized. We notice that the graviphoton term

$$\int d^5x \sqrt{-G} \frac{1}{4\pi^2 r} \frac{r}{4} \varepsilon^{\mu\nu\lambda\kappa\tau} \omega(A)_{\mu\nu\lambda} W_{\kappa\tau} = \frac{1}{16\pi^2} \int dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\kappa \wedge dx^\tau \omega(A)_{\mu\nu\lambda} W_{\kappa\tau} \tag{2.1}$$

makes no reference to the 5d metric. The supersymmetry variations are

$$\begin{aligned}
\delta\phi^A &= i\bar{\varepsilon}\Gamma^A \psi \\
\delta A_\mu &= i\bar{\varepsilon}\Gamma_\mu \Gamma^{\widehat{\psi}} \psi \\
\delta\psi &= \frac{1}{2}\Gamma^{\mu\nu}\Gamma^{\widehat{\psi}} \varepsilon F_{\mu\nu} + \Gamma^\mu \Gamma^A \varepsilon \left(D_\mu \phi^A + \frac{\partial_\mu r}{r}\phi^A\right) \\
&\quad + \frac{r}{2}\Gamma^A \Gamma^{\mu\nu}\Gamma^{\widehat{\psi}} \varepsilon W_{\mu\nu}\phi^A - \frac{i}{2}\Gamma^{AB}\Gamma^{\widehat{\psi}}\varepsilon[\phi^A, \phi^B]
\end{aligned}$$

where the supersymmetry parameter satisfies the following Killing spinor equation

$$\begin{aligned}
\nabla_\mu \varepsilon &= M_\mu \varepsilon \\
M_\mu &= \frac{1}{2r}\Gamma_\mu \Gamma^\nu \partial_\nu r - \frac{r}{8}\Gamma_\mu \Gamma^{\rho\sigma}\Gamma^{\widehat{\psi}} W_{\rho\sigma} - \frac{r}{4} W_{\mu\nu}\Gamma^\nu \Gamma^{\widehat{\psi}}
\end{aligned} \tag{2.2}$$

We may introduce a Weyl covariant derivative

$$\begin{aligned}
\mathcal{D}_\mu \phi &= D_\mu \phi^A + \frac{\partial_\mu r}{r}\phi^A \\
\mathcal{D}_\mu \psi &= D_\mu \psi + \frac{3}{2}\psi \frac{\partial_\mu r}{r} - \frac{1}{2}\Gamma_\mu{}^\nu \psi \frac{\partial_\nu r}{r}
\end{aligned}$$

In terms of this derivative the Lagrangian becomes

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}^2 + \frac{r}{4}\varepsilon^{\mu\nu\lambda\kappa\tau}\omega(A)_{\mu\nu\lambda}W_{\kappa\tau} - \frac{1}{2}(\mathcal{D}_\mu \phi^A)^2 + \frac{i}{2}\bar{\psi}\Gamma^\mu \mathcal{D}_\mu \psi \\
&\quad + \frac{ir}{16}\bar{\psi}\Gamma^{\mu\nu}\Gamma^{\widehat{\psi}}\psi W_{\mu\nu} - \frac{1}{2}\bar{\psi}\Gamma^A \Gamma^{\widehat{\psi}}[\psi, \phi^A] + \frac{1}{4}[\phi^A, \phi^B]^2
\end{aligned}$$

We now notice that the complicated mass term for the scalars has got completely absorbed into the Weyl covariant derivative. The Weyl transformations act as

$$\begin{aligned}
G_{\mu\nu} &\to e^{2\Omega} G_{\mu\nu} \\
r &\to e^{\Omega} r
\end{aligned}$$

$$\begin{aligned} \phi^A &\to e^{-\Omega}\phi^A \\ \psi &\to e^{-\frac{3}{2}\Omega}\psi \\ A_\mu &\to A_\mu \end{aligned}$$

Under these transformations the Weyl covariant derivative transforms Weyl covariantly as

$$\begin{aligned} \mathcal{D}_\mu\phi^A &\to e^{-\Omega}\mathcal{D}_\mu\phi^A \\ \mathcal{D}_\mu\psi &\to e^{-\frac{3}{2}\Omega}\mathcal{D}_\mu\psi \end{aligned}$$

Using this, it can be easily seen that the action is Weyl invariant. In addition to this, one may also show that if we transform the supersymmetry parameter as

$$\varepsilon \to e^{\frac{1}{2}\Omega}\varepsilon$$

then the supersymmetry variations are also Weyl invariant.

3 Conditions on the geometry

The Killing spinor equation (2.2) is derived from the 6d conformal Killing spinor equation of the M5 brane by imposing the condition $\partial_\psi\varepsilon = 0$. Solutions to the 6d conformal Killing spinor equation have been summarized in [2], but the condition $\partial_\psi\varepsilon = 0$ was not considered there.

The most general condition on the geometry from the Killing spinor equation is obtained by analyzing $[\nabla_\mu, \nabla_\nu]\varepsilon$. But not only will this lead to fairly complicated computations, but also we will not need this strong condition for our purposes here. For the purpose of checking supersymmetry of the action, we will only need the weaker conditions that arises from

$$\Gamma^{\mu\nu}\nabla_\mu\nabla_\nu\varepsilon = -\frac{R}{4}\varepsilon \tag{3.1}$$

This equation limis our search for possible geometries, but since (3.1) gives a weaker condition on the geometry than the Killing spinor equation itself, we will still need to show the existence of solutions to the Killing spinor equation. From (3.1) we obtain the following conditions

$$20\left(\frac{\nabla_\mu r}{r}\right)^2 - 8\frac{\nabla^2 r}{r} + \frac{r^2}{4}W_{\mu\nu}^2 = R \tag{3.2}$$

$$\nabla^\nu\left(\frac{1}{r}W_{\nu\mu}\right) + \frac{1}{4}\mathcal{E}_\mu{}^{\nu\lambda\rho\sigma}W_{\nu\lambda}W_{\rho\sigma} = 0 \tag{3.3}$$

Now this geometry is best understood not as a five-manifold, but as the six-manifold that is a circle bundle over the base five-manifold. We may express the first integrability condition in terms of the curvature scalar of the six-manifold if we note the relation

$$R_{6d} = R - \frac{r^2}{4}W_{\mu\nu}^2 - 2\frac{\nabla^2 r}{r}$$

that relates the curvature scalars R_{6d} of the six-manifold with the curvature scalar R of the base five-dimensional base-manifold. By using this relation, the integrability condition (3.2) can be expressed as

$$R_{6d} = 20\left(\frac{\nabla_\mu r}{r}\right)^2 - 10\frac{\nabla^2 r}{r} \tag{3.4}$$

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3.1 Conformally flat spacetimes

The constraint (3.4) is satisfied for a conformally flat metric, where in order to preserve supersymmetry under dimensional reduction, we need to restrict such a metric to be on the form

$$ds^2 = r^2\eta_{\mu\nu}dx^\mu dx^\nu + r^2 d\psi^2$$

where $r = r(x^\mu)$ does not depend on the fiber direction parametrized by $\psi \sim \psi + 2\pi$. To show this, we use the a standard formula for how the curvature scalar transforms under a Weyl rescaling $G_{\mu\nu} \to r^2 G_{\mu\nu}$, [6]

$$R \to \frac{1}{r^2}R - 2(D-1)\frac{1}{r^3}G^{\mu\nu}\nabla_\mu\nabla_\nu r - (D-1)(D-4)\frac{1}{r^4}G^{\mu\nu}\nabla_\mu r\nabla_\nu r$$

for a D-dimensional manifold. Here we take $D = 6$ and $G_{\mu\nu} = \eta_{\mu\nu}$. This formula then gives

$$R \to -10\frac{1}{r^3}\eta^{\mu\nu}\partial_\mu\partial_\nu r - 10\frac{1}{r^4}\eta^{\mu\nu}\partial_\mu r\partial_\nu r$$

In order to see that this corresponds to the constraint (3.4), we need to express the expressions on the right-hand side that are to be computed with respect to the metric $G_{\mu\nu} = r^2\eta_{\mu\nu}$ on the 5d base manifold, in terms of the flat matric $\eta_{\mu\nu}$. We have

$$\begin{aligned}(\nabla_\mu r)^2 &= \frac{1}{r^2}\eta^{\mu\nu}\partial_\mu r\partial_\nu r\\ \nabla^2 r &= \frac{1}{r^2}\eta^{\mu\nu}\partial_\mu r\partial_\nu r + \frac{3}{r^3}\eta^{\mu\nu}\partial_\mu r\partial_\nu r\end{aligned}$$

Using these results we find the result

$$R \to 20\left(\frac{\nabla_\mu r}{r}\right)^2 - 10\frac{\nabla^2 r}{r}$$

for the curvature scalar of the Weyl transformed metric, in agreement with the integrability constraint (3.4).

This, however, is not sufficient to prove the existence of a Killing spinor, since we only study a weaker version of the integrability condition. But it is not difficult to construct Killing spinor solutions to (2.2) explicitly for conformally flat metrics. The Killing spinor equation is

$$\partial_\mu\varepsilon = \frac{1}{2r}\varepsilon\partial_\mu r$$

It has the general solution

$$\varepsilon = \sqrt{r}\xi$$

for a constant spinor ξ. This solution can also be obtained by starting from the flat metric $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + d\psi^2$ and the Killing spinor ξ and then making a Weyl rescaling $\xi \to \sqrt{r}\xi$.

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3.2 $\mathbb{R}^{1,1}\times$ multi-Taub-NUT

Another class of six-manifolds that satisfy the constraints (3.3) and (3.4) are of the form $\mathbb{R}^{1,1} \times X$ where we take X to be a hyper-Kahler manifold with the metric

$$ds_X^2 = U d\vec{x}\cdot d\vec{x} + \frac{1}{U}\left(d\psi + \vec{\kappa}\cdot d\vec{x}\right)^2 \tag{3.5}$$

Here $\vec{x}$ parametrizes $\mathbb{R}^3$ and U is a function on $\mathbb{R}^3$. Since X is Ricci flat we have that $R_{6d} = 0$ and (3.4) reduces to

$$\frac{2}{r^2}G^{\mu\nu}\nabla_\mu r\nabla_\nu r = \frac{1}{r}G^{\mu\nu}\nabla_\mu\nabla_\nu r \tag{3.6}$$

where the radius is $r = \frac{1}{\sqrt{U}}$. The left-hand side of (3.6) is

$$\frac{1}{2U^3}\vec{\nabla}U\cdot\vec{\nabla}U$$

and the right-hand side is

$$\frac{1}{2U^3}\vec{\nabla}U\cdot\vec{\nabla}U - \frac{1}{2U^2}\nabla^2 U$$

We see that (3.6) is satisfied if U is harmonic everywhere on $\mathbb{R}^3$

$$\nabla^2 U = 0$$

except for points where $\frac{1}{U^2}$ vanishes. Let us now look at the constraint (3.3). With the metric (3.5) this constraint becomes

$$\partial_i(UW_{ij}) = 0 \tag{3.7}$$

in Cartesian coordinates on $\mathbb{R}^3$. In addition, it is necessary for W_{ij} to be closed everywhere outside the singular points. This leads to the solution

$$W_{ij} = \epsilon_{ijk}\partial_k U$$

which is automatically closed

$$\epsilon_{ijl}\partial_l W_{ij} = \partial_k\partial_k U = 0$$

outside singular points, and moreover it satisfies (3.7),

$$\partial_i(UW_{ij}) = \epsilon_{ijk}\partial_i U\partial_k U + U\epsilon_{ijk}\partial_i\partial_k U = 0$$

A general harmonic function on $\mathbb{R}^3$ has the form

$$U = \frac{1}{R^2} + \frac{1}{2}\sum_{i=1}^{N}\frac{1}{|\vec{x}-\vec{x}_i|}$$

which leads to the multi Taub-NUT metric [1]. There are singularities at $\vec{x} = \vec{x}_i$ for $i = 1,\ldots,N$. One may notice that such singularities are fine since close to a singularity we have $\frac{1}{U^2}\nabla^2 U \sim r^2\nabla^2\frac{1}{r} = 0$. Let us now take a closer look at a singularity, starting with $N = 1$ for which we have the Taub-NUT metric

$$ds_X^2 = U dx^i dx^i + \frac{1}{U}\left(d\psi + \kappa_i dx^i\right)^2$$

where

$$U = \frac{1}{R^2} + \frac{1}{2|\vec{x}|}$$

We can view the Taub-NUT space as a circle bundle over a base-manifold that is a rescaled version of $\mathbb{R}^3$ with the metric $G_{ij} = U\delta_{ij}$. If we use polar coordinates on $\mathbb{R}^3$ then the Taub-NUT metric is

$$ds_X^2 = U\left(dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right) + \frac{1}{U}\left(d\psi + \frac{1}{2}\cos\theta d\varphi\right)^2$$

and the radius of the circle fiber is $1/\sqrt{U}$ where

$$U = \frac{1}{R^2} + \frac{1}{2r}$$

The radius of the circle fibration (not to be confused with the radius r of $\mathbb{R}^3$) vanishes at $r = 0$ so the circle fibration is singular. But the manifold is nonetheless smooth at $r = 0$. To see that, we may look at the metric close to $r = 0$. When $r\ell R^2$ we may neglect the constant term in U so the metric is

$$ds_X^2 = \frac{1}{2r}\left(dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right) + 2r\left(d\psi + \frac{1}{2}\cos\theta d\varphi\right)^2$$

The interpretation of this metric becomes clearer if we define

$$r = \frac{\rho^2}{2}$$

Then the metric becomes

$$ds_X^2 = d\rho^2 + \frac{\rho^2}{4}\left(d\theta^2 + \sin^2\theta d\varphi^2\right) + \frac{\rho^2}{4}\left(2d\psi + \cos\theta d\varphi\right)^2 \tag{3.8}$$

This is now the metric of flat $\mathbb{R}^4$. For the details of the construction of this flat metric we refer to appendix B. The important point is that ψ is 2π periodic.

If we look at the Taub-NUT metric in the other limit when $r \gg R^2$ we may approximate $U \approx \frac{1}{R^2}$ and the metric is

$$ds_X^2 = \frac{1}{R^2}dx^i dx^i + R^2\left(d\psi + \kappa_i dx^i\right)^2$$

which describes a cylinder of radius R. The Taub-NUT space thus interpolates between flat $\mathbb{R}^4$ at the origin and a cylinder at infinity.

The graviphoton one-form is

$$\kappa = \frac{1}{2}\cos\theta d\varphi$$

Its curvature two-form $w = d\kappa$ is

$$W = -\frac{1}{2}\sin\theta d\theta \wedge d\varphi \tag{3.9}$$

that is integrated over S^2 to

$$\oint_{S^2} W = -2\pi$$

Another way of expressing this curvature two-form is as

$$W_{ij} = \epsilon_{ijk}\partial_k U$$

in flat $\mathbb{R}^3$ with metric δ_{ij} where $\epsilon_{123} = 1$ and totally antisymmetric. It is also useful to express this same relation in a covariant form by using the metric $G_{ij} = U\delta_{ij}$ of the base. Covariantly we then have

$$W_{ij} = \frac{1}{G^{1/6}}\varepsilon_{ijk}G^{k\ell}\partial_\ell U \tag{3.10}$$

where we define the covariant tensor $\varepsilon_{ijk} = \sqrt{G}\epsilon_{ijk}$. We may now confirm the equivalence of the two expressions (3.9) and (3.10) by choosing polar coordinates on the base. We then need to study the expression

$$W_{\theta\varphi} = \frac{1}{G^{1/6}}\varepsilon_{\theta\varphi r}G^{rr}\partial_r U$$

We start by rewriting everything in terms of U using

$$G = U^3$$
$$G^{rr} = \frac{1}{U}$$

Then

$$W_{\theta\varphi} = \frac{1}{U\sqrt{U}}\varepsilon_{\theta\varphi r}\partial_r U$$

Next we notice that

$$\partial_r U = -\frac{1}{2r^2}$$

We then need to address the quesion of finding an explicit expression for the antisymmetric tensor component

$$\varepsilon_{\theta\varphi r} = \sqrt{G}\epsilon_{\theta\varphi r} = U^{3/2}\epsilon_{\theta\varphi r}$$

Here $\epsilon_{\theta\varphi r} = r^2\sin\theta$ is the determinant of the Jacobian when we go from Cartesian to Polar coordinates. We now have all ingredients. Putting them together, we obtain

$$W_{\theta\varphi} = -\frac{1}{2}\sin\theta$$

which is in perfect agreement with (3.9).

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For multi-Taub-NUT we take

$$U = \frac{1}{R^2} + \frac{1}{2}\sum_{I=1}^{N} \frac{1}{|\vec{x} - \vec{x}_I|}$$

and the gravi-photon is implicitly defined through

$$W_{ij} = \varepsilon_{ijk}\partial_k U$$

This is a sum of terms,

$$\begin{aligned} W_{ij} &= \sum_I W_{ij}^I \\ W_{ij}^I &= \varepsilon_{ijk}\partial_k U^I \\ U^I &= \frac{1}{2}\frac{1}{|\vec{x} - \vec{x}_I|} \end{aligned}$$

which means that the gravi-photon itself is a sum of terms,

$$\kappa_i = \sum_I \kappa_i^I$$

When $\vec{x} \approx \vec{x}_I$ the geometry is locally that of flat $\mathbb{R}^4$ if no other points $\vec{x}_J$ coincide with $\vec{x}_I$. If Q_I points coincide at $\vec{x}_I$, then we have, locally near that point, the metric

$$ds_X^2 = \frac{Q}{2r}\left(dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right) + \frac{2r}{Q}\left(d\psi + \frac{Q}{2}\cos\theta d\varphi\right)^2$$

If we put $r = \frac{\rho^2}{2}$ we get

$$\frac{1}{Q}ds_X^2 = d\rho^2 + \frac{\rho^2}{4}\left(d\theta^2 + \sin^2\theta d\varphi^2\right) + \frac{\rho^2}{4Q^2}\left(2d\psi + Q\cos\theta d\varphi\right)^2$$

If we put $\psi = Q\widetilde{\psi}$ we get

$$\frac{1}{Q}ds_X^2 = d\rho^2 + \frac{\rho^2}{4}\left(d\theta^2 + \sin^2\theta d\varphi^2\right) + \frac{\rho^2}{4}\left(2d\widetilde{\psi} + \cos\theta d\varphi\right)^2$$

where $\widetilde{\psi} \sim \widetilde{\psi} + \frac{2\psi}{Q}$. This is the metric of the orbifold $\mathbb{C}^2/\mathbb{Z}_Q$ with the identification $(z_1, z_2) \sim (e^{2\pi i/Q}z_1, e^{2\pi i/Q}z_2)$.

We conclude that the multi-Taub-NUT space TN_N is everywhere smooth for N non-coinciding singular points. But when Q singular points coincide we get an orbifold singularity of the type $\mathbb{C}^2/\mathbb{Z}_Q$.

Having found these geometries of multi-Taub-NUT, it remains to establish the existence of Killing spinor solutions. To this end, we will simply review an argument from [5] that shows that such six-manifolds support 8 real covariantly constant spinors. The existence of a covariantly constant spinor on X implies that the Ricci tensor must vanish. For hyper-Kahler manifolds the Ricci tensor vanishes and the generic $\mathrm{SO}(4) = \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$ holonomy group is reduced to $\mathrm{SU}(2)_+$. According to the holonomy principle, a covariantly

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constant spinor is a singlet under the holonomy group $SU(2)_+$. Let us represent gamma matrices in the SO(4) tangent space group of X as

$$\begin{aligned}\gamma^i &= \sigma^i \otimes \sigma^1 \\ \gamma^4 &= 1 \otimes \sigma^2\end{aligned}$$

Then the embedding of the $SU(2)_\pm$ generators into SO(4) is done as

$$\sigma^i P_\pm = -\frac{i}{2}\gamma^{i4} \mp \frac{i}{4}\varepsilon^{ijk}\gamma^{jk}$$

where $P_\pm = \frac{1}{2}\left(1 \otimes 1 \pm 1 \otimes \sigma^3\right)$. From this, we conclude that spinors that are not rotated by the holonomy group $SU(2)_+$ satisfy $P_+\psi = 0$ so they are anti-Weyl spinors. The six-manifold has a tangent space group $SO(1,5)$. A Weyl spinor under this tangent space group has 4 complex components and the constraint that it shall be invariant under $SU(2)_+$ imposes another Weyl projection leading to 2 complex components. For the M5 brane there is in addition an SO(5) R-symmetry group and the spinor has 4 internal SO(5) R-symmetry spinor components that leads to in total $2 \times 4 = 8$ complex spinor components, but there is a Majorana condition that one can impose on a spinor in $SO(1,5) \times SO(5) \subset SO(1,10)$ resulting in 8 real components. These correspond to 8 real supercharges [5].

4 Gauge and supersymmetry anomalies

Let us assume that the circle fiber vanishes on a two-dimensional submanifold Σ with Lorentzian signature. Because the circle fibration degenerates on Σ, it is somewhat difficult to analyze what happens there directly. One way to circumvent this difficulty is by considering a tubular neighborhood around Σ. We then consider a five-manifold with a boundary four-manifold of the form $\Sigma \times S^2_{r_0}$ where $S^2_{r_0}$ is a small sphere that is enclosing Σ and then in the end we shall take the limit $r_0 \to 0$. Let us assume that the gauge group is Abelian. Since $S^2_{r_0}$ has a nonvanishing second cohomology group, it can support a nonzero magnetic flux. Since there are two gauge fields, A_μ and κ_μ there could apriori be two types of magnetic fluxes going through $S^2_{r_0}$ as well. For the Yang-Mills gauge field such a magnetic flux would have to come from D1 branes ending on the D4 brane. By uplift to the M5 brane, we would need M2 branes ending on the M5 brane. But there is no natural place for the M2 branes to end because the M5 brane geometry is completely smooth.[1] We may also understand this from the fact that there is no three-cycle in the M5 brane geometry close to the singular locus. The geometry close to the singular locus is $R^{1,1} \times \mathbb{R}^4$ for the Taub-NUT space and similarly for the multi-Taub-NUT case as long as this is regular manifold, in which case there is clearly no three-cycle and hence no possibility of having a magnetic flux. In this paper we will for simplicity not consider situations where the multl-Taub-NUT space develops orbifold singularities as singular loci coincide.

The gravi-photon gauge field does have a nonzero flux through $S^2_{r_0}$. It is entirely fixed by the geometry and can be obtained directly from the metric. By allowing r_0 to take

[1] I would like to thank the referee for providing this argument.

any value greater than zero, we find that this magnetic flux has to be produced by a delta function localized on Σ. We thus have the following modifications of the Bianchi identity,

$$\partial_\mu W_{\nu\lambda} + \partial_\lambda W_{\mu\nu} + \partial_\nu W_{\lambda\mu} = -2\pi \sum_{I=I}^{n} Q^I \delta^I_{\mu\nu\lambda} \tag{4.1}$$

The charge Q^I denotes the charge of submanifold Σ_I. For the multi-Taub-NUT, these charges sum up to N

$$\sum_{I=I}^{n} Q^I = N$$

where $N = 1, 2, 3, \ldots$ is the integer charactarizing the multi-Taub-NUT. Further, $\delta^I_{\mu\nu\lambda}$ denotes the Poincare dual of Σ_I, defined as

$$\int d^5x\sqrt{-G}\varepsilon^{\mu\nu\lambda\rho\sigma}\frac{1}{6}\delta^I_{\mu\nu\lambda}\frac{1}{2}\omega_{\rho\sigma} = \int_{\Sigma_I} d^2\sigma\sqrt{-\eta}\varepsilon^{\alpha\beta}\frac{1}{2}\omega_{\alpha\beta} \tag{4.2}$$

for an arbitrary test-two-form $\omega_{\mu\nu}$. Here $\eta_{\alpha\beta}$ denotes the induced metric on Σ_I that we parametrize by coordinates σ^α.

There are two options for writing the gravi-photon term. Either

$$\mathcal{L}_{grav} = \frac{r}{4}\varepsilon^{\mu\nu\lambda\rho\sigma}\omega(A)_{\mu\nu\lambda}W_{\rho\sigma} \tag{4.3}$$

or

$$\mathcal{L}'_{grav} = -\frac{r}{8}\varepsilon^{\mu\nu\lambda\rho\sigma}F_{\mu\nu}F_{\lambda\rho}\kappa_\sigma \tag{4.4}$$

The two ways differ by a total derivative,

$$\mathcal{L}_{grav} = \mathcal{L}'_{grav} + r\nabla_\rho\left(\frac{1}{2}\varepsilon^{\mu\nu\lambda\rho\sigma}\omega(A)_{\mu\nu\lambda}\kappa_\sigma\right)$$

inconsequential for the equation of motion, but $\mathcal{L}'_{grav}$ is gauge invariant while $\mathcal{L}_{grav}$ is not, so one might prefer to use $\mathcal{L}'_{grav}$. But $\mathcal{L}'_{grav}$ is not invariant under a reparametrization of the fiber coordinate. Under a reparametrization $\psi \to \psi' = \psi + f$ the graviphoton field transforms as $\kappa_\mu \to \kappa'_\mu - \partial_\mu f$. So this way of writing the gravi-photon term is not invariant under such a 'geometric' gauge transformations.

The gauge field equation of motion is

$$\nabla_\nu\left(\frac{1}{r}F^{\nu\mu}\right) + \frac{1}{4}\varepsilon^{\mu\nu\lambda\rho\sigma}F_{\nu\lambda}W_{\rho\sigma} = 0 \tag{4.5}$$

This equation of motion holds irrespectively of whether we use the gravi-photon term (4.3) or (4.4) in the action. If we act by ∇_μ on the left-hand side, then we get

$$\frac{2}{r}R_{\mu\nu}F^{\mu\nu} + \frac{1}{4}\varepsilon^{\mu\nu\lambda\rho\sigma}F_{\nu\lambda}\nabla_\mu W_{\rho\sigma} = 0 \tag{4.6}$$

The first term is zero because the Ricci scalar is symmetric while $F_{\mu\nu}$ is antisymmetric. But if Q is nonzero, then the second term is not zero. In this case, the equation of motion (4.6)

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becomes inconsistent, and must be modified somehow. Following [4], we make the following ansatz for such a modified equation of motion,

$$\nabla_\nu \left(\frac{1}{r} F^{\nu\mu}\right) + \frac{1}{4}\varepsilon^{\mu\nu\lambda\rho\sigma} F_{\nu\lambda} W_{\rho\sigma} = J^\mu \tag{4.7}$$

Now if we act on both sides by ∇_μ we get

$$2\pi\varepsilon^{\mu\nu\lambda\rho\sigma} \frac{1}{2} F_{\nu\lambda} \frac{1}{6} \sum_{I=1}^{n} Q^I \delta^I_{\mu\rho\sigma} = \nabla_\mu J^\mu \tag{4.8}$$

and if the left-hand side is nonzero, then this shows that 5d SYM can not be the full story. Something more is needed that can produce J_μ.

Let us then use the gravi-photon term (4.3), which is not gauge invariant but diffeomorphism invariant. If we assume that the gauge group is non-Abelian, then under a finite gauge transformation

$$A^g = g^{-1} A_\mu g + i g^{-1} \partial_\mu g \tag{4.9}$$

the Chern-Simons three-form transforms as

$$\omega(A^g)_{\mu\nu\lambda} = \omega(A)_{\mu\nu\lambda} + \partial_\nu \left(i \partial_\mu g g^{-1} A_\lambda\right) + \frac{1}{3} g^{-1} \partial_\mu g g^{-1} \partial_\nu g g^{-1} \partial_\lambda g \tag{4.10}$$

The last term gives rise to the following term in the action,

$$\frac{1}{16\pi^2} \oint_{S^2} dx^i \wedge dx^j W_{ij} \int_{\mathbb{R}_+ \times \mathbb{R}^{1,1}} dx^\mu \wedge dx^\nu \wedge dx^\lambda \frac{1}{3} \mathrm{tr}\left(g^{-1} \partial_\mu g g^{-1} \partial_\nu g g^{-1} \partial_\lambda g\right)$$

Using that the magnetic charge is $-2\pi Q$ gives

$$-\frac{Q}{12\pi} \int_{\mathbb{R}_+ \times \mathbb{R}^{1,1}} dx^\mu \wedge dx^\nu \wedge dx^\lambda \mathrm{tr}\left(g^{-1} \partial_\mu g g^{-1} \partial_\nu g g^{-1} \partial_\lambda g\right)$$

The manifold over which this is to be integrated is $\mathbb{R}_+ \times \mathbb{R}^{1,1}$ where $\mathbb{R}_+$ is the radial direction outwards from the S^2 over which we integrated W_{ij}. So this three-manifold has a boundary S^2. The metric on this manifold does not enter since the term is topological. Let us Wick rotate $\mathbb{R}^{1,1}$ into $\mathbb{R}^2$ that we subsequently compactify into S^2. Then the manifold over which we integrate has turned into $\mathbb{R}^3$ where we have removed a small ball at the center. Expecting nothing particular happens to the field g at the origin, we can let this ball shrink to zero size and integrate over the full $\mathbb{R}^3$ space. Assuming that g falls off sufficiently fast at infinity, this amounts to integrating over S^3 as we may then identify all points at infinity and make a one-point compactification of $\mathbb{R}^3$ to S^3 by adding the point at infinity. When integrating over S^3 this term is quantized in integer multiples of $2\pi i$ where the integer is the winding number as we map S^3 into an SU(2) subgroup of the gauge group by the field g.

For the second term in (4.10) we applying (4.2) after making an integration by parts. The final result is that the gravi-photon term transforms as

$$S_{grav}(A^g) = S_{grav}(A) - \sum_{I=1}^{n} \frac{iQ^I}{4\pi} \int_{\Sigma_I} dx^\mu \wedge dx^\nu \mathrm{tr}\left(\partial_\mu g g^{-1} A_\nu\right) - 2\pi n_w$$

where n_w is an integer winding number. This integer plays no role in the quantum theory where one considers the exponentiated action e^{iS} in Lorentzian signature.

The action is also not supersymmetric. But to study this problem it is advantagous to first consider a more general setup of a five-manifold with a generic four-manifold boundary. Then under a supersymmetry variation of the action, we will pick up the following boundary terms,

$$\begin{aligned}\delta S &= \frac{1}{4\pi^2}\int d^5x\partial_\mu\left(\sqrt{-G}b^\mu\right)\\ b^\mu &= -\frac{1}{r}F^{\mu\nu}\delta A_\nu - \frac{1}{4}\varepsilon^{\mu\nu\lambda\rho\sigma}A_\nu\delta A_\lambda W_{\rho\sigma} - \frac{1}{r}D^\mu\phi^A\delta\phi^A - \frac{i}{2r}\bar{\psi}\Gamma^\mu\delta\psi\end{aligned}$$

Explicitly we get

$$\begin{aligned}b^\mu &= -\frac{i}{4r}\bar{\psi}\Gamma^{\rho\sigma}\Gamma^\mu\varepsilon\Gamma^{\widehat{\psi}}\varepsilon F_{\rho\sigma} + \frac{i}{2r}\bar{\psi}\Gamma^\rho\Gamma^\mu\Gamma^A\varepsilon D_\rho\phi^A - \frac{1}{4}\varepsilon^{\mu\nu\lambda\rho\sigma}A_\nu i\bar{\varepsilon}\Gamma_\lambda\Gamma^{\widehat{\psi}}\psi W_{\rho\sigma}\\ &\quad -\frac{i}{2r}\bar{\psi}\Gamma^\mu\Gamma^\rho\Gamma^A\varepsilon\frac{\partial_\rho r}{r}\phi^A - \frac{i}{4}\bar{\psi}\Gamma^\mu\Gamma^{\rho\sigma}\Gamma^A\Gamma^{\widehat{\psi}}\varepsilon W_{\rho\sigma}\phi^A - \frac{1}{4r}\bar{\psi}\Gamma^\mu\Gamma^{AB}\Gamma^{\widehat{\psi}}\varepsilon[\phi^A,\phi^B]\end{aligned}$$

We would now have liked to proceed along the lines of reference [7] and find boundary degrees that we add so that the total action becomes supersymmetric without imposing boundary conditions. This strategy works nicely for low-dimensional super Yang-Mills. But for 5d SYM this strategy fails. So instead we will use a different approach. In the end we are not interested in the 4d boundary theory, but in a 2d submanifold theory. So we want to take the limit where the radius r_0 goes to zero. As we are still integrating over $S^2_{r_0}$, taking the limit r_0 to zero, amounts to averaging over the radial directions. For most terms, such an averaging will produce zero net result because the fields are not expected to vary very much close to the singular point. From a 6d viewpoint, this singular point is perfectly regular and we expect the fields to be smooth close to the singularity. We then expect that the only terms that will survive the integration over $S^2_{r_0}$ will be those that arise as magnetic charges when we integrate the two-form $W_{\mu\nu}$. So we may isolate the terms that involve this two-form and only consider those terms.

Let us analyze this problem on $\mathbb{R}^{1,1}\times TN$. We use the relation (A.1) which leads to

$$b^\mu = \frac{i}{8}\bar{\psi}\Gamma^A\Gamma^\mu\Gamma^{\rho\sigma}\Gamma^{\widehat{\psi}}\varepsilon W_{\rho\sigma}\phi^A - \frac{1}{4}\varepsilon^{\mu\nu\lambda\rho\sigma}i\bar{\varepsilon}\Gamma_\lambda\Gamma^{\widehat{\psi}}\psi A_\nu W_{\rho\sigma} + \dots$$

where we extracted terms proportional to $W_{\mu\nu}$. Now we use the fact that $W_{\mu\nu}$ has components only in the $\mathbb{R}^3$ base of TN where it is a magnetic monopole of strength $-2\pi Q$. The boundary is taken to be $\mathbb{R}^{1,1}\times S^2$ and the normal direction is the radial direction. Thus we are interested in the radial component

$$b^r = \frac{i}{4}\bar{\psi}\Gamma^A\Gamma^{r\theta\varphi}\Gamma^{\widehat{\psi}}\varepsilon W_{\theta\varphi}\phi^A + \frac{1}{2}\mathcal{E}^{\mu\nu}\mathcal{E}^{r\theta\varphi}i\bar{\psi}\Gamma^{\widehat{\psi}}\Gamma_\mu\varepsilon A_\nu W_{\theta\varphi} + \dots$$

Now we need to examine the Weyl projections. On TN we have the Weyl projection (A.2)

$$\Gamma^{123}\Gamma^{\widehat{\psi}}\varepsilon = \varepsilon$$

By combining that we the 6d Weyl projection

$$\Gamma^0\Gamma^{123}\Gamma^4\Gamma^{\widehat{\psi}}\varepsilon = -\varepsilon$$

we get

$$\Gamma^{04}\varepsilon = -\varepsilon$$

We define $\varepsilon^{01234\widehat{\psi}} = \varepsilon^{01234} = 1$, and we put $\varepsilon^{\mu\nu ijk} = \varepsilon^{\mu\nu}\varepsilon^{ijk}$ where we put $\varepsilon^{04} = -1$ and $\epsilon^{123} = 1$. Then we have

$$\epsilon^{\nu\lambda}\Gamma_\nu\varepsilon = \Gamma^\lambda\varepsilon$$

Using this relation, we get

$$b^r = \frac{i}{4}\mathcal{E}^{r\theta\varphi}\bar{\psi}\Gamma^A\varepsilon W_{\theta\varphi}\phi^A + \frac{1}{2}\mathcal{E}^{r\theta\varphi}i\bar{\psi}\Gamma^{\widehat{\psi}}\Gamma^\nu\varepsilon A_\nu W_{\theta\varphi} + \dots$$

This leads to a variation of the action given by

$$\delta S_{5d} = -\sum_I \frac{Q_I}{8\pi}\int_{\Sigma_I} d^2x i\bar{\psi}\Gamma^A\varepsilon\phi^A - \sum_I \frac{Q_I}{4\pi}\int_{\Sigma_I} d^2x i\bar{\psi}\Gamma^{\widehat{\psi}}\Gamma^\mu\varepsilon A_\mu$$

In a 5d reduced notation

$$\delta S_{5d} = \sum_I \frac{Q_I}{8\pi}\int_{\Sigma_I} d^2x i\chi\tau^A\mathcal{E}\phi^A - \sum_I \frac{Q_I}{4\pi}\int_{\Sigma_I} d^2x\chi\gamma^\mu\mathcal{E}A_\mu \tag{4.11}$$

where $\mathcal{E}^{\alpha\dot{\alpha}} = \varepsilon^{\alpha-\dot{\alpha}}$ and $\chi^{\alpha\dot{\alpha}} = \psi^{\alpha+\dot{\alpha}}$. The 5d conjugate spinor is defined as $\bar{\chi}_{\beta\dot{\beta}} = \chi^{\alpha\dot{\alpha}}C_{\alpha\beta}C_{\dot{\alpha}\dot{\beta}}$. We further reduce to 2d notation by decomposing $\alpha = (u, m)$ and put $\gamma^\mu = (\gamma^\mu)^u{}_v\delta^m_n$. Then $\chi\gamma^\mu\mathcal{E} = \chi_{um\dot{\alpha}}(\gamma^\mu)^u{}_v\mathcal{E}^{vm\dot{\alpha}}$. In the 2d reduced notation, the sum over m is trivial and gives three identical copies. For more details on our spinor notations, we refer to appendix D.

5 WZW theories on Σ_I

A supersymmetric WZW theory has been constructed in [13] using a superfield formulation. Here we will use a component formulation instead. Let us start by analyzing the following supersymmetric WZW Lagrangian (where $\varepsilon^{01} = 1$),

$$\begin{aligned}\mathcal{L}_{WZW}(g, A_\mu) = &\frac{1}{8\pi}\mathrm{tr}\left(g^{-1}\left(\partial_\mu - iA_\mu\right)g\right)^2 - \frac{i}{4\pi}\varepsilon^{\mu\nu}\mathrm{tr}\partial_\mu g g^{-1}A_\nu - \frac{i}{8\pi}\mathrm{tr}\left(\bar{\lambda}\gamma^\mu\partial_\mu\lambda\right)\\ &+\frac{1}{12\pi}\varepsilon^{\mu\nu\lambda}\mathrm{tr}\left(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\lambda g\right)\end{aligned}$$

To get the action, we should integrated this over Σ_I, except for the last term that should be integrated over some three-manifold whose boundary is Σ_I. This Lagrangian is invariant under the supersymmetry variations

$$\begin{aligned} g^{-1}\delta g &= -\bar{\varepsilon}\lambda\\ \delta\lambda &= -i\gamma^\mu\varepsilon g^{-1}\partial_\mu g\end{aligned}$$

provided we impose the following chiral projection on the supersymmetry parameter,

$$\gamma^{01}\varepsilon = \varepsilon$$

Under a variation of A_μ, keeping g and λ fixed, we have

$$\delta\mathcal{L}_{WZW}(g, A_\mu) = -\frac{i}{4\pi}\partial_\mu g g^{-1}\left(\delta A^\mu + \varepsilon^{\mu\nu}\delta A_\nu\right) - \frac{1}{4\pi}A_\mu\delta A^\mu$$

If we make the specific variation

$$\delta A_\mu = -\chi\gamma^\mu\mathcal{E}$$

as induced from 5d, then we get

$$\delta\mathcal{L}_{WZW}(g, A_\mu) = \frac{i}{4\pi}\partial_\mu g g^{-1}\chi\left(\gamma^\mu + \varepsilon^{\mu\nu}\gamma_\nu\right)\mathcal{E} + \frac{1}{4\pi}A_\mu\chi\gamma^\mu\mathcal{E}$$

The first term vanishes by the projection $\gamma^{01}\mathcal{E} = \mathcal{E}$. So if we add the following 2d action,

$$S_{WZW} = \sum_I Q_I \int_{\Sigma_I} d^2x\mathcal{L}_{WZW}(g_I, A_\mu) \tag{5.1}$$

then we cancel the second term in (4.11).

The resulting Lagrangian is also not fully gauge invariant, where the gauge variation acts on the fields as

$$\begin{aligned} A_\mu &\to hA_\mu h^{-1} - i\partial_\mu h h^{-1} \\ g &\to hg \end{aligned} \tag{5.2}$$

for a gauge parameter h. One obviously gauge noninvariant term is $\mathcal{L}_{\text{non}} = -\frac{i}{4\pi}\varepsilon^{\mu\nu}\text{tr}\partial_\mu g g^{-1}A_\nu$, which transforms into

$$\begin{aligned} \mathcal{L}_{\text{non}} \to \mathcal{L}_{\text{non}} &- \frac{i}{4\pi}\varepsilon^{\mu\nu}\text{tr}\left(h^{-1}\partial_\mu h A_\nu\right) \\ &- \frac{1}{4\pi}\varepsilon^{\mu\nu}\left(\partial_\mu h h^{-1}\partial_\nu h h^{-1} + \partial_\mu g g^{-1}h^{-1}\partial_\nu h\right) \end{aligned}$$

The other gauge noninvariant term is the WZ term $\mathcal{L}_{WZ} = \frac{1}{12\pi}\varepsilon^{\mu\nu\lambda}\text{tr}\left(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\lambda g\right)$ that transforms into

$$\begin{aligned} \mathcal{L}_{WZ} \to \mathcal{L}_{WZ} &+ \frac{1}{12\pi}h^{-1}\partial_\mu h h^{-1}\partial_\nu h h^{-1}\partial_\lambda h \\ &+ \frac{1}{4\pi}\varepsilon^{\lambda\mu\nu}\partial_\lambda\left(\partial_\mu h h^{-1}\partial_\nu h h^{-1} + \partial_\mu g g^{-1}h^{-1}\partial_\nu h\right) \end{aligned}$$

We now see that many terms cancel between $\delta\mathcal{L}_{\text{non}}$ and δ_{WZ} and we are left with the gauge variation

$$\delta S_{2d} = -\sum_I \frac{iQ_I}{4\pi}\int dx^\mu \wedge dx^\nu\text{tr}\left(h^{-1}\partial_\mu h A_\nu\right)$$

In order to match the gauge variation (5.2) with the gauge transformation (4.9) we shall substitute h here by g^{-1}, in which case we get

$$\delta S_{2d} = \sum_I \frac{Q_I}{4\pi}\int dx^\mu \wedge dx^\nu\text{tr}\left(i\partial_\mu g g^{-1}A_\nu\right)$$

that is cancelling the gauge variation of S_{5d}.

6 The WZW current

If we would vary the gauge potential in the WZW theory, then we would get the equation of motion

$$\partial_\mu g g^{-1} - \varepsilon_{\mu\nu}\partial^\nu g g^{-1} - iA_\mu = 0 \tag{6.1}$$

This equation is not gauge invariant. But that is not so surprising. The gauge potential in the WZW theory is a background field and we are not supposed vary a background field. However, it is a dynamical field in the 5d SYM theory and so if we vary the gauge potential in the combined system of 5d SYM plus the WZW theory, then we should recover a gauge invariant equation of motion. The gauge invariant completion of the left-hand side in (6.1) is

$$D_\mu g g^{-1} - \varepsilon_{\mu\nu} D^\nu g g^{-1} \tag{6.2}$$

Thus we are looking for a missing term $i\varepsilon_{\mu\nu}A^\nu$ that should come from the 5d SYM upon variation of the gauge potential, so that (6.1) is completed into (6.2). Let us examine the gravi-photon term, and the following term

$$\frac{1}{16\pi^2} A_\mu \partial_\nu \delta A_\lambda W_{\rho\sigma}$$

when we vary the gauge potential. To derive the Euler-Lagrange equation of motion, we would make an integration by parts. But here, when we make an integration by parts we must be careful because of (4.1). It is not too hard to see that this will exactly produce our missing 2d term that will complete (6.1) into (6.2).

Let us follow [3] and put

$$J_\mu = D_\mu g g^{-1} - \varepsilon_{\mu\nu} D^\nu g g^{-1}$$

Then the 5d SYM equation of motion is modified to

$$D^\nu \left(\frac{1}{r} F_{\nu\mu}\right) + \frac{1}{4}\varepsilon^{\mu\nu\lambda\rho\sigma} F_{\nu\lambda} W_{\rho\sigma} = \sum_I i\pi Q^I \delta^I_{123} J_\mu \tag{6.3}$$

Let us now assume the gauge group is Abelian for simplicity. Nothing essential changes in the argument we will make below when the gauge group is non-Abelian. When the gauge group is Abelian, we put

$$g = e^{i\phi}$$

and then ϕ will be a 2π periodic scalar field, and the WZW action is given by

$$S_{WZW} = -\frac{Q}{8\pi}\int d^2x \left((\partial_\mu\phi - A_\mu)^2 + \varepsilon^{\mu\nu}\phi F_{\mu\nu} + i\bar{\lambda}\gamma^\mu\partial_\mu\lambda\right)$$

that is invariant under the supersymmetry variations

$$\begin{aligned}\delta\phi &= i\bar{\varepsilon}\lambda \\ \delta\lambda &= \gamma^\mu\varepsilon(\partial_\mu\phi - A_\mu)\end{aligned}$$

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for $\gamma\varepsilon = \varepsilon$. Under the gauge variation

$$\delta\phi = \Lambda$$
$$\delta A_\mu = \partial_\mu \Lambda$$

we have the variation

$$\delta S_{WZW} = \frac{Q}{8\pi}\int d^2x \Lambda \varepsilon^{\mu\nu} F_{\mu\nu}$$

that is canceling against the gauge variation of the 5d SYM action,

$$\delta S_{5d} = -\frac{Q}{8\pi}\int d^2x \Lambda \varepsilon^{\mu\nu} F_{\mu\nu}$$

The equation of motion for ϕ is given by

$$\nabla^2\phi - \nabla^\mu A_\mu = \frac{1}{2}\varepsilon^{\mu\nu}F_{\mu\nu} \tag{6.4}$$

For the gauge field, we should consider the combined system of 5d SYM coupled to 2d WZW, and then we find the equation of motion

$$\nabla_\nu\left(\frac{1}{r}F^{\nu\mu}\right) + \frac{1}{4}\varepsilon^{\mu\nu\lambda\rho\sigma}F_{\nu\lambda}W_{\rho\sigma} = -\pi Q\delta_{123}\left(\nabla^\mu\phi - \varepsilon^{\mu\nu}\nabla_\nu\phi - A^\mu + \varepsilon^{\mu\nu}A_\nu\right)$$

Acting on both sides by ∇_μ we get

$$\frac{1}{4}\varepsilon^{\mu\nu\lambda\rho\sigma}F_{\nu\lambda}\partial_\mu W_{\rho\sigma} = -\pi Q\delta_{123}\left(\nabla^2\phi - \nabla_\mu A^\mu + \frac{1}{2}\varepsilon^{\mu\nu}F_{\mu\nu}\right)$$

Consistency with (6.4) implies that

$$\partial_i W_{jk} + \partial_k W_{ij} + \partial_j W_{ki} = -2\pi Q\delta_{123}$$

which is a consistency check that everything fits together nicely.

In addition to this WZW theory, if we also add a mass term for the five scalar fields that is localized to Σ_I,

$$S_{\text{mass}} = -\frac{Q_I}{16\pi}\int d^2x \phi^A\phi^A$$

then that has the supersymmetry variation

$$\delta S_{\text{mass}} = -\frac{Q_I}{8\pi}\int d^2x i\chi\tau^A\mathcal{E}\phi^A$$

that cancels the first term in (4.11).

7 Discussion

Assuming an $\mathbb{R}^{1,1}\times$multi-Taub-NUT geometry that is smooth everywhere, we do not expect anything particular to happen at the singular locus of the circle fibration in the tensor multiplet theory describing a single M5 brane. In particular, we do not expect new degrees of freedom shall be added there to the tensor multiplet. However, on $\mathbb{R}^{1,1}\times$multi-Taub-NUT there are also solitonic solutions of the form

$$H = \sum_I h^I_\mu dx^\mu \wedge \Omega^I \tag{7.1}$$

where Ω^I are the selfdual harmonic two-forms on multi-Taub-NUT [4, 12]. These solitonic solutions preserve all 8 real supercharges that are already present when we put the M5 brane on this geometry. Upon dimensional reduction along the circle fiber of multi-Taub-NUT, these solitons describe a particular gauge field configuration in the 5d SYM that have as 'moduli parameters' the selfdual one-forms $h^I_\mu dx^\mu$. By evaluating the 5d SYM action on the solitonic solution (7.1) an effective WZW action for $h^I_\mu dx^\mu$ is obtained [12]. This effective WZW action is derived from 5d SYM, it is not a gauged WZW theory since the gauge field is gone in the process of deriving the effective action, so this is clearly a different WZW theory from the gauged WZW theory that we have added to 5d SYM in this paper. However, we may still expect that there is some, but perhaps not so direct, relation between the two WZW theories. An analogous example would be to take two dyons in four dimensions coupled to a Maxwell action. We may describe this system by adding to the Maxwell action some interaction terms and mass terms for the dyons integrated over their worldlines. But we may also describe the dynamics of these dyons, at least in the low-energy limit, by a low-energy particle action where the particles are moving on a moduli space. This latter description is insufficient to describe the dynamics of the gauge field itself, but it suffices to describe the low-energy dynamics of the dyons. The analogous way to think of the result in [12] would be that we have the effective WZW action describing the D4-D6 strings and the original 5d SYM action is entirely gone in the process of deriving that effective WZW action. What we do in this paper corresponds to the other description, where we consider both 5d SYM and an added WZW theory for the D4-D6 strings. In our analogy, this added WZW theory would correspond to the particle actions for the dyons that we added to the Maxwell theory, whereas the effective WZW action in [12] would correspond to the effective particle action on their moduli space.

In this paper we added a WZW theory to the 5d SYM. One reason for adding the WZW theory is to restore gauge symmetry and supersymmetry. Another reason could be that, if the graviphoton has a magnetic flux, we need to modify the Yang-Mills equation of motion by adding a source term J_μ as in eq. (4.7). The source term will then be identified as the WZW current. If we can use the equations of motion to argue for the necessity of adding a WZW theory, then that would have the advantage that we would avoid the problem of whether we should use the graviphoton term (4.3) or (4.4). In [4, 12] the M5 brane equations of motion were solved on $\mathbb{R}^{1,1}\times$Taub-NUT and the solutions they found are of the form (7.1). Upon dimensional reduction, one then finds that the original Yang-Mills equations are satisfied without the source term J_μ. Around eq. (7.3) below, we show that, contrary to the claim in [4], no delta function source J_μ is produced on the right-hand side of the Yang-Mills equations. We may also look at eq. (4.8). If the three-form dW has nonvanishing components only in the three transverse directions, then for the left-hand side to be non-zero, F must have a non-vanishing component along the intersection brane. But for the on-shell solution (7.2) that arises from dimensional reduction of the on-shell solution (7.1), that component is zero, which shows that $\nabla_\mu J^\mu = 0$. This is the current conservation law that we would expect if there is no gauge anomaly. So this argument alone is insufficient to conclude that the WZW current itself would have to vanish on-shell. But we show that J_μ actually does vanish on-shell around eq. (7.3) below. This means that the argument that was used in [4] based on the equation of motion, falls apart upon closer inspection.

We have an older argument [5] that is based on restoring gauge invariance of the graviphoton term in the Yang-Mills action by adding a WZW action. This argument does not require the fields to be on-shell as it is based on the action rather than the equations of motion. One could still argue that this argument too could fall apart if we could use the gauge invariant graviphoton action in eq. (4.4) in place of the gauge non-invariant action (4.3) that we have used in our paper. But the action (4.4) is not reparametrization invariant as it depends explicity on κ_μ. Such an action may be used for a restricted set of computations where κ_μ does not show up explicitly in the final resuls. But as a general action principle, it is an unsatisfactory resolution of the problem, as it is just moving the gauge anomaly into a reparametrization anomaly and does not cancel anyone of these anomalies. To cancel the anomaly we need to add the WZW theory. Even if we were to start out with the gauge-invariant graviphoton term, then by cancerling the reparametrization anomaly we would discover that we need to add the boundary term that is the difference between the two graviphoton terms, effectively bringing it over to the gauge non-invariant graviphoton term and then we would be back again to the WZW terms that we need to add to cancel the gauge anomaly. In that sense there is no ambiguity in the construction of the action.

Finally it seems very difficult to invalidate our argument that is based on supersymmetry of the action. What we have seen in this paper is that the Yang-Mills action is not supersymmetric in the presence of an intersection brane and that we can restore supersymmetry by adding the WZW theory.

If one just wants to derive an effective theory on the moduli space of the BPS solutions (7.1), then since these BPS solutions satisfy the unmodified Yang-Mills equations, there is no need to modify those equations of motion by adding a source term. That source term would be zero and on-shell the WZW theory on the intersection brane is decoupled from the Yang-Mills in the bulk. There is also no need to add the decoupled WZW action because all the WZW fields can be taken to vanish on the BPS solution as one may see from eq. (6.4) where the right-hand side is zero on-shell as F has no component along the intersection brane. So the WZW scalar field ϕ can be put to zero on-shell. Presumably this argument can be straightforwardly generalized to the non-Abelian case as well. This line of argument could justify the approach that was taken in [12] to derive the effective WZW theory.

The reference [12] shows that the solution found in [4] is supersymmetric. It preserves all supersymmetries. There are no fermionic zero modes and no broken supersymmetries. The intersection brane was shown to carry an electric charge and has a tension, both of which are expressed in terms of the function h_μ (that was denoted ν_+ in [12]). This has a non-Abelian generaltzaion and also a generalization to multi-Taub-NUT [12]. It would be interesting to show that the mass saturates a BPS bound determined by a central charge. Presumably that central charge would be proportional to the electric charge.

In [12] it was objected that the effective WZW for multi-Taub-NUT could not be the usual WZW with one three-manifold with N different boundaries whenever $Q > 1$. But there is no need for the three-manifold to connect all the N different intersection branes. Some three-manifold could extend from one intersection brane out to infinity. Then if Q intersection branes would coincide, then we just take the WZW level to be equal to Q and extend the three-manifold to infinity, and it would not affect the WZW theories on the other intersection branes if we extend the three-manifold to infinity in such a way that it does not cross some of the other submanifolds. So by allowing the three-manifold to extend to infinity rather than joining different intersection branes, we are able to avoid the problem that was raised in [12]. We notice that there is plenty of room to draw three-manifolds that extend to infinity. These three-manifolds are lines in $\mathbb{R}^3$ and extended along $\mathbb{R}^{1,1}$. Singular points where the circle fiber vanishes in multi-Taub-NUT corresponds to points in $\mathbb{R}^3$. So we can always find a line from any singular point in $\mathbb{R}^3$ that extends to infinity. There is no need to connect the different singular points with lines. These lines may instead extend to infinity and then we have genuinely different WZW theories for each singular point. But there appears to be some ambiguity in how we may choose to draw these lines. We expect that different choices will lead to equivalent physical descriptions.

In [4] it was proposed that we may extract the degrees of freedom of the gauged WZW theory directly from the tensor gauge field and more specifically as the selfdual $h_\mu dx^\mu$ component in the tensor gauge field. Then upon dimensional reduction, this degree of freedom will remain in the Yang-Mills gauge field strength as we will show below in eq. (7.2). But by the argument above, the selfdual $h_\mu dx^\mu$ that we extract from the selfdual tensor gauge field will enter in the effective WZW action. So it will not appear in the gauged WZW action that we shall add to the 5d theory. The added gauged WZW theory contains new degrees of freedom that we can not extract from the degrees of freedom of 5d SYM. So the argument in [4] must be wrong. But since it is wrong in a subtle and interesting way, let us here carefully review the argument in [4] to see exactly where it goes wrong.

An on-shell solution for the gauge field was constructed in [4] by making use of the harmonic two-form on Taub-NUT. The strategy was to first solving the 6d equations of motion

$$
\begin{aligned}
dH &= 0 \\
H &= *H
\end{aligned}
$$

for the selfdual tensor field of the Abelian M5 brane on $\mathbb{R}^{1,1}\times$ Taub-NUT. The solution is given by

$$H = h_\mu dx^\mu \wedge \Omega$$

Here Ω is the unique antiselfdual harmonic two-form on Taub-NUT,

$$\Omega = \frac{\partial_k U}{U^2}\left(-e^k \wedge e^4 + \frac{1}{2}\varepsilon^{ijk} e^i \wedge e^j\right)$$

where

$$\begin{aligned} e^i &= \sqrt{U} dx^i \\ e^4 &= \frac{1}{\sqrt{U}}\left(d\psi + \kappa^i dx^i\right) \end{aligned}$$

Now let us perform the dimensional reduction and ket us assume that the gauge group is Abelian (just for simplicity), and let us make the following ansatz for an on-shell solution

$$F_{i\mu} = \frac{\partial_i U}{U^2} h_\mu \tag{7.2}$$

with all the other components vanishing. To see whether this satisfies the equation of motion, we shall start by computing

$$\nabla^i\left(\frac{1}{r}F_{i\mu}\right) + \frac{1}{2}\varepsilon_\mu{}^{i\nu jk} F_{i\nu} W_{jk} = -\frac{1}{U^{3/2}}\frac{1}{U^2}\partial_i U \partial_i U\left(h_\mu + \varepsilon_{\mu\nu} h^\nu\right) + \frac{1}{U}\frac{1}{U^{3/2}}\partial_i\partial_i U h_\mu \tag{7.3}$$

The first term vanishes if we demand that

$$h_\mu + \varepsilon_{\mu\nu} h^\nu = 0$$

The second term is identically zero. First we note that $\frac{1}{U^{3/2}}\partial_i\partial_i U$ is a delta function with respect to the integartion measure $d^3xU^{3/2}$. Second, $1/U$ evaluated at $|x| = 0$ is zero, which is killing the whole thing. Thus we find that this solves the equation of motion (6.3) with $J_\mu = 0$. From the equation of motion $\partial^\mu F_{\mu i} = 0$ we find $\partial^\mu h_\mu = 0$. By combining this with $h_\mu = -\varepsilon_{\mu\nu} h^\nu$ we get $\partial_\mu h_\nu - \partial_\nu h_\mu = 0$ that is locally solved by $h_\mu = \partial_\mu \varphi$ for some scalar field φ. Our computation shows a different result from [4] where it was found that $J_\mu \sim h_\mu$, from which one would conclude that the scalar field ϕ in the WZW theory would be the same as the component φ that sits in the super Yang-Mills field $F_{\mu i}$ rather than a new degree of freedom.

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A Differential geometry

A.1 A formula for the spin connection

Given a vielbein $e^a{}_\mu$, the spin connection $(\omega_\mu)^a{}_b$ is implicitly defined by two equations. It is covariantly constant, and the torsion is vanishing,

$$\begin{aligned} \partial_\mu e^a{}_\nu + (\omega_\mu)^a{}_b e^b{}_\nu - e^a{}_\rho \Gamma^\rho_{\mu\nu} &= 0 \\ \Gamma^\rho_{[\mu\nu]} &= 0 \end{aligned}$$

Then we get

$$(de^a)_{\mu\nu} + (\omega_\mu)^a{}_b e^b{}_\nu - (\omega_\nu)^a{}_b e^b{}_\mu = 0$$

We contract this equation by $e^{c\nu}$,

$$\begin{aligned} (de^{[d})_{\mu\nu} e^{c]\nu} + (\omega_\mu)^{[dc]} + (\omega_\nu)^{a[d} e^{c]\nu} e^a{}_\mu &= 0 \\ \frac{1}{2}(de^a)_{\mu\nu} e^{c\mu} e^{d\nu} e^a{}_\rho + (\omega_\mu)^{a[d} e^{c]\mu} e^a{}_\rho &= 0 \end{aligned}$$

Subtracting these equations leaves us with

$$(\omega_\mu)^{dc} = e^{\nu[d}(\partial_\mu e^{c]}_\nu - \partial_\nu e^{c]}_\mu) - \frac{1}{2} e^{d\kappa} e^{c\tau} (\partial_\kappa e^a_\tau - \partial_\tau e^a_\kappa) e^a_\mu$$

This is a formula to compute the spin connection directly from the vielbein.

A.2 Covariant derivatives on a circle bundle

We denote 6d objects with hats, 5d base manifold objects without the hat. So the 6d metric is

$$ds^2 = \widehat{G}_{MN} dx^M dx^N$$

We assume this metric has the circle-bundle form

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu + r^2 (d\theta + \kappa_\mu dx^\mu)^2$$

The manifest Killing vector is $\widehat{v} = \partial_\theta$. For its lower components we have $\widehat{v}_\theta = r^2$ and $\widehat{v}_\mu = r^2 \kappa_\mu$. The vielbein is

$$\begin{aligned} \widehat{e}^{\widehat{\theta}} &= r (d\theta + \kappa_\mu dx^\mu) \\ \widehat{e}^{\widehat{\mu}} &= e^{\widehat{\mu}}{}_\nu dx^\nu \end{aligned}$$

The inverse vielbein is

$$\begin{aligned} \widehat{e}_{\widehat{\theta}} &= \frac{1}{r} \partial_\theta \\ \widehat{e}_{\widehat{\mu}} &= e^\nu{}_{\widehat{\mu}} (\partial_\nu - \kappa_\nu \partial_\theta) \end{aligned}$$

The inverse metric is

$$\widehat{G}^{MN} \partial_M \phi \partial_N \phi = \frac{1}{r^2} \partial_\theta \phi \partial_\theta \phi + G^{\mu\nu} (\partial_\mu \phi - \kappa_\mu \partial_\theta \phi)(\partial_\nu \phi - \kappa_\nu \partial_\theta \phi)$$

If we perform dimensional reduction along the fiber, then we put $\partial_\theta = 0$ and we get

$$\widehat{G}^{MN}\partial_M\phi\partial_N\phi = G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

This corresponds to the metric

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu$$

on the base-manifold. The simple form of this dimensionally reduced metric is why this form of the fiber-bundle metric is a preferred choice when we perform dimensional reduction.

The covariant derivatives of a vector field $\widehat{\nabla}_M\widehat{v}_N = \partial_M\widehat{v}_N - \widehat{\Gamma}^P_{MN}\widehat{v}_P$ are

$$\begin{aligned}
\widehat{\nabla}_\theta v_\theta &= \partial_\theta v_\theta - r\nabla^\rho r v_\rho - r\kappa_\rho\nabla^\rho r v_\theta\\
\widehat{\nabla}_\theta v_\mu &= \partial_\theta v_\mu - \frac{1}{2}r^2 W_\mu{}^\rho v_\rho + r\kappa_\mu\nabla^\rho r v_\rho + \frac{1}{2}r^2 W_\mu{}^\rho\kappa_\rho v_\theta - r\kappa_\mu\kappa_\rho\nabla^\rho r v_\theta - \frac{1}{r}\nabla_\mu r v_\theta\\
\widehat{\nabla}_\mu v_\nu &= \nabla_\mu v_\nu + r^2 w^\rho{}_{(\mu}\kappa_{\nu)}v_\rho + r\nabla^\rho r\kappa_\mu\kappa_\nu v_\rho\\
&\quad -\nabla_{(\mu}\kappa_{\nu)}v_\theta - r^2\kappa_\rho w^\rho{}_{(\mu}\kappa_{\nu)}v_\theta - r\kappa_\rho\nabla^\rho r\kappa_\mu\kappa_\nu v_\theta - \frac{2}{r}\nabla_{(\mu}r\kappa_{\nu)}v_\theta
\end{aligned}$$

The covariant derivatives of a spinor field $\widehat{\nabla}_M\psi = \partial_M\psi + \frac{1}{4}\omega_M^{\widehat{M}\widehat{N}}\Gamma_{\widehat{M}\widehat{N}}\psi$ are

$$\begin{aligned}
\widehat{\nabla}_\theta\psi &= \partial_\theta\psi - \frac{r^2}{8}\Gamma^{\mu\nu}\psi W_{\mu\nu} - \frac{1}{2}\Gamma^\mu\Gamma^{\widehat{\theta}}\psi\partial_\mu r\\
\widehat{\nabla}_\mu\psi &= \nabla_\mu\psi + \kappa_\mu\left(\widehat{\nabla}_\theta\psi - \partial_\theta\psi\right) + \frac{r}{4}\Gamma^\nu\Gamma^{\widehat{\theta}}\psi W_{\mu\nu}
\end{aligned}$$

Here we define $\Gamma_\mu = \Gamma_{\widehat{\nu}}e^{\widehat{\nu}}{}_\mu$ and $\widehat{\Gamma}_M = \Gamma_{\widehat{N}}\widehat{e}^{\widehat{N}}{}_M$.

A.3 Reducing the conformal Killing spinor equation

Let us now analyze the conformal Killing spinor equations

$$\begin{aligned}
\widehat{\nabla}_\mu\varepsilon &= \widehat{\Gamma}_\mu\eta\\
\widehat{\nabla}_\theta\varepsilon &= \widehat{\Gamma}_\theta\eta
\end{aligned}$$

We expand the covariant derivatives and the gamma matrices in 5d quantities,

$$\begin{aligned}
\nabla_\mu\varepsilon + \kappa_\mu\left(\widehat{\nabla}_\theta\varepsilon - \partial_\theta\varepsilon\right) + \frac{r}{4}\Gamma^\nu\Gamma^{\widehat{\theta}}\varepsilon W_{\mu\nu} &= \Gamma_\mu\eta + r\kappa_\mu\Gamma_{\widehat{\theta}}\eta\\
\partial_\theta\varepsilon - \frac{r^2}{8}\Gamma^{\mu\nu}\varepsilon W_{\mu\nu} - \frac{1}{2}\Gamma^\mu\Gamma^{\widehat{\theta}}\varepsilon\partial_\mu r &= r\Gamma_{\widehat{\theta}}\eta
\end{aligned}$$

We put $\partial_\theta\varepsilon = 0$ and get

$$\begin{aligned}
\nabla_\mu\varepsilon + \frac{r}{4}\Gamma^\nu\Gamma^{\widehat{\theta}}\varepsilon W_{\mu\nu} &= \Gamma_\mu\eta\\
-\frac{r^2}{8}\Gamma^{\mu\nu}\varepsilon W_{\mu\nu} - \frac{1}{2}\Gamma^\mu\Gamma^{\widehat{\theta}}\varepsilon\partial_\mu r &= r\Gamma_{\widehat{\theta}}\eta
\end{aligned}$$

We get

$$\nabla_\mu\varepsilon + \frac{r}{4}\Gamma^\nu\Gamma^{\widehat{\theta}}\varepsilon W_{\mu\nu} = -\frac{r}{8}\Gamma_\mu\Gamma^{\rho\sigma}\Gamma^{\widehat{\theta}}\varepsilon W_{\rho\sigma} + \frac{1}{2r}\Gamma_\mu\Gamma^\rho\varepsilon\partial_\rho r$$

that we may also write as

$$\nabla_\mu\varepsilon = M_\mu\varepsilon$$

where we define

$$M_\mu = \frac{1}{2r}\Gamma_\mu\Gamma^\rho\partial_\rho r - \frac{r}{8}\Gamma_\mu\Gamma^{\rho\sigma}\Gamma^{\widehat{\theta}}W_{\rho\sigma} - \frac{r}{4}\Gamma^\nu\Gamma^{\widehat{\theta}}W_{\mu\nu}$$

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A.4 Covariantly constant spinors on Taub-NUT

The conditions that both the derivatives $\widehat{\nabla}_\theta \varepsilon = 0$ and $\partial_\theta \varepsilon = 0$ are vanishing imply that

$$\frac{r^2}{4}\Gamma^{\mu\nu}\varepsilon W_{\mu\nu} + \Gamma^{\mu}\Gamma^{\widehat{\theta}}\varepsilon \partial_\mu r = 0 \tag{A.1}$$

Let us study this condition in the context of Taub-NUT. There we have

$$\frac{1}{2}\Gamma^{ij}\varepsilon W_{ij} = \frac{1}{U^{1/2}}\Gamma^{i}\Gamma^{\widehat{\theta}}\varepsilon \partial_i U$$

We may express $W_{ij} = \epsilon_{ijk}\partial_k U$ in a covariant way as

$$W_{ij} = \frac{1}{\sqrt{U}} G^{k\ell}\varepsilon_{ijk}\partial_\ell U$$

where $\varepsilon_{ijk} = \sqrt{G}\epsilon_{ijk}$ is the covariant form of the antisymmetric tensor where $\varepsilon_{123} = 1$ and totally antisymmetric. Then

$$\frac{1}{2}\Gamma^{ij}\varepsilon G^{k\ell}\varepsilon_{ijk}\partial_\ell U = \Gamma^{\ell}\Gamma^{\widehat{\theta}}\varepsilon \partial_\ell U$$

Let us cancel out $\partial_\ell U$ on both sides and use $\Gamma_k\Gamma^k = 3$ to get

$$\frac{1}{6}\Gamma^{ijk}\varepsilon\varepsilon_{ijk} = \Gamma^{\widehat{\theta}}\varepsilon \tag{A.2}$$

This is a Weyl projection condition that reduces the amount of supersymmetry by half and assures that ε is invariant under the $\mathrm{SU}(2)_+$ holonomy group.

Let us notice that for

$$U = \frac{1}{R^2} + \frac{1}{2|\vec{x}|}$$

we have

$$w = -\frac{x_k}{4|\vec{x}|^3}\epsilon_{ijk} dx^i \wedge dx^j$$

which is a monopole with charge $\oint_{S^2} w = -2\pi$.

B Flat metric on $\mathbb{R}^4$

The flat metric on $\mathbb{R}^4 = \mathbb{C}^2$ is $ds^2 = |dz_1|^2 + |dz_2|^2$. We may parametrize the space by Euler anges as

$$z_1 = \rho\cos\frac{\theta}{2}e^{-\frac{i}{2}(2\psi+\varphi)}$$
$$z_2 = \rho\sin\frac{\theta}{2}e^{-\frac{i}{2}(2\psi-\varphi)}$$

where $\psi \sim \psi + 2\pi$. These coordinates can be obtained by acting on the spin-up state with the Euler rotation

$$g = e^{-\frac{i}{2}\phi\sigma_3}e^{-\frac{i}{2}\theta\sigma_2}e^{-\frac{i}{2}2\psi\sigma_3} \tag{B.1}$$

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In this form it is clear that $g^{-1} = g^\dagger$ so the rotation operator is unitary. The Maurer-Cartan forms are

$$g^{-1}dg = -\frac{i}{2}\sigma_a\omega^a$$

where

$$\begin{aligned}\omega^1 &= \sin 2\psi d\theta - \sin\theta\cos 2\psi d\phi \\ \omega^2 &= \cos 2\psi d\theta + \sin\theta\sin 2\psi d\phi \\ \omega^3 &= 2d\psi + \cos\theta d\phi\end{aligned}$$

C Gauge group normalization

We assume the gauge group Lie algebra is

$$[T_a, T_b] = i f_{ab}{}^c T_c$$

with the metric

$$\mathrm{tr}(T_a T_b) = h_{ab} = \frac{1}{2}\delta_{ab}$$

We write the gauge potential as $A_\mu = A_\mu^a T_a$. The field strength is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

or in component form

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\mu A_\nu^a + f_{bc}{}^a A_\mu^b A_\nu^c$$

We fix the normalization by taking SU(2) gauge group with group element g as in (B.1) parametrized by the Euler angles. Then we get

$$\begin{aligned}\mathrm{tr}(g^{-1}dg) &= \frac{3}{2}\omega^1\omega^2\omega^3 \\ &= -3\sin\theta d\theta d\varphi d\psi\end{aligned}$$

which gives

$$\int \mathrm{tr}\left(g^{-1}dg\right)^3 = -24\pi^2$$

and consequently we shall normalize the Wess-Zumino term as

$$\frac{1}{12\pi}\int \mathrm{tr}\left(g^{-1}dg\right)^3 = \frac{1}{12\pi}\int d^3x\sqrt{-G}\varepsilon^{\mu\nu\lambda}\mathrm{tr}\left(g^{-1}\partial_\mu g g^{-1}\partial_\nu g^{-1}\partial_\lambda g\right)$$

and it will be quantized in units of 2π where the integer measures the winding number as we map S^3 into SU(2).

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D Gamma matrices

D.1 Two dimensions

In 2d we have the Majorana representation

$$\begin{aligned}\gamma^0 &= i\sigma^2 \\ \gamma^1 &= \sigma^1\end{aligned}$$

The chirality matrix is then $\gamma = \gamma^{01} = \sigma^3$. The charge conjugation matrix is $C = \gamma^0 = i\sigma^2 = \varepsilon$. The Dirac conjugate spinor is $\bar{\psi} = \psi^\dagger\gamma^0$. Writing out the spinor components, we have the spinor ψ^u acted on by the gamma matrices $(\gamma^\alpha)^u{}_v$, however, the charge conjugatation matrix is ε_{uv} with component $\varepsilon_{+-} = 1$. The Dirac conjugate is $\bar{\psi}_v = (\psi^u)^* i(\sigma^2)^u{}_v$. We note that $(\psi^u)^*$ transforms under Lorentz rotations like $\psi_u = \psi^v\varepsilon_{vu}$. For the components, we have $\bar{\psi}_+ = -(\psi^-)^*$ and $\bar{\psi}_- = (\psi^+)^*$. We may impose the Majorana condition $\bar{\psi}_u = \psi^v\varepsilon_{vu}$. In the Majorana representation it amounts to a spinor with real components, $(\psi^+)^* = \psi^+$ and $(\psi^-)^* = \psi^-$. We note that $C\gamma^\mu$ are symmetric.

D.2 Five dimensions

We define 5d gamma matrices as

$$\begin{aligned}\gamma^0 &= (\gamma^0)^u{}_v\delta^m_n \\ \gamma^4 &= (\gamma^1)^u{}_v\delta^m_n \\ \gamma^i &= \gamma^u{}_v(\sigma^i)^m{}_n\end{aligned}$$

The charge conjugation matrix is

$$C = (\gamma^1)^u{}_v\varepsilon_{mn}$$

We have

$$\begin{aligned}\gamma^{04} &= \gamma^u{}_v\delta^m_n \\ \gamma^{01234} &= -i\delta^u_v\delta^m_n\end{aligned}$$

We note that both C and $C\gamma^\mu$ are antisymmetric, consistent with the decomposition of a product of two spinors into a scalar, a vector and an antisymmetric tensor,

$$4 \otimes 4 = 1_{anti} \oplus 5_{anti} \oplus 10_{symm}$$

D.3 Eleven dimensions

We define the gamma matrices as

$$\begin{aligned}\Gamma^\mu &= \gamma^\mu \otimes \sigma^1 \otimes 1 \\ \Gamma^\psi &= 1 \otimes \sigma^2 \otimes 1 \\ \Gamma^A &= 1 \otimes \sigma^3 \otimes \tau^A\end{aligned}$$

where $A = 6, 7, 8, 9, 10$ and $\tau^{12345} = 1$. The 6d chirality matrix is

$$\Gamma = \Gamma^0\Gamma^{123}\Gamma^4\Gamma^\psi = 1 \otimes \sigma^3 \otimes 1$$

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The charge conjugatation matrix is

$$C = C_5 \otimes \varepsilon \otimes C_5'$$

where $\varepsilon = i\sigma^2$. Thus C is antisymmetric while $C\Gamma^M$ for $M = (\mu, \psi)$ and $C\Gamma^A$ are symmetric. The Majorana condition is

$$\bar{\psi} = \psi^T C$$

where $\bar{\psi} = \psi^\dagger \Gamma^0$.

E 5d SYM in reduced notation

Expressed in terms of 5d gamma matrices and 5d spinors, we have

$$\begin{aligned} \delta\phi^A &= -i\mathcal{E}\tau^A\chi \\ &= i\chi\tau^A\mathcal{E} \\ \delta A_\mu &= \mathcal{E}\gamma_\mu\chi \\ &= -\chi\gamma_\mu\mathcal{E} \\ \delta\chi &= -\frac{i}{2}\gamma^{\mu\nu}\mathcal{E}F_{\mu\nu} - \gamma^\mu\tau^A\mathcal{E}\mathcal{D}_\mu\phi^A - \frac{ir}{2}\gamma^{\mu\nu}\tau^A\mathcal{E}W_{\mu\nu}\phi^A - \frac{1}{2}\tau^{AB}\mathcal{E}[\phi^A, \phi^B] \end{aligned}$$

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