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Effective Lagrangian and stability analysis in warped space

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Abstract: In the warped space model, the inter-brane distance can be stabilized by the Goldberger-Wise mechanism. Of particular importance, the stabilization potential calls for a proper identification of the dynamical degree of freedom. In this paper, we provided a complete calculation of the effective Lagrangian till the quadratic order that is generic for the Randall-Sundrum model and its N-brane ($N \geq 4$) extensions. By applying the variation principle to a specific perturbation field, we derived the equations of motion and orthogonal conditions for decoupling the graviton. This approach is demonstrated to be equivalent to the analysis using the linearized Einstein equation. Our derivation clarifies that in the N-brane set up, just one degree of freedom for the radion field is dynamical, with the other modes eliminated by the gauge fixings. Thus we can directly generalize the GW stabilization to the N-brane model in a way similar to the RS1 scenario.

Keywords: Effective Field Theories, Extra Dimensions, Field Theories in Higher Dimensions

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Contents

1 Introduction

The original Randall-Sundrum (RS) model with 2 branes at the orbifold fixed points [1, 2] was proposed to address the hierarchy problem using a warped factor. Also the localization of gravity near the UV brane naturally explains the weakness of coupling at the large distance. An interesting generalization of one slice of Anti-de-sitter (AdS) space is to build RS-like models with extra branes [3–5]. Such construction is attractive for the existence of ultra-light massive gravitons and potential new phenomenology. A general warped multibrane model was described in [6], where two non-fixed point branes were added given that the bulk cosmology constants are different in two spatial regions. For the RS1 model, the inter-brane distance has to be stabilized by the Goldberger-Wise (GW) mechanism [7, 8], that requires at least a single bulk scalar coupling to the gravitons. As a result, the fluctuation of the bulk scalar becomes entangled with the metric modulus field. Meanwhile the effective potential of the scalar develops a minimum after an integration over the fifth dimension so that the radion obtains a mass.

Recently we have proved that this stabilization mechanism can be generalized to a multibrane set-up in a straightforward manner [9]. In that paper, with the addition of a new perturbation ϵ in the metric, we derived the linearized Einstein equation in a multibrane

RS model with the junction conditions matched at all the branes. However the preliminary analysis shows that the perturbation ϵ simply plays the role of gauge fixing. Following the strategy of [10], we can solve a single equation of motion (EOM) as an eigenvalue problem in the limit of stiff brane potentials and find that with a small back-reaction the mass of radion is significantly suppressed compared to its KK excitations [9]. In this paper, in order to strengthen the argument of dynamical degree of freedom, we expanded the 5d action into the quadratic order of perturbations. Despite of the complexity, by applying the variation principle to the effective Lagrangian, we derived the same EOMs and orthogonal conditions as from the Einstein equation. Note that the impact of perturbation ϵ on the scalar EOM can only be explored in the framework of effective Lagrangian. Of particular interest is that the dependence of ϵ in the effective Lagrangian can be eliminated after imposing the orthogonal conditions (gauge fixings). This constitutes a stronger demonstration that a unique radion field with its profile $F \propto e^{2A}$ at the zeroth order is the legitimate solution to the Einstein equation in a stabilized N-brane model.

As a consistency check of stability, we further examined the tadpole behavior of the lowest mode of radion-scalar system after the GW stabilization. The result shows that the linear terms of ϵ and bulk scalar fluctuation are removed by the radion EOMs and background equations, while the remaining term related to the 5d profile is automatically zero at the leading order.

2 5d model and diffeomorphism

We start with a brief review of N-brane ($N \in$ even integer) model, considering the 5d action with the graviton minimally coupling to a single bulk scalar field:

$$-\frac{1}{2\kappa^2}\int d^5x\sqrt{g}\,\mathcal{R} + \int d^5x\sqrt{g}\left(\frac{1}{2}g^{IJ}\partial_I\phi\partial_J\phi - V(\phi)\right) - \int d^5x\frac{\sqrt{g}}{\sqrt{-g_{55}}}\sum_i \lambda_i(\phi)\delta(y-y_i)\,, \tag{2.1}$$

where $\mathcal{R}$ is the Ricci scalar and the Latin indices I, J run over $(\mu, 5)$, with the Greek one designated for 4d Minkowski space $\mu = 0, \cdots 3$ and y being the coordinator of extra dimension. The $\kappa^2 \equiv 1/(2M_5^3)$ is related to the 5d Planck mass. Note that the orbifold symmetry is imposed for the Lagrangian where all the fields satisfy $Q(y) = Q(-y)$, thus the integration of y is conducted in the region of $y \in [-L, L]$ that is equivalent to a circle S^1 under the diffeomorphism. In eq. (2.1), the first line contains the Einstein-Hilbert action and the bulk scalar action that is responsible for the GW mechanism. While the second line is composed of the brane terms determined by the *jump*[1] of the derivative fields as well as the brane potentials of ϕ. With the appropriate potentials $V(\phi)$ and $\lambda_i(\phi)$, $i = 1, \cdots N$, the bulk scalar will develop a VEV i.e. $\phi(x, y) = \phi_0(y) + \varphi(x, y)$, so that the radion field is stabilized. In an N-brane set up, the fifth dimension can be divided into $N/2$ subregions with different curvatures $k_\alpha^2 = -\frac{\kappa^2}{6}\Lambda_\alpha$, $\alpha = 1, \cdots \frac{N}{2}$, where Λ_α is the cosmology constant in

[1]The *jump* of a given quantity Q cross the brane located at $y = y_i$ is defined as $[Q]|_{y=y_i} \equiv Q(y_i + \varepsilon) - Q(y_i - \varepsilon)|_{\varepsilon \to 0}$.

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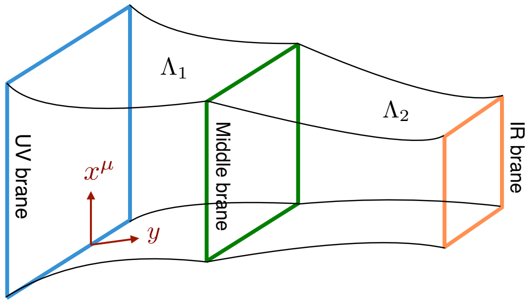

Figure 1. The 4-brane RS model visualized in the range of $0 < |y| < L$. Due to the orbifold symmetry, there are two intermediate branes located at $y = \pm r$.

each subregion. Henceforth besides the UV and IR branes at the fixed points of $y = \{0, L\}$ like the RS1 model, $N - 2$ copies of intermediate branes (not dynamical as later proven) arise at $y = \pm r_a$ $(0 < r_a < L)$ with energy densities to match the junction conditions of the metric.[2] As a concrete example, the 4-brane RS model is displayed in figure 1.

The general metric ansatz on an S^1/Z_2 orbifold that can decouple the transverse graviton from the physical radion field is [10, 11]:

$$ds^2 = e^{-2A(y)-2F(x,y)} \left[\eta_{\mu\nu} + 2\epsilon(y)\partial_\mu\partial_\nu f(x) + h_{\mu\nu}(x,y)\right] dx^\mu dx^\nu - [1 + G(x,y)]^2 \, dy^2 \,, \tag{2.2}$$

with $A(y)$ being the background metric. Among the perturbations, $h_{\mu\nu}$ is a symmetric tensor standing for the graviton. While $F, G, \epsilon f(x)$ plus the scalar perturbation φ are not independent, they all belong to the radion excitation. Compared with the previous paper [9], the definitions of F, G absorb the $f(x)$ that is factorized outside for $\epsilon(y)$. We would like to mention that eq. (2.2) is enforced an implicit constraint $g_{\mu 5} = 0$ otherwise there is one more scalar mode that could be removed by the gauge fixing [12].

Before the radius stabilization, it is inspiring to investigate the transformation property of these metric perturbations under a class of infinitesimal coordinate shift $X^I \to X^I + \xi^I(X)$. As a result, the metric transforms accordingly:

$$\delta g_{IJ} = -\xi^K \, \partial_K g^{(0)}_{IJ} - \partial_I \xi^K \, g^{(0)}_{KJ} - \partial_J \xi^K \, g^{(0)}_{IK} \,. \tag{2.3}$$

Note that the diffeomorphism symmetry keeps the Einstein-Hilbert action to be invariant. Since the Ricci scalar is purely constructed by the metric, the diffeomorphism will retain the metric in its original structure after the appropriate field redefinition. This requires the transformation to be of the specific form [6, 12]:

$$\xi^\mu(x,y) = \hat{\xi}^\mu(x) + \eta^{\mu\nu}\partial_\nu\zeta(x,y) \,, \quad \xi^5(x,y) = e^{-2A} \, \zeta'(x,y) \,. \tag{2.4}$$

[2]The boundary terms in eq. (2.1) with $\delta(y - y_i)$ do not describe dynamical branes, instead spacetime defects located at coordinates y_i. Such spacetime defects can for example arise in string theory as orbifold or orientifold planes. Here these objects will continue to be referred as branes like in the original RS model. The analysis in section 3 shows that the intermediate one is not allowed to fluctuate, with no extra radion associated.

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where the prime denotes the derivative $\partial_5 \equiv \partial/\partial y$. Due to the presence of the brane terms, the fifth coordinate shift is subject to a constraint $\zeta'(x,y)|_{y=\{0,\pm r_a,L\}} = 0$, $a = 1, \cdots \frac{N}{2} - 1$. Substituting the ξ^K of eq. (2.3) in terms of eq. (2.4), one can extract the transformation rules for all component fields in the metric expansion:

$$\begin{aligned} \delta h_{\mu\nu} &= -\partial_\mu \hat{\xi}_\nu - \partial_\nu \hat{\xi}_\mu \\ \delta F &= -A'\zeta' e^{-2A} \\ \delta G &= -\left(\zeta'' - 2A'\zeta'\right) e^{-2A} \\ \delta \epsilon f(x) &= -\zeta\,. \end{aligned} \tag{2.5}$$

As we can see, $\hat{\xi}^\mu(x)$ represents the usual 4d diffeomorphism under which $h_{\mu\nu}$ transforms as a spin-2 tensor. For the spin-0 modes G, F and $\epsilon f(x)$, their transformations are fixed by the metric A and the parameter ζ. In particular, the last ansatz in eq. (2.5) indicates that without stabilization the $\epsilon f(x)$ can be eliminated as a gauge fixing by choosing $\zeta = \epsilon f(x)$.

In general, the system of metric and scalar perturbations can be studied using the variation principle since the physical path evolves along the one minimizing the action. Varying of the action (2.1) with respect to the 5d metric g_{IJ} will give the Einstein Equation:

$$R_{IJ} = \kappa^2 \left(T_{IJ} - \frac{1}{3} g_{IJ} T^m_m \right) = \kappa^2 \tilde{T}_{IJ}\,, \tag{2.6}$$

with $T_{IJ} = 2\delta\left(\sqrt{g}\,\mathcal{L}_\phi\right) / \left(\sqrt{g}\,\delta g^{IJ}\right)$ being the energy-momentum tensor. Similarly by minimizing the scalar action with respect to ϕ, one can derive the scalar EOM that is not included in eq. (2.6). Grouping the zeroth order of eq. (2.6) with the equation of ϕ_0 together, the background (BG) equations in an N-brane RS model are:

$$\phi_0'' = 4A'\phi_0' + \frac{\partial V(\phi_0)}{\partial \phi} + \sum_i \frac{\partial \lambda_i(\phi_0)}{\partial \phi} \delta(y - y_i)\,, \tag{2.7}$$

$$A'' = \frac{\kappa^2}{3} \phi_0'^2 + \frac{\kappa^2}{3} \sum_i \lambda_i(\phi_0) \delta(y - y_i)\,, \tag{2.8}$$

$$A'^2 = \frac{\kappa^2 {\phi_0}'^2}{12} - \frac{\kappa^2}{6} V(\phi_0)\,. \tag{2.9}$$

The delta functions in eq. (2.7) and eq. (2.8) signal the discontinuity of ϕ_0' and A' at the boundaries. Note that the three BG equations are not independent due to the Bianchi identity. First let us take a ∂_5 operation on eq. (2.9). Then by inserting eq. (2.8) to the differentiated ansatz, we will arrive the scalar BG equation eq. (2.7). The coupled BG equations can be analytically solved in terms of a single super-potential function $W(\phi)$ [13, 14], with the solutions written as:

$$\phi_0' = \frac{1}{2} \frac{\partial W(\phi_0)}{\partial \phi_0}\,, \quad A' = \frac{\kappa^2}{6} W(\phi_0)\,, \quad V(\phi) = \frac{1}{8} \left(\frac{\partial W(\phi)}{\partial \phi} \right)^2 - \frac{\kappa^2}{6} W(\phi)^2\,. \tag{2.10}$$

In this approach the back-reaction of the bulk scalar on the metric is automatically included. It is well known that the linear expansion of Einstein equation (2.6) gives the EOMs of graviton ($h_{\mu\nu}$) and radion fields (F and G) [9]. However, one must expand the 5d action till the

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quadratic order, so that the scalar EOM (modified by a shift of $\phi_0'\epsilon'\Box f(x)$ in the metric of eq. (2.2)) can be obtained by the variation principle. Furthermore, an effective Lagrangian with explicit kinetic terms is indispensable for phenomenology study, thus necessary for working out. In the next section, we will demonstrate that all the EOMs and orthogonal conditions can be derived with more clarity in the formalism of effective Lagrangian.

3 The effective Lagrangian

The 5d action eq. (2.1) can be expanded in terms of the metric and scalar perturbations. We will postpone the discussion of tadpole term in next section. The effective Lagrangian at the quadratic order is:

$$\begin{aligned}\mathcal{L}_{\text{eff}} = \int dy \frac{e^{-2A}}{2\kappa^2}\Big\{ & \frac{e^{-2A}}{4}\left[(\partial_5 h)^2 - \partial_5 h_{\mu\nu}\,\partial_5 h^{\mu\nu}\right] - \mathcal{L}_{FP} \\ & - \left[G - 2F - e^{2A}\partial_5\left(\epsilon' f(x) e^{-4A}\right)\right](\partial_\mu\partial_\nu h^{\mu\nu} - \Box h) \\ & -3e^{-2A}\left[F' - A'G - \frac{\kappa^2}{3}\phi_0'\varphi\right]\partial_5 h - \kappa^2 e^{-2A}\mathcal{L}_{5m} \\ & +\kappa^2\partial_\mu\varphi\partial^\mu\varphi - 6\Big[\partial_\mu F\partial^\mu(F-G) \\ & -e^{-2A}\epsilon'\partial_\mu\left[F' - A'G - \frac{\kappa^2}{3}\phi_0'\varphi\right]\partial^\mu f(x)\Big]\Big\}\end{aligned} \tag{3.1}$$

where $\mathcal{L}_{FP}$ is the Fierz-Pauli Lagrangian in eq. (A.8). Note that all the terms with graviton and the kinetic term of radion are derived in eqs. (A.11), (A.24), (A.27) in appendix A. Specifically $\mathcal{L}_{5m}$ contains the quadratic terms of radion without the ∂_μ operator. We define that:

$$\begin{aligned}\mathcal{L}_{5m} = \frac{4}{\kappa^2}\Big[& 2(G+4F)F'' - 10(G+4F)A'F' \\ & + 5F'^2 - 4(2F+G)A'G' + 2F'G' \\ & + G^2(5A'^2 - 2A'') + 16F^2(A'^2 - A'') \\ & + 8FG(4A'^2 - A'')\Big] + \varphi'^2 + \left[G^2 + 16F^2\right]\phi_0'^2 \\ & -2(G+4F)\phi_0'\varphi' + \left[2(G-4F)\frac{\partial V}{\partial\phi_0}\varphi + \frac{\partial^2 V}{\partial\phi_0^2}\varphi^2\right] \\ & + \sum_i\left[8F\left(2F\lambda_i - \frac{\partial\lambda_i}{\partial\phi_0}\varphi\right) + \frac{\partial^2\lambda_i}{\partial\phi_0^2}\varphi^2\right]\delta(y-y_i)\end{aligned} \tag{3.2}$$

For convenience, the terms in $\mathcal{L}_{5m}$ will be classified. In fact the expansions of Einstein-Hilbert action $S_{EH} \sim \int d^5x\sqrt{g}\,\mathcal{R}$ are put in the first square parenthesis. While the remaining items are from other origins that are not absorbable to the kinetic term. As we emphasized earlier that the y-integration path is along a circle S^1, hence any total differentiation term can be set to be zero. Using the tricks of partial integration, the terms in $\mathcal{L}_{5m}$ from the Ricci scalar can be recasted into a concise expression:

$$-\frac{12}{\kappa^2}\left[F'^2 + G^2A'^2 - 2F'GA' + 4F^2A''\right] \tag{3.3}$$

where the A'' will be substituted by the expression in eq. (2.8). Then in eq. (3.2) the two terms in the forms of $F^2\phi_0'^2$ and $\lambda_i F^2\delta(y-y_i)$ are precisely eliminated. Hence $\mathcal{L}_{5m}$ is further simplified to be:

$$\begin{aligned}\mathcal{L}_{5m} = &-\frac{12}{\kappa^2}\Big[F'^2+G^2A'^2-2F'GA'\Big]+\varphi'^2+G^2\phi_0'^2-2(G+4F)\phi_0'\varphi' \\ &+\left[2(G-4F)\frac{\partial V}{\partial\phi_0}\varphi+\frac{\partial^2 V}{\partial\phi_0^2}\varphi^2\right]-\sum_i\left[8\frac{\partial\lambda_i}{\partial\phi_0}F\varphi-\frac{\partial^2\lambda_i}{\partial\phi_0^2}\varphi^2\right]\delta(y-y_i)\end{aligned} \tag{3.4}$$

Eq. (3.4) shows that only $G^2\phi_0'^2$ survives after the stabilization while other metric expansions proportional to $\phi_0'^2$ are all cancelled.

Now we are ready to practice the variation principle without imposing any gauge fixing in advance. One can vary the effective Lagrangian eq. (3.1) with respect to G, F, φ. This will give 3 equations of motion:

$$\begin{aligned}&\phi_0'\varphi'-G\phi_0'^2-\frac{\partial V}{\partial\phi_0}\varphi \\ &\quad=\frac{3}{\kappa^2}\left[4A'(F'-A'G)+\Box\Big(Fe^{2A}-A'\epsilon' f(x)\Big)\right],\end{aligned} \tag{3.5}$$

$$\begin{aligned}&\left(\phi_0'\varphi'+\frac{\partial V}{\partial\phi_0}\varphi\right)+\sum_i\left(\lambda_i G+\frac{\partial\lambda_i}{\partial\phi_0}\varphi\right)\delta(y-y_i) \\ &\quad=\frac{3}{\kappa^2}\left[F''-G'A'-4A'F'\right]-2GV \\ &\qquad+\frac{3}{4\kappa^2}e^{2A}\Box\Big(G-2F-e^{-2A}\left[\epsilon''-4A'\epsilon'\right]f(x)\Big),\end{aligned} \tag{3.6}$$

$$\begin{aligned}&(G'+4F')\phi_0'+4A'\varphi'+\sum_i\left(\frac{\partial\lambda_i}{\partial\phi_0}G+\frac{\partial^2\lambda_i}{\partial\phi_0^2}\varphi\right)\delta(y-y_i) \\ &\quad=\varphi''-\left(2\frac{\partial V}{\partial\phi_0}G+\frac{\partial^2 V}{\partial\phi_0^2}\varphi\right)-\Box\Big(\varphi e^{2A}-\phi_0'\epsilon' f(x)\Big).\end{aligned} \tag{3.7}$$

In addition by requiring no mixing between the graviton and radion, the 3rd and 4th terms in eq. (3.1) lead to two orthogonal conditions:

$$F'-A'G-\frac{\kappa^2}{3}\phi_0'\varphi=0, \tag{3.8}$$

$$G-2F-e^{-2A}\left[\epsilon''-4A'\epsilon'\right]f(x)=0. \tag{3.9}$$

Eqs. (3.8)–(3.9) are precisely the transverse and traceless gauge fixings derived in [9] that can decouple the graviton. Note that the appearance of ϵ' in EOMs (3.5)–(3.7) comes purely from the variation of a term $-\frac{3}{\kappa^2}\int dye^{-4A}\epsilon'\partial_\mu\left[F'-A'G-\frac{\kappa^2}{3}\phi_0'\varphi\right]\partial^\mu f(x)$, i.e. ϵ' times the first orthogonal condition.[3]

In the following, we will prove that the formalism of effective Lagrangian is equivalent to the linearized Einstein equation, by providing the same set of correlated EOMs. Firstly we can identify that eq. (3.7) is just the EOM of bulk scalar. Compared with the case

[3] $\epsilon' f(x)$ behaves like a Lagrange multiplier. Varying eq. (3.1) with respect to $\epsilon' f(x)$ gives back eq. (3.8).

without ϵ perturbation, the scalar EOM is modified with a shift of $\phi_0'\epsilon'\Box f(x)$. Next using eqs. (2.7)–(2.8) and eq. (3.8), the first EOM (3.5) can be transformed to be:

$$\begin{aligned}\Box\left[Fe^{2A}-A'\epsilon' f(x)\right]+A''G-\frac{\kappa^2}{3}\left(\phi_0'\varphi'-\phi_0''\varphi\right)\\=\frac{\kappa^2}{3}\sum_i\left(\lambda_i(\phi_0)G+\frac{\partial\lambda_i(\phi_0)}{\partial\phi}\varphi\right)\delta(y-y_i)\end{aligned}\tag{3.10}$$

Then after taking the differentiation of eq. (3.8), i.e.

$$\frac{\kappa^2}{3}\phi_0''\varphi=(F''-A'G'-A''G)-\frac{\kappa^2}{3}\phi_0'\varphi'\tag{3.11}$$

we'll insert eq. (3.11) into eq. (3.10) and obtain:

$$\begin{aligned}\Box\left[Fe^{2A}-A'\epsilon' f(x)\right]+(F''-A'G')-\frac{2\kappa^2}{3}\phi_0'\varphi'\\=\frac{\kappa^2}{3}\sum_i\left(\lambda_i(\phi_0)G+\frac{\partial\lambda_i(\phi_0)}{\partial\phi}\varphi\right)\delta(y-y_i)\end{aligned}\tag{3.12}$$

that is exactly the EOM of radion derived in [9]. We would like to remark that an exact correspondence can be established between the first two EOMs and the Einstein equation eq. (2.6). In fact one can find that eq. (3.5) is from the assembling of $\frac{1}{\kappa^2}\left[e^{2A}R_{\mu\nu}/\eta_{\mu\nu}+R_{55}\right]$, while eq. (3.6) corresponds to the combination of $-\frac{1}{2\kappa^2}\left[2e^{2A}R_{\mu\nu}/\eta_{\mu\nu}-R_{55}\right]$. The correlation between the $(\mu\nu)$ and (55) components of eq. (2.6) is demonstrated in appendix B. Hence one can obtain eq. (3.6) directly from eq. (3.5) after some lengthy algebra, with the necessity to remove the last term in eq. (3.6) by enforcing the second orthogonal condition.

Finally the scalar EOM (3.7) is not independent to the Einstein equation. Dropping out the brane terms, one can first construct an ansatz of $\partial_5\left[e^{-2A}\text{eq. (3.12)}\right]$, then subtract it with $e^{-2A}A'\left[R_{55}-\kappa^2\tilde{T}_{55}\right]=0$. The resulting equation containing a term of $\Box(F'-A'G)$ is actually the scalar EOM (3.7) times $\kappa^2\phi_0'e^{-2A}/3$ (see appendix B).

Therefore although we start with 4 radion related scalars, only a single EOM eq. (3.12) plus the two gauge fixings eqs. (3.8)–(3.9) are independent ones, indicating that one perturbation is not dynamical. Recalling the diffeomorphism in eq. (2.5), before stabilization one can always set $\zeta=\epsilon f(x)$ to eliminate the arbitrariness in the bulk. Simultaneously ϵ' will inherit the zeros of ζ' at all the branes. Is this gauge fixing still operative in the presence of stabilization? With the constraint $\epsilon'|_{y_i}=\epsilon''|_{y_i}=0$, $y_i=\{0,\pm r_a,L\}$, $a=1,\cdots\frac{N}{2}-1$, it is viable to conduct the field redefinition according to a spurious symmetry,

$$\begin{aligned}\tilde{F}&=F-A'\epsilon' f(x)e^{-2A}\\\tilde{G}&=G-\left(\epsilon''-2A'\epsilon'\right)f(x)e^{-2A}\\\tilde{\varphi}&=\varphi-\phi_0'\epsilon' f(x)e^{-2A}\end{aligned}\tag{3.13}$$

so that the ϵ' can be fully removed from the EOMs (3.5)–(3.7) and eqs. (3.8)–(3.9) become $\tilde{F}'-A'\tilde{G}=\frac{\kappa^2}{3}\phi_0'\tilde{\varphi}$ and $\tilde{G}=2\tilde{F}$. But the justification of this transformation should be investigated. Inspecting the 5d action first, the kinetic term is shifted by:

$$\frac{3}{\kappa^2}\int dx^5\left(e^{-4A}\partial_\mu\tilde{F}\left[\epsilon''-2A'\epsilon'\right]+\frac{A'}{2}\frac{d}{dy}\left[\epsilon'^2e^{-6A}\right]\partial_\mu f(x)\right)\partial^\mu f(x)+\Box(\delta\varphi)\text{ terms} \tag{3.14}$$

which depends on the bulk value of $\epsilon'(y)$ for $\phi_0'\neq 0$. This implies that the ζ-symmetry (relic of 5d diffeomorphism) in eq. (2.5) is spontaneously broken if the GW scalar develops a y-dependent VEV. Actually as φ must be invariant, $\delta\varphi=0$ in eq. (3.13) will force $\phi_0'\,\epsilon'=0$ to preserve the 4d Poincaré invariance, otherwise ambiguity will enter in the radion kinetic term. Thus two cases are permitted according to the symmetry principle:

(a) For $\phi_0'(y)\neq 0$, one will get $\epsilon'(y)=0$, i.e. $\epsilon=$ *constant* is a pure gauge in eq. (2.2) without impact on the dynamics. This option can provide a static radion solution, while signals the breaking of ζ-symmetry.

(b) For $\phi_0'(y)=0$, $\epsilon'(y)\neq 0$ is allowed, that preserves the 5d diffeomorphism if $\epsilon'|_{y_i}=0$. With $\phi_0'=0$, one can also relax the BC to be $\epsilon'(r_a)\neq 0$ and eq. (3.14) is a nonzero surface term because of $\tilde{F}\sim e^{2A}$ and $A'\sim$ *constant*. However the second option offers no radion stabilization.[4]

Consequently case (a) is the correct option for a stabilized radion. In fact, simply imposing the two orthogonal conditions (3.8)–(3.9) on the effective Lagrangian eq. (3.1), we find that no any $\epsilon' f(x)$ could remain. This leaves the kinetic terms of the graviton and radion to be:

$$\mathcal{L}_{kin}=-\frac{1}{2\kappa^2}\int dy\,e^{-2A}\Big\{\mathcal{L}_{FP}-\kappa^2\partial_\mu\varphi\partial^\mu\varphi+6\left[\partial_\mu F\partial^\mu(F-G)\right]\Big\} \tag{3.15}$$

Therefore the solvable radion EOM (respecting 4d diffeomorphism) should be eq. (3.12) gauged with $\epsilon'=0$ and $G=2F$. Notice that this property deduced from the symmetry principle is valid for any N-brane ($N\geq 2$) RS model.

4 The tadpole term

Now we will discuss the tadpole term that is the linear order expansion of eq. (2.1), since a sizable tadpole might disturb the radion EOM. In the literature the radion tadpole is linked to one of the sum rules in the brane worlds [15], derived from the background Einstein equation in a spatially periodic extra dimension. Assuming the internal curvature of Minkowski space is zero and $F\sim e^{2A}$, the paper [16] claimed that the coefficient of the linear term F is proportional to

$$\oint dye^{-2A}\left[\phi_0'^2+2V(\phi_0)+2\sum_i\lambda_i\delta(y-y_i)\right]=0 \tag{4.1}$$

[4]Case (b) is discussed in ref. [6] for a 4-brane model with no GW stabilization, where one radion is presumably chosen with $\epsilon_1'(r)=0$ and the other one with $\epsilon_2'(r)\neq 0$. But please note there are other solutions for two massless modes with $\epsilon_1'(r)\neq 0$ and $\epsilon_2'(r)\neq 0$. The excitation created by borrowing the BC of a gauge fixing field does not seem to be physical as they are not well defined.

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that is zero and can be immediately verified using eq. (2.10) (see appendix C). However the weakness in that argument is the scalar perturbation φ is fully ignored. Indeed for $\phi_0 = 0$ and $V(\phi_0) = -\frac{6}{\kappa^2}A'^2$, the tadpole of radion field is bound to vanish due to eq. (4.1) [9]. In this section we will provide a rigorous calculation for the radion tadpole, that does not align to eq. (4.1) after the GW stabilization.

The derivation is proceeded by employing the Einstein equation eq. (2.6) to transform the Ricci scalar $\mathcal{R}$ into the forms of $V(\phi_0)$ or $\lambda_i(\phi_0)$ and their first derivatives with respect to ϕ_0. From the compact expression of the modified energy momentum tensor $\tilde{T}_{IJ} = T_{IJ} - \frac{1}{3}g_{IJ}T^m_m$, one can obtain the following expansions [9]:

$$\begin{aligned}
\tilde{T}^{(0)}_{\mu\nu} &= -\frac{e^{-2A}}{3}\eta_{\mu\nu}\left(2V(\phi_0) + \sum_i \lambda_i(\phi_0)\delta(y-y_i)\right)\\
\tilde{T}^{(0)}_{55} &= \phi_0'^2 + \frac{2}{3}\left(V(\phi_0) + 2\sum_i \lambda_i(\phi_0)\delta(y-y_i)\right)\\
\tilde{T}^{m(0)}_{m} &= -\phi_0'^2 - \frac{10}{3}V(\phi_0) - \frac{8}{3}\sum_i \lambda_i(\phi_0)\delta(y-y_i)
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
\tilde{T}^{(1)}_{\mu\nu} &= -\frac{2e^{-2A}}{3}\left[\eta_{\mu\nu}\left(\frac{\partial V}{\partial\phi_0}\varphi - 2VF\right) + 2\epsilon\partial_\mu\partial_\nu f(x)V\right]\\
&\quad -\frac{e^{-2A}}{3}\sum_i \eta_{\mu\nu}\left(\frac{\partial\lambda_i}{\partial\phi_0}\varphi - \lambda_i(G+2F)\right)\delta(y-y_i)\\
&\quad -\frac{2e^{-2A}}{3}\sum_i \lambda_i\,\epsilon\partial_\mu\partial_\nu f(x)\delta(y-y_i)\\
\tilde{T}^{(1)}_{55} &= 2\phi_0'\varphi' + \frac{2}{3}\left[\sum_i 2\left(\lambda_i G + \frac{\partial\lambda_i}{\partial\phi_0}\varphi\right)\delta(y-y_i) + 2VG + \frac{\partial V}{\partial\phi_0}\varphi\right]
\end{aligned} \tag{4.3}$$

Note that although the ϵ is kept in $\tilde{T}^{(1)}_{IJ}$, one can anticipate the tadpole term does not depend on the relic gauge. Substituting $\kappa^2\times$ eqs. (4.2)–(4.3) into eq. (2.1), we will first arrive:

$$\begin{aligned}
\mathcal{L}_{\rm tad} = \frac{2}{3}&\oint dy e^{-4A}\left[(G-4F) + \epsilon\Box f(x)\right]V(\phi_0)\\
-\frac{1}{3}&\oint dy e^{-4A}\sum_i \lambda_i(\phi_0)\left(4F - \epsilon\Box f(x)\right)\delta(y-y_i)\\
+\frac{1}{3}&\oint dy e^{-4A}\varphi\left(2\frac{\partial V}{\partial\phi_0} + \sum_i \frac{\partial\lambda_i}{\partial\phi_0}\delta(y-y_i)\right)
\end{aligned} \tag{4.4}$$

where the terms proportional to $\phi_0'^2$ and $\phi'\varphi'$ are cancelled in the linear order expansion. Applying eqs. (2.8)–(2.9) and eq. (3.8), we can perform the transformation:

$$\begin{aligned}
&\frac{2}{3}\oint dy e^{-4A}\left[(G-4F)V - \sum_i 2\lambda_i F\delta(y-y_i)\right]\\
&\quad = \frac{1}{3}\oint dy e^{-4A}\left[G\phi_0'^2 + 4A'\phi_0'\varphi\right].
\end{aligned} \tag{4.5}$$

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Then substituting eq. (4.5) into eq. (4.4), the radion tadpole becomes:

$$\begin{aligned}\mathcal{L}_{\rm tad} = &\frac{1}{3}\oint dy e^{-4A}\left[G\phi_0'^2 + \left(4A'\phi_0' + \frac{\partial V}{\partial \phi_0}\right)\varphi\right] \\ &+\frac{1}{3}\oint dy e^{-4A}\left[\frac{\partial V}{\partial \phi_0} + \sum_i \frac{\partial \lambda_i}{\partial \phi_0}\delta(y-y_i)\right]\varphi \\ &+\frac{1}{3}\oint dy e^{-4A}\left[2V + \sum_i \lambda_i \delta(y-y_i)\right]\epsilon\Box f(x)\end{aligned} \tag{4.6}$$

Now we apply the eqs. (3.5), (3.8) and BG equation (2.7), the first two lines in eq. (4.6) can be simplified to be:

$$\begin{aligned}&\frac{1}{\kappa^2}\oint dy\left[\frac{\kappa^2}{3}\frac{d}{dy}\left(\phi_0'\varphi e^{-4A}\right) - e^{-2A}\Box\left(F - \frac{A'\epsilon' f(x)}{e^{2A}}\right)\right] \\ &= \frac{m^2}{\kappa^2}\oint dy\, e^{-4A}\left[Fe^{2A} - A'\epsilon' f(x)\right]\end{aligned} \tag{4.7}$$

where the integration of the total differential term is zero. And again using the BG equations (2.8)–(2.9), the third line in eq. (4.6) is rewritten as:

$$\begin{aligned}&\frac{1}{\kappa^2}\oint dy\left[A'\frac{d}{dy}\left(\epsilon\Box f(x) e^{-4A}\right) - e^{-4A}A'\epsilon'\Box f(x)\right] \\ &+\frac{1}{3}\oint dy\, e^{-4A}\left[\phi_0'^2 + \sum_i \lambda_i\delta(y-y_i)\right]\epsilon\Box f(x) \\ &= \frac{m^2}{\kappa^2}\oint dy\, e^{-4A}A'\epsilon' f(x)\end{aligned} \tag{4.8}$$

Combining eq. (4.7) and eq. (4.8), the final expression for the radion tadpole reads:

$$\begin{aligned}\mathcal{L}_{\rm tad} &= \frac{m^2}{\kappa^2}\oint dy\, e^{-4A}\left[\left[Fe^{2A} - A'\epsilon' f(x)\right] + A'\epsilon' f(x)\right] \\ &= \frac{m^2}{\kappa^2}\oint dy\, e^{-2A}\, F\end{aligned} \tag{4.9}$$

Impressively the tadpole term is proportional to $m^2 e^{-2A}F$ after the stabilization, with other spin-0 perturbations eliminated by the EOM and BG equations. Further investigation requires the knowledge of the 5d profile by solving the radion EOM with proper boundary conditions. In the limit of small back-reaction, the mass squared of radion is parameterized as $m^2 = \tilde{m}^2 l^2$ with $l = \frac{\kappa}{\sqrt{2}}\phi_0|_{y=0}$. Therefore to evaluate the tadpole term at the $\mathcal{O}(l^2)$ order, substituting the zeroth order profile i.e. $F = c\, e^{2A}$ into eq. (4.9), we find that the tadpole of the lowest mode automatically vanishes. While if the solution of F contains a ϵ' part as shown in [6], eq. (4.9) is nonzero in general at the $\mathcal{O}(l^2)$ order.

5 Radion stabilization

As argued in the previous sections, only one dynamical degree of freedom exists for the radion in an N-brane RS model governed by the action eq. (2.1). In fact our analysis is consistent with the naive counting of degree of freedom, since the 5d metric contains $5 = 1_0 \oplus 2_{\pm1} \oplus 2_{\pm2}$ dynamical fields, where the spin-0 scalar 1_0 plays the role of radion. A direct consequence of one radion is that just the UV-IR brane distance L is stabilized by the potential. Expanding the 5d action at the zeroth order, its derivative with respect to L is exactly zero if eq. (2.10) holds true and the coordinates r_a of intermediate branes are fixed. This implies that the intermediate branes are not dynamical, thus will not generate additional radion-like modes. In such a way, the GW mechanism is generalized into an N-brane setup. For $N = 4$, we will choose the following superpotential for stabilization:

$$W(\phi) = \begin{cases} \frac{6k_1}{\kappa^2} - u_1\phi^2\,, & 0 < y < r \\ \frac{6k_2}{\kappa^2} - u_2\phi^2\,, & r < y < L \end{cases} \tag{5.1}$$

where the discontinuity in the first term originates from $\Lambda_1 \neq \Lambda_2$ and the mass parameters $u_{1,2}$ are assumed to be unequal at first. By matching the singular terms in eq. (3.6)–(3.7) and eq. (2.7)–(2.8), the boundary conditions (BC) with $\epsilon' = 0$ and $G = 2F$ are derived as:

$$[F']_i - 2\,[A']_i\,F = \frac{\kappa^2}{3}\,[\phi_0']_i\,\varphi\,, \tag{5.2}$$

$$[\varphi']_i - 2\,[\phi_0']_i\,F = \frac{\partial^2\lambda_i}{\partial\phi^2}\varphi\,. \tag{5.3}$$

Similar to RS1, we can impose a stiff potential at the UV and IR branes, leading to $\varphi(y)|_{y=\{0,L\}} = 0$. But due to a single radion field, one can set $\frac{\partial^2\lambda_r}{\partial\phi^2} = 0$ at $y = r$ and this will result in a constraint on $u_{1,2}$. Note that eq. (5.3) corresponds to the BC of scalar EOM, hence needs to be consistent with the radion EOM. The left hand of eq. (5.3) can be transformed from eq. (3.10) to be:

$$[\varphi']_r - 2\,[\phi_0']_r\,F = \left[\frac{\phi_0''}{\phi_0'}\right]_r \varphi + \frac{3e^{2A}}{\kappa^2}\Box F\left[\frac{1}{\phi_0'}\right]_r \tag{5.4}$$

Using the specific superpotential in eq. (5.1), one can derive the relevant jumps at $y = r$:

$$\left[\frac{\phi_0''}{\phi_0'}\right]_r = u_2 - u_1\,, \qquad \left[\frac{1}{\phi_0'}\right]_r = \frac{1}{\phi_0(r)}\left(\frac{1}{u_1} - \frac{1}{u_2}\right)\,. \tag{5.5}$$

Then substituting eq. (5.5) into eq. (5.4), the junction condition becomes:

$$(u_1 - u_2)\left(\phi_0\varphi - \frac{3}{\kappa^2}\frac{1}{u_1u_2}e^{2A}\Box F\right)\Big|_{y=r} = 0\,, \tag{5.6}$$

A trivial solution that satisfies eq. (5.6) is $u_1 = u_2$. With this choice $\lambda_{\pm r}$ gets no ϕ dependence, i.e.$\frac{\partial\lambda_r}{\partial\phi} = [\phi_0']_r = 0$, and eq. (5.2) determines the BC at $y = r$ to be $[F']_r = 2\,[A']_r\,F$. Thus by solving the EOM (3.12) with the prescribed BC, one can obtain a stabilized radion with its mass below the cutoff scale of IR brane [9].

6 Conclusion

The main goal of this paper serves to clarify the degree of freedom in an N-brane RS model. First of all, we provided a complete calculation of the effective Lagrangian till the quadratic order. Practicing the variation principle in the EFT delivers 3 EOMs and two gauge fixings (3.8)–(3.9) originate from the non-mixing condition for the graviton and radion. Note that eq. (3.9) should be enforced such that an exact correspondence between the first two EOMs and the linearized Einstein equation can be established. Moreover, we illustrated that the scalar EOM removing away brane terms can be derived from the linearized Einstein equation and a single EOM is actually independent. Thus only one dynamical degree freedom is allowed for the radion in an N-brane RS model. For consistency, we investigate the linear order expansion in the effective Lagrangian and proved that the radion tadpole vanishes at the $\mathcal{O}(l^2)$ order. Hence the radion EOM is valid to be derived from the quadratic expansion, without including the tadpole effect.

This paper also clearly explains whether $\epsilon\,\partial_\mu\partial_\nu f(x)$ can be added in $g_{\mu\nu}$ as a radion perturbation. Without the radion stabilization, the ζ-symmetry (relic of 5d diffeomorphism) in eq. (2.5), keeping the Einstein-Hilbert action invariant, can remove the ϵ perturbation in the bulk. By relaxing one BC at an intermediate brane $y = r_a$, at most a surface kinetic term is induced by $\epsilon'(r_a)$ in the 5d action. However in the scenario that the GW scalar develops a y-dependent VEV, the ζ-symmetry needs to be broken in order to preserve the 4d Poincaré symmetry. This symmetry analysis is universal for an N-brane RS model and excludes the possibility to use $\epsilon'(r_a) \neq 0$ at the intermediate branes to gain another radion excitation as proposed by [6]. In the presence of radion stabilization, the arbitrariness of $\epsilon(y)$ will cause ambiguity in the radion kinetic term [9] and physical observables, e.g. the quartic interaction of $\epsilon f(x)$ coupling with one radion (or one graviton $h_{\mu\nu}$) plus two SM particles off the branes is not zero in general. Then $\epsilon'(y)$ has to be zero in the bulk and branes due to the 5d ζ-symmetry breaking.

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A Quadratic expansion of the 5d action

In this section, we provide the intermediate steps to derive the effective Lagrangian at the quadratic order eq. (3.1) except for the $\mathcal{L}_{5m}$ term.

A.1 The kinetic and ∂_5^2 terms of graviton

First of all we will calculate the kinetic and ∂_5^2 terms of the graviton. We expand the Ricci tensor till the second order i.e. $R_{IJ} = R_{IJ}^{(0)} + R_{IJ}^{(1)} + R_{IJ}^{(2)}$, with the number inside a pair of parenthesis representing the expansion order. The zeroth order part is:

$$R_{\mu\nu}^{(0)} = e^{-2A}\,\eta_{\mu\nu}\left(4A'^2 - A''\right)\,, \quad R_{55}^{(0)} = 4\left(A'' - A'^2\right) \tag{A.1}$$

And the parts at the higher order including only the $h_{\mu\nu}$ terms read:

$$\begin{aligned}R^{(1)}_{\mu\nu} \supset\ & \frac{1}{2}\left(\partial_\mu\partial_\lambda h^\lambda_\nu + \partial_\nu\partial_\lambda h^\lambda_\mu - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right)\\ & + \frac{1}{2}e^{-2A}\left(\partial_5^2 h_{\mu\nu} - 4A'\partial_5 h_{\mu\nu}\right)\\ & + e^{-2A}\left[4A'^2 - A''\right]h_{\mu\nu} - \frac{1}{2}e^{-2A}\eta_{\mu\nu}A'\partial_5 h \end{aligned} \tag{A.2}$$

$$R^{(1)}_{55} \supset -\frac{1}{2}\left(\partial_5^2 h - 2A'\partial_5 h\right) \tag{A.3}$$

$$\begin{aligned}R^{(2)}_{\mu\nu} \supset\ & \frac{1}{2}h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} - \frac{1}{2}\partial_\alpha\left(h^{\alpha\beta}\left(\partial_\mu h_{\nu\beta} + \partial_\nu h_{\mu\beta} - \partial_\beta h_{\mu\nu}\right)\right) + \frac{1}{4}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\\ & + \frac{1}{2}\partial^\alpha h^\beta_\nu\left(\partial_\alpha h_{\beta\mu} - \partial_\beta h_{\alpha\mu}\right) + \frac{1}{4}\left(\partial^\alpha h\right)\left(\partial_\mu h_{\nu\alpha} + \partial_\nu h_{\mu\alpha} - \partial_\alpha h_{\mu\nu}\right)\\ & + \frac{1}{4}e^{-2A}\left[\eta_{\mu\nu}\left(2A'h^{ij}\partial_5 h_{ji}\right) + \left(\partial_5 h_{\mu\nu} - 2A'h_{\mu\nu}\right)\partial_5 h - 2\partial_5 h_{\mu\alpha}\partial_5 h^\alpha_\nu\right] \end{aligned} \tag{A.4}$$

$$R^{(2)}_{55} \supset -\frac{1}{4}\partial_5 h^{ij}\partial_5 h_{ij} - A'h^{i\beta}\partial_5 h_{\beta i} + \frac{1}{2}\partial_5\left(h^{ij}\partial_5 h_{ji}\right) \tag{A.5}$$

Note that the linear order expansion was given in [9]. With eq. (A.1)–(A.5), the Ricci scalar can be calculated at each order, e.g. at the second order:

$$R^{(2)} = g^{IJ(0)}R^{(2)}_{IJ} + g^{IJ(1)}R^{(1)}_{IJ} + g^{IJ(2)}R^{(0)}_{IJ}\,. \tag{A.6}$$

Some parts of quadratic terms originate from $\sqrt{g}^{(0)}R^{(2)} + \sqrt{g}^{(1)}R^{(1)} + \sqrt{g}^{(2)}\left(R^{(0)} + 2\kappa^2 V(\phi_0)\right)$ in the 5d action eq. (2.1):

$$\begin{aligned}\sqrt{g}^{(0)}R^{(2)} + \sqrt{g}^{(1)}R^{(1)} \supset\ & e^{-2A}\left[\mathcal{L}_{FP} + \partial_\alpha\left(h_{\mu\nu}\partial^\alpha h^{\mu\nu} - \partial_\mu\left(h^{\mu\nu}h^\alpha_\nu\right) - \frac{1}{2}h\partial_\beta h^{\alpha\beta} - \frac{1}{2}h\partial^\alpha h\right)\right]\\ & + e^{-4A}\left[\frac{1}{4}\left[\partial_5 h_{\mu\nu}\partial_5 h^{\mu\nu} - \left(\partial_5 h\right)^2\right] + A'h^{ij}\partial_5 h_{ij} - \frac{1}{2}A'h\partial_5 h\right]\\ & - \partial_5\left(e^{-4A}h_{\mu\nu}\partial_5 h^{\mu\nu}\right) + \frac{1}{2}\partial_5\left(e^{-4A}h\partial_5 h\right) \end{aligned} \tag{A.7}$$

with $\mathcal{L}_{FP}$ standing for the Fierz-Pauli Lagrangian:

$$\mathcal{L}_{FP} = \frac{1}{2}\partial_\nu h_{\mu\alpha}\,\partial^\alpha h^{\mu\nu} - \frac{1}{4}\partial_\mu h_{\alpha\beta}\,\partial^\mu h^{\alpha\beta} - \frac{1}{2}\partial_\alpha h\,\partial_\beta h^{\alpha\beta} + \frac{1}{4}\partial_\alpha h\,\partial^\alpha h \tag{A.8}$$

and

$$\sqrt{g}^{(2)}\left(R^{(0)} + 2\kappa^2 V(\phi_0)\right) \supset -\frac{1}{4}e^{-4A}\left[h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2\right]\left[8\left(A'^2 - A''\right) + \kappa^2\phi_0'^2\right] \tag{A.9}$$

While the scalar kinetic term and the brane potentials in eq. (2.1) contribute as well:

$$\begin{aligned}& -\sqrt{g}^{(2)}\left(\frac{g^{IJ(0)}}{2}\left(\partial_I\phi\partial_J\phi\right)^{(0)}\right) + \sqrt{g_4}^{(2)}\sum_i\lambda_i\delta(y-y_i)\\ & \supset -\frac{1}{4}e^{-4A}\left[h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2\right]\left(\frac{1}{2}\phi_0'^2 + \sum_i\lambda_i\delta(y-y_i)\right) \end{aligned} \tag{A.10}$$

Note that the total differential terms with ∂_α or ∂_5 in eq. (A.7) vanish after the integration. Thus calculating the quantity of $-\frac{1}{2\kappa^2}\int dy$ [eq. (A.7) + eq. (A.9) + $2\kappa^2$ eqs. (A.10)], at the second order the effective Lagrangian contains:

$$\mathcal{L}^{(2)} \supset -\frac{1}{2\kappa^2}\int dy \left(e^{-2A}\left[\mathcal{L}_{FP} - \frac{e^{-2A}}{4}\left[(\partial_5 h)^2 - \partial_5 h_{\mu\nu}\partial_5 h^{\mu\nu}\right]\right] + \mathcal{L}_{h^2}\right) \tag{A.11}$$

with

$$\begin{aligned}\mathcal{L}_{h^2} = \int dy \Bigg[& e^{-4A}\left[h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2\right]\left(-\frac{\kappa^2}{2}\left(\phi_0'^2 + \sum_i \lambda_i \delta(y-y_i)\right) + 2A''\right) \\ &+\frac{1}{2}A'\partial_5\left(e^{-4A}\left[h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2\right]\right)\Bigg] = 0\end{aligned} \tag{A.12}$$

where the last term can be applied a partial integration and then using eq. (2.8) sets $\mathcal{L}_{h^2}$ to equal zero.

A.2 The radion and graviton mixing terms

The four radion fields F, G, φ and $\epsilon(y)\partial_\mu\partial_\nu f(x)$ will mix with the graviton field. The mixing via the Ricci tensor can only proceed with two ∂_μ or two ∂_5 derivatives. Due to the conformal flatness of RS metric, the ∂_μ operator can not differentiate the radion and graviton perturbations. Therefore by replacing only one graviton field in the Fierz-Pauli Lagrangian $\mathcal{L}_{FP}$ to be $h_{\mu\nu} \to -2F\eta_{\mu\nu}$, $h_{55} \to 2G\eta_{55}$ and $h \to 2(G-4F)$, we can obtain the mixing part:

$$\begin{aligned}-\frac{1}{2\kappa^2}\int dy e^{-2A}\mathcal{L}_{FP} \Rightarrow & -\frac{1}{2\kappa^2}\int dy e^{-2A}\Big[2F\partial_\mu\partial_\nu h^{\mu\nu} - 2F\Box h \\ &\qquad +(G-4F)\partial_\alpha\partial_\beta h^{\alpha\beta} - \Box h(G-4F)\Big] \\ = & -\frac{1}{2\kappa^2}\int dy e^{-2A}(G-2F)\left[\partial_\mu\partial_\nu h^{\mu\nu} - \Box h\right]\end{aligned} \tag{A.13}$$

For the mixing through two ∂_5, the perturbations of F and G should be treated in different approach. Firstly we can pick the term $-\frac{1}{4}\left[(\partial_5 h)^2 - \partial_5 h_{\mu\nu}\partial_5 h^{\mu\nu}\right]$, and make a single substitution of $h_{\mu\nu} \to 2\left[\epsilon\partial_\mu\partial_\nu f(x) - F\eta_{\mu\nu}\right]$ and $h \to 2\left[\epsilon\Box f(x) - 4F\right]$ to obtain:

$$\begin{aligned}&\frac{1}{2\kappa^2}\int dy \frac{e^{-4A}}{4}\left[(\partial_5 h)^2 - \partial_5 h_{\mu\nu}\partial_5 h^{\mu\nu}\right] \\ &\Rightarrow \frac{1}{2\kappa^2}\int dy e^{-4A}\left[\partial_5(\epsilon\Box f(x))\partial_5 h - \partial_5(\epsilon\partial_\mu\partial_\nu f(x))\partial_5 h^{\mu\nu})\right] \\ &\quad -\frac{1}{2\kappa^2}\int dy e^{-4A}\left[\partial_5(4F)\partial_5 h - (F\eta_{\mu\nu})\partial_5 h^{\mu\nu})\right] \\ &= \frac{1}{2\kappa^2}\int dy e^{-4A}\left[(\epsilon'\Box f(x)\partial_5 h - \epsilon'\partial_\mu\partial_\nu f(x)\partial_5 h^{\mu\nu}) - 3F'\partial_5 h\right]\end{aligned} \tag{A.14}$$

Next the mixings between G, φ and $\partial_5 h$ have to be directly calculated. For clarity, we will list the Ricci tensor that contributes to the mixing of G and $\partial_5 h$ as:

$$R^{(1)}_{\mu\nu} \supset e^{-2A}\eta_{\mu\nu}\left[A'G' - 2G\left(4A'^2 - A''\right)\right] \tag{A.15}$$

$$R^{(1)}_{55} \supset -\frac{1}{2}\left(\partial_5^2 h - 2A'\partial_5 h\right) \tag{A.16}$$

$$\begin{aligned} R^{(2)}_{\mu\nu} \supset e^{-2A}&[3A'G\left(\partial_5 h_{\mu\nu} - 2A'h_{\mu\nu}\right) + GA'\left(\partial_5 h\eta_{\mu\nu} - \partial_5 h_{\mu\nu}\right) \\ &-\frac{1}{2}G'\left(\partial_5 h_{\mu\nu} - 2A'h_{\mu\nu}\right) \\ &-G\partial_5\left(\partial_5 h_{\mu\nu} - 2A'h_{\mu\nu}\right) - 2A'^2 G h_{\mu\nu}] \end{aligned} \tag{A.17}$$

$$R^{(2)}_{55} \supset \frac{1}{2}G'\partial_5 h \tag{A.18}$$

Combining eqs. (A.15)–(A.18), we can extract the relevant terms in $\sqrt{g}^{(0)}R^{(2)} + \sqrt{g}^{(1)}R^{(1)}$:

$$\begin{aligned} &\sqrt{g}^{(0)}R^{(2)} + \sqrt{g}^{(1)}R^{(1)} \\ &\supset e^{-4A}\left[-G\partial_5^2 h + 5GA'\partial_5 h - G'\partial_5 h + 4A'G'h - 4Gh\left(5A'^2 - 2A''\right)\right] \end{aligned} \tag{A.19}$$

There are similar terms originating from the expansion of $\sqrt{g}^{(2)}\left(R^{(0)} + 2\kappa^2 V(\phi_0)\right)$:

$$\sqrt{g}^{(2)}\left(R^{(0)} + 2\kappa^2 V(\phi_0)\right) \supset \frac{1}{2}e^{-4A}Gh\left[8\left(A'^2 - A''\right) + \kappa^2\phi_0'^2\right] \tag{A.20}$$

Then we add the contribution from the scalar kinetic term:

$$-\frac{1}{2}\left(\sqrt{g}^{(2)}g^{IJ(0)} + \sqrt{g}^{(1)}g^{IJ(1)}\right)\left(\partial_I\phi\partial_J\phi\right)^{(0)} \supset -\frac{1}{4}e^{-4A}Gh\phi_0'^2 \tag{A.21}$$

Finally substituting eqs. (A.19)–(A.21) into eq. (2.1), we find that the terms proportional to $\phi_0'^2$ are all cancelled, and the mixing term of $G\partial_5 h$ is derived to be:

$$\begin{aligned} \mathcal{L}_{Gh} &= -\frac{1}{2\kappa^2}\int dy\left[-3e^{-4A}GA'\partial_5 h - \partial_5\left(e^{-4A}G\partial_5 h\right)\right] \\ &\quad -\frac{2}{\kappa^2}\int dy\left[e^{-4A}GhA'' + A'\partial_5\left(e^{-4A}Gh\right)\right] \\ &= \frac{3}{2\kappa^2}\int dy e^{-4A}GA'\partial_5 h \end{aligned} \tag{A.22}$$

Now we will show how to calculate the mixing term involving $\varphi\partial_5 h$. Such type of term comes from the following combination:

$$\begin{aligned} &-\int dy\left[-\sqrt{g}^{(1)}\left(\frac{g^{IJ(0)}}{2}\left(\partial_I\phi\partial_J\phi\right)^{(1)}\right) + \sqrt{g}^{(1)}V^{(1)} + \sum_i\sqrt{g_4}^{(1)}\lambda_i^{(1)}\delta(y-y_i)\right] \\ &\supset -\int dy e^{-4A}\frac{1}{2}h\left[\phi_0'\varphi' + \frac{\partial V}{\partial\phi_0}\varphi + \sum_i\frac{\partial\lambda_i}{\partial\phi_0}\varphi\delta(y-y_i)\right] \\ &= -\int dy e^{-4A}\frac{1}{2}h\left[\partial_5\left(\phi_0'\varphi\right) - \left(\phi_0'' - \frac{\partial V}{\partial\phi_0} - \sum_i\frac{\partial\lambda_i}{\partial\phi_0}\delta(y-y_i)\right)\varphi\right] \\ &= -\int dy e^{-4A}\frac{1}{2}h\left[\partial_5\left(\phi_0'\varphi\right) - 4A'\phi_0'\varphi\right] = \frac{1}{2}\int dy e^{-4A}\phi_0'\varphi\partial_5 h \end{aligned} \tag{A.23}$$

Adding up eqs. (A.13)–(A.14) and eqs. (A.22)–(A.23), we obtain the final forms of the mixing between the radion and graviton:

$$\mathcal{L}_{\rm mix} = -\frac{1}{2\kappa^2}\int dy e^{-2A}\left[\left[G-2F-e^{2A}\partial_5\left(\epsilon' f(x)e^{-4A}\right)\right]\left(\partial_\mu\partial_\nu h^{\mu\nu}-\Box h\right)\right.$$
$$\left.+3e^{-2A}\left[F'-A'G-\frac{\kappa^2}{3}\phi_0'\varphi\right]\partial_5 h\right] \tag{A.24}$$

A.3 The kinetic term of radion

The kinetic term of radion contains three parts: 1) involving only F and G perturbations, 2) with one $\epsilon'\Box f(x)$, and 3) involving only the scalar perturbation φ. For the first part, we can derive it from the kinetic term of graviton $-\frac{1}{2\kappa^2}\int dy e^{-2A}\mathcal{L}_{FP}$ by replacing $h_{\mu\nu}\to -2F\eta_{\mu\nu}$, $h_{55}\to 2G\eta_{55}$ and $h\to 2\left(G-4F\right)$:

$$\begin{aligned}\mathcal{L}_{\rm rad} \supset & -\frac{1}{2\kappa^2}\int dy e^{-2A}\left[2\partial_\mu F\partial^\mu F-\left(4\partial_\mu F\partial^\mu F+\partial_\mu G\partial^\mu G\right)\right.\\ & \left.+2\partial_\mu\left(G-4F\right)\partial^\mu F+\partial_5(G-4F)\partial^\mu\left(G-4F\right)\right]\\ = & -\frac{3}{\kappa^2}\int dy e^{-2A}\partial_\mu F\partial^\mu\left(F-G\right)\end{aligned} \tag{A.25}$$

And the second part can be calculated from eq. (A.24) by substituting $h_{\mu\nu}\to 2\epsilon\partial_\mu\partial_\nu f(x)$ and $h\to 2\epsilon\Box f(x)$:

$$\begin{aligned}\mathcal{L}_{\rm rad} \supset & -\frac{3}{\kappa^2}\int dy e^{-4A}\left[F'-A'G-\frac{\kappa^2}{3}\phi_0'\varphi\right]\epsilon'\partial_\mu\partial^\mu f(x)\\ = & \frac{3}{\kappa^2}\int dy e^{-4A}\epsilon'\partial_\mu\left[F'-A'G-\frac{\kappa^2}{3}\phi_0'\varphi\right]\partial^\mu f(x)+\text{the surface term}\end{aligned} \tag{A.26}$$

Combining eqs. (A.25)–(A.26) and the part 3), the total expression for the kinetic term of radion field is:

$$\begin{aligned}\mathcal{L}_{\rm rad} = \frac{1}{2}\int dy e^{-2A}\Bigg[\partial_\mu\varphi\partial^\mu\varphi-\frac{6}{\kappa^2}\Bigg[&\partial_\mu F\partial^\mu\left(F-G\right)\\ &-e^{-2A}\epsilon'\partial_\mu\left[F'-A'G-\frac{\kappa^2}{3}\phi_0'\varphi\right]\partial^\mu f(x)\Bigg]\Bigg]\end{aligned} \tag{A.27}$$

B Correlation of the scalar EOM and Einstein equation

The scalar EOM (3.7) can be derived from the linearized Einstein equations. Analogously the $(\mu\nu)$ and (55) components of eq. (2.6) are not independent. We start with providing the detail to show the correlation between the scalar EOM and Einstein equation. Firstly we need to evaluate $\partial_5\left[e^{-2A}\text{eq. (3.12)}\right]$. By transforming eq. (3.12) back into eq. (3.10), this gives:

$$\begin{aligned}&\Box F'-\partial_5\left(e^{-2A}\left(A'\epsilon'\Box f(x)-\frac{\kappa^2}{3}\phi_0'^2 G\right)\right)\\ &-\frac{\kappa^2}{3}e^{-2A}\left(\phi_0'\varphi''-\phi_0'''\varphi-2A'\left(\phi_0'\varphi'-\phi_0''\varphi\right)\right)=0\end{aligned} \tag{B.1}$$

Dropping the boundary terms, $A'e^{-2A}\left(R_{55}-\kappa^2\tilde{T}_{55}\right)=0$ gives [9]:

$$\left[-(\epsilon''-2A'\epsilon')\Box f(x)+e^{2A}\Box G+4F''-4A'(G'+2F')\right]A'e^{-2A} \\ =\kappa^2\left[\frac{4}{3}GV+2\phi_0'\varphi'+\frac{2}{3}\frac{\partial V}{\partial\phi}\varphi\right]A'e^{-2A} \tag{B.2}$$

Now one can immediately calculate the quantity of (eq. (B.1)–(B.2)) to be:

$$\Box\left(F'-A'G\right)-e^{-2A}A''\epsilon'\Box f(x)+\frac{\kappa^2}{3}\phi_0'\left(4A'\varphi'-\varphi''e^{-2A}+\frac{\partial^2 V}{\partial\phi_0^2}\varphi\right)e^{-2A} \\ +\left[\frac{\kappa^2}{3}\left(2\phi_0'\phi_0''G+\phi_0'^2G'+4A''\phi_0'\varphi\right)-4A'A''G\right]e^{-2A} \\ +8\left[\frac{\kappa^2}{6}GA'\left(V-\frac{1}{2}\phi_0'^2\right)+A'^2F'\right]e^{-2A}+\frac{2\kappa^2}{3}\left(\frac{\partial V}{\partial\phi}-\phi_0''\right)\varphi A'e^{-2A}=0 \tag{B.3}$$

where the first two term can be combined as:

$$\Box\left(F'-A'G\right)-e^{-2A}A''\epsilon'\Box f(x)=\frac{\kappa^2}{3}\phi_0'e^{-2A}\Box\left(\varphi e^{2A}-\phi_0'\epsilon' f(x)\right) \tag{B.4}$$

and applying eqs. (2.7)–(2.8) for the terms in the second line, we can rewrite:

$$\left[\frac{\kappa^2}{3}\left(2\phi_0'\phi_0''G+\phi_0'^2G'+4A''\phi_0'\varphi\right)-4A'A''G\right]e^{-2A} \\ =\frac{\kappa^2}{3}\phi_0'\left[2\frac{\partial V}{\partial\phi}G+\phi_0'(G'+4F')\right]e^{-2A} \tag{B.5}$$

Then using eq. (2.9) and eq. (3.8), one can obtain:

$$\left[\frac{\kappa^2}{6}GA'\left(V-\frac{1}{2}\phi_0'^2\right)+A'^2F'\right]e^{-2A}=\frac{\kappa^2}{3}A'^2\phi_0'\varphi e^{-2A} \tag{B.6}$$

Combining eq. (B.6) with the last term in eq. (B.3) and applying eq. (2.7), we can find that all the terms in the third line of eq. (B.3) are exactly cancelled. Finally substituting eqs. (B.4)–(B.5) into eq. (B.3), that equation reproduces the scalar EOM (3.7) times $\kappa^2\phi_0'e^{-2A}/3$.

Next we will prove that the $(\mu\nu)$ part of eq. (2.6) can be derived from its (55) part. Expanding to the linear order, one can extract out the $\eta_{\mu\nu}$ part in $R_{\mu\nu}=\kappa^2\tilde{T}_{\mu\nu}$ [9]:

$$e^{2A}\Box\left[F-A'\epsilon' e^{-2A}f(x)\right]-F''+A'(8F'+G') \\ =-\frac{\kappa^2}{3}\left[4GV+2\frac{\partial V}{\partial\phi_0}\varphi+\sum_i\left(\frac{\partial\lambda_i}{\partial\phi_0}\varphi+\lambda_i G\right)\delta(y-y_i)\right] \tag{B.7}$$

as the $\partial_\mu\partial_\nu$ part is a gauge fixing. On the other hand, the linearized $R_{55}=\kappa^2\tilde{T}_{55}$ is [9]:

$$e^{2A}\Box\left[G-\left(\epsilon''-2A'\epsilon'\right)e^{-2A}f(x)\right]+4F''-4A'(G'+2F') \\ =\frac{2\kappa^2}{3}\left[2GV+\frac{\partial V}{\partial\phi_0}\varphi+3\phi_0'\varphi'+2\sum_i\left(\frac{\partial\lambda_i}{\partial\phi_0}\varphi+\lambda_i G\right)\right], \tag{B.8}$$

The derivative of the first orthogonal condition eq. (3.8) gives:

$$\phi_0'\phi' = -\phi_0''\phi + \frac{3}{\kappa^2}\left(F'' - A'G' - A''G\right) \tag{B.9}$$

By substituting eq. (B.9) and the second orthogonal condition eq. (3.9) into eq. (B.8), one can arrive that:

$$\begin{aligned} &e^{2A}\Box\left(F - A'\epsilon' e^{-2A} f(x)\right) - F'' + A'(8F' + G') + 3(A'' - 4A'^2)G \\ &= \frac{\kappa^2}{3}\left[2GV - 2\frac{\partial V}{\partial \phi_0}\varphi - \sum_i \left(\frac{\partial \lambda_i}{\partial \phi_0}\varphi - 2\lambda_i G\right)\delta(y - y_i)\right] \end{aligned} \tag{B.10}$$

Then applying eq. (2.8)–(2.9) to eq. (B.10), the (55) part of Einstein equation is explicitly transformed into the $(\mu\nu)$ one in eq. (B.7).

C Proof of sum rule

The absence of tadpole was claimed in the literature as the consequence of the following ansatz:

$$\oint dy e^{-2A} T^\mu_\mu = \oint dy e^{-2A}\left[-2\phi_0'^2 - 4V(\phi_0) - 4\sum_i \lambda_i \delta(y - y_i)\right] = 0\,, \tag{C.1}$$

and we are going to provide an alternative proof for this sum rule. Due to $\phi_0' = \frac{1}{2}\frac{\partial W(\phi_0)}{\partial \phi_0}$ in eq. (2.10) and counting the discontinuity of the superpotential $W(\phi)$ at the junctions, one gets:

$$\phi_0'^2 = \frac{1}{2}\frac{\partial W}{\partial \phi_0}\frac{\partial \phi_0}{\partial y} = \frac{1}{2}\left[\frac{\partial W}{\partial y} - 2\sum_i \lambda_i \delta(y - y_i)\right] \tag{C.2}$$

Applying $V(\phi) = \frac{1}{8}\left(\frac{\partial W(\phi)}{\partial \phi}\right)^2 - \frac{\kappa^2}{6}W(\phi)^2$ in eq. (2.10), we can rewrite eq. (C.1) in the following form:

$$\begin{aligned} &\oint dy e^{-2A}\left[-2\phi_0'^2 - 4V(\phi_0) - 4\sum_i \lambda_i \delta(y - y_i)\right] \\ &= \oint dy e^{-2A}\left[-2\phi_0'^2 - \frac{1}{2}\left(\frac{\partial W(\phi_0)}{\partial \phi_0}\right)^2 + \frac{2\kappa^2}{3}W(\phi_0)^2 - 4\sum_i \lambda_i \delta(y - y_i)\right] \\ &= \oint dy e^{-2A}\left[-4\phi_0'^2 + \frac{2\kappa^2}{3}W(\phi_0)^2 - 4\sum_i \lambda_i \delta(y - y_i)\right] \end{aligned} \tag{C.3}$$

Now we can insert eq. (C.2) to remove the singular term and after partial integration this gives:

$$\begin{aligned} \oint dy e^{-2A} T^\mu_\mu &= \oint dy \left[e^{-2A}\left(-4A'W + \frac{2\kappa^2}{3}W^2\right) - 2\frac{d}{dy}\left(e^{-2A}W\right)\right] \\ &= \oint dy e^{-2A}\left[-4A'W + \frac{2\kappa^2}{3}W^2\right] = 0 \end{aligned} \tag{C.4}$$

where the total differential term vanishes and in the last line we used $A' = \frac{\kappa^2}{6}W$ in eq. (2.10).

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