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From orthosymplectic structure to super topological matter

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Abstract

Topological supermatter is given by ordinary topological matter constrained by supersymmetry or graded supergroups such as OSP(2N|2N). Using results on super oscillators and lattice QFT_d, we construct a super tight binding model on hypercubic super lattice with supercharge $\boldsymbol{Q} = \sum_{\mathbf{k}} \hat{F}_{\mathbf{k}}.\boldsymbol{q}_{\mathbf{k}}.\hat{B}_{\mathbf{k}}$. We first show that the algebraic triplet (Ω, G, J) of super oscillators can be derived from the $OSp(2N|2N)$ supergroup containing the symplectic $Sp(2N)$ and the orthogonal $SO(2N)$ as even subgroups. Then, we apply the obtained result on super oscillating matter to super bands and investigate its topological obstructions protected by TPC symmetries. We also give a classification of the Bose/Fermi coupling matrix $\boldsymbol{q}_{\mathbf{k}}$ in terms of subgroups of $OSp(2N|2N)$ and show that there are $2P_N$ (partition of N) classes $\boldsymbol{q}_{\mathbf{k}}$ given by unitary subgroups of $U(2) \times U(N)$. Other features are also given.

1. Introduction

Supersymmetry in Lorentzian 4D space time and higher dimensions plays a crucial role in the study of relativistic superfield theory [1] and superstrings [2]. This Bose/Fermi symmetry is

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not an exact symmetry at our energy scale; if it existed in early universe, it is now broken. For that, several efforts have been made to find footprints of supersymmetry in particle physics at high energies; but still without direct experimental evidence. Supersymmetry in non relativistic physics has also received some attention [3–10]; supersymmetric models have been constructed in numerous fields such as condensed matter [11], statistical and quantum mechanics [12–14], optics [15,16] and cosmology [17].

Recently, a special attention has been given to supersymmetric topological phases of matter [18–25] and supersymmetric entanglement [26–30]. There, super models [31,32] were constructed by using fermionic $\hat{f}^{\alpha}/\hat{f}_{\alpha}^{\dagger}$ and bosonic $\hat{b}^{\alpha}/\hat{b}_{\beta}^{\dagger}$ oscillators coupled as

$$Q=\sum_{\alpha,\beta=1}^{N}\hat{f}_{\alpha}^{\dagger}\left[A_{\beta}^{\alpha}\right]\hat{b}^{\beta}\qquad,\qquad Q^{\dagger}=\sum_{\alpha,\beta=1}^{N}\hat{b}_{\beta}^{\dagger}\left[\left(A^{\dagger}\right)_{\alpha}^{\beta}\right]\hat{f}^{\alpha} \tag{1.1}$$

A simple coupling matrix A_{β}^{α} is given by the diagonal frequency matrix $\delta_{\beta}^{\alpha}\sqrt{\nu}$ leading to the familiar quantum supercharge $Q=\sqrt{\nu}\hat{f}_{\alpha}^{\dagger}\hat{b}^{\alpha}$. The Hamiltonian H of this super oscillator is given by the quadratic composite $QQ^{\dagger}+Q^{\dagger}Q$ generating four contributions $H_{ff}+H_{bb}+H_{fb}+H_{bf}$ namely:

(**i**) a quadratic fermionic $H_{ff}=\hat{f}_{\alpha}^{\dagger}\left(h_{f}\right)_{\beta}^{\alpha}\hat{f}^{\beta}$ with coupling matrix $h_{f}=AA^{\dagger}$.

(**ii**) a quadratic bosonic $H_{bb}=\hat{b}_{\alpha}^{\dagger}\left(h_{b}\right)_{\beta}^{\alpha}\hat{b}^{\beta}$ with coupling matrix $h_{b}=A^{\dagger}A$; and

(**iii**) two *quartic* terms H_{fb} and H_{bf}, compensating each other, given by the coupling $\pm\hat{f}^{\dagger}\hat{b}^{\dagger}\left[A\otimes A^{\dagger}\right]\hat{f}\hat{b}$.

On the other hand, it is quite well known that, besides supersymmetry, the bosonic and fermionic oscillators share several common features; but show also different behaviours due to their different quantum statistics. In this regard, the bosonic oscillator operators combined like $\hat{B}^{A}=(\hat{b}^{\alpha},\hat{b}_{\alpha}^{\dagger})$ are described by the symplectic symmetry $Sp(2N)$ of the phase space $\mathcal{E}_{ph}$ of statistical physics [33]-[34]; they are characterised by the symplectic structure Ω^{AB} given by the commutator $\hat{B}^{A}\hat{B}^{B}-\hat{B}^{B}\hat{B}^{A}$. However, the properties of the fermionic oscillator operators, combined as $\hat{F}^{A}=(\hat{f}^{\alpha},\hat{f}_{\alpha}^{\dagger})$, are described by the orthogonal symmetry $SO(2N)$ and are characterised by the metric G^{AB} given by the anti-commutator $\hat{F}^{A}\hat{F}^{B}+\hat{F}^{B}\hat{F}^{A}$ [35–37]. Because of the Bose/Fermi symmetry of the oscillating $\hat{B}/\hat{F}$ system, it is legitimate to ask on: (i) the relationship between supersymmetry and the combination of the orthogonal and the symplectic structures, (termed below as the orthosymplectic structure). (ii) the seek of lattice super QFTs governed by the orthosymplectic property for probing supersymmetric effect in non relativistic settings; and (iii) the use of the orthosymplectic idea to approach super topological matter while imposing discrete TPC symmetries in the same spirit as in the derivation of the periodic table classifying the topological insulators and superconductors in tenfold way classes [38].

In this paper, we contribute to super topological matter and super band theory from the view of the orthosymplectic structure represented by the triplet (Ω,G,J) with Ω and G as above; the J is a complex structure found to be given by the intersection $\Omega.G^{-1}$ and described by the intersection $U(N)$ symmetry of $Sp(2N)$ and $SO(2N)$. We show that the algebraic properties of the super oscillators are nicely described by the super group $OSp(2N|2N)$; which when combined with lattice super QFT methods and TPC symmetries, allow to approach the topological properties of the super bands. Among our results, we cite the following:

(**1**) the construction of a super tight binding model (TBM) for lattice supermatter described by a supercharge $\boldsymbol{Q}$ quadratic in local oscillator field operators $\hat{f}^{\alpha}(\mathbf{r}_{i})$, $\hat{b}^{\beta}(\mathbf{r}_{i})$ as follows

$$
\begin{aligned}
\boldsymbol{Q} = & \sum_{i,j} \left[\hat{f}^{\dagger}_{\alpha}(\mathbf{r}_i) \left(\kappa_{ij}\right)^{\alpha}_{\beta} \hat{b}^{\beta}\left(\mathbf{r}_j\right) + \hat{f}^{\alpha}(\mathbf{r}_i) \left(\varkappa_{ij}\right)_{\alpha\beta} \hat{b}^{\beta}\left(\mathbf{r}_j\right) \right] + \\
& \sum_{i,j} \left[\hat{b}^{\dagger}_{\alpha}(\mathbf{r}_i) \left(\kappa'_{ij}\right)^{\alpha}_{\beta} \hat{f}^{\beta}\left(\mathbf{r}_j\right) + \hat{b}^{\dagger}_{\alpha}(\mathbf{r}_i) \left(\varkappa'_{ij}\right)^{\alpha\beta} \hat{f}^{\dagger}_{\beta}\left(\mathbf{r}_j\right) \right]
\end{aligned}
\tag{1.2}
$$

with coupling matrices $\kappa\left(\mathbf{r}_i - \mathbf{r}_j\right)$ and $\varkappa\left(\mathbf{r}_i - \mathbf{r}_j\right)$ assumed translation invariant. This supercharge can be presented shortly as $\hat{F}_{\mathbf{r}_i} \boldsymbol{J}_{ij} \hat{B}_{\mathbf{r}_j}$; its square generates the super hamiltonian H_{super} given by (i) the sum $\hat{F}_{\mathbf{r}_i} \left(h_f\right)_{ij} \hat{F}_{\mathbf{r}_j} + \hat{B}_{\mathbf{r}_i} \left(h_b\right)_{ij} \hat{B}_{\mathbf{r}_j}$, with coupling $\left(h_f\right)_{ij}$ and $\left(h_b\right)_{ij}$ quadratic in $\boldsymbol{J}_{ij}$, and (ii) two extra terms that kill each other (Bose/Fermi compensation effect).

(**2**) the determination of the intrinsic properties of the super H_{super} and its topological distortions. By using the Fourier transform, the supercharge operator expands as $\sum Q_{\mathbf{k}}$ with Fourier modes $Q_{\mathbf{k}}$ given by the quadratic form $\hat{F}_{\mathbf{k}}.\boldsymbol{q}_{\mathbf{k}}.\hat{B}_{\mathbf{k}}$. The local $\boldsymbol{q}_{\mathbf{k}}$ living on the Brillouin torus $\mathbb{T}^d$ is a $2N \times 2N$ coupling matrix valued in the bi-fundamental representation of $Sp(2N) \times SO(2N)$ with sub-bloc matrices $N \times N$ given by the Fourier transforms of $\boldsymbol{\kappa}_{ij}, \boldsymbol{\kappa}'_{ij}, \varkappa_{ij}, \varkappa'_{ij}$. Here, we show that the super Hamiltonian $H_{\mathbf{k}}$ has two main contributions: (**i**) A fermion dependent $(H_{\mathbf{k}})_f$ given by $\hat{F}_{\mathbf{k}}.\boldsymbol{h}_f.\hat{F}_{\mathbf{k}}$ with coupling matrix as $\boldsymbol{h}_f = \boldsymbol{q}_{\mathbf{k}} Z_f \boldsymbol{q}^{\dagger}_{\mathbf{k}}$ and $Z_f = \sigma_z \otimes I_N$. It can carry non-trivial topological distortions as for non super Altland-Zirnbauer (AZ) matter. (**ii**) A boson dependent H_b given by $\hat{B}_{\mathbf{k}}.\boldsymbol{h}_b.\hat{B}_{\mathbf{k}}$ with coupling matrix $\boldsymbol{h}_b = \boldsymbol{q}^{\dagger}_{\mathbf{k}} \boldsymbol{q}_{\mathbf{k}}$; it has a trivial topology.

(**3**) the derivation of constraints on the coupling matrix $\boldsymbol{q}_{\mathbf{k}}$ that are required by the embedding of $\mathcal{N} = 2$ supersymmetric quantum mechanical matter ($\mathcal{N} = 2$ super QM) within tight binding models based on the orthosymplectic osp(2N|2N). In these regards, we distinguish two supermatter systems: orthosymplectic (ORTIC) and supersymmetric (SUSY). We also give solutions for $\boldsymbol{q}_{\mathbf{k}}$ which turn out to be classified by (i) the complex structure J represented by the diagonal $U(N)$ subsymmetry of the OSP(2N|2N) supergroup; and (ii) the TPC symmetries of AZ matter.

The presentation is as follows: In section 2, we study the structure triplet $(\Omega, G, J) \equiv \boldsymbol{\tau}$ of supersymmetric oscillator operators from the view of symplectic $Sp(2N)$ and orthogonal $SO(2N)$ symmetries. In section 3, we give the relation of the triplet $\boldsymbol{\tau}$ with the orthosymplectic $OSp(2N2|N)$ super group; and its link with the graded algebra of quantum super oscillator. In section 4, we study the link between the orthosymplectic osp(2N|2M) Lie superalgebra and $\mathcal{N} = 2$ super QM. Using properties of osp(2N|2M), we build out of $\boldsymbol{Q}$ several observables including two interesting ones termed as orthosymplectic (ORTIC) Hamiltonian H_{ortic} and supersymmetric (SUSY) Hamiltonian H_{susy}. In section 5, we develop the study of super tight binding model by using the orthosymplectic supergroup representations and methods of lattice super QFT$_d$. In section 6, we investigate the topological properties of the super TBM. In section 7, we give a conclusion and make comments. Last section is devoted to two appendices: In appendix A, we deepen the study on the link between: (i) $\mathcal{N} = 2$ supersymmetry on world line, (ii) orthosymplectic osp(2|2) superalgebra and (iii) $\mathcal{N} = 2$ $U(1)$ superconformal invariance. In appendix B, we develop the oscillator realisation of the osp(2N|2M) used in our tight binding modelling and in the construction of the supersymmetric fivefold ways of [20].

2. Orthosymplectic structure

In this section, we study two building bloc structures $\boldsymbol{\tau}$ and $\tilde{\boldsymbol{\tau}}$ of "orthosymplectic" manifolds $\mathcal{K}$ [39] with the aim of using them later to investigate topological supermatter. These building blocs are given by the following triplets,

$$\boldsymbol{\tau} = (g, \omega, J) \quad , \quad \tilde{\boldsymbol{\tau}} = (G, \Omega, J^T) \tag{2.1}$$

They have an interpretation in the quantized phase space of free supersymmetric oscillators. The g and G will be associated with fermions $\digamma$, the ω and Ω with bosons ξ; and the J and J^T give the link between them.
First, we introduce the above triplets as recently formulated in [40]. Then, we investigate their useful properties. We show that they constitute main pillars in dealing with the superalgebra of supersymmetric quantum oscillators and its representations; thus offering a new way to think about (2.1).

2.1. Revisiting the (g, ω, J) and (G, Ω, J^T) triplets

Following [40], the triplet $\boldsymbol{\tau} = (g, \omega, J)$ and its dual $\tilde{\boldsymbol{\tau}} = (G, \Omega, J^T)$ play an important role in the construction of the bosonic and fermionic Gaussian states. These are algebraic structures that have representations in terms of non degenerate real rank 2 tensors $\mathcal{T}$ in its three variants: contravariant $\mathcal{T}^{AB}$, covariant $\mathcal{T}_{AB}$ and mixed $\mathcal{T}_B^A$ as follows

$$\begin{array}{ccccc} g_{AB} & , & \omega_{AB} & , & J_A{}^B \\ G^{AB} & , & \Omega^{AB} & , & J^A{}_B \end{array} \tag{2.2}$$

with $(J_A{}^B)^T = J^A{}_B$. For later use, we give below those interesting properties regarding the tensor realisation of the triplet (g, ω, J). Similar things can be written down for (G, Ω, J^T).
The real g_{AB} is symmetric ($g^T = g$) and positive defined ($\det g > 0$). It acts as an orthogonal metric, it maps $SO\,(2N, \mathbb{R})$ contravariant vectors $\digamma^B$ (fermions) to the covariant ones like

$$\digamma_A = g_{AB}\digamma^B \quad , \quad \digamma^B = G^{BC}\digamma_C \quad , \quad G^{BC}g_{CA} = \delta_A^B \tag{2.3}$$

This metric is invariant under orthogonal transformation; i.e.: $\Lambda g \Lambda^T = g$ with transformation matrix Λ belonging to the orthogonal group $SO\,(2N, \mathbb{R})$.
The ω_{AB} is the usual antisymmetric symplectic structure ($\omega^T = -\omega, \det\omega \neq 0$). It acts as a symplectic metric on contravariant vectors ξ^B (bosons) to map them to covariant ones like

$$\xi_A = \omega_{AB}\xi^B \quad , \quad \xi^B = \Omega^{BC}\xi_C \quad , \quad \Omega^{BC}\omega_{CA} = \delta_A^B \tag{2.4}$$

This symplectic metric is invariant under symplectic transformations; i.e.: $\mathcal{S}\omega\mathcal{S}^T = \omega$ with matrix $\mathcal{S}$ belonging to the group $SP\,(2N, \mathbb{R})$ [41].
The mixed tensor J is the complex structure living on the Kahler space $\mathcal{K}$ with square $J^2 = -I_{id}$ showing that J behaves as the pure imaginary number unit i ($i^2 = -1$). An interesting definition of J_B^A is given by

$$J_B^A = \Omega^{AC}g_{CB} \tag{2.5}$$

from which we learn that it can act on both the orthogonal vectors $\digamma$ (fermions) and the symplectic vectors $\boldsymbol{\xi}$ (bosons). For example, we have

$$\begin{array}{lcl} J_B^A F^B & = & \Omega^{AC} F_C \\ \xi_A J_B^A & = & -\xi^C g_{CB} \end{array} \tag{2.6}$$

and

$$\begin{array}{lcl} \xi_A J_B^A F^B & = & \Omega^{AC} \xi_A F_C \\ & = & -g_{CB} \xi^C F^B \end{array} \tag{2.7}$$

as well as

$$\begin{array}{lcl} F_A J_B^A F^B & = & \Omega^{AC} F_A F_C \\ \xi_A J_B^A \xi^B & = & -g_{CB} \xi^C \xi^B \end{array} \tag{2.8}$$

To distinguish the labels of the $SO(2N)$ and $Sp(2N)$ representations, we use below dotted labels for $SO(2N)$; for example F^A, g_{AB} and $J_B^A = \Omega^{AC} g_{CB}$ will be replaced by $F^{\dot{A}}$, $g_{\dot{A}\dot{B}}$ and $J_{\dot{B}}^A = \Omega^{AC} g_{\dot{C}\dot{B}}$. The last relation requires the identification of the labels C and $\dot{C}$; this feature will be discussed later.

2.1.1. Duality and constraint relations

The rank 2 tensors (2.2) realising the 3+3 components of the triplets (2.1) are not all of them free; they are subject to relationships and constraint equations that we describe here after:

- **Duality relations**
  As noticed before, the (g, ω, J) and (G, Ω, J^T) are dual between them. The duality relations are given by

$$\begin{array}{lclclcl} g_{\dot{A}\dot{C}} G^{\dot{C}\dot{B}} & = & +\delta_{\dot{A}}^{\dot{B}} & , & J_{\dot{B}}^A J_C^{\dot{B}} & = & \delta_C^A \\ \omega_{AC} \Omega^{CB} & = & +\delta_A^B & , & J_C^{\dot{A}} J_{\dot{B}}^C & = & \delta_{\dot{B}}^{\dot{A}} \end{array} \tag{2.9}$$

  These relations read in a condensed way as follows

$$\begin{array}{lclclcl} g.G & = & +I_{2N} & , & J.J^T & = & +I_{2N} \\ \omega.\Omega & = & +I_{2N} & , & J^T.J & = & +I_{2N} \end{array} \tag{2.10}$$

- **Constraint Equations**
  Amongst the above mentioned 3 basis components of the two triplets, only two of them which are really basic ones. For example the symplectic ω_{AB} of the bosonic sector and the $g_{\dot{A}\dot{B}}$ of the fermionic sector. The following relations express one element of the triplet as a product of two others:

$$\begin{array}{lclclcl} J & = & +\Omega.g & , & J^T & = & -g.\Omega \\ J & = & -G.\omega & , & J^T & = & +\omega.G \end{array} \tag{2.11}$$

  and

$$\begin{array}{lclclcl} \Omega & = & +J.G & , & \Omega & = & -G.J^T \\ G & = & -J.\Omega & , & G & = & +\Omega.J^T \end{array} \tag{2.12}$$

  Eq. (2.11) captures the feature that the complex structure J and its underlying unitary group $U(N)$ is given by the intersection of the orthogonal and the symplectic structures as depicted by the Fig. 1.

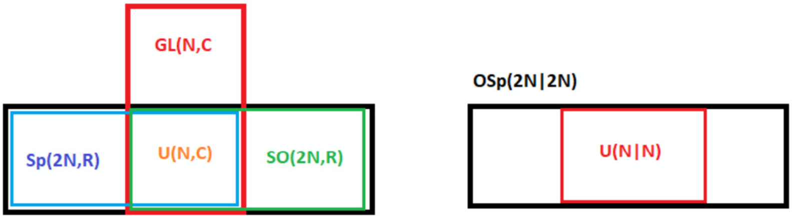

Fig. 1. On the left, the unitary group $U(N)$ as the Intersection of three groups: the $GL(N,\mathbb{C})$ in green color, the $SP(2N,\mathbb{R})$ in blue; and the $SO(2N,\mathbb{R})$ in red. On the right the $U(N|N)$ supergroup as a graded subsymmetry of $OSP(2N|2N)$.

- **Refining the constraints**
  From eqs. (2.11)-(2.12), we deduce remarkable relationships; in particular the two following:
  (**i**) the triple intersection relation

$$\omega.J.G = I_{id} \tag{2.13}$$

  which can be also expressed in different, but equivalent, ways.
  (**ii**) the transformations

$$\begin{array}{ccccccc} J.G.J^T & = & G & , & J.\Omega.J^T & = & \Omega \\ J^T.g.J & = & g & , & J^T\omega.J & = & \omega \end{array} \tag{2.14}$$

  which are useful for explicit calculations.

2.1.2. Canonical representation of the triplets

If thinking about the symmetric matrices g and G as given by the identity matrix I_{2N}, then the complex structure J coincides with the symplectic structure as shown by substituting in (2.13) $g = G = I_{2N}$; thus leading to

$$\omega.J = I_{id} \tag{2.15}$$

For this canonical choice, we have $J = \Omega$ and $J^T = \omega$. In matrix notation [26],

$$G = \begin{pmatrix} I_N & 0 \\ 0 & I_N \end{pmatrix} \quad , \quad J = \Omega = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix} \tag{2.16}$$

Because of the property $J^2 = -I_{2N}$, one can define two interesting quantities:

- The two projectors

$$P_\pm = \frac{1}{2}(I_{id} \pm iJ) \tag{2.17}$$

  with

$$(P_\pm)^2 = P_\pm \quad , \quad P_+ + P_- = I_{id} \tag{2.18}$$

- Given a real contravariant (resp. covariant) vector ξ^A (resp. ξ_A) with 2N components, one can construct two chiral vectors $\xi_\pm^A$ (resp. $\xi_A^\pm$) given by

$$\begin{array}{ccc} \xi_+^A & = & \xi^A + iJ_B^A\xi^B \\ \xi_-^A & = & \xi^A - iJ_B^A\xi^B \end{array} \tag{2.19}$$

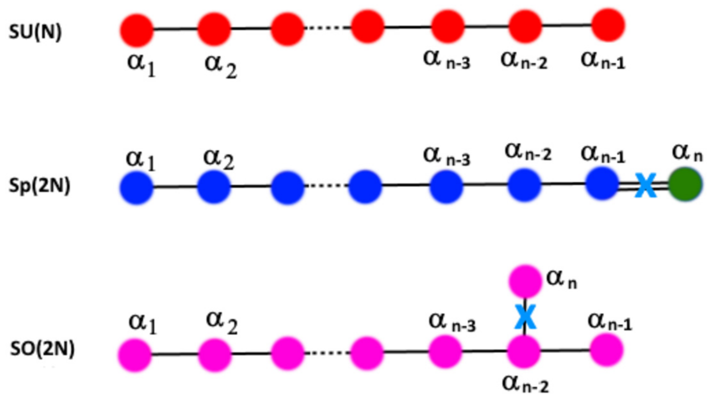

Fig. 2. The Dynkin diagrams of the unitary $SU(N)$, the $SP(2N)$ and the $SO(2N)$ groups. By cutting the simple root α_N in $SP(2N)$ and the $SO(2N)$, we obtain the diagram of $SU(N)$.

and

$$\begin{array}{lcl} \xi_B^+ & = & \xi_B + iJ_B^A \xi_A \\ \xi_B^- & = & \xi_B - iJ_B^A \xi_A \end{array} \tag{2.20}$$

2.2. *Supersymmetric phase space* $\mathcal{E}_{ph}^{2N|2N}$

So far we have used variables and structures carrying charges of the symmetry groups $Sp(2N)$, $SO(2N)$ and $SU(N)$. The unitary group $SU(N)$ is a subgroup of $Sp(2N)$ and of $SO(2N)$ as schematized by the Fig. 1. This feature can be also exhibited by using the Dynkin diagrams of the Lie algebras of these Lie groups as given by the Fig. 2. The diagram of $SU(N)$ has $(N-1)$ nodes while those of $Sp(2N)$ and $SO(2N)$ have N nodes [42]; by cutting the node α_N of $Sp(2N)$ and $SO(2N)$, one obtains the graph of $SU(N)$. This unitary group $SU(N)$ plays an important role in our present study; we put it aside for the moment; we will take it up later. Moreover, we will think of the symplectic (bosons) and the orthogonal (fermions) groups as

$$G_{\bar{0}} = Sp(2N) \times SO(2N) \tag{2.21}$$

that is the even part of the orthosymplectic group [43]

$$OSp(2N|2N) = G_{\bar{0}} \ltimes G_{\bar{1}} \tag{2.22}$$

with $G_{\bar{1}}$ given by $(2N, 2\dot{N})$, the bi-fundamental representation of $G_{\bar{0}}$ with $2\mathbf{N}$ standing for the fundamental representation of $Sp(2N)$ and the $2\dot{\mathbf{N}}$ referring to the fundamental representation of $SO(2N)$.

Among the relevant features of $OSp(2N|2N)$ and the realisation of the orthosymplectic manifold $\mathcal{K}$ for our present study, we cite the supersymmetric phase space $\mathcal{E}_{ph}^{2N|2N}$ with graded coordinates like

$$Z^{\underline{A}} = \begin{pmatrix} \xi^A \\ \lambda^{\dot{A}} \end{pmatrix} \qquad \leftrightarrow \qquad \begin{pmatrix} bosons \\ fermions \end{pmatrix} \tag{2.23}$$

and super label $\underline{A} = (A, \dot{A})$. The bosonic ξ^A is a vector of $Sp(2N)$ with label taking values as $A = 1, ..., 2N$; it parameterises $\mathcal{E}_{ph}^{2N|0}$. The fermionic $\lambda^{\dot{A}}$ is a vector of $SO(2N)$ with label

$\dot{A} = 1, ..., 2\dot{N}$; it parameterises $\mathcal{E}_{ph}^{0|2N}$.
Because of the $\mathbb{Z}_2$- grading of supersymmetry, the orthosymplectic space $\mathcal{K}$ has two kinds of variables: bosonic phase space variables ξ^A and fermionic ones $\lambda^{\dot{A}}$ splitting as follows

$$\xi^A = \begin{pmatrix} x^I \\ p_I \end{pmatrix} \quad , \quad \lambda^{\dot{A}} = \begin{pmatrix} \gamma_x^{\dot{I}} \\ \gamma_p^{\dot{I}} \end{pmatrix} \tag{2.24}$$

where x^I and p_I refer to the position variables and their momentums. The $\gamma_x^{\dot{I}}$ and $\gamma_p^{\dot{I}}$ are real fermionic variables; they are the super partners of x^I and p_I.

2.2.1. Bosonic sector of $\mathcal{K}$

The $\mathcal{K}$ contains the bosonic phase space $\mathcal{E}_{ph}^{2N|0}$ of Hamiltonian systems as a subspace where live the $Sp(2N)$ symplectic symmetry. The $\mathcal{E}_{ph}^{2N|0}$ is coordinated by the $N+N$ Darboux variables x^I, p_I with label $I = 1, ..., N$. These real symplectic variables can be combined into two interesting ways: (i) $Sp(2N, \mathbb{R})$ vectors; or (ii) $U(N, \mathbb{C})$ complex vectors.

(**1**) As $Sp(2N, \mathbb{R})$ vectors ξ^A and ξ_A given by

$$\xi^A = \begin{pmatrix} x^I \\ p_I \end{pmatrix} \quad , \quad \xi_A = \left(-p_I, x^I\right) \tag{2.25}$$

with $x^I = \xi^I$ and $p_I = \xi^{I+N}$. The contravariant ξ^A and the covariant ξ_A vectors have 2N components and are related to each other by the metric as

$$\xi_A = \omega_{AB}\xi^B \quad , \quad \xi^A = \Omega^{AB}\xi_B \tag{2.26}$$

with antisymmetric ω_{AB} and its inverse Ω^{AB} thought of in terms of their canonical form as follows

$$\omega_{AB} = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix} \quad , \quad \Omega^{AB} = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix} \tag{2.27}$$

Using two vectors ξ^A and ξ'^B, their symplectic invariant is given by $\xi_A\xi'^A = \omega_{AB}\xi^B\xi'^A$. By substituting (2.25), we get

$$\xi_A\xi'^A = x^I p'_I - p_I x'^I \tag{2.28}$$

capturing manifestly the property $\omega_{AB}\xi^A\xi^B = 0$ because of the commutativity property of the bosonic variable namely $\xi^A\xi^B = \xi^B\xi^A$ and anti-commutativity of the symplectic metric $\omega_{AB} = -\omega_{BA}$.

(**2**) As $U(N, \mathbb{C})$ vectors by using complex coordinates like

$$z^I = x^I + ip_I \quad , \quad z_I^* = x^I - ip_I \tag{2.29}$$

in term of which the symplectic invariant $\xi_A\xi'^A$ is given by the imaginary part of $z_I^* z'^I$,

$$\xi_A\xi'^A = \mathrm{Im}\left(z_I^* z'^I\right) \tag{2.30}$$

So, the real 2N dimensional bosonic phase space $\mathcal{E}_{ph}^{2N|0}$ is homomorphic to a N-dimensional complex space that we imagine it here as $\mathbb{C}^N$ but with metric (2.30). Notice that the complex variables can be also combined as

$$\zeta^A = \begin{pmatrix} z^I \\ z_I^* \end{pmatrix} \quad , \quad \zeta_A = \left(-z_I^*, z^I\right) \tag{2.31}$$

with reality condition $\left(\zeta^A\right)^* = \zeta_A$ given by $\zeta_A = \omega_{AB}\zeta^B$. The passage from the ζ-coordinate basis to the ξ-coordinate is given by

$$\zeta^A = \mathcal{P}_B^A \xi^B \tag{2.32}$$

with

$$\mathcal{P}_B^A = \begin{pmatrix} \delta_J^I & i\delta_J^I \\ \delta_J^I & -i\delta_J^I \end{pmatrix} \tag{2.33}$$

2.2.2. *Fermionic sector of $\mathcal{K}$*

In addition to the bosonic $\mathcal{E}_{ph}^{2N|0}$, the space $\mathcal{K}$ contains also a super extension $\mathcal{E}_{ph}^{0|2N}$. This is an odd part which is coordinated by the $N+N$ fermionic variables given by $(\gamma_x^I, \gamma_{pI})$. For the particular case $N=1$, the odd space $\mathcal{E}_{ph}^{0|2}$ has two real variables (γ_x, γ_p) while the full space $\mathcal{E}_{ph}^{2|2}$ has four real variables: two bosonic (x, p) and the two fermionic (γ_x, γ_p); that is super coordinates as

$$\tilde{Z} = \left(x, p; \gamma_x, \gamma_p\right) \tag{2.34}$$

As far as the generic odd sub-superspace $\mathcal{E}_{ph}^{0|2N}$ is concerned, notice that its coordinate variables can be also combined into two ways, the same as for the bosonic $\mathcal{E}_{ph}^{2N|0}$:

(**1**) As real orthogonal vectors $\lambda^{\dot{A}}$ and $\lambda_{\dot{A}}$ like

$$\lambda^{\dot{A}} = \begin{pmatrix} \gamma_x^{\dot{I}} \\ \gamma_{p\dot{I}} \end{pmatrix} \quad , \quad \lambda_{\dot{A}} = \left(\gamma_{x\dot{I}}, \gamma_p^{\dot{I}}\right) \tag{2.35}$$

where we have used dotted labels $\dot{A}$, $\dot{I}$ in order to distinguish them from the labels A, I of eq. (2.25). The contravariant $\lambda^{\dot{A}}$ and the covariant $\lambda_{\dot{A}}$ vectors are 2N dimensional; they are linked by

$$\lambda_{\dot{A}} = g_{\dot{A}\dot{B}}\lambda^{\dot{B}} \quad , \quad \lambda^{\dot{A}} = G^{\dot{A}\dot{B}}\lambda_{\dot{B}} \tag{2.36}$$

with symmetric metric $g_{\dot{A}\dot{B}}$ and its inverse $G^{\dot{A}\dot{B}}$ whose canonical forms are as

$$g_{\dot{A}\dot{B}} = \begin{pmatrix} I_N & 0 \\ 0 & I_N \end{pmatrix} \quad , \quad G^{\dot{A}\dot{B}} = \begin{pmatrix} I_N & 0 \\ 0 & I_N \end{pmatrix} \tag{2.37}$$

In terms of these orthogonal vectors, we can calculate interesting quantities like

$$g_{\dot{A}\dot{B}}\lambda^{\dot{B}}\lambda'^{\dot{A}} = \gamma_{x\dot{I}}\gamma_x'^{\dot{I}} + \gamma_p^{\dot{I}}\gamma'_{p\dot{I}} \tag{2.38}$$

from which we learn the property $g_{\dot{A}\dot{B}}\lambda^{\dot{A}}\lambda^{\dot{B}} = 0$ because of the anti-commutativity property fermionic coordinates ($\lambda^{\dot{A}}\lambda^{\dot{B}} = -\lambda^{\dot{B}}\lambda^{\dot{A}}$) and the symmetry $g_{\dot{A}\dot{B}} = g_{\dot{B}\dot{A}}$.

(**2**) By using complex fermionic variables

$$\beta^{\dot{I}} = \gamma_x^{\dot{I}} + i\gamma_{p\dot{I}} \quad , \quad \beta_{\dot{I}}^* = \gamma_x^{\dot{I}} - i\gamma_{p\dot{I}} \tag{2.39}$$

in term of which we can also define

$$\chi^{\dot{A}} = \begin{pmatrix} \beta^{\dot{I}} \\ \beta^{*}_{\dot{I}} \end{pmatrix} \qquad , \qquad \chi_{\dot{A}} = \begin{pmatrix} \beta_{\dot{I}}, \beta^{*\dot{I}} \end{pmatrix} \tag{2.40}$$

and

$$g_{\dot{A}\dot{B}}\chi^{\dot{B}}\tilde{\chi}^{\dot{A}} = \beta_{\dot{I}}\tilde{\beta}^{\dot{I}} + \beta^{*\dot{I}}\tilde{\beta}^{*}_{\dot{I}} \tag{2.41}$$

with the property $g_{\dot{A}\dot{B}}\chi^{\dot{B}}\chi^{\dot{A}} = 0$.

3. Supersymmetric oscillators and observables

In this section, we give the relationship between the triplets (2.1) and the free supersymmetric oscillator algebra and its observables. We also construct the super oscillator realisation of the orthosymplectic symmetry and derive a family of observables $\mathcal{O}_{\eta}(\hat{\xi}, \hat{\lambda})$ which includes the supercharge $Q = Q(\hat{\xi}, \hat{\lambda})$ operator and the supersymmetric Hamiltonian

$$Q^2 = H \qquad , \qquad H = H_b + H_f \tag{3.1}$$

For that we begin by studying the algebra of the $N + N$ quantum super oscillators. We refer to this supersymmetric algebra as the generalised super Heisenberg algebra (for short $\mathrm{SHA}^{2N|2N}$).

3.1. *The super algebra:* $\mathrm{SHA}^{2N|2N}$

The supersymmetric oscillator algebra is an extension of the usual Heisenberg algebra of quantum bosonic oscillator $\hat{b} = (\hat{x} + i\hat{p})/\sqrt{2}$ by harmonic fermionic operators $\hat{c} = (\hat{\gamma}_x + i\hat{\gamma}_p)/\sqrt{2}$. Here, the $\hat{b}/\hat{b}^{\dagger}$ and the $\hat{c}/\hat{c}^{\dagger}$ satisfy graded commutations relations; in particular $[\hat{b}, \hat{b}^{\dagger}] = 1$ and $\{\hat{c}, \hat{c}^{\dagger}\} = 1$ describing $\mathrm{SHA}^{2|2}$. For the generators of $\mathrm{SHA}^{2N|2N}$, we have $2N + 2N$ operators

$$\hat{x}^{I}, \quad \hat{p}_{I}; \quad \hat{\gamma}_{x}^{\dot{I}}, \quad \hat{\gamma}_{p\dot{I}} \qquad \Leftrightarrow \qquad \hat{b}, \quad \hat{b}^{\dagger}, \quad \hat{c}, \quad \hat{c}^{\dagger} \tag{3.2}$$

in one to one with the even coordinates x^{I}, p_{I} $(\hat{b}, \hat{b}^{\dagger})$; and the odd $\gamma_{x}^{\dot{I}}$, $\gamma_{p\dot{I}}$ $(\hat{c}, \hat{c}^{\dagger})$. These operators obey the following graded commutators [44],

$$\begin{array}{lclclcl} \left[\hat{x}^{I}, \hat{p}_{J}\right] & = & i\delta^{I}_{J} & , & \{\hat{\gamma}_{x}^{\dot{I}}, \hat{\gamma}_{x}^{\dot{J}}\} & = & \delta^{\dot{I}\dot{J}} \\ \left[\hat{x}^{I}, \hat{x}^{J}\right] & = & 0 & , & \{\hat{\gamma}_{\dot{I}p}, \hat{\gamma}_{p\dot{J}}\} & = & \delta_{\dot{I}\dot{J}} \\ \left[\hat{p}_{I}, \hat{p}_{J}\right] & = & 0 & , & \{\hat{\gamma}_{x}^{\dot{I}}, \hat{\gamma}_{p\dot{J}}\} & = & 0 \end{array} \tag{3.3}$$

and vanishing crossed relations. For the particular case $N = 1$, the four generators of $\mathrm{SHA}^{2|2}$ are given by $\hat{x}$, $\hat{p}$, $\hat{\gamma}_x$, $\hat{\gamma}_p$; they obey the following non trivial relations

$$\begin{aligned} &\left[\hat{x}, \hat{p}\right] = i \\ &\left(\hat{\gamma}_x\right)^2 = \left(\hat{\gamma}_p\right)^2 = \frac{1}{2} I_{id} \\ &\hat{\gamma}_x\hat{\gamma}_p = -\hat{\gamma}_p\hat{\gamma}_x \end{aligned} \tag{3.4}$$

A typical realisation of these relations is given by

$$\begin{array}{lllllll} \hat{x} & = & x & , & \hat{\gamma}_x & = & \frac{1}{\sqrt{2}}\boldsymbol{\sigma}_1 \\ \hat{p} & = & i\frac{\partial}{\partial x} & , & \hat{\gamma}_p & = & \frac{1}{\sqrt{2}}\boldsymbol{\sigma}_2 \end{array} \tag{3.5}$$

where $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_x$ and $\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_y$ are Pauli matrices. For the generalised super Heisenberg algebra $\mathrm{SHA}^{2N|2N}$, the above realisation extends as follows

$$\hat{x}^I = x^I \qquad , \qquad \hat{p}_I = i\frac{\partial}{\partial x^I} \tag{3.6}$$

and

$$\hat{\gamma}_x^{\dot{i}} = \frac{1}{\sqrt{2}}\Gamma^{2\dot{i}-1} \qquad , \qquad \hat{\gamma}_{p\dot{i}} = \frac{1}{\sqrt{2}}\Gamma^{2\dot{i}} \tag{3.7}$$

where $\Gamma^{\dot{A}} = \left(\Gamma^{2\dot{i}-1}, \Gamma^{2\dot{i}}\right)$ are 2N dimensional Clifford algebra

$$\Gamma^{\dot{A}}\Gamma^{\dot{B}} + \Gamma^{\dot{B}}\Gamma^{\dot{A}} = 2\delta^{\dot{A}\dot{B}} \tag{3.8}$$

with $\delta^{\dot{A}\dot{B}}$ thought of as the canonical form of the orthogonal metric $G^{\dot{A}\dot{B}}$. In this regard, notice that

$$\Sigma^{\dot{A}\dot{B}} = \frac{1}{2i}(\Gamma^{\dot{A}}\Gamma^{\dot{B}} - \Gamma^{\dot{B}}\Gamma^{\dot{A}}) \tag{3.9}$$

are the generators of the spinor representation of $SO(2N, \mathbb{R})$.

3.2. *The* $\mathrm{SHA}^{2N|2N}$ *and the triplets (2.1)*

We study two aspects of the $\mathrm{SHA}^{2N|2N}$ superalgebra. First, we show that the graded commutation relations (3.3) defining the $\mathrm{SHA}^{2N|2N}$ are intimately related with the triplets (2.1) which we recombine into three pairs as follows.

$$(\boldsymbol{\tau}, \tilde{\boldsymbol{\tau}}) = \begin{pmatrix} \Omega^{AB} & & \omega_{AB} & & J^{\dot{A}}_{B} \\ G^{\dot{A}\dot{B}} & , & g_{\dot{A}\dot{B}} & , & J^{B}_{\dot{A}} \end{pmatrix} \tag{3.10}$$

This link indicates that eqs. (3.3) can be expressed in three different, but equivalent, ways depending on the algebraic structure we want to exhibit. Second, we construct the orthosymplectic symmetry underlying the quantum super oscillators and the super Hamiltonian underlying their dynamics.

3.2.1. *Three bases for* $\mathrm{SHA}^{2N|2N}$

Below, we give the three bases we can use to write down the graded commutators of the $\mathrm{SHA}^{2N|2N}$ Lie superalgebra. These three bases are distinguished by the tensorial properties of the generators namely (i) contravariant, (ii) covariant and (iii) mixed.

(1) Contravariant basis $\{\hat{\xi}^A, \hat{\lambda}^{\dot{A}}\}$

Using the contravariant supersymmetric oscillators operators $\hat{\xi}^A$ and $\hat{\lambda}^{\dot{A}}$, associated with super coordinate variables ξ^A (2.25) and $\lambda^{\dot{A}}$ (2.35), the graded commutation relations defining the $\mathrm{SHA}^{2N|2N}$ read as follows

$$\begin{array}{lll} [\hat{\xi}^A, \hat{\xi}^B] & = & i\Omega^{AB} \\ \{\hat{\lambda}^{\dot{A}}, \hat{\lambda}^{\dot{B}}\} & = & G^{\dot{A}\dot{B}} \end{array} \tag{3.11}$$

and vanishing others. In this definition, the right hand side of (3.11) are given by the contravariant symplectic Ω^{AB} and orthogonal $G^{\dot{A}\dot{B}}$ structures.

(2) Covariant basis $\{\hat{\xi}_A, \hat{\lambda}_{\dot{A}}\}$

The super $\text{SHA}^{2N|2N}$ (3.3) can be also defined by using the covariant operators $\hat{\xi}_A$ and $\hat{\lambda}_{\dot{A}}$, associated with the phase space super variables ξ_A and $\lambda_{\dot{A}}$. The non vanishing graded commutation relations of $\hat{\xi}_A$ and $\hat{\lambda}_{\dot{A}}$ are given by

$$\begin{aligned} [\hat{\xi}_A, \hat{\xi}_B] &= i\omega_{AB} \\ \{\hat{\lambda}_{\dot{A}}, \hat{\lambda}_{\dot{B}}\} &= g_{\dot{A}\dot{B}} \end{aligned} \tag{3.12}$$

In this definition, the right hand side of (3.12) is given by the covariant symplectic Ω_{AB} and the orthogonal $G_{\dot{A}\dot{B}}$ structures.

(3) Mixed basis

The superalgebra $\text{SHA}^{2N|2N}$ in the mixed basis is defined as follows,

$$\begin{aligned} [\hat{\xi}^{\dot{A}}, \hat{\xi}_B] &= -iJ_B^{\dot{A}} \\ \{\hat{\lambda}_{\dot{A}}, \hat{\lambda}^B\} &= J_{\dot{A}}^B \end{aligned} \tag{3.13}$$

where the right hand side is given by the complex structure $J_B^{\dot{A}}$ and its transpose $J_{\dot{A}}^B$. This complex structure is related to the symplectic and the orthogonal structures by (2.11) namely,

$$J_B^{\dot{A}} = -\sum_{\dot{C}=C=1}^{2N} G^{\dot{A}\dot{C}}\omega_{CB} \qquad , \qquad J_{\dot{A}}^B = \sum_{\dot{C}=C=1}^{2N} \Omega^{BC} g_{\dot{C}\dot{A}} \tag{3.14}$$

Notice that the requirement of the condition $\dot{C} = C$ breaks the $Sp\,(2N) \times SO\,(2N)$ down to the diagonal $S[U\,(2) \times U\,(N)]$. The mixed basis can be derived from (3.11)-(3.12) by using eq. (2.11). Indeed, starting for example from the superalgebra (3.12); then multiplying both sides of the super commutators by $G^{\dot{A}\dot{C}}$ and Ω^{AC} like

$$\begin{aligned} G^{\dot{A}\dot{C}}[\hat{\xi}_C, \hat{\xi}_B] &= iG^{\dot{A}\dot{C}}\omega_{CB} \\ \Omega^{AC}\{\hat{\lambda}_{\dot{C}}, \hat{\lambda}_{\dot{B}}\} &= \Omega^{AC} g_{\dot{C}\dot{B}} \end{aligned} \tag{3.15}$$

and setting

$$\hat{\xi}^{\dot{A}} = \sum_{\dot{C}=C=1}^{2N} G^{\dot{A}\dot{C}}\hat{\xi}_C \qquad , \qquad \hat{\lambda}^A = \sum_{\dot{C}=C=1}^{2N} \Omega^{AC}\hat{\lambda}_{\dot{C}} \tag{3.16}$$

we end up exactly with eq. (3.13).

3.2.2. *From* $\text{SHA}^{2N|2N}$ *to orthosymplectic* $osp\,(2N|2N)$

Given the super oscillators operators $\hat{\xi}^A$ and $\hat{\lambda}^{\dot{A}}$ obeying the $\text{SHA}^{2N|2N}$ (3.11), we can construct quantum observables $\mathcal{O}(\hat{\xi}, \hat{\lambda})$ given by polynomials of $\hat{\xi}^A$ and $\hat{\lambda}^{\dot{A}}$ as

$$\mathcal{O}_{\text{N+M}} = \sum_{k=1}^{\text{N}}\sum_{l=1}^{\text{M}} a_{A_1...A_k\dot{A}_1...\dot{A}_l}(\hat{\xi}^{A_1}...\hat{\xi}^{A_k})(\hat{\lambda}^{\dot{A}_1}...\hat{\lambda}^{\dot{A}_l}) \tag{3.17}$$

As illustrations, we have for $\text{N+M}= 1$, the two basic $\mathcal{O}_{(1,0)} = \hat{\xi}^A$ and $\mathcal{O}_{(0,1)} = \hat{\lambda}^{\dot{A}}$. For $\text{N+M}= 3$, we have the four following generators

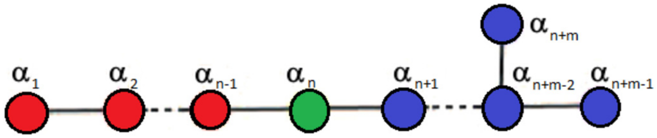

Fig. 3. Distinguished super Dynkin diagram of the orthosymplectic Lie algebra $osp(2m|2n)$. The α_i's on the nodes are the simple roots of the Lie superalgebra. The green node is fermionic while the (symplectic) red and (orthogonal) blue ones are bosonic.

$$\begin{array}{lllllll} \mathcal{O}_{(3,0)} & : & \hat{\xi}^A\hat{\xi}^B\hat{\xi}^C & , & \mathcal{O}_{(2,1)} & : & \hat{\xi}^A\hat{\xi}^B\hat{\lambda}^{\dot{C}} \\ \mathcal{O}_{(1,2)} & : & \hat{\xi}^A\hat{\lambda}^{\dot{B}}\hat{\lambda}^{\dot{C}} & , & \mathcal{O}_{(0,3)} & : & \hat{\lambda}^{\dot{A}}\hat{\lambda}^{\dot{B}}\hat{\lambda}^{\dot{C}} \end{array} \tag{3.18}$$

In what follow, we study the interesting set of observables $\mathcal{O}_{(\mathrm{N},\mathrm{M})}$ given by quadratic monomials; that is observables $\mathcal{O}_{(\mathrm{N},\mathrm{M})}$ with labels constrained like $\mathrm{N}+\mathrm{M}=2$. This set is generated by

$$\mathcal{O}^{AB} = \hat{\xi}^A\hat{\xi}^B, \qquad \mathcal{O}^{A\dot{B}} = \hat{\xi}^A\hat{\lambda}^{\dot{B}}, \qquad \mathcal{O}^{\dot{A}\dot{B}} = \hat{\lambda}^{\dot{A}}\hat{\lambda}^{\dot{B}} \tag{3.19}$$

We show amongst others that these observables generate the $osp\,(2N|2N)$ orthosymplectic Lie superalgebra.

Recall that the $osp\,(2N|2N)$ Lie superalgebra has two sectors: (i) an even sector $\mathcal{G}_{\bar{0}}$ given by $sp\,(2N) \oplus so\,(2N)$ having $4N^2$ bosonic generators. (ii) an odd sector $\mathcal{G}_{\bar{1}}$ given by the $(2N, 2\dot{N})$ representation of $\mathcal{G}_{\bar{0}}$. This module $\mathcal{G}_{\bar{1}}$ has also $4N^2$ Fermionic generators. The distinguished Dynkin super diagram of the orthosymplectic group $OSP\,(2N|2N)$ is given by the Fig. 3.

A) Symplectic $sp\,(2N)$ **and orthogonal** $so\,(2N)$

We begin by noticing that the quadratic monomials of the quantum bosonic oscillators $\hat{\xi}^A$ give the so-called oscillator representation of the symplectic Lie algebra $sp\,(2N, \mathbb{R})$. Similarly, the quadratic monomials of the fermionic oscillators $\hat{\lambda}^A$ give the oscillator representation of the orthogonal Lie algebra $so\,(2N, \mathbb{R})$ Lie algebras. The $sp\,(2N)$ and $so\,(2N)$ are bosonic Lie algebras with respective generators $\mathcal{K}^{(AB)}$ and $\mathcal{J}^{[AB]}$ realised in terms of the super oscillator operators as given below

Lie algebra	bosonic generators	dim
$sp\,(2N)$	$2\mathcal{K}^{(AB)} = \hat{\xi}^A\hat{\xi}^B + \hat{\xi}^B\hat{\xi}^A$	$N\,(2N+1)$
$so\,(2N)$	$2\mathcal{J}^{[\dot{A}\dot{B}]} = \hat{\lambda}^{\dot{A}}\hat{\lambda}^{\dot{B}} - \hat{\lambda}^{\dot{B}}\hat{\lambda}^{\dot{A}}$	$N\,(2N-1)$

(3.20)

These generators $\mathcal{K}^{(AB)}$ and $\mathcal{J}^{[AB]}$ can be also presented in terms of the usual creation $\hat{b}^\dagger/\hat{c}^\dagger$ and annihilations $\hat{b}/\hat{c}$ as follows

$$\mathcal{K}^{AB} = \begin{pmatrix} \mathcal{K}^{IJ} & \mathcal{K}_J^I \\ \mathcal{K}_I^J & \mathcal{K}_{IJ} \end{pmatrix} \quad , \quad \mathcal{J}^{\dot{A}\dot{B}} = \begin{pmatrix} \mathcal{J}^{\dot{I}\dot{J}} & \mathcal{J}_{\dot{J}}^{\dot{I}} \\ \mathcal{J}_{\dot{I}}^{\dot{J}} & \mathcal{J}_{\dot{I}\dot{J}} \end{pmatrix} \tag{3.21}$$

with

$$\begin{array}{lllllll} \mathcal{K}^{IJ} & = & \hat{b}^I\hat{b}^J & , & \mathcal{J}^{\dot{I}\dot{J}} & = & \hat{c}^{\dot{I}}\hat{c}^{\dot{J}} \\ \mathcal{K}_I^J & = & \hat{b}_I^\dagger\hat{b}^J + \frac{1}{2}\delta_I^J & , & \mathcal{J}_{\dot{I}}^{\dot{J}} & = & \hat{c}_{\dot{I}}^\dagger\hat{c}^{\dot{J}} - \frac{1}{2}\delta_{\dot{I}}^{\dot{J}} \\ \mathcal{K}_{IJ} & = & \hat{b}_I^\dagger\hat{b}_J^\dagger & , & \mathcal{J}_{\dot{I}\dot{J}} & = & \hat{c}_{\dot{I}}^\dagger\hat{c}_{\dot{J}}^\dagger \end{array} \tag{3.22}$$

Notice that the $\mathcal{K}_I^J$, $\mathcal{K}_J^I$ operators are the generators of $U\,(N) \subset Sp(2N)$; and the $\mathcal{J}_{\dot{I}}^{\dot{J}}$, $\mathcal{J}_{\dot{J}}^{\dot{I}}$ operators are the generators of $U\,(N)' \subset SO(2N)$.

B) Fermionic operators $\mathcal{F}^{A\dot{B}}$

These operators $\mathcal{F}^{A\dot{B}}$ are given by crossing products of the bosonic $\hat{\xi}^{A}$ and fermionic $\hat{\lambda}^{\dot{B}}$ oscillator operators like

$$\mathcal{F}^{A\dot{B}} = \hat{\xi}^{A}\hat{\lambda}^{\dot{B}} \tag{3.23}$$

They split in terms of $U(N) \times U(N)'$ labels as follows

$$\mathcal{F}^{A\dot{B}} = \begin{pmatrix} \hat{b}^{I}\hat{c}^{j} & \hat{b}^{I}\hat{c}^{\dagger}_{j} \\ \hat{b}^{\dagger}_{I}\hat{c}^{j} & \hat{b}^{\dagger}_{I}\hat{c}^{\dagger}_{j} \end{pmatrix} = \begin{pmatrix} \boldsymbol{F}^{Ij} & \boldsymbol{G}^{I}_{j} \\ \bar{\boldsymbol{G}}^{j}_{I} & \bar{\boldsymbol{F}}_{Ij} \end{pmatrix} \tag{3.24}$$

The interesting features of these operators are listed below:

(**i**) they are fermionic generators.

(**ii**) they relate bosons and fermions. We have

$$\{\mathcal{F}^{A\dot{B}}, \hat{\lambda}^{\dot{C}}\} = G^{\dot{B}\dot{C}}\hat{\xi}^{A} \qquad , \qquad \left[\mathcal{F}^{A\dot{B}}, \hat{\xi}^{C}\right] = i\Omega^{AC}\hat{\lambda}^{\dot{B}} \tag{3.25}$$

(**iii**) The set of the operators $\mathcal{K}^{AB}$, $\mathcal{J}^{\dot{A}\dot{B}}$ and $\mathcal{F}^{A\dot{B}}$ generate the osp$(2N|2N)$ orthosymplectic Lie superalgebra; it contains $u(N|N)$ as a sub- superalgebra. The even sector of these superalgebras read as

$$\begin{array}{lcl} osp\,(2N|2N)_{\bar{0}} & = & sp\,(2N) \oplus so\,(2N) \\ u\,(N|N)_{\bar{0}} & = & u\,(N) \oplus u\,(N) \end{array} \tag{3.26}$$

while the odd sectors are respectively given by the bi-fundamentals $\left(2N, 2\dot{N}\right)$ for $osp\,(2N|2N)_{\bar{1}}$; and $(N, \overline{\dot{N}})$ and $\left(\bar{N}, \dot{N}\right)$ for $u\,(N|N)_{\bar{1}}$.

4. From osp(2N|2M) to super QM

In this section, we use properties of the orthosymplectic osp(2N|2M) Lie superalgebra to build Hamiltonians H_{susy} modelling supersymmetric quantum mechanical systems with two supersymmetric charges ($\mathcal{N} = 2$ super QM)

$$Q = Q_1 + iQ_2 \qquad , \qquad Q^{\dagger} = Q_1 - iQ_2 \tag{4.1}$$

First, we describe the osp(2|2) orthosymplectic model as a theory enveloping $\mathcal{N} = 2$ super QM. Then we embed the $\mathcal{N} = 2$ super QM into the larger osp(2N|2M) theory with integers $M \geq N \geq 1$.

To that purpose, we begin by studying osp(2|2) theory having four conserved fermionic charges obeying general graded commutation relations to be given later. To avoid confusion between the osp(2|2) orthosymplectic (ORTIC) superalgebra and supersymmetric (SUSY) algebra, we refer to the osp(2|2) supercharges as $Q_{osp_{2|2}}$ (sometimes also as Q_{ortic}) and to the supersymmetric ones like Q_{susy} because the latter obeys extra constraints. The $Q_{osp_{2|2}}$ characterising the osp(2|2) theory is given by a fermionic operator valued in the odd sector of osp(2|2); by using (3.24), we have $Q_{osp_{2|2}} = \sum \Lambda_{\dot{B}A}\mathcal{F}^{\dot{B}A}$ expanding like

$$Q_{osp_{2|2}} = t_{Ij}\boldsymbol{F}^{jI} + r^{j}_{I}\boldsymbol{G}^{I}_{j} + w^{I}_{j}\bar{\boldsymbol{G}}^{j}_{I} + s^{Ij}\bar{\boldsymbol{F}}_{jI} \in osp_{(2|2)_{\bar{1}}} \tag{4.2}$$

where $\Lambda_{\dot{B}A}$ are complex numbers. The Hamiltonian $H_{osp_{2|2}}$ of this theory is given by the anticommutator $\{Q_{osp_{2|2}}, Q^{\dagger}_{osp_{2|2}}\}$; it is valued in the even sector of osp(2|2). The $Q^{\dagger}_{osp_{2|2}}$ is the adjoint conjugate of $Q_{osp_{2|2}}$; in general it is different from $Q_{osp_{2|2}}$; but we may also have $Q^{\dagger}_{osp_{2|2}} = Q_{osp_{2|2}}$. So, given $Q_{osp_{2|2}}$, we can construct a family of observables in the osp(2|2) theory by using the anticommutators as

$$\begin{array}{lll} \{Q_{osp_{2|2}}, Q^{\dagger}_{osp_{2|2}}\} & = & H_{osp_{2|2}} \\ \{Q_{osp_{2|2}}, Q_{osp2|2}\} & = & Z_{osp_{2|2}} \\ \{Q^{\dagger}_{osp_{2|2}}, Q^{\dagger}_{osp_{2|2}}\} & = & Z^{\dagger}_{osp_{2|2}} \end{array} \tag{4.3}$$

valued into $osp(2|2)_{\bar{0}}$. We also have commutators like

$$\left[H_{osp_{2|2}}, Q_{osp_{2|2}}\right] = \tilde{Q}_{osp_{2|2}} \tag{4.4}$$

valued into $osp(2|2)_{\bar{1}}$. To engineer supersymmetric quantum mechanical models (super QM) out of the osp(2|2) theory, we have to impose constraints required by supersymmetry (SUSY). A basic set of such constraints is given by

$$\left\{Q_{osp_{2|2}}, Q_{osp_{2|2}}\right\} = 0 \qquad , \qquad [H_{osp_{2|2}}, Q_{osp_{2|2}}] = 0 \tag{4.5}$$

Under these SUSY constraints, the supercharge $Q_{osp_{2|2}}$ and $H_{osp_{2|2}}$ reduce respectively down to Q_{susy} and H_{susy}; they sit in particular subspaces of the odd and even sectors of osp(2|2). Notice that for the case of one complex SUSY charge ($Q^{\dagger}_{susy} \neq Q_{susy}$), we talk about $\mathcal{N} = 2$ super QM while for hermitian Q_{susy}, we have $\mathcal{N} = 1$ super QM; for further details see appendix A. Notice also that the construction given in this section can be also viewed as a front matter towards the building of tight binding models for super AZ matter living in the Brillouin Zone [45–48]; see next sections. There, the $Q_{osp_{2|2}}$ and $H_{osp_{2|2}}$ (resp. Q_{susy} and H_{susy}) should be read as $Q^{osp_{2|2}}_{\mathbf{k}}$ and $H^{osp_{2|2}}_{\mathbf{k}}$ (resp. $Q^{susy}_{\mathbf{k}}$ and $H^{susy}_{\mathbf{k}}$) with $\mathbf{k}$ standing from the momentum variable in the Brillouin torus.

4.1. ***The osp(2|2)*** *model*

Here, we first construct the osp(2|2) orthosymplectic model based on $Q_{osp_{2|2}}$; then we derive the constraint relations towards $\mathcal{N} = 2$ super QM resting on Q_{susy}. After that, we work out typical solutions for Hamiltonians H_{susy} descending from Q_{susy}.

4.1.1. Oscillator realisation of ***the osp(2|2) structure***

The engineering of a simple orthosymplectic model that is invariant under the graded OSP(2|2) symmetry relies on using one fermionic $\hat{c}/\hat{c}^{\dagger}$ oscillator and one bosonic $\hat{b}/\hat{b}^{\dagger}$ describing super particle excitations super QM. In terms of these quantum oscillators, the eight generators of the osp(2|2) Lie superalgebra are realised as follows:

- **Generators of the even sector**
  The four bosonic operators generating the $SO(2) \times SP(2)$ subsymmetry of OSP(2|2) are given by

$$\begin{array}{ccccccc} \text{bosonic generators} & : & \boldsymbol{J}_0 & \boldsymbol{S}_0 & \boldsymbol{S}_+ & \boldsymbol{S}_- \\ \text{oscillator realisation} & : & \frac{1}{4}\left(\hat{c}^{\dagger}\hat{c} - \hat{c}\hat{c}^{\dagger}\right) & \frac{1}{4}(\hat{b}^{\dagger}\hat{b} + \hat{b}\hat{b}^{\dagger}) & \frac{1}{2}\hat{b}^{\dagger}\hat{b}^{\dagger} & \frac{1}{2}\hat{b}\hat{b} \end{array} \tag{4.6}$$

  with $2\boldsymbol{J}_0 = (\mathcal{N}_f - 1/2)$ generating $SO(2)$; and $2\boldsymbol{S}_0 = (\mathcal{N}_b + 1/2)$ being the Cartan charge operator of $SP(2, \mathbb{R})$. These four generators obey the usual commutations relations of the $so(2) \oplus sp(2)$ Lie algebra; see appendix A.

- **Generators of the odd sector**
  The four fermionic operators generating the odd sector of osp(2|2) transform in the (2; 2) representation of $SO(2) \times SP(2)$; they are denoted like $\boldsymbol{F}_p^q$ with $q = \pm$ labelling the charges of $so(2)$ while $p = \pm$ the charges of $sp(2)$. The oscillator realisation of the $\boldsymbol{F}_p^q$'s reads as follows

$$\begin{array}{ccccccc} \text{fermionic} & : & \boldsymbol{F}_-^+ & \boldsymbol{F}_+^- & \boldsymbol{F}_+^+ & \boldsymbol{F}_-^- \\ \text{realisation} & : & \hat{c}^\dagger \hat{b} & \hat{c}\hat{b}^\dagger & \hat{c}^\dagger \hat{b}^\dagger & \hat{c}\hat{b} \end{array} \tag{4.7}$$

  with (**i**) the adjoint conjugations $\left(\boldsymbol{F}_-^-\right)^\dagger = \boldsymbol{F}_+^+$ and $\left(\boldsymbol{F}_-^+\right)^\dagger = \boldsymbol{F}_+^-$; and (**ii**) the nilpotency $\left(\boldsymbol{F}_p^q\right)^2 = 0$ due to $\hat{c}\hat{c} = 0$. By using the notation $\hat{c}^q$ and $\hat{b}_p$, the above fermionic generators can be also presented collectively like $\boldsymbol{F}_p^q = \hat{c}^q \hat{b}_p$ with $\hat{\upsilon}^+ = \hat{\upsilon}^\dagger$ and $\hat{\upsilon}^- = \hat{\upsilon}$.

A) Graded commutations

The anticommutation relations between the fermionic operators are given by

$$\begin{array}{lcl} \left\{\boldsymbol{F}_-^-, \boldsymbol{F}_+^+\right\} & = & 2S_0 - 2J_0 \\ \left\{\boldsymbol{F}_-^-, \boldsymbol{F}_-^+\right\} & = & 2S_- \\ \left\{\boldsymbol{F}_-^-, \boldsymbol{F}_+^-\right\} & = & 0 \end{array} \quad , \quad \begin{array}{lcl} \left\{\boldsymbol{F}_-^+, \boldsymbol{F}_+^-\right\} & = & 2S_0 + 2J_0 \\ \left\{\boldsymbol{F}_+^-, \boldsymbol{F}_+^+\right\} & = & 2S_+ \\ \left\{\boldsymbol{F}_-^+, \boldsymbol{F}_+^+\right\} & = & 0 \end{array} \tag{4.8}$$

where appear the observables $S_0 \pm J_0$ and where $\boldsymbol{F}_q^q$ does not anticommute with $\boldsymbol{F}_q^{-q}$ because they are equal to S_q. The commutation relations between the four fermionic $\boldsymbol{F}_p^q$ and the bosonic step operators $S_\pm$ read as follows

$$\begin{array}{lcl} \left[S_-, \boldsymbol{F}_-^q\right] & = & 0 \\ \left[S_+, \boldsymbol{F}_+^q\right] & = & 0 \end{array} \quad , \quad \begin{array}{lcl} \left[S_+, \boldsymbol{F}_-^q\right] & = & -\boldsymbol{F}_+^q \\ \left[S_-, \boldsymbol{F}_+^q\right] & = & +\boldsymbol{F}_-^q \end{array} \tag{4.9}$$

while the commutators of the $\boldsymbol{F}_p^q$'s with the Cartan charge operators are given by $[J_0, \boldsymbol{F}_p^q] = q\boldsymbol{F}_p^q/2$ and $[S_0, \boldsymbol{F}_p^q] = -p\boldsymbol{F}_p^q/2$ or equivalently

$$\begin{array}{lcl} [J_0 + S_0, \boldsymbol{F}_p^q] & = & \frac{q-p}{2}\boldsymbol{F}_p^q \\ [J_0 - S_0, \boldsymbol{F}_p^q] & = & \frac{q+p}{2}\boldsymbol{F}_p^q \end{array} \tag{4.10}$$

B) Consequences of eqs. (4.8)-(4.10)

We begin by noticing that in general the four fermionic $\boldsymbol{F}_p^q$ do not commute with the Cartan operators $J_0 \pm S_0$; so they cannot diagonalise in the same basis. This feature is read from (4.10); for $p = q$, we find that $\boldsymbol{F}_q^q$ commutes with $J_0 + S_0$; but does not commute with $J_0 - S_0$ as shown below

$$\begin{array}{lcl} [J_0 + S_0, \boldsymbol{F}_q^q] & = & 0 \\ [J_0 - S_0, \boldsymbol{F}_q^q] & = & q\boldsymbol{F}_q^q \end{array} \tag{4.11}$$

Similarly for $p = -q$, we have a vanishing $[J_0 - S_0, \boldsymbol{F}_{-q}^q] = 0$ but a non vanishing commutator $[J_0 + S_0, \boldsymbol{F}_{-q}^q] = q\boldsymbol{F}_{-q}^q$. This violation of the commutation relation of $J_0 \pm S_0$ with the four fermionic $\boldsymbol{F}_p^q$ is a feature of the orthosymplectic model.

To find bosonic operators commuting with the four fermionic $\boldsymbol{F}_\pm^\pm$, we need to either reduce osp(2|2) to particular sub- superalgebras; or go to the enveloping algebra of osp(2|2); for example by considering the Casimirs of osp(2|2). A tricky way to engineer a quadratic operator $\mathcal{C}$ commuting with the four fermionic generators, that is obeying

$$[\mathcal{C}, \boldsymbol{F}_p^q] = 0 \tag{4.12}$$

is by taking advantage of the structure of eq. (4.10). Thinking of $\mathcal{C}$ as given by the product of the Cartans $(S_0 + J_0)$ and $(S_0 - J_0)$; we end up with $\mathcal{C} = S_0^2 - J_0^2$; thanks to $S_0 J_0 = J_0 S_0$. By computing the commutator between $\boldsymbol{F}_p^q$ and $S_0^2 - J_0^2$, we find that it vanishes identically; this is because of the property $q^2 = p^2 = 1$.

4.1.2. From osp(2|2) towards $\mathcal{N} = 2$ super QM

The odd sector of osp(2|2) has four fermionic charges given by the two complex $\boldsymbol{F}_+^+$, $\boldsymbol{F}_-^+$, and their adjoint conjugates $\boldsymbol{F}_-^-$, $\boldsymbol{F}_+^-$. This sector is suggestive for building (**i**) orthosymplectic models with diagonal observables proportional to $S_0 + J_0$ and $S_0 - J_0$; and (**ii**) deriving $\mathcal{N} = 2$ super QM models with supersymmetric hamiltonians H_{susy} obtained by constraining the OSP(2|2) invariance.

To build observables $\mathcal{O}_{osp_{2|2}}$ characterising the osp(2|2) models, we consider a complex fermionic charge $Q_{osp_{2|2}}$ which is given by a generic linear combination of the osp(2|2) fermionic generators as

$$Q_{osp_{2|2}} = \sum_{p=\pm} \sum_{q=\pm} X_q^p \boldsymbol{F}_p^q \qquad , \qquad Q_{osp_{2|2}}^\dagger = \sum_{p=\pm} \sum_{q=\pm} \bar{\boldsymbol{F}}_p^q \bar{X}_q^p \tag{4.13}$$

with $\boldsymbol{F}_p^q$ as in (4.7), $\bar{\boldsymbol{F}}_p^q = (\boldsymbol{F}_p^q)^\dagger$ and where the X_q^p is a complex 2×2 matrix which in tight binding modelling is interpreted in terms of hoppings. Using this fermionic charge and its adjoint $Q^\dagger$, we can construct the Hamiltonian $H_{osp_{2|2}}$ describing the OSP(2|2) model. It is given by

$$H_{osp_{2|2}} = Q_{osp_{2|2}} Q_{osp_{2|2}}^\dagger + Q_{osp_{2|2}}^\dagger Q_{osp_{2|2}} \tag{4.14}$$

By substituting (4.13) into (4.14) while using the short notation $H_{pr}^{qs} = \{\boldsymbol{F}_p^q, \boldsymbol{F}_r^s\}$ with anticommutators $\{\boldsymbol{F}_p^q, \boldsymbol{F}_r^s\}$ valued in $so\,(2) \oplus sp\,(2)$ as shown by eq. (4.8), we can present the $H_{osp_{2|2}}$ as follows

$$H_{osp_{2|2}} = \sum X_q^p H_{pr}^{qs} \bar{X}_s^r \tag{4.15}$$

So the hamiltonian $H_{osp_{2|2}}$ is generally valued in $so\,(2) \oplus sp\,(2)$; and as such it has the typical expansion $f_0 \boldsymbol{S}_0 + f_+ \boldsymbol{S}_- + f_- \boldsymbol{S}_+ + g_0 \boldsymbol{J}_0$ with $f_{0,\pm}$, g_0 some coupling parameters that can be read from $X_q^p \bar{X}_s^r$; and where $\boldsymbol{S}_{0,\pm}$, $\boldsymbol{J}_0$ are the generators of $so\,(2) \oplus sp\,(2)$ given by eq. (4.6). Notice also that for osp(2|2) models, the $Q_{osp_{2|2}}$ is in general not nilpotent ($Q_{osp_{2|2}}^2 \neq 0$) and the Hamiltonian $H_{osp_{2|2}}$ does not commute with $Q_{osp_{2|2}}$; i.e.:

$$H_{osp_{2|2}} Q_{osp_{2|2}} \neq Q_{osp_{2|2}} H_{osp_{2|2}} \qquad , \qquad Q_{osp_{2|2}}^2 \neq 0 \tag{4.16}$$

To construct $\mathcal{N} = 2$ super QM models out of the osp(2|2) orthosymplectic ones based on $Q_{osp_{(2|2)}}$, we impose the $\mathcal{N} = 2$ supersymmetric algebra on 1D world line (1D $\mathcal{N} = 2$) which is defined by the following graded commutation relations

$$\begin{array}{lll} \left\{Q, Q^\dagger\right\} & = & H_{susy} \\ \{Q, Q\} & = & 0 \\ \left[H_{susy}, Q\right] & = & 0 \end{array} \tag{4.17}$$

In these regards, notice the following: (**a**) the graded relations (4.17) can be also interpreted in terms of the $\mathcal{N} = 1$ supersymmetry in 2D world sheet generated by the 2D Majorana operator $Q_\alpha = (Q_1, Q_2)$; for details see appendix A. (**b**) For $Q^\dagger = Q$, the relations (4.17) reduce to

$2Q^2 = H_{susy}$.
A simple realisation of the above 1D $\mathcal{N} = 2$ superalgebra (4.17) in terms of the quantum oscillators $\hat{c}$ and $\hat{b}$ is given by the following family of supercharges

$$Q = \hat{c}^\dagger \hat{B}, \qquad Q^\dagger = \hat{B}^\dagger \hat{c} \tag{4.18}$$

where Q has no dependence into $\hat{c}$ and where the bosonic operator $\hat{B}$ and its useful properties are given are as follows

$$\begin{array}{lll} \hat{B} & = & X\hat{b} + \hat{b}^\dagger Y \\ \hat{B}^\dagger & = & \hat{b}^\dagger \bar{X} + \bar{Y}\hat{b} \end{array} \quad , \quad \begin{array}{lll} [\hat{B}, \hat{B}^\dagger] & = & Z \\ X\bar{X} - Y\bar{Y} & = & Z \end{array} \tag{4.19}$$

Here, the X and Y are two complex parameters and $Z = X\bar{X} - Y\bar{Y}$ is the discriminant of the mapping $\hat{b} \to \hat{B}$. The quadratic $\hat{B}^\dagger \hat{B}$ and $\hat{B}\hat{B}^\dagger$ operators descending from (4.19) read as follows

$$\begin{array}{lll} \hat{B}\hat{B}^\dagger & = & X\bar{X}\hat{b}\hat{b}^\dagger + Y\bar{Y}\hat{b}^\dagger\hat{b} + X\bar{Y}\hat{b}\hat{b} + Y\bar{X}\hat{b}^\dagger\hat{b}^\dagger \\ \hat{B}^\dagger\hat{B} & = & \bar{X}X\hat{b}^\dagger\hat{b} + Y\bar{Y}\hat{b}\hat{b}^\dagger + X\bar{Y}\hat{b}\hat{b} + Y\bar{X}\hat{b}^\dagger\hat{b}^\dagger \end{array} \tag{4.20}$$

Notice that the relations (4.20) are just linear combinations of the generators $\boldsymbol{S}_{0,\pm}$ of the symplectic sp(2,R) subsymmetry of osp(2|2); they read as follows,

$$\begin{array}{lll} \hat{B}\hat{B}^\dagger & = & 2\left(X\bar{X} + Y\bar{Y}\right)\boldsymbol{S}_0 + 2X\bar{Y}\boldsymbol{S}_- + 2Y\bar{X}\boldsymbol{S}_+ + \frac{1}{2}Z \\ \hat{B}^\dagger\hat{B} & = & 2\left(X\bar{X} + Y\bar{Y}\right)\boldsymbol{S}_0 + 2X\bar{Y}\boldsymbol{S}_- + 2Y\bar{X}\boldsymbol{S}_+ - \frac{1}{2}Z \\ \hat{B}^\dagger\hat{B} + \hat{B}\hat{B}^\dagger & = & 4\left(X\bar{X} + Y\bar{Y}\right)\boldsymbol{S}_0 + 4X\bar{Y}\boldsymbol{S}_- + 4Y\bar{X}\boldsymbol{S}_+ \end{array} \tag{4.21}$$

where in addition to $\boldsymbol{S}_{0,\pm}$ given by (4.6) we have a central charge Z commuting with them. From the particular oscillator realisation (4.18), we can perform several calculations and derive first results of the embedding of 1D $\mathcal{N} = 2$ supersymmetry into osp(2|2). Particular results are as listed below:

- *Algebra of $\hat{B}$ and $\hat{B}^\dagger$*:
  Using eq. (4.19), we calculate the useful commutation relations
  $$\left[\hat{B}^\dagger\hat{B}, \hat{B}^\dagger\right] = \hat{B}^\dagger Z \qquad , \qquad \left[\hat{B}^\dagger\hat{B}, \hat{B}\right] = -Z\hat{B} \tag{4.22}$$
  They reduce to the usual $[\hat{b}^\dagger\hat{b}, \hat{b}^\dagger] = \hat{b}^\dagger$ and $[\hat{b}^\dagger\hat{b}, \hat{b}] = -\hat{b}$ for the case $Z = 1$ corresponding to $X\bar{X} = 1$ and $Y = 0$. From these relations, we learn that $\pm Z$ are some how charges of the new bosonic oscillator under the bosonic operator number $\hat{B}^\dagger\hat{B}$.
- *Nilpotency constraint equation $\{Q, Q\} = 0$*:
  We can also check that we have indeed the nilpotency condition $Q^2 = 0$ required by 1D $\mathcal{N} = 2$ supersymmetry. This feature follows from the nilpotency of the fermionic oscillator namely $\left(\hat{c}^\dagger\right)^2 = 0$ and the commutativity $XY = YX$. Though trivial in this example, the last commutativity relation is required when embedding $\mathcal{N} = 2$ super QM models into osp(2N|2M).
- *Supersymmetric Hamiltonian H_{susy}*:
  The realisation of the supersymmetric Hamiltonian operator H_{susy} in terms of the oscillators $\hat{b}/\hat{b}^\dagger$ and $\hat{c}/\hat{c}^\dagger$ reads as follows
  $$\begin{array}{lll} H_{susy} & = & \hat{B}^\dagger\hat{B} + \hat{c}^\dagger Z\hat{c} \\ & = & \frac{1}{2}\left(\hat{B}^\dagger\hat{B} + \hat{B}^\dagger\hat{B}\right) + \hat{c}^\dagger Z\hat{c} - \frac{1}{2}Z \end{array} \tag{4.23}$$

with $\hat{B}^\dagger \hat{B}$ given by (4.20). In terms of the $SO(2, R) \times SP(2, R)$ generators $\boldsymbol{J}_0$ and $\boldsymbol{S}_{0,\pm}$, the above supersymmetric Hamiltonian is given by the following linear combination

$$H_{susy} = 2\left(X\bar{X} + Y\bar{Y}\right)\boldsymbol{S}_0 + 2X\bar{Y}\boldsymbol{S}_- + 2Y\bar{X}\boldsymbol{S}_+ + 2Z\boldsymbol{J}_0 \tag{4.24}$$

with $Z = X\bar{X} - Y\bar{Y}$. From this supersymmetric Hamiltonian, we learn the bosonic and the fermionic contributions to H_{susy} namely

$$\begin{array}{lll} H_{bose} & = & 2\left(X\bar{X} + Y\bar{Y}\right)\boldsymbol{S}_0 + 2X\bar{Y}\boldsymbol{S}_- + 2Y\bar{X}\boldsymbol{S}_+ \\ H_{fermi} & = & 2Z\boldsymbol{J}_0 \end{array} \tag{4.25}$$

- *The commutation $H_{susy}Q = QH_{susy}$:*
Using the oscillator realisation (4.18)-(4.19) of the supercharge Q and the supersymmetric Hamiltonian H_{susy}, we calculate the commutator $[H_{susy}, Q]$; we find that is equal to $\hat{c}^\dagger Z\hat{B} - \hat{c}^\dagger Z\hat{B}$ which vanishes identically. In this regard, notice the two following: (**i**) Given a supersymmetric highest weight state $|\phi\rangle$ (ground state) constrained as

$$Q\,|\phi\rangle = 0, \qquad H\,|\phi\rangle = \varepsilon_\phi\,|\phi\rangle \tag{4.26}$$

its super partner $|\psi\rangle$ is given by $Q^\dagger\,|\phi\rangle$ with the property $H_{susy}\,|\psi\rangle = \varepsilon_\phi\,|\psi\rangle$; thanks to the commutation $H_{susy}Q = QH_{susy}$. Obviously such property does not hold for $Q_{osp_{2|2}}$ because $H_{osp_{2|2}}Q_{osp_{2|2}}$ differs from $Q_{osp_{2|2}}H_{osp_{2|2}}$. (**ii**) By thinking about H_{susy} in terms of its bosonic contribution $H_{bose} = \hat{B}^\dagger\hat{B}$ and the fermionic $H_{fermi} = \hat{c}^\dagger Z\hat{c}$ as well as on free total energy ε_ϕ as the sum $\varepsilon_\phi^{fermi} + \varepsilon_\phi^{bose}$, we learn that neither H_{bose} commutes with Q nor H_{fermi} commutes with Q since we have

$$[H_{bose}, Q] = -ZQ \qquad , \qquad \left[H_{fermi}, Q\right] = +ZQ \tag{4.27}$$

But, from these remarkable relations, we learn that

$$\left[H_{bose}^2, Q\right] = \left[H_{fermi}^2, Q\right] = Z^2 Q$$

with $\left[D^2, Q\right]$ given by the adjoint action $[D, [D, Q]]$ and where here the operator D stands for H_{bose} and H_{fermi}. Moreover, using (4.26), we end up with

$$\begin{array}{lll} H_{bose}\,|\psi\rangle & = & \left(\varepsilon_\phi^{bose} - Z\right)|\psi\rangle \\ H_{fermi}\,|\psi\rangle & = & \left(\varepsilon_\phi^{fermi} + Z\right)|\psi\rangle \end{array} \tag{4.28}$$

So for bosonic ground states with $\varepsilon_\phi^{bose} = \varepsilon_\phi^{fermi} = 0$, their fermionic partners $|\psi\rangle$ have also vanishing energy ε_ϕ but with opposite contributions $\pm Z$ from bosonic and fermionic excitations.

4.2. $\mathcal{N} = 2$ super QM within OSp(2N|2M)

A more involved realisation of orthosymplectic osp(2|2) models and consequently the $\mathcal{N} = 2$ super QM ones is given by embedding osp(2|2) into OSp(2N|2M) with $M \geq N \geq 1$. Here, the modelling relies on using N fermionic $\hat{c}^i/\hat{c}_i^\dagger$ oscillators and M bosonic $\hat{b}^\alpha/\hat{b}_\alpha^\dagger$ with label i running from 1 to N and label $\alpha = 1, ..., M$. But later, we will restrict the study to $M = N$.

4.2.1. Realisation of OSp(2N|2M)

In terms of these N+M quantum graded oscillators $\hat{c}^i$ and $\hat{b}^\alpha$, the generators of the osp(2N|2M) Lie superalgebra are realised as follows:

1) Even sector of OSp(2N|2M)

The even part of OSp(2N|2M) is given by $SO(2N) \times Sp(2M)$; the oscillator realisation of the orthogonal $SO(2N)$ and the symplectic $Sp(2M)$ generators is given by

$SO(2N)$	$Sp(2M)$
$\mathcal{O}_i^j = \frac{1}{4}\left(\hat{c}_i^\dagger \hat{c}^j - \hat{c}^j \hat{c}_i^\dagger\right)$	$\mathcal{S}_\alpha^\beta = \frac{1}{4}\left(\hat{b}_\alpha^\dagger \hat{b}^\beta + \hat{b}^\beta \hat{b}_\alpha^\dagger\right)$
$\mathcal{O}^{[ij]} = \frac{1}{4}\left(\hat{c}^i \hat{c}^j - \hat{c}^j \hat{c}^i\right)$	$\mathcal{S}^{[\alpha\beta]} = \frac{1}{4}\left(\hat{b}^\alpha \hat{b}^\beta + \hat{b}^\beta \hat{b}^\alpha\right)$
$\mathcal{O}_{[ij]}^\dagger = \frac{1}{4}\left(\hat{c}_i^\dagger \hat{c}_j^\dagger - \hat{c}_j^\dagger \hat{c}_i^\dagger\right)$	$\mathcal{S}_{[\alpha\beta]}^\dagger = \frac{1}{4}\left(\hat{b}_\alpha^\dagger \hat{b}_\beta^\dagger + \hat{b}_\beta^\dagger \hat{b}_\alpha^\dagger\right)$

(4.29)

The $N+M$ commuting Cartan generators are as follows

$$J_i = \frac{1}{4}\left(\hat{c}_i^\dagger \hat{c}^i - \hat{c}^i \hat{c}_i^\dagger\right), \qquad S_\alpha = \frac{1}{4}(\hat{b}_\alpha^\dagger \hat{b}^\alpha + \hat{b}^\alpha \hat{b}_\alpha^\dagger) \tag{4.30}$$

with no summation on the labels.

2) Odd sector of OSp(2N|2M)

The 4NM fermionic generators of OSp(2N|2M) are given by

$$\begin{array}{ccccccc} \boldsymbol{G}_i^\alpha & = & \hat{c}_i^\dagger \hat{b}^\alpha & , & \boldsymbol{F}^{i\alpha} & = & \hat{c}^i \hat{b}^\alpha \\ \bar{\boldsymbol{G}}_\alpha^i & = & \hat{c}^i \hat{b}_\alpha^\dagger & , & \bar{\boldsymbol{F}}_{i\alpha} & = & \hat{c}_i^\dagger \hat{b}_\alpha^\dagger \end{array} \tag{4.31}$$

they carry eigenvalue charges under the J_i and the S_α operators generating the Cartan subsymmetry of OSp(2N|2M). For example, we have $[J_i, \boldsymbol{G}_l^\alpha] = +\frac{1}{2}\delta_{il}\boldsymbol{G}_i^\alpha$ and $[S_\alpha, \boldsymbol{G}_l^\beta] = -\frac{1}{2}\delta_{\alpha\beta}\boldsymbol{G}_i^\beta$.

4.2.2. 1D $\mathcal{N} = 2$ supersymmetry

A particular realisation of the $1D\ \mathcal{N} = 2$ supersymmetric charges Q and $Q^\dagger$ is given by the extension of the representation (4.18) reading as follows

$$Q = \hat{c}_i^\dagger \hat{B}^i \qquad , \qquad Q^\dagger = \hat{B}_i^\dagger \hat{c}^i \tag{4.32}$$

constrained by the supersymmetric constraints $Q^2 = 0$ and $[H_{susy}, Q] = 0$ with hamiltonian. $H_{susy} = \left\{Q, Q^\dagger\right\}$. In these expressions, the bosonic operators $\hat{B}^i$ and $\hat{B}_i^\dagger$ are defined by the following linear combinations

$$\hat{B}^i = X_\alpha^i \hat{b}^\alpha + \hat{b}_\alpha^\dagger Y^{\alpha i} \qquad , \qquad \hat{B}_i^\dagger = \bar{Y}_{i\alpha} \hat{b}^\alpha + \hat{b}_\alpha^\dagger \bar{X}_i^\alpha \tag{4.33}$$

where the complex coupling tensors X_α^i and $Y^{\alpha i}$ are respectively $N \times M$ and $M \times N$ rectangular matrices; they will be constrained below by imposing the superalgebra (4.17). From the relations (4.32)-(4.33), we can calculate useful quantities; in particular the following ones:

- *Algebra of $\hat{B}^i$'s and $\hat{B}_j^\dagger$'s*

  It is given by the following commutations

$$[\hat{B}^i, \hat{B}_j^\dagger] = Z_j^i \qquad , \qquad Z_j^i = X_\alpha^i \bar{X}_j^\alpha - \bar{Y}_{j\alpha} Y^{\alpha i} \tag{4.34}$$

  and

$$\begin{array}{lllllll} [\hat{B}^i, \hat{B}^j] & = & \Delta^{[ij]} & , & \Delta^{[ij]} & = & X^i_\alpha Y^{\alpha j} - X^j_\alpha Y^{\alpha i} \\ [\hat{B}^\dagger_i, \hat{B}^\dagger_j] & = & \bar{\Delta}_{ij} & , & \bar{\Delta}_{ij} & = & \bar{Y}_{i\alpha} \bar{X}^\alpha_j - \bar{Y}_{j\alpha} \bar{X}^\alpha_i \end{array} \tag{4.35}$$

As far as these commutations are concerned, notice the two following: (**i**) the commutation $\hat{B}^i \hat{B}^j = \hat{B}^j \hat{B}^i$ requires $\Delta^{[ij]} = 0$. (**ii**) By expressing this antisymmetric tensor like

$$\Delta^{[ij]} = (X.Y)^{ij} - (X.Y)^{ji} \tag{4.36}$$

with $(X.Y)^{ij} = X^i_\alpha Y^{\alpha j}$, the vanishing condition $\Delta^{[ij]} = 0$ can be solved by taking $(X.Y)^{ij} = \eta_{XY} G^{ij}$ with G^{ij} a symmetric tensor and η_{XY} a complex parameter.

- *Supersymmetric Hamiltonian*
The supersymmetric Hamiltonian is defined by $\{Q, Q^\dagger\} = H_{susy}$; by substituting (4.32)-(4.33) and using the algebra of the $\hat{B}^i$'s, we obtain

$$H_{susy} = \hat{B}^\dagger_i \hat{B}^i + \hat{c}^\dagger_i Z^i_j \hat{c}^j \tag{4.37}$$

having the property $H_{bose} + H_{fermi}$ with

$$\begin{array}{lll} H_{bose} & = & \hat{B}^\dagger_i \hat{B}^i \\ H_{fermi} & = & \hat{c}^\dagger_i Z^i_j \hat{c}^j \end{array} \tag{4.38}$$

The bosonic Hamiltonian $\hat{B}^\dagger_i \hat{B}^i$ can be also presented in other ways like: (**i**) in terms of $\hat{b}^\alpha$'s and $\hat{b}^\dagger_\beta$s as follows

$$\begin{array}{lll} \hat{B}^\dagger_i \hat{B}^i & = & \hat{b}^\dagger_\beta (\bar{X}^\beta_i X^i_\alpha) \hat{b}^\alpha + \hat{b}^\dagger_\beta \left(Y^{\beta i} \bar{X}^\alpha_i\right) \hat{b}^\dagger_\alpha + \\ & & \hat{b}^\beta \left(X^i_\beta \bar{Y}_{i\alpha}\right) \hat{b}^\alpha + \hat{b}^\beta \left(\bar{Y}_{i\beta} Y^{\alpha i}\right) \hat{b}^\dagger_\alpha \end{array} \tag{4.39}$$

(**ii**) in the matrix language as

$$H_{bose} = \left(\hat{b}^\dagger_\beta, \hat{b}^\beta\right) \begin{pmatrix} \bar{X}^\beta_i X^i_\alpha & Y^{\beta i} \bar{X}^\alpha_i \\ X^i_\beta \bar{Y}_{i\alpha} & \bar{Y}_{i\beta} Y^{\alpha i} \end{pmatrix} \begin{pmatrix} \hat{b}^\alpha \\ \hat{b}^\dagger_\alpha \end{pmatrix} \tag{4.40}$$

and (**iii**) by using the generators of $SO(2N) \times SP(2M)$,

$$\begin{array}{lll} \hat{B}^\dagger_i \hat{B}^i & = & 2\left(X^i_\alpha \bar{Y}_{i\beta}\right) \boldsymbol{S}^{(\alpha\beta)} + 2Y^{\alpha i} \bar{X}^\beta_i \bar{\boldsymbol{S}}_{(\alpha\beta)} \\ & & +2\left(\bar{X}^\alpha_i X^i_\beta + Y^{\alpha i} \bar{Y}_{i\beta}\right) \boldsymbol{S}^\beta_\alpha - \frac{1}{2} Z \end{array} \tag{4.41}$$

with $Z = tr\left(Z^i_j\right)$ given by

$$Z = \bar{X}^\alpha_i X^i_\alpha - Y^{\alpha i} \bar{Y}_{i\alpha} \tag{4.42}$$

- *the conditions* $\{Q, Q\} = \left[H_{susy}, Q\right] = 0$
First, the nilpotency property $Q^2 = 0$ of the supersymmetric algebra follows from two things: (**i**) the anticommutations $\hat{c}^\dagger_i \hat{c}^\dagger_j = -\hat{c}^\dagger_j \hat{c}^\dagger_i$ which usually hold; and (**ii**) the commutations $\hat{B}^i \hat{B}^j = \hat{B}^j \hat{B}^i$ which are ensured by demanding $\Delta^{[ij]} = 0$ solved by $(X.Y)^{ij} = \eta_{XY} G^{(ij)}$.
Regarding the vanishing of the commutation relation $\left[H_{susy}, Q\right]$, we use the splitting $H_{susy} = H_{bose} + H_{fermi}$; then calculate first $[H_{bose}, Q]$, which by using the previous relationships, leads to $-\hat{c}^\dagger_l Z^l_i \hat{B}^i$ with Z^l_i as in (4.34). Doing the same thing for $[H_{fermi}, Q]$, we end up with the value $\hat{c}^\dagger_i Z^i_j \hat{B}^j$ which cancels the previous contribution.

4.3. Towards tight binding modelling

In this subsection, we give a useful parameterisation of the fermionic and the bosonic oscillators to be used in the construction of tight binding modelling of super AZ matter.

4.3.1. The $\hat{c}/\hat{c}^{\dagger}$ and $\hat{b}/\hat{b}^{\dagger}$ as local field operators

Here, we will think about the fermionic $\hat{c}/\hat{c}^{\dagger}$ and the bosonic $\hat{b}/\hat{b}^{\dagger}$ operators, used in the building of the fermionic charges Q, in terms of local field operators as $\hat{c}\,(\mathbf{r}_i)\,/\hat{c}^{\dagger}\,(\mathbf{r}_i)$ and $\hat{b}\,(\mathbf{r}_i)\,/\hat{b}^{\dagger}\,(\mathbf{r}_i)$. These local fields living on lattice with coordinates $\mathbf{r}_i$ will be used later for the study of the tight binding modelling of super AZ matter.

A) Local fermionic oscillators

The complex fermionic $\sqrt{2}\hat{c}^{\dot{I}}(\mathbf{r}) = \hat{\gamma}^{\dot{I}}(\mathbf{r}) + i\hat{\eta}_{\dot{I}}(\mathbf{r})$ and its adjoint $\sqrt{2}\hat{c}^{\dagger}_{\dot{I}}(\mathbf{r}) = \hat{\gamma}^{\dot{I}}(\mathbf{r}) - i\hat{\eta}_{\dot{I}}(\mathbf{r})$ combine into the orthogonal $SO(2N)$ vector $\hat{\lambda}^{\dot{A}}(\mathbf{r}) = (\hat{c}^{\dot{I}}(\mathbf{r}), \hat{c}^{\dagger}_{\dot{I}}(\mathbf{r}))$ with $\dot{I} = 1, ..., N$ and off diagonal metric $g_{\dot{A}\dot{B}}$. In this parametrisation, the $\hat{c}^{\dot{I}}(\mathbf{r})$ transforms in the fundamental representation of the maximal unitary $U\,(N)$ contained in $SO(2N)$ and the $\hat{c}^{\dagger}_{\dot{I}}(\mathbf{r})$ transforms in the anti-fundamental. Notice that, it is the hermitian vector operator

$$\hat{F}^{\dot{A}}(\mathbf{r}) = \begin{pmatrix} \hat{\gamma}^{\dot{I}}(\mathbf{r}) \\ \hat{\eta}_{\dot{I}}(\mathbf{r}) \end{pmatrix} \tag{4.43}$$

made of the Majoranas that transforms under the vector representation of $SO(2N)$ with diagonal metric $\delta_{\dot{A}\dot{B}}$. The passage between the two frames is given by $\hat{\lambda}^{\dot{A}}(\mathbf{r}) = V^{\dot{A}}_{\dot{B}}\hat{F}^{\dot{B}}(\mathbf{r})$ with

$$V^{\dot{A}}_{\dot{B}} = \frac{1}{\sqrt{2}}\begin{pmatrix} \delta^{\dot{I}}_{\dot{J}} & i\delta^{\dot{J}}_{\dot{I}} \\ \delta^{\dot{I}}_{\dot{J}} & -i\delta^{\dot{I}}_{\dot{J}} \end{pmatrix} \tag{4.44}$$

B) Local bosonic oscillators

The complex bosonic $\sqrt{2}\hat{b}^{I}(\mathbf{r}) = \hat{X}^{I}(\mathbf{r}) + i\hat{P}_{I}(\mathbf{r})$ and its adjoint $\sqrt{2}\hat{b}^{\dagger}_{I}(\mathbf{r}) = \hat{X}^{I}(\mathbf{r}) - i\hat{P}_{I}(\mathbf{r})$ form the symplectic $SP(2M)$ vector $\hat{\xi}^{A}(\mathbf{r}) = (\hat{b}^{I}(\mathbf{r}), \hat{b}^{\dagger}_{I}(\mathbf{r}))$ with label I running from 1 to M. In this parametrisation, the $\hat{b}^{I}(\mathbf{r})$ transforms in the fundamental representation of the maximal unitary $U\,(M)$ contained in $SP(2M)$ and the $\hat{b}^{\dagger}_{I}(\mathbf{r})$ transforms in the anti-fundamental. Here also notice that, it is the real vector

$$\hat{\phi}^{A}\,(\mathbf{r}) = \begin{pmatrix} \hat{X}^{I}\,(\mathbf{r}) \\ \hat{P}_{I}\,(\mathbf{r}) \end{pmatrix} \tag{4.45}$$

that transforms with the usual antisymmetric symplectic ω_{AB}. The bridge between the two frames is given by $\hat{\xi}^{A}(\mathbf{r}) = U^{A}_{B}\hat{\phi}^{B}(\mathbf{r})$ with

$$U^{A}_{B} = \frac{1}{\sqrt{2}}\begin{pmatrix} \delta^{I}_{J} & i\delta^{J}_{I} \\ \delta^{I}_{J} & -i\delta^{J}_{I} \end{pmatrix} \tag{4.46}$$

C) Oscillators on lattice $\mathbb{L}$

The graded symmetry of the above super oscillator system $(\hat{c}^{\dot{I}}, \hat{b}^{I})$ is given by the orthosymplectic OSP(2N|2M). It contains the $SP(2N) \times SP(2M)$ invariance and the $U(N) \times U(M)$ group as bosonic subsymmetries. Moreover, to build tight binding super models, we have to think about the bosonic $\hat{\xi}^{A}$ and the fermionic $\hat{\lambda}^{\dot{A}}$ oscillators as local lattice QFT$_d$ operators labelled like

$$\hat{\xi}^{A}(\mathbf{r}_i)=\begin{pmatrix}\hat{b}^{I}(\mathbf{r}_i)\\ \hat{b}_{I}^{\dagger}(\mathbf{r}_i)\end{pmatrix}\quad,\quad\hat{\lambda}^{\dot{A}}(\mathbf{r}_i)=\begin{pmatrix}\hat{c}^{\dot{I}}(\mathbf{r}_i)\\ \hat{c}_{\dot{I}}^{\dagger}(\mathbf{r}_i)\end{pmatrix}\tag{4.47}$$

and

$$\hat{\xi}_{A}^{\dagger}(\mathbf{r}_i)=\left(\hat{b}_{I}^{\dagger}(\mathbf{r}_i),\hat{b}^{I}(\mathbf{r}_i)\right)\quad,\quad\hat{\lambda}_{\dot{A}}^{\dagger}(\mathbf{r}_i)=\left(\hat{c}_{\dot{I}}^{\dagger}(\mathbf{r}_i),\hat{c}^{\dot{I}}(\mathbf{r}_i)\right)\tag{4.48}$$

with variables $\mathbf{r}_i$ referring to the oscillators' positions in the hyper cubic lattice $\mathbb{L}$ as in the Fig. 4. By using the hermitian phase space operators $(\hat{X}^{I}(\mathbf{r}_i),\hat{P}_{I}(\mathbf{r}_i))$ and the Majoranas $(\hat{\gamma}^{\dot{I}}(\mathbf{r}_i),\hat{\eta}_{\dot{I}}(\mathbf{r}_i))$ that make the local complex bosonic $\hat{b}^{I}(\mathbf{r}_i)$ and the local complex fermionic $\hat{c}^{\dot{I}}(\mathbf{r}_i)$, we can also express the above $\hat{\xi}^{A}(\mathbf{r}_i)$ and $\hat{\lambda}^{\dot{A}}(\mathbf{r}_i)$ in terms of the hermitian $sp(2N,\mathbb{R})$ field $\hat{\phi}^{A}(\mathbf{r}_i)$ and the $so(2N,\mathbb{R})$ Majorana $\hat{F}^{\dot{A}}(\mathbf{r}_i)$ given by

$$\hat{\phi}^{A}(\mathbf{r}_i)=\begin{pmatrix}\hat{X}^{I}(\mathbf{r}_i)\\ \hat{P}_{I}(\mathbf{r}_i)\end{pmatrix}\quad,\quad\hat{F}^{\dot{A}}(\mathbf{r}_i)=\begin{pmatrix}\hat{\gamma}^{\dot{I}}(\mathbf{r}_i)\\ \hat{\eta}_{\dot{I}}(\mathbf{r}_i)\end{pmatrix}\tag{4.49}$$

4.3.2. Restriction to OSp(2N|2N)

In the analysis given below, we restrict the orthosymplectic symmetric OSP(2N|2M) down to the particular case where $M=N$. This constraint has been motivated by the modelling of supermatter living on the hypercubic super lattice of the Fig. 4 for which the $\mathbb{L}_{fermi}$ is isomorphic to $\mathbb{L}_{bose}$. However, our analysis can be also used for super lattices $\mathbb{L}_{fermi}^{(N)}/\mathbb{L}_{bose}^{(M)}$ with $M\neq N$. Indeed, following [20], there exist several super lattice constructions involving different numbers of fermionic and bosonic oscillators ($M\neq N$). To fix the ideas, we cite here after four examples of such lattice pairs,

$\mathbb{L}_{fermi}^{2D}$	$\mathbb{L}_{bose}^{2D}$
Honeycomb	Kagome
Square-octagon	squagome

;

$\mathbb{L}_{fermi}^{3D}$	$\mathbb{L}_{bose}^{3D}$
Hyper-honeycomb	Hyperkagome
Diamond	Pyrochlore

(4.50)

For explicit details regarding the super bands associated with the super lattices in eq (4.50), we refer to [20]. Moreover it is interesting to notice that the quantity $\nu=M-N$ defining the super-dimension of the graded space $\mathbb{R}^{N|M}$ has interesting interpretations; in particular: **(i)** as the so-called Maxwell-Callading index of topological mechanics given by eq(5) of the work [20]. **(ii)** like zero modes of the supersymmetric Hamiltonian (flat bands); and **(iii)** as values of non vanishing Witten index $\mathrm{Tr}(-)^{N_F}$ [57].

From the above description, it follows that models with orthosymplectic OSP(2N|2N) have a vanishing index $\nu=0$ and then no flat bands. Within this picture, we demand the two following conditions for our OSP(2N|2N) model:

(1) **Local OSP(2N|2N) invariance on Lattice**:
The bosonic $\hat{\xi}(\mathbf{r}_i)$ and the fermionic $\hat{\lambda}(\mathbf{r}_i')$ —or equivalently the symplectic $\hat{\phi}^{A}(\mathbf{r}_i)$ and the orthogonal $\hat{F}^{\dot{A}}(\mathbf{r}_i')$— live on identical hypercubic lattices Θ_{ξ} and Θ_{λ} isomorphic to $\mathbb{Z}^{d}$ with size $|\Theta|=L^{d}$. This feature insures that the number of degrees of freedom of the bosonic $\hat{\xi}$'s is equal to the fermionic $\hat{\lambda}$'s; thus the OSP(2N|2N) invariance. The two $\Theta_{\xi}\equiv\mathbb{L}_{Bose}$ and $\Theta_{\lambda}\equiv\mathbb{L}_{fermi}$ are as illustrated in the Fig. 4 for the example 2D square lattice. So, the super lattice $\Theta_{\xi}/\Theta_{\lambda}$ has 2N local bosonic $(\hat{b}^{I}(\mathbf{r}),\hat{b}_{I}^{\dagger}(\mathbf{r}))$ and 2N local fermionic $(\hat{c}^{\dot{I}}(\mathbf{r}),\hat{c}_{\dot{I}}^{\dagger}(\mathbf{r}))$ operators; thus inducing a local OSP(2N|2N) symmetry on the Brillouin Zone.

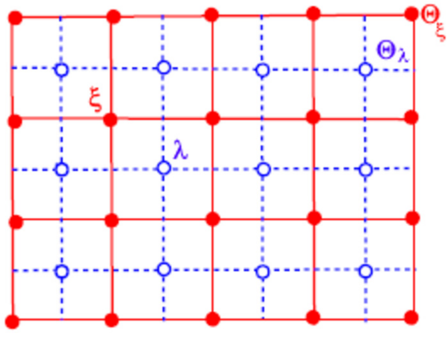

Fig. 4. 2D lattice made of two identical sublattices: Bosonic operators live on red sublattice Θ_ξ. Fermionic operators live on the blue sublattice Θ_λ.

(**2**) **Fourier modes of field operators$\hat{\phi}^A(\mathbf{r})$ and $\hat{F}^{\dot{A}}(\mathbf{r})$**
The 4N local hermitian oscillator operators $\hat{X}^I(\mathbf{r})$, $\hat{P}_I(\mathbf{r})$, $\hat{\gamma}^{\dot{I}}(\mathbf{r})$, $\hat{\eta}_{\dot{I}}(\mathbf{r})$ —or equivalently the complex $\hat{b}^I(\mathbf{r}), \hat{c}^{\dot{I}}(\mathbf{r})$ and their adjoints $\hat{b}_I^\dagger(\mathbf{r}), \hat{c}_{\dot{I}}^\dagger(\mathbf{r})$ — sit in a unit cell of the super lattice subject to a periodic boundary condition. It has 2N bosonic $(\hat{b}^I(\mathbf{r}), \hat{b}_I^\dagger(\mathbf{r}))$ and 2N fermionic $(\hat{c}^{\dot{I}}(\mathbf{r}), \hat{c}_{\dot{I}}^\dagger(\mathbf{r}))$ operators in agreement with the OSP(2N|2N) symmetry requiring an equal number of degrees of freedom.
The momentum modes $\hat{\phi}_{\mathbf{k}}^A = (\hat{X}_{\mathbf{k}}^I, \hat{P}_{I\mathbf{k}})$ and $\hat{F}_{\mathbf{k}}^{\dot{A}} = (\hat{\gamma}_{\mathbf{k}}^{\dot{I}}, \hat{\eta}_{\dot{I}\mathbf{k}})$ descending from the Fourier transform of the $\hat{\phi}^A(\mathbf{r})$ and $\hat{F}^{\dot{A}}(\mathbf{r})$ (equivalently $\hat{b}^I(\mathbf{r}), \hat{c}^{\dot{I}}(\mathbf{r})$) are given by the usual relation

$$F(\mathbf{k}) = \frac{1}{\sqrt{|\Theta|}} \sum_{\mathbf{r}\in\boldsymbol{\Theta}} e^{-i\mathbf{k}.\mathbf{r}} F(\mathbf{r}) \tag{4.51}$$

where $F(\mathbf{k})$ stands for $\hat{\phi}^A(\mathbf{k}) \equiv \hat{\phi}_{\mathbf{k}}^A$ and for $\hat{F}^{\dot{A}}(\mathbf{k}) \equiv \hat{F}_{\mathbf{k}}^{\dot{A}}$. Similar relations are valid for the $\hat{b}_{\mathbf{k}}^I$ and $\hat{c}_{\mathbf{k}}^{\dot{I}}$ descending from $\hat{b}^I(\mathbf{r})$ and $\hat{c}^{\dot{I}}(\mathbf{r})$; they read as $\hat{b}_{\mathbf{k}}^I = (\hat{X}_{\mathbf{k}}^I + i\hat{P}_{I\mathbf{k}})/\sqrt{2}$ and $\hat{c}_{\mathbf{k}}^{\dot{I}} = (\hat{\gamma}_{\mathbf{k}}^{\dot{I}} + i\hat{\eta}_{\dot{I}\mathbf{k}})/\sqrt{2}$. These Fourier modes will be used below.

5. ORTIC and SUSY tight binding models

In this section, we use the orthosymplectic group properties (3.20)-(3.23) to develop the study of the osp(2N|2N) orthosymplectic (ORTIC) and the $\mathcal{N}$- supersymmetric (SUSY) tight binding models respectively based on the fermionic Q_{orth} and Q_{susy} charges. The section is organised in three subsections; the first subsection concerns the ORTIC and the SUSY observables on lattice. The second regards the ORTIC and the SUSY tight binding models. The third subsection deals with the building of osp(2N|2N) and $\mathcal{N}=2$ super TBMs.

5.1. ORTIC and SUSY observables on lattice

Here, we extend the observables (3.17)-(3.19) of the orthosymplectic algebra to band theory. We focus on two particular observables Q_{ortho} and H_{ortho} as well as Q_{susy} and H_{susy}; they are quadratic in the super oscillator operators and are related as described here below:

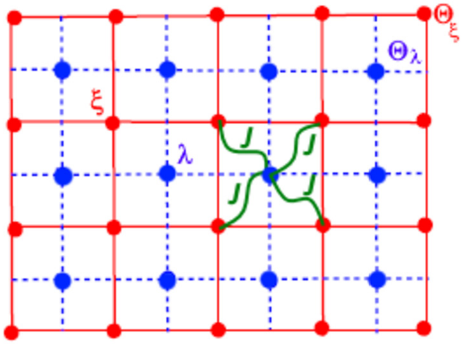

Fig. 5. Coupling of fermions and bosons. Here, the fermionic $\hat{\lambda}$ (in blue) interacts with its four closed neighbours $\hat{\xi}$ (in red). The coupling is given by J (in green). Similar couplings can be drawn for a bosonic site.

(**1**) **the orthosymplectic** Q_{ortho}
This is a fermionic charge given by $\sum_{i,j} \hat{\lambda}_{\dot{A}}(\mathbf{r}_i)[\boldsymbol{J}_{ij}]^{\dot{A}}_B \hat{\xi}^B(\mathbf{r}_j)$ with local field operators $\hat{\lambda}_{\dot{A}} = (\hat{c}_i^\dagger, \hat{c}^{\dot{I}})$ and $\hat{\xi}^A = (\hat{b}^I, \hat{b}_I^\dagger)$ as in eqs. (4.48)-(4.47). It reads explicitly as

$$\begin{aligned} Q_{\text{ortho}} \quad = \quad & \hat{c}_i^\dagger(\mathbf{r}_i)[(\boldsymbol{J}_1)_{ij}]^i_J \hat{b}^J(\mathbf{r}_j) + \hat{c}^{\dot{I}}(\mathbf{r}_i)[(\boldsymbol{J}_4)_{ij}]^J_{\dot{I}} \hat{b}_J^\dagger(\mathbf{r}_j) + \\ & \hat{c}^{\dot{I}}(\mathbf{r}_i)[(\boldsymbol{J}_3)_{ij}]_{\dot{I}J} \hat{b}^J(\mathbf{r}_j) + \hat{c}_i^\dagger(\mathbf{r}_i)[(\boldsymbol{J}_2)_{ij}]^{iJ} \hat{b}_J^\dagger(\mathbf{r}_j) \end{aligned} \tag{5.1}$$

where $[(\boldsymbol{J}_1)_{ij}]^i_J$, $[(\boldsymbol{J}_2)_{ij}]^{iJ}$, $[(\boldsymbol{J}_3)_{ij}]_{\dot{I}J}$, and $[(\boldsymbol{J}_4)_{ij}]^i_J$ are coupling tensors. This fermionic Q_{ortho} which reads also as

$$Q_{\text{ortho}} = (\hat{c}_i^\dagger(\mathbf{r}_i), \hat{c}^{\dot{I}}(\mathbf{r}_i)) \begin{pmatrix} (\boldsymbol{J}_1)_{ij} & (\boldsymbol{J}_2)_{ij} \\ (\boldsymbol{J}_3)_{ij} & (\boldsymbol{J}_4)_{ij} \end{pmatrix} \begin{pmatrix} \hat{b}^I(\mathbf{r}_j) \\ \hat{b}_I^\dagger(\mathbf{r}_j) \end{pmatrix} \tag{5.2}$$

defines the *orthosymplectic* TBM; it is characterised by the translation invariant coupling matrix $\boldsymbol{J}_{ij} = J(\mathbf{r}_i - \mathbf{r}_j)$. For a hermitian fermionic charge ($Q_{\text{ortho}}^\dagger = Q_{\text{ortho}}$), the coupling tensors $\boldsymbol{J}_1$, $\boldsymbol{J}_2$, $\boldsymbol{J}_3$ and $\boldsymbol{J}_4$ are related like $\boldsymbol{J}_4^\dagger = \boldsymbol{J}_1$ and $\boldsymbol{J}_3^\dagger = \boldsymbol{J}_2$. An illustration of this coupling is given by the Fig. 5. By performing the Fourier transform, the (5.1) can put Q_{ortho} into the form $\sum_{\mathbf{k}} Q_{\mathbf{k}}$ with Fourier modes $Q_{\mathbf{k}}$ constituting the basic object of the orthosymplectic TBM.

(**2**) **Orthosymplectic** H_{ortho} **and supersymmetric** H_{susy}
The orthosymplectic H_{ortho} is related to the fermionic Q_{ortho} as follows

$$2H_{\text{ortho}} = [Q_{\text{ortho}}, Q_{\text{ortho}}^\dagger] \tag{5.3}$$

Notice that for a fermionic charge that is nilpotent ($Q_{\text{ortho}}^2 = 0$) and commuting with the orthosymplectic Hamiltonian ($[H_{\text{ortho}}, Q_{\text{ortho}}] = 0$), eq(5.3) should be read like

$$2H_{\text{susy}} = [Q_{\text{susy}}, Q_{\text{susy}}^\dagger] \tag{5.4}$$

with

$$\begin{aligned} \{Q_{\text{susy}}, Q_{\text{susy}}\} \quad &= \quad 0 \\ [H_{\text{susy}}, Q_{\text{susy}}] \quad &= \quad 0 \end{aligned} \tag{5.5}$$

For $Q_{\text{susy}}^\dagger \neq Q_{\text{susy}}$, eqs(5.4)-(5.5) define the $\mathcal{N} = 2$ supersymmetric band theory while for $Q_{\text{susy}}^\dagger = Q_{\text{susy}}$, it defines $\mathcal{N} = 1$ theory. Eq. (5.5) defines the supersymmetric constraints that distinguishes Q_{susy} from Q_{orth}; the SUSY models form then a subfamily of ORTIC models. Moreover, being an even operator and quadratic in the super oscillator operators; the H_{ortho} (resp. H_{susy}) splits as the sum $H_f + H_b$ with (**i**) fermionic contribution having the form $H_f = \sum \hat{\lambda}_{\mathbf{r}_i} \left(h_f\right)_{ij} \hat{\lambda}_{\mathbf{r}_j}$ that can be interpreted as in the realisation of the AZ table; and (**ii**) bosonic contribution like $H_b = \sum \hat{\xi}_{\mathbf{r}_i} [(h_b)_{ij}] \hat{\xi}_{\mathbf{r}_j}$. Furthermore, using translation invariance and the Fourier modes $Q_{\mathbf{k}}^{\dagger} = Q_{-\mathbf{k}}$, we have $Q_{\text{ortho}}^{\dagger} = \sum_{\mathbf{k}} Q_{-\mathbf{k}}$ which by renaming the variable as $\mathbf{p} = -\mathbf{k}$ is equal Q_{ortho}; then we also have $H_{\text{ortho}} = Q_{\text{ortho}}^2$. The same feature holds for hermitian Q_{susy} leading to $H_{\text{susy}} = Q_{\text{susy}}^2$.

To deal with the Q_{ortho} (resp. Q_{susy}) and the H_{ortho} (resp. H_{susy}), we start by the oscillator operators on lattice represented by the Fourier modes $\hat{\xi}_{\mathbf{k}}^{A}$ and $\hat{\lambda}_{\mathbf{k}\dot{C}}$ with symplectic/orthogonal labels $A/\dot{A}$ ranging from 1 to 2N. They read in terms of the usual $\hat{b}/\hat{c}$ operators as follows

$$\hat{\xi}_{\mathbf{k}}^{A} = \begin{pmatrix} \hat{b}_{\mathbf{k}}^{I} \\ \hat{b}_{\mathbf{k}I}^{\dagger} \end{pmatrix} \quad , \quad \hat{\lambda}_{\mathbf{k}\dot{C}} = \left(\hat{c}_{\mathbf{k}\dot{K}}^{\dagger}, \hat{c}_{\mathbf{k}}^{\dot{K}}\right) \tag{5.6}$$

with momentum vector $\mathbf{k} = (k_1, ..., k_d)$ parameterising the Brillouin torus $\mathbb{T}^d$. For later calculations, we use the adjoint conjugation property $\hat{b}_{\mathbf{k}I}^{\dagger} = \hat{b}_{-\mathbf{k}I}$ and $\hat{c}_{\mathbf{k}\dot{I}}^{\dagger} = \hat{c}_{-\mathbf{k}\dot{I}}$ relating Fourier modes at $\mathbf{k}$ and $-\mathbf{k}$. Using the operators $\hat{\xi}_{\mathbf{k}}^{A}$ and $\hat{\lambda}_{\mathbf{k}\dot{C}}$, we calculate the following graded commutators

$$[\hat{\xi}_{\mathbf{k}}^{A}, \hat{\xi}_{-\mathbf{k}}^{B}] = Z^{AB} \quad , \quad \{\hat{\lambda}_{\mathbf{k}\dot{C}}, \hat{\lambda}_{-\mathbf{k}\dot{D}}\} = G_{\dot{C}\dot{D}} \tag{5.7}$$

with $2N \times 2N$ matrices Z^{AB} and $G_{\dot{C}\dot{D}}$ as follows

$$Z^{AB} = \begin{pmatrix} \delta^{IJ} & 0 \\ 0 & -\delta_{IJ} \end{pmatrix} \quad , \quad G_{\dot{C}\dot{D}} = \begin{pmatrix} \delta_{\dot{K}\dot{L}} & 0 \\ 0 & \delta^{\dot{K}\dot{L}} \end{pmatrix} \tag{5.8}$$

they read in a condensed form as the tensor products $Z = \sigma_z \otimes I_N$ and $G = \sigma_0 \otimes I_N$ with σ_μ referring to the 2×2 Pauli matrices. Notice that to get the relations (5.7), we used the properties $[\hat{b}_{\mathbf{k}}^{I}, \hat{b}_{-\mathbf{k}}^{J}] = \delta^{IJ}$ and $[\hat{b}_{\mathbf{k}I}^{\dagger}, \hat{b}_{-\mathbf{k}I}^{\dagger}] = -\delta_{IJ}$ as well $\{\hat{c}_{\mathbf{k}\dot{K}}^{\dagger}, \hat{c}_{-\mathbf{k}\dot{L}}^{\dagger}\} = \delta_{\dot{K}\dot{L}}$ and $\{\hat{c}_{\mathbf{k}}^{\dot{K}}, \hat{c}_{-\mathbf{k}}^{\dot{L}}\} = \delta^{\dot{K}\dot{L}}$. From these expressions, we learn the interesting relations

$$\hat{\xi}_{\mathbf{k}}^{A} \hat{\xi}_{-\mathbf{k}}^{B} = Z^{AB} + \hat{\xi}_{-\mathbf{k}}^{B} \hat{\xi}_{\mathbf{k}}^{A} \quad , \quad \hat{\lambda}_{\mathbf{k}\dot{C}} \hat{\lambda}_{-\mathbf{k}\dot{D}} = G_{\dot{C}\dot{D}} - \hat{\lambda}_{-\mathbf{k}\dot{D}} \hat{\lambda}_{\mathbf{k}\dot{C}} \tag{5.9}$$

Notice that at a given $\mathbf{k}$, we also have

$$[\hat{\xi}_{\mathbf{k}}^{A}, \hat{\xi}_{\mathbf{k}}^{B}] = Y^{AB} \quad , \quad \{\hat{\lambda}_{\mathbf{k}\dot{C}}, \hat{\lambda}_{\mathbf{k}\dot{D}}\} = X_{\dot{C}\dot{D}} \tag{5.10}$$

with

$$Y^{AB} = \begin{pmatrix} 0 & \delta_J^I \\ -\delta_I^J & 0 \end{pmatrix} \quad , \quad X_{\dot{C}\dot{D}} = \begin{pmatrix} 0 & \delta_{\dot{K}}^{\dot{L}} \\ \delta_{\dot{L}}^{\dot{K}} & 0 \end{pmatrix} \tag{5.11}$$

with $YX = Z$.

5.1.1. ORTIC and SUSY charges on lattice

The translation invariant supercharge operator Q (5.1) on the hypercubic super lattice leads to the expansion $Q = \sum_{\mathbf{k}} Q_{\mathbf{k}}$. The Fourier modes $Q_{\mathbf{k}}$ are complex fermionic operators expressed in terms of the super oscillator operators $\hat{\xi}_{\mathbf{k}}^{A}$ and $\hat{\lambda}_{\mathbf{k}\dot{C}}$ as follows

$$Q_{\mathbf{k}} = \hat{\lambda}_{\mathbf{k}\dot{C}}[\boldsymbol{q}_{\mathbf{k}}]_A^{\dot{C}}\hat{\xi}_{\mathbf{k}}^A \qquad , \qquad [(\boldsymbol{q}_{\mathbf{k}})_A^{\dot{C}}]^{\dagger} = (\boldsymbol{q}_{\mathbf{k}}^{*})_{\dot{C}}^{A} \tag{5.12}$$

where the complex coupling tensor

$$(\boldsymbol{q}_{\mathbf{k}})_A^{\dot{C}} = \begin{pmatrix} (\boldsymbol{q}_1)_I^{\dot{K}}(\mathbf{k}) & (\boldsymbol{q}_2)^{\dot{K}I}(\mathbf{k}) \\ (\boldsymbol{q}_3)_{\dot{K}I}(\mathbf{k}) & (\boldsymbol{q}_4)_{\dot{K}}^{I}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} (\boldsymbol{q}_{1\mathbf{k}})_I^{\dot{K}} & (\boldsymbol{q}_{2\mathbf{k}})^{\dot{K}I} \\ (\boldsymbol{q}_{3\mathbf{k}})_{\dot{K}I} & (\boldsymbol{q}_{4\mathbf{k}})_{\dot{K}}^{I} \end{pmatrix} \tag{5.13}$$

is a $2N \times 2N$ matrix; it is the Fourier transform of $J\left(\mathbf{r}_i - \mathbf{r}_j\right)$. This coupling matrix plays an important role in our super TBM as it captures the information on the physical properties of the super bands; it is a complex function of momentum $\mathbf{k}$ and takes values in the $(2N, 2\dot{N})$ bi-fundamental representation of $Sp(2N) \times SO(2N)$. In this regard, recall that $\hat{\xi}_{\mathbf{k}}^A$ transforms in the $2N$ representation of $Sp(2N)$ while $\hat{\lambda}_{\mathbf{k}\dot{C}}$ transforms in the $2\dot{N}$ representation of $Sp(2N)$. Below, we give other useful features of this coupling matrix. Notice that by setting $\hat{\Phi}_{\mathbf{k}}^{\dot{C}} = \left(\boldsymbol{q}_{\mathbf{k}}\right)_A^{\dot{C}}\hat{\xi}_{\mathbf{k}}^A$, the supercharge (5.12) reads simply as

$$Q_{\mathbf{k}} = \hat{\lambda}_{\mathbf{k}\dot{C}}\hat{\Phi}_{\mathbf{k}}^{\dot{C}} \tag{5.14}$$

Notice also that being a local matrix in the Brillouin torus, the coupling $\boldsymbol{q}_{\mathbf{k}}$ satisfies symmetry properties that we present as a list of five points:

1) Periodicity of $\boldsymbol{q}_A^{\dot{C}}(\mathbf{k})$

The $\boldsymbol{q}_{\mathbf{k}}$ lives on the d-torus $\mathbb{T}^d$; it obeys the periodicity conditions which are given by

$$\boldsymbol{q}_A^{\dot{C}}\left(\mathbf{k} + 2\pi\mathbf{e}_j\right) = \boldsymbol{q}_A^{\dot{C}}(\mathbf{k}) \tag{5.15}$$

with $\mathbf{e}_j$ being the unit vector in the k_j direction of $\mathbf{k}$. Explicit expressions of $\boldsymbol{q}_{\mathbf{k}}$ involve $\sin k_j$ and $\cos k_j$ functions in addition to constant moduli,

$$\boldsymbol{q}_{\mathbf{k}} = \boldsymbol{q}\left(\sin k_j, \cos k_j\right) \tag{5.16}$$

For the example of the Brillouin torus $\mathbb{T}^2$ parameterised by $\left(k_x, k_y\right)$; and for the case of two bands, an interesting $\boldsymbol{q}_{\mathbf{k}}$ is given by

$$\boldsymbol{q}_{\mathbf{k}} = \begin{pmatrix} e^{-i\varphi_{\mathbf{k}}}\sqrt{\varepsilon + (M - \cos k_x)} & \sqrt{\varepsilon - (M - \cos k_x)} \\ \sqrt{\varepsilon - (M - \cos k_x)} & -e^{+i\varphi_{\mathbf{k}}}\sqrt{\varepsilon + (M - \cos k_x)} \end{pmatrix} \tag{5.17}$$

with M constant and

$$\begin{aligned} \varepsilon &= (\sin^2 k_x + \sin^2 k_y + (M - \cos k_x)^2)^{1/2} \\ e^{-i\varphi_{\mathbf{k}}} &= \frac{\sin k_x - i \sin k_y}{\sqrt{\sin^2 k_x + \sin^2 k_y}} \end{aligned} \tag{5.18}$$

The (5.17) obeys the property (5.15). As far as the coupling matrix (5.17) is concerned, we give the following comments. (**i**) the SUSY models based on eq. (5.17) are just super extensions of the 2D tight binding hamiltonian $h_{\mathbf{k}}^{fermi}$ given by $d_x\sigma^x + d_y\sigma^y + d_z\sigma^z$ with $d_x = \sin k_x$, $d_y = \sin k_y$ and $d_z = M - \cos k_x$; it is invariant under the symmetry generated by $P = \sigma_y K$. (**ii**) The structure of this hamiltonian $h_{\mathbf{k}}^{fermi}$ may be also used to study the super extension of higher order topological insulators along the construction of [22,23].

2) Invariance of the ORTIC charge

It transforms as $Sp(2N) \times SO(2N)$ vectors with changes given by $\Lambda_{\dot{D}}^{\dot{C}} q_B^{\dot{D}} S_A^B$ where $\Lambda_{\dot{D}}^{\dot{C}}$ and S_A^B are orthogonal and symplectic matrices. Invariance of the ORTIC Q under the orthosymplectic symmetry requires the constraint relation

$$\boldsymbol{q}_A^{\dot{C}} = \Lambda_{\dot{D}}^{\dot{C}} \boldsymbol{q}_B^{\dot{D}} S_A^B \tag{5.19}$$

This condition is too strong to fulfil; so the $Sp(2N) \times SO(2N)$ symmetry will be broken down to subgroups.

3) Algebraic structure of $\boldsymbol{q}_A^{\dot{C}}(\mathbf{k})$

Under the particular breakings $Sp(2N) \to SU(N)$ and $SO(2N) \to SU(N)'$, we have the following representations decomposition:

(**i**) Adjoint representations

$$\begin{array}{lcl} sp(2N) & \to & u(N) \oplus \boldsymbol{n}_+ \oplus \boldsymbol{n}_- \\ N(2N+1) & \to & N^2 + \frac{N(N+1)}{2} + \frac{N(N+1)}{2} \end{array} \tag{5.20}$$

and

$$\begin{array}{lcl} so(2N) & \to & u(N)' \oplus \boldsymbol{n}'_+ \oplus \boldsymbol{n}'_- \\ N(2N-1) & \to & N^2 + \frac{N(N-1)}{2} + \frac{N(N-1)}{2} \end{array} \tag{5.21}$$

(**ii**) The bi-fundamental $(2N, 2\dot{N})$ as direct sum of four blocs of $U(N) \times U(N)'$ as follows

$$(2N, 2\dot{N}) = (N_+, \dot{N}_-) \oplus (N_-, \dot{N}_+) \oplus (N_+, \dot{N}_+) \oplus (N_-, \dot{N}_-) \tag{5.22}$$

This reduction, corresponds to the decomposition of $\boldsymbol{q}_A^{\dot{C}}(\mathbf{k})$ in terms of four blocs of $N \times N$ matrices like in (5.13). There, the blocs $\boldsymbol{q}_2^{\dot{K}I}$ and $(\boldsymbol{q}_3)_{\dot{K}I}$ take values in the $(N_+, \dot{N}_+)$ and the $(N_-, \dot{N}_-)$ representations while the $(\boldsymbol{q}_1)_I^{\dot{K}}$ and $(\boldsymbol{q}_4)_{\dot{K}}^I$ are respectively valued in $(N_+, \dot{N}_-)$ and $(N_-, \dot{N}_+)$. The transformation (5.19) splits as follows

$$\begin{array}{lclclcl} (\boldsymbol{q}_1)_I^{\dot{K}} & = & V_{\dot{L}}^{\dot{K}} (\boldsymbol{q}_1)_J^{\dot{L}} \bar{U}_I^J & , & (\boldsymbol{q}_2)^{\dot{K}I} & = & V_{\dot{L}}^{\dot{K}} (\boldsymbol{q}_2)^{\dot{L}J} U_J^I \\ (\boldsymbol{q}_4)_{\dot{K}}^I & = & \bar{V}_{\dot{K}}^{\dot{L}} (\boldsymbol{q}_4)_{\dot{L}}^J U_J^I & , & (\boldsymbol{q}_3)_{\dot{K}I} & = & \bar{V}_{\dot{K}}^{\dot{L}} (\boldsymbol{q}_3)_{\dot{L}J} \bar{U}_I^J \end{array} \tag{5.23}$$

with unitary matrices obeying $U_J^I \bar{U}_K^J = \delta_K^I$ and $V_{\dot{L}}^{\dot{K}} \bar{V}_{\dot{J}}^{\dot{L}} = \delta_{\dot{J}}^{\dot{K}}$.

4) TPC transformations of $\boldsymbol{q}_A^{\dot{C}}(\mathbf{k})$

For the breaking down to $S[U(2) \times U(N)]$ materialised by the double label notations $A = (\alpha, I)$ and $\dot{A} = (\dot{\alpha}, \dot{I})$, the splitting (5.13) is expressed like

$$\boldsymbol{q}_A^{\dot{B}}(\mathbf{k}) = q_{\alpha I}^{\dot{\beta}\dot{J}}(\mathbf{k}) \qquad , \qquad \alpha, \dot{\alpha} = 1, 2 \tag{5.24}$$

This tensor can be remarkably expanded in terms of Pauli matrices as follows

$$q_{\alpha I}^{\dot{\beta}\dot{J}}(\mathbf{k}) = \sum_{\mu=0}^{3} (\sigma^\mu)_\alpha^{\dot{\beta}} [\mathrm{q}_\mu(\mathbf{k})]_I^{\dot{J}} \tag{5.25}$$

with $\mathrm{q}_\mu(\mathbf{k})$ four $N \times N$ matrices whose topological properties are given by requiring discrete symmetries like $\mathcal{T} = K$, $\mathcal{P} = X \circ K$ and $\mathcal{C}$ with $X = \sigma_x \otimes I_N$. In this spinless matter case, the $q_A^{\dot{B}}(\mathbf{k})$ must be constrained as

$$\begin{array}{lclcl} \mathcal{T} & : & q(\mathbf{k})^* & = & +q(-\mathbf{k}) \\ \mathcal{P} & : & Xq(\mathbf{k})^* X & = & +q(-\mathbf{k}) \\ \mathcal{C} & : & \mathcal{C}q(\mathbf{k})\mathcal{C}^{-1} & = & +q(\mathbf{k}) \end{array} \tag{5.26}$$

as it will be checked later on when considering the Hamiltonian language. By substituting $q(\mathbf{k}) = \sigma^\mu \mathrm{q}_\mu(\mathbf{k})$, we get the following constraint relations

$$\begin{array}{lllll}\mathcal{T} & : & \mathfrak{q}_2(\mathbf{k})^* = -\mathfrak{q}_2(-\mathbf{k}) & , & \mathfrak{q}_{0,1,3}(\mathbf{k})^* = \mathfrak{q}_{0,1,3}(-\mathbf{k}) \\ \mathcal{P} & : & \mathfrak{q}_3(\mathbf{k})^* = -\mathfrak{q}_3(-\mathbf{k}) & , & \mathfrak{q}_{0,1,2}(\mathbf{k})^* = \mathfrak{q}_{0,1,2}(-\mathbf{k}) \\ \mathcal{C} & : & \mathfrak{q}_{2,3}(\mathbf{k}) = 0 & & \end{array} \tag{5.27}$$

5) Oscillator realisation of (5.14)

Using (5.13) we can express the supercharge (5.12) in terms the bosonic $\hat{b}_{\mathbf{k}}$ and the fermionic $\hat{c}_{\mathbf{k}}$ operators. First, we have for $\hat{\Phi}_{\mathbf{k}}^{\dot{C}} = (\boldsymbol{q}_{\mathbf{k}})_A^{\dot{C}} \hat{\xi}_{\mathbf{k}}^A$, the following

$$\hat{\Phi}_{\mathbf{k}}^{\dot{C}} = \begin{pmatrix} (\boldsymbol{q}_{1\mathbf{k}})_I^{\dot{K}} \hat{b}_{\mathbf{k}}^I + \hat{b}_{\mathbf{k}I}^{\dagger} (\boldsymbol{q}_{2\mathbf{k}})^{I\dot{K}} \\ (\boldsymbol{q}_{3\mathbf{k}})_{\dot{K}I} \hat{b}_{\mathbf{k}}^I + \hat{b}_{\mathbf{k}I}^{\dagger} (\boldsymbol{q}_{4\mathbf{k}})_{\dot{K}}^I \end{pmatrix} \tag{5.28}$$

By putting into (5.12), we get the supercharge at $\mathbf{k}$ namely

$$\begin{array}{lll} Q_{\mathbf{k}} & = & \hat{c}_{\mathbf{k}\dot{K}}^{\dagger} (\boldsymbol{q}_{1\mathbf{k}})_I^{\dot{K}} \hat{b}_{\mathbf{k}}^I + \hat{c}_{\mathbf{k}}^{\dot{K}} (\boldsymbol{q}_{4\mathbf{k}})_{\dot{K}}^I \hat{b}_{\mathbf{k}I}^{\dagger} + \\ & & \hat{c}_{\mathbf{k}\dot{K}}^{\dagger} (\boldsymbol{q}_{2\mathbf{k}})^{\dot{K}I} \hat{b}_{\mathbf{k}I}^{\dagger} + \hat{c}_{\mathbf{k}}^{\dot{K}} (\boldsymbol{q}_{3\mathbf{k}})_{\dot{K}I} \hat{b}_{\mathbf{k}}^I \end{array} \tag{5.29}$$

It has four block terms generated by the fermionic operators $\hat{c}_{\mathbf{k}}^{\dagger}\hat{b}_{\mathbf{k}}$, $\hat{c}_{\mathbf{k}}\hat{b}_{\mathbf{k}}^{\dagger}$, $\hat{c}_{\mathbf{k}}^{\dagger}\hat{b}_{\mathbf{k}}^{\dagger}$ and $\hat{c}_{\mathbf{k}}\hat{b}_{\mathbf{k}}$. Notice that by using the notation **(5.14)**, the above relation (5.29) reads simply as

$$Q_{\mathbf{k}} = \hat{c}_{\mathbf{k}\dot{A}}^{\dagger} \hat{B}_{\mathbf{k}}^{\dot{A}} + \hat{c}_{\mathbf{k}}^{\dot{A}} D_{\mathbf{k}\dot{A}} \tag{5.30}$$

where we have set

$$\begin{array}{lll} \hat{B}_{\mathbf{k}}^{\dot{A}} & = & (\boldsymbol{q}_{1\mathbf{k}})_I^{\dot{A}} \hat{b}_{\mathbf{k}}^I + (\boldsymbol{q}_{2\mathbf{k}})^{\dot{A}I} \hat{b}_{\mathbf{k}I}^{\dagger} \\ D_{\mathbf{k}\dot{A}} & = & (\boldsymbol{q}_{3\mathbf{k}})_{\dot{A}I} \hat{b}_{\mathbf{k}}^I + (\boldsymbol{q}_{4\mathbf{k}})_{\dot{A}}^I \hat{b}_{\mathbf{k}I}^{\dagger} \end{array} \tag{5.31}$$

Eq. (5.31) is a mapping from the $\hat{b}_{\mathbf{k}}^I/\hat{b}_{\mathbf{k}I}^{\dagger}$ to linear local combinations $\hat{B}_{\mathbf{k}}^{\dot{A}}$ and $D_{\mathbf{k}\dot{A}}$. These new operators obey the commutation relations

$$\begin{array}{lll} \left[\hat{B}_{\mathbf{k}}^{\dot{A}}, \hat{B}_{\mathbf{k}}^{\dot{D}}\right] & = & (\boldsymbol{q}_{1\mathbf{k}}\boldsymbol{q}_{2\mathbf{k}}^T)^{\dot{A}\dot{D}} - (\boldsymbol{q}_{1\mathbf{k}}\boldsymbol{q}_{2\mathbf{k}}^T)^{\dot{D}\dot{A}} \\ \left[D_{\mathbf{k}\dot{A}}, D_{\mathbf{k}\dot{B}}\right] & = & (\boldsymbol{q}_{3\mathbf{k}}\boldsymbol{q}_{4\mathbf{k}}^T)_{\dot{A}\dot{B}} - (\boldsymbol{q}_{3\mathbf{k}}\boldsymbol{q}_{4\mathbf{k}}^T)_{\dot{B}\dot{A}} \\ \left[\hat{B}_{\mathbf{k}}^{\dot{A}}, D_{\mathbf{k}\dot{C}}\right] & = & (\boldsymbol{q}_{1\mathbf{k}}\boldsymbol{q}_{4\mathbf{k}}^T)_{\dot{C}}^{\dot{A}} - (\boldsymbol{q}_{3\mathbf{k}}\boldsymbol{q}_{2\mathbf{k}}^T)_{\dot{C}}^{\dot{A}} \end{array} \tag{5.32}$$

Notice that a necessary condition to go from orthosymplectic (5.3) to the supersymmetric (5.4)-(5.5) is given by the nilpotency condition $Q_{\mathbf{k}}^2 = 0$. This demands the vanishing of the commutators (5.32) requiring in turns that the $\boldsymbol{q}_{1\mathbf{k}}\boldsymbol{q}_{2\mathbf{k}}^T$ and $\boldsymbol{q}_{3\mathbf{k}}\boldsymbol{q}_{4\mathbf{k}}^T$ must be symmetric matrices and $\boldsymbol{q}_{1\mathbf{k}}\boldsymbol{q}_{4\mathbf{k}}^T = \boldsymbol{q}_{3\mathbf{k}}\boldsymbol{q}_{2\mathbf{k}}^T$.

5.1.2. Classification of coupling matrix $q_{\mathbf{k}}$ by solving (5.19)

The super band models have definite supercharges $Q_{\mathbf{k}}$; as such they are completely characterised by the coupling matrix $(\boldsymbol{q}_{\mathbf{k}})_A^{\dot{C}}$; so ORTIC and SUSY TBMs can be classified by those $\boldsymbol{q}_{\mathbf{k}}$'s solving (5.19) and their topological properties by the discrete TPC (5.26).

In what follows, we focus on the constraint $\boldsymbol{q}_A^{\dot{C}} = \Lambda_{\dot{D}}^{\dot{C}} \boldsymbol{q}_B^{\dot{D}} S_A^B$ whose structure indicates that non trivial solutions require relating the orthogonal $\Lambda_{\dot{D}}^{\dot{C}}$ and the symplectic S_A^B matrices. This is achieved by breaking

$$Sp(2N) \times SO(2N) \to \mathcal{G}$$

with maximal $\mathcal{G}$ symmetry given by $U(2) \times U(N)$ containing the $U(N)$ describing the complex structure J. Below, we look for solutions of (5.19) by focusing on the case of unitary groups $\mathcal{G}$ is contained in $U(N)$ while the general situation will be considered in the discussion section. For concreteness, we give here after three families of solutions to the constraint relations.

A) Super Model with $U(N)$ symmetry

One of the interesting coupling matrices $(\boldsymbol{q}_{\mathbf{k}})_A^{\dot{C}}$ solving the constraint (5.19), while using the splitting (5.13), is given by

$$\begin{array}{llllll} (\boldsymbol{q}_{1\mathbf{k}})_I^{\dot{K}} & = & \mu_{\mathbf{k}}\delta_I^{\dot{K}} & , & (\boldsymbol{q}_{2\mathbf{k}})^{\dot{K}I} & = & 0 \\ (\boldsymbol{q}_{4\mathbf{k}})_{\dot{K}}^I & = & \nu_{\mathbf{k}}\delta_{\dot{K}}^I & , & (\boldsymbol{q}_{3\mathbf{k}})_{\dot{K}I} & = & 0 \end{array} \tag{5.33}$$

It characterised by two complex scalars $\mu_{\mathbf{k}}$ and $\nu_{\mathbf{k}}$ which are functions of the momentum $\mathbf{k}$. For this model, the fermionic charge (5.29) reduces to

$$\begin{array}{lll} Q_{\mathbf{k}} & = & \mu_{\mathbf{k}}(\hat{c}_{\mathbf{k}I}^{\dagger}\hat{b}_{\mathbf{k}}^I) + \nu_{\mathbf{k}}(\hat{b}_{\mathbf{k}I}^{\dagger}\hat{c}^I) \\ Q_{\mathbf{k}}^{\dagger} & = & \bar{\nu}_{\mathbf{k}}(\hat{c}_{\mathbf{k}I}^{\dagger}\hat{b}_{\mathbf{k}}^I) + \bar{\mu}_{\mathbf{k}}(\hat{b}_{\mathbf{k}I}^{\dagger}\hat{c}^I) \end{array} \tag{5.34}$$

For this super TBM, the coupling matrix $\boldsymbol{q}_{\mathbf{k}}$ is proportional to the $N \times N$ identity $\delta_I^{\dot{K}}$; so it has a $U(N)$ symmetry as manifestly exhibited by (5.34). In this regard, recall that generally speaking, the $(\boldsymbol{q}_{\mathbf{k}})_A^{\dot{C}}$ has $4N^2$ complex functions. The choice (5.33) corresponds just to the diagonal

$$\boldsymbol{q}_{\mathbf{k}} = \begin{pmatrix} \mu_{\mathbf{k}}I_N & 0 \\ 0 & \nu_{\mathbf{k}}I_N \end{pmatrix} = \begin{pmatrix} \mu_{\mathbf{k}} & 0 \\ 0 & \nu_{\mathbf{k}} \end{pmatrix} \otimes I_N \tag{5.35}$$

reading also like

$$\boldsymbol{q}_{\mathbf{k}} = \frac{1}{2}(\mu_{\mathbf{k}} + \nu_{\mathbf{k}})\,\sigma^0 \otimes I_N + \frac{1}{2}(\mu_{\mathbf{k}} - \nu_{\mathbf{k}})\,\sigma^z \otimes I_N \tag{5.36}$$

To get more information on the functions $\mu_{\mathbf{k}}$ and $\nu_{\mathbf{k}}$, we think about them in terms of $\cos k_j$ and $\sin k_j$ and impose TPC symmetries acting as

$$\begin{array}{llll} \mathcal{T} & : & \mu(\mathbf{k})^*, \nu(\mathbf{k})^* & = & \mu(-\mathbf{k}), \nu(-\mathbf{k}) \\ \mathcal{P} & : & \mu(\mathbf{k})^* & = & \nu(-\mathbf{k}) \\ \mathcal{C} & : & \mu(\mathbf{k}) & = & \nu(\mathbf{k}) \end{array} \tag{5.37}$$

For the case of time reversal invariance, examples of the $\mu_{\mathbf{k}}$ and the $\nu_{\mathbf{k}}$ are given by

$$\begin{aligned} \mu_{\mathbf{k}} &= \left(M - \sum_{j=1}^{d} \Delta_j \cos k_j\right) + i\sum_{j=1}^{d} t_j \sin k_j \\ \nu_{\mathbf{k}} &= \left(M' - \sum_{j=1}^{d} \Delta'_j \cos k_j\right) + i\sum_{j=1}^{d} t'_j \sin k_j \end{aligned} \tag{5.38}$$

If in addition to time reversal, we demand moreover particle-hole symmetry, the condition $\mu(\mathbf{k})^* = \nu(-\mathbf{k})$ requires the identification of the coupling parameters; that is $(M, \Delta, t) = (M', \Delta', t')$.

B) Super TBMs with $\prod U(n_i)$ symmetries

Starting from the above $U(N)$ super TBM, we can engineer other super TBMs having symmetries $\mathcal{G}$ contained into it. These subsymmetries are given by the tensor product group $\prod U(n_i)$ with the condition $\sum_{i=1}^{\eta} n_i = N$; i.e.:

$$\mathcal{G} = \prod_{i=1}^{\eta} U(n_i) \subset U(N) \tag{5.39}$$

A particular super TBM model belonging to this family is given by $U(1)^N$ which is the maximal breaking of the $U(N)$ symmetry. For this super TBM, the coupling matrix $\boldsymbol{q}_A^{\dot{C}}(\mathbf{k})$ of (5.13) is given by

$$\begin{array}{ccccccc} (\boldsymbol{q}_{1\mathbf{k}})_I^{\dot{K}} & = & (\mu_\mathbf{k})_I \, \delta_I^{\dot{K}} & , & (\boldsymbol{q}_{2\mathbf{k}})^{\dot{K}I} & = & 0 \\ (\boldsymbol{q}_{4\mathbf{k}})_{\dot{K}}^I & = & (\nu_\mathbf{k})_I \, \delta_{\dot{K}}^I & , & (\boldsymbol{q}_{3\mathbf{k}})_{\dot{K}I} & = & 0 \end{array} \tag{5.40}$$

having 2N parameters given by $(\mu_\mathbf{k})_I$ and $(\nu_\mathbf{k})_I$. The supercharges are

$$\begin{aligned} Q_\mathbf{k} &= \sum_I (\mu_\mathbf{k})_I \left(\hat{c}_{\mathbf{k}I}^\dagger \hat{b}_\mathbf{k}^I\right) + (\nu_\mathbf{k})_I \left(\hat{b}_{\mathbf{k}I}^\dagger \hat{c}^I\right) \\ Q_\mathbf{k}^\dagger &= \sum_I (\bar{\nu}_\mathbf{k})_I \left(\hat{c}_{\mathbf{k}I}^\dagger \hat{b}_\mathbf{k}^I\right) + (\bar{\mu}_\mathbf{k})_I \left(\hat{b}_{\mathbf{k}I}^\dagger \hat{c}^I\right) \end{aligned} \tag{5.41}$$

By equating all the $(\mu_\mathbf{k})_I$'s and equating all the $(\nu_\mathbf{k})_I$'s, we recover the $U(N)$ model described above. For this super TBM, the coupling matrix is still diagonal as shown below

$$\boldsymbol{q}_\mathbf{k} = \begin{pmatrix} (\mu_\mathbf{k})_1 & & & & & \\ & \ddots & & & & \\ & & (\mu_\mathbf{k})_N & & & \\ & & & (\nu_\mathbf{k})_1 & & \\ & & & & \ddots & \\ & & & & & (\nu_\mathbf{k})_N \end{pmatrix} \tag{5.42}$$

Notice that by equating a part of the $(\mu_\mathbf{k})_I$'s and the $(\nu_\mathbf{k})_I$'s in above (5.42), we get new super models with $\prod U_{(n_i)}$ symmetry. Examples of such symmetries are listed in the following table

parameters	unitary symmetry groups		
1	$U(N)$	-	-
2	$U(1) \times U(N-1)$	-	-
$(3,2)$	$U(1)^2 \times U(N-2)$	$U(2) \times U(N-2)$	-
$(4,3,2)$	$U(1)^3 \times U(N-3)$	$U(1) \times U(2) \times U(N-3)$	$U(3) \times U(N-3)$
$\vdots$	$\vdots$		
$N-1$	$U(1)^{N-2} \times U(2)$	-	-
N	$U(1)^N$	-	-

(5.43)

C) Super models III with $U(N)$ symmetry

Another interesting choice of the coupling $(\boldsymbol{q}_\mathbf{k})_A^{\dot{C}}$ is that given by the following non diagonal matrix

$$\boldsymbol{q}_\mathbf{k} = \begin{pmatrix} \mu_\mathbf{k} I_N & \rho_\mathbf{k} I_N \\ \varsigma_\mathbf{k} I_N & \nu_\mathbf{k} I_N \end{pmatrix} = \begin{pmatrix} \mu_\mathbf{k} & \rho_\mathbf{k} \\ \varsigma_\mathbf{k} & \nu_\mathbf{k} \end{pmatrix} \otimes I_N \tag{5.44}$$

It has four complex functions $\mu_{\mathbf{k}}$, $\nu_{\mathbf{k}}$, $\rho_{\mathbf{k}}$ and $\varsigma_{\mathbf{k}}$. It describes the following supercharge

$$\begin{aligned} Q_{\mathbf{k}} &= \mu_{\mathbf{k}}(\hat{c}^{\dagger}_{\mathbf{k}I}\hat{b}^{I}_{\mathbf{k}}) + \nu_{\mathbf{k}}(\hat{b}^{\dagger}_{\mathbf{k}I}\hat{c}^{I}) + \\ & \rho_{\mathbf{k}}(\hat{c}^{\dagger}_{\mathbf{k}\dot{K}}\delta^{\dot{K}I}\hat{b}^{\dagger}_{\mathbf{k}I}) + \varsigma_{\mathbf{k}}(\hat{c}^{\dot{K}}_{\mathbf{k}}\delta_{\dot{K}I}\hat{b}^{I}_{\mathbf{k}}) \end{aligned} \tag{5.45}$$

By comparing (5.44) with (5.36), we learn that they have different numbers of entries. This indicates that (5.36) has a bigger symmetry than (5.44) which is given by $U(1)^2 \times U(N)$.

5.2. The super TBM Hamiltonian

On the lattice, the ORTIC hamiltonian $H^{ortic}_{\mathbf{k}}$ is given by the anticommutator of the ORTIC supercharge $Q^{ortic}_{\mathbf{k}}$ with its adjoint namely $(Q_{\mathbf{k}}Q^{\dagger}_{\mathbf{k}} + Q^{\dagger}_{\mathbf{k}}Q_{\mathbf{k}})/2$. By using $Q^{\dagger}_{\mathbf{k}} = Q_{-\mathbf{k}}$, we also have

$$H^{ortic}_{\mathbf{k}} = \frac{1}{2}\{Q_{\mathbf{k}}, Q_{-\mathbf{k}}\} \tag{5.46}$$

The SUSY hamiltonian $H^{susy}_{\mathbf{k}}$ is given by (5.46) constrained by the nilpotency condition $Q^2_{\mathbf{k}} = 0$ and the commutativity $\left[H^{susy}_{\mathbf{k}}, Q_{\pm\mathbf{k}}\right] = 0$. Unitary symmetries of the super $Q_{\pm\mathbf{k}} = U Q_{\pm\mathbf{k}} U^{\dagger}$ are also symmetries of the SUSY Hamiltonian $H_{\mathbf{k}} = U H_{\mathbf{k}} U^{\dagger}$.

5.2.1. ORTIC Hamiltonian matrices

To get the realisation of the $H^{ortic}_{\mathbf{k}}$ in terms of the super oscillators, we substitute $Q_{\pm\mathbf{k}}$ by their expressions given by (5.12). We get a bosonic operator with a *quartic* dependence into the super oscillators as shown here below,

$$H^{ortic}_{\mathbf{k}} = \frac{1}{2}\left(\boldsymbol{q}_{\mathbf{k}}\right)^{\dot{C}}_{A}\left(\boldsymbol{q}_{-\mathbf{k}}\right)^{\dot{D}}_{B}\left(\hat{\lambda}_{\mathbf{k}\dot{C}}\hat{\lambda}_{-\mathbf{k}\dot{D}}\hat{\xi}^{A}_{\mathbf{k}}\hat{\xi}^{B}_{-\mathbf{k}} + \hat{\lambda}_{-\mathbf{k}\dot{D}}\hat{\lambda}_{\mathbf{k}\dot{C}}\hat{\xi}^{B}_{-\mathbf{k}}\hat{\xi}^{A}_{\mathbf{k}}\right) \tag{5.47}$$

By using eq. (5.9), we can bring the above relation to the following form

$$\begin{aligned} H^{ortic}_{\mathbf{k}} &= \tfrac{1}{2}\left(\boldsymbol{q}_{\mathbf{k}}\right)^{\dot{C}}_{A}\left(\boldsymbol{q}_{-\mathbf{k}}\right)^{\dot{D}}_{B}\left[Z^{AB}\hat{\lambda}_{\mathbf{k}\dot{C}}\hat{\lambda}_{-\mathbf{k}\dot{D}} + \hat{\xi}^{B}_{-\mathbf{k}}\hat{\xi}^{A}_{\mathbf{k}}\hat{\lambda}_{\mathbf{k}\dot{C}}\hat{\lambda}_{-\mathbf{k}\dot{D}}\right] \\ & +\tfrac{1}{2}\left(\boldsymbol{q}_{\mathbf{k}}\right)^{\dot{C}}_{A}\left(\boldsymbol{q}_{-\mathbf{k}}\right)^{\dot{D}}_{B}\left[G_{\dot{C}\dot{D}}\hat{\xi}^{B}_{-\mathbf{k}}\hat{\xi}^{A}_{\mathbf{k}} - \hat{\lambda}_{\mathbf{k}\dot{C}}\hat{\lambda}_{-\mathbf{k}\dot{D}}\hat{\xi}^{B}_{-\mathbf{k}}\hat{\xi}^{A}_{\mathbf{k}}\right] \end{aligned} \tag{5.48}$$

This expression is remarkable and deserves a comment. First it can be presented like the sum of two contributions $H^{(I)}_{\mathbf{k}} + H^{(II)}_{\mathbf{k}}$ with

$$\begin{aligned} H^{(I)}_{\mathbf{k}} &= \tfrac{1}{2}\hat{\lambda}_{\mathbf{k}\dot{C}}\left(\mathcal{H}^{I}_{\mathbf{k}}\right)^{\dot{C}\dot{D}}\hat{\lambda}_{-\mathbf{k}\dot{D}} \\ H^{(II)}_{\mathbf{k}} &= \tfrac{1}{2}\hat{\xi}^{B}_{-\mathbf{k}}\left(\mathcal{H}^{II}_{\mathbf{k}}\right)_{AB}\hat{\xi}^{A}_{\mathbf{k}} \end{aligned} \tag{5.49}$$

and

$$\begin{aligned} \left(\mathcal{H}^{I}_{\mathbf{k}}\right)^{\dot{C}\dot{D}} &= \left(\boldsymbol{q}_{\mathbf{k}}\right)^{\dot{C}}_{A} Z^{AB}\left(\boldsymbol{q}_{-\mathbf{k}}\right)^{\dot{D}}_{B} + \left(\boldsymbol{q}_{\mathbf{k}}\right)^{\dot{C}}_{A}\left[\hat{\xi}^{B}_{-\mathbf{k}}\hat{\xi}^{A}_{\mathbf{k}}\right]\left(\boldsymbol{q}_{-\mathbf{k}}\right)^{\dot{D}}_{B} \\ \left(\mathcal{H}^{II}_{\mathbf{k}}\right)_{AB} &= \left(\boldsymbol{q}_{\mathbf{k}}\right)^{\dot{C}}_{A} G_{\dot{C}\dot{D}}\left(\boldsymbol{q}_{-\mathbf{k}}\right)^{\dot{D}}_{B} - \left(\boldsymbol{q}_{\mathbf{k}}\right)^{\dot{C}}_{A}\left[\hat{\lambda}_{\mathbf{k}\dot{C}}\hat{\lambda}_{-\mathbf{k}\dot{D}}\right]\left(\boldsymbol{q}_{-\mathbf{k}}\right)^{\dot{D}}_{B} \end{aligned} \tag{5.50}$$

Second, the coupling matrices $\left(\mathcal{H}^{I}_{\mathbf{k}}\right)^{\dot{C}\dot{D}}$ and $\left(\mathcal{H}^{II}_{\mathbf{k}}\right)_{AB}$ have interesting features that we describe here after.

- **(a)** As required by orthosymplectic invariance (and then supersymmetry), the coupling matrices $\mathcal{H}^{I}_{\mathbf{k}}$ and $\mathcal{H}^{II}_{\mathbf{k}}$ are quadratic in the Fermi-Bose coupling matrix $\boldsymbol{q}_{\mathbf{k}}$.

(**b**) The $\left(\mathcal{H}_{\mathbf{k}}^{I}\right)^{\dot{C}\dot{D}}$ matrix gives the coupling between $\hat{\lambda}_{\mathbf{k}\dot{C}}$ and $\hat{\lambda}_{-\mathbf{k}\dot{D}}$; it has two terms: (**i**) a field independent term

$$\left(h_f\right)^{\dot{C}\dot{D}}=\left(\boldsymbol{q}_{\mathbf{k}}\right)_A^{\dot{C}} Z^{AB}\left(\boldsymbol{q}_{-\mathbf{k}}\right)_B^{\dot{D}} \tag{5.51}$$

with constant Z^{AB}. (**ii**) a field dependent term

$$\left(h_f'\right)^{\dot{C}\dot{D}}=\left(\boldsymbol{q}_{\mathbf{k}}\right)_A^{\dot{C}}[\hat{\xi}_{-\mathbf{k}}^{B}\hat{\xi}_{\mathbf{k}}^{A}]\left(\boldsymbol{q}_{-\mathbf{k}}\right)_B^{\dot{D}} \tag{5.52}$$

with local field dependence given by $\hat{\xi}_{-\mathbf{k}}^{B}\hat{\xi}_{\mathbf{k}}^{A}$.

(**c**) The $\left(\mathcal{H}_{\mathbf{k}}^{II}\right)_{AB}$ coupling matrix gives the interaction between $\hat{\xi}_{-\mathbf{k}}^{B}$ and $\hat{\xi}_{\mathbf{k}}^{A}$; it also has two terms: (**i**) a field independent term

$$\left(h_b\right)_{AB}=\left(\boldsymbol{q}_{\mathbf{k}}\right)_A^{\dot{C}} G_{\dot{C}\dot{D}}\left(\boldsymbol{q}_{-\mathbf{k}}\right)_B^{\dot{D}} \tag{5.53}$$

with constant $G_{\dot{C}\dot{D}}$. (**ii**) a field dependent term

$$\left(h_b'\right)_{AB}=\left(\boldsymbol{q}_{\mathbf{k}}\right)_A^{\dot{C}}[\hat{\lambda}_{\mathbf{k}\dot{C}}\hat{\lambda}_{-\mathbf{k}\dot{D}}]\left(\boldsymbol{q}_{-\mathbf{k}}\right)_B^{\dot{D}} \tag{5.54}$$

with local field dependence given by $\hat{\lambda}_{\mathbf{k}\dot{C}}\hat{\lambda}_{-\mathbf{k}\dot{D}}$.

(**d**) *Cancellation effect*: Expressing eq(5.48) as

$$\begin{array}{rcl} H_{\mathbf{k}}^{ortic} & = & \frac{1}{2}\left[\hat{\lambda}_{\mathbf{k}\dot{C}}\left(h_f\right)^{\dot{C}\dot{D}}\hat{\lambda}_{-\mathbf{k}\dot{D}}+\hat{\xi}_{-\mathbf{k}}^{B}\left(h_b\right)_{AB}\hat{\xi}_{\mathbf{k}}^{A}\right]+ \\ & & \frac{1}{2}\left[\hat{\lambda}_{\mathbf{k}\dot{C}}\left(h_f'\right)^{\dot{C}\dot{D}}\hat{\lambda}_{-\mathbf{k}\dot{D}}-\hat{\xi}_{-\mathbf{k}}^{B}\left(h_b'\right)_{AB}\hat{\xi}_{\mathbf{k}}^{A}\right] \end{array} \tag{5.55}$$

and using the commutation relation $\hat{\xi}_{\mathbf{k}}\hat{\lambda}_{\mathbf{k}}=\hat{\lambda}_{\mathbf{k}}\hat{\xi}_{\mathbf{k}}$, we see that the second line in the above relation vanishes identically due to the following compensation property

$$\hat{\lambda}_{\mathbf{k}\dot{C}}\left(h_f'\right)^{\dot{C}\dot{D}}\hat{\lambda}_{-\mathbf{k}\dot{D}}-\hat{\xi}_{-\mathbf{k}}^{B}\left(h_b'\right)_{AB}\hat{\xi}_{\mathbf{k}}^{A}=\hat{\xi}_{-\mathbf{k}}^{B}\hat{\xi}_{\mathbf{k}}^{A}\hat{\lambda}_{\mathbf{k}\dot{C}}\hat{\lambda}_{-\mathbf{k}\dot{D}}-\hat{\lambda}_{\mathbf{k}\dot{C}}\hat{\lambda}_{-\mathbf{k}\dot{D}}\hat{\xi}_{-\mathbf{k}}^{B}\hat{\xi}_{\mathbf{k}}^{A}=0 \tag{5.56}$$

This feature indicates that the presence of bosons in topological supermatter is not without effect the quantum physics.

By taking into account the compensation property, we find that the supersymmetric Hamiltonian $H_{\mathbf{k}}^{ortic}$ can be presented like H_f+H_b with fermionic part

$$\begin{array}{rcl} H_f & = & \frac{1}{2}\hat{\lambda}_{\mathbf{k}\dot{C}}\left[h_f\left(\mathbf{k}\right)\right]^{\dot{C}\dot{D}}\hat{\lambda}_{-\mathbf{k}\dot{D}} \\ & = & \frac{1}{2}\hat{\boldsymbol{\lambda}}_{\mathbf{k}}\left[\mathbf{h}_f\left(\mathbf{k}\right)\right]\hat{\boldsymbol{\lambda}}_{-\mathbf{k}} \end{array} \tag{5.57}$$

and bosonic

$$\begin{array}{rcl} H_b & = & \frac{1}{2}\hat{\xi}_{\mathbf{k}}^{A}\left[h_b\left(\mathbf{k}\right)\right]_{AB}\hat{\xi}_{-\mathbf{k}}^{B} \\ & = & \frac{1}{2}\hat{\boldsymbol{\xi}}_{\mathbf{k}}\left[h_b\left(\mathbf{k}\right)\right]\hat{\boldsymbol{\xi}}_{-\mathbf{k}} \end{array} \tag{5.58}$$

The coupling matrices $h_f(\mathbf{k})$ and $h_b(\mathbf{k})$ are quadratic in the coupling $q_{\mathbf{k}}$ as given below

$$\begin{array}{rcl} \left[h_f\left(\mathbf{k}\right)\right]^{\dot{C}\dot{D}} & = & \left(\boldsymbol{q}_{\mathbf{k}}\right)_A^{\dot{C}} Z^{AB}\left(\boldsymbol{q}_{-\mathbf{k}}\right)_B^{\dot{D}} \\ \left[h_b\left(\mathbf{k}\right)\right]_{AB} & = & \left(\boldsymbol{q}_{\mathbf{k}}\right)_A^{\dot{C}} G_{\dot{C}\dot{D}}\left(\boldsymbol{q}_{-\mathbf{k}}\right)_B^{\dot{D}} \end{array} \tag{5.59}$$

They read shortly as follows

$$h_f(\mathbf{k}) = \boldsymbol{q}_{\mathbf{k}} Z \boldsymbol{q}_{\mathbf{k}}^{\dagger} \qquad , \qquad h_b(\mathbf{k}) = \boldsymbol{q}_{\mathbf{k}}^{\dagger} \boldsymbol{q}_{\mathbf{k}} \tag{5.60}$$

with $\det \boldsymbol{h}_b = |\det_{\mathbf{k}} \boldsymbol{q}|^2$ and $\det \boldsymbol{h}_f = (\det Z) \, |\det_{\mathbf{k}} \boldsymbol{q}|^2$.
By using eq. (5.13), we have the following explicit relations

$$\begin{aligned} h_f &= \begin{pmatrix} (q_{\mathbf{k}})_I^{\dot{K}} (q_{-\mathbf{k}})_I^{\dot{L}} - (q_{\mathbf{k}})^{\dot{K}I} (q_{-\mathbf{k}})_{I\dot{L}} & (q_{\mathbf{k}})_I^{\dot{K}} (q_{-\mathbf{k}})^{I\dot{L}} - (q_{\mathbf{k}})^{\dot{K}I} (q_{-\mathbf{k}})_{\dot{L}}^{I} \\ (q_{\mathbf{k}})_{\dot{K}I} (q_{-\mathbf{k}})_I^{\dot{L}} - (q_{\mathbf{k}})_{\dot{K}}^{I} (q_{-\mathbf{k}})_{I\dot{L}} & (q_{\mathbf{k}})_{\dot{K}I} (q_{-\mathbf{k}})^{I\dot{L}} - (q_{\mathbf{k}})_{\dot{K}}^{I} (q_{-\mathbf{k}})_{\dot{L}}^{I} \end{pmatrix} \\ h_b &= \begin{pmatrix} (q_{\mathbf{k}})_I^{\dot{K}} (q_{-\mathbf{k}})_I^{\dot{L}} + (q_{\mathbf{k}})^{\dot{K}I} (q_{-\mathbf{k}})_{I\dot{L}} & (q_{\mathbf{k}})_I^{\dot{K}} (q_{-\mathbf{k}})^{I\dot{L}} + (q_{\mathbf{k}})^{\dot{K}I} (q_{-\mathbf{k}})_{\dot{L}}^{I} \\ (q_{\mathbf{k}})_{\dot{K}I} (q_{-\mathbf{k}})_I^{\dot{L}} + (q_{\mathbf{k}})_{\dot{K}}^{I} (q_{-\mathbf{k}})_{I\dot{L}} & (q_{\mathbf{k}})_{\dot{K}I} (q_{-\mathbf{k}})^{I\dot{L}} + (q_{\mathbf{k}})_{\dot{K}}^{I} (q_{-\mathbf{k}})_{\dot{L}}^{I} \end{pmatrix} \end{aligned} \tag{5.61}$$

5.2.2. *Massless modes from singular couplings*

Here, we give properties of the massless states in ORTIC and SUSY TBMs while focusing on the coupling matrix $\left(\boldsymbol{q}_{\mathbf{k}}\right)_A^{\dot{C}}$ given by the $U(N)$ super family. These are the topological super states that are protected by discrete symmetries.

• **Zeros of the Hamiltonian** H_{tot}

By setting $H_{tot} = H_b + H_f$ given by eqs. (5.57)-(5.58) and using the $osp(2N|2N)$ matrix notation with 4N dimensional super vector basis $(\hat{\boldsymbol{\xi}}_{-\mathbf{k}}, \hat{\boldsymbol{\lambda}}_{-\mathbf{k}})$, the total Hamiltonian can be presented like

$$(H_{\mathbf{k}})_{tot} = \frac{1}{2} \left(\hat{\boldsymbol{\xi}}_{\mathbf{k}}, \hat{\boldsymbol{\lambda}}_{\mathbf{k}} \right) (\mathbf{h}_{\mathbf{k}})_{tot} \begin{pmatrix} \hat{\boldsymbol{\xi}}_{-\mathbf{k}} \\ \hat{\boldsymbol{\lambda}}_{-\mathbf{k}} \end{pmatrix} \tag{5.62}$$

with $4N \times 4N$ matrix as follows

$$(\mathbf{h}_{\mathbf{k}})_{tot} = \begin{pmatrix} (\mathbf{h}_{\mathbf{k}})_b & 0 \\ 0 & (\mathbf{h}_{\mathbf{k}})_f \end{pmatrix} \tag{5.63}$$

This matrix has $4N$ eigenstates states: 2N of them for the bosonic $(\mathbf{h}_{\mathbf{k}})_b$ and the other 2N for the fermion $(\mathbf{h}_{\mathbf{k}})_f$. Because of the decoupling of $(\mathbf{h}_{\mathbf{k}})_b$ and $(\mathbf{h}_{\mathbf{k}})_f$, the determinant $\det \mathbf{h}_{tot}$ is given by the product $(\det \mathbf{h}_b) \, . \left(\det \mathbf{h}_f\right)$. Moreover, substituting (5.60), we get

$$\det \mathbf{h}_{tot} = (-)^N \, |\det q_{\mathbf{k}}|^4 \tag{5.64}$$

Zero modes of $\mathbf{h}_{tot}$ are given by the zeros of $\det q_{\mathbf{k}}$; so massless states of $\mathbf{h}_{tot}$ have multiplicity 4; two fermionic modes with vanishing gap and two bosonic partners. For an illustration, we give in the Fig. 6 the four super band energies $\epsilon_{\pm}^{(f)} = \pm[(1 - \cos k)^2 + \sin^2 k]$ and $\epsilon_{\pm}^{(b)} = \pm[(1 - \cos k)^2 + \sin^2 k]$.

• **Supersymmetric bands in the** $U(1)$ **super model**

For the super model (5.44) with $N = 1$, the $\boldsymbol{q}_{\mathbf{k}}$-matrix has four entries and the hamiltonian matrices h_f and h_b read as follows

$$\begin{aligned} h_f &= \begin{pmatrix} |\mu_{\mathbf{k}}|^2 - |\rho_{\mathbf{k}}|^2 & \mu_{\mathbf{k}} \bar{\varsigma}_{\mathbf{k}} - \rho_{\mathbf{k}} \bar{\nu}_{\mathbf{k}} \\ \varsigma_{\mathbf{k}} \bar{\mu}_{\mathbf{k}} - \nu_{\mathbf{k}} \bar{\rho}_{\mathbf{k}} & |\varsigma_{\mathbf{k}}|^2 - |\nu_{\mathbf{k}}|^2 \end{pmatrix} \\ h_b &= \begin{pmatrix} |\mu_{\mathbf{k}}|^2 + |\varsigma_{\mathbf{k}}|^2 & \bar{\mu}_{\mathbf{k}} \rho_{\mathbf{k}} + \bar{\varsigma}_{\mathbf{k}} \nu_{\mathbf{k}} \\ \bar{\rho}_{\mathbf{k}} \mu_{\mathbf{k}} + \bar{\nu}_{\mathbf{k}} \varsigma_{\mathbf{k}} & |\rho_{\mathbf{k}}|^2 + |\nu_{\mathbf{k}}|^2 \end{pmatrix} \end{aligned} \tag{5.65}$$

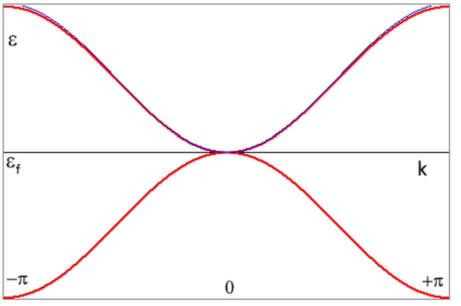

Fig. 6. The four bands $\epsilon_{\pm}^{(f)} = \pm |\mu_{\mathbf{k}}|^2$, $\epsilon_{\pm}^{(b)} = |\mu_{\mathbf{k}}|^2$ of the $U(1)$ super model given by eq. (5.72). In red, the two symmetric bands of the h_f with respect to ε_F. In blue, the degenerate bosonic bands having positive energies. The parameter M is taken around 1.

Referring to these two matrices formally like

$$h = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \tag{5.66}$$

with entries as (5.65); we can determine their eigenstates and eigenvalues. The eigenvalues $E_{\pm}$ read like

$$E_{\pm} = \frac{1}{2}(A+D) \pm \frac{1}{2}\sqrt{(A-D)^2 + 4|B|^2} \tag{5.67}$$

and the corresponding eigenstates $|\upsilon_{\pm}\rangle$ as

$$|\upsilon_{\pm}\rangle = \frac{1}{\sqrt{\mathcal{N}_{\pm}}} \begin{pmatrix} A - D \pm \sqrt{(A-D)^2 + 4|B|^2} \\ 2B^* \end{pmatrix} \tag{5.68}$$

with $\mathcal{N}_{\pm}$ given by the normalisation $\langle \upsilon_i | \upsilon_j \rangle = \delta_{ij}$. By demanding charge conjugation invariance (particle-hole symmetry), the number of functions gets reduced as $\bar{\mu}_{-\mathbf{k}} = \nu_{\mathbf{k}}$ and $\bar{\rho}_{-\mathbf{k}} = \varsigma_{\mathbf{k}}$. By substituting, we have

$$\begin{aligned} h_f &= \begin{pmatrix} |\mu_{\mathbf{k}}|^2 - |\rho_{\mathbf{k}}|^2 & \mu_{\mathbf{k}}\rho_{-\mathbf{k}} - \rho_{\mathbf{k}}\mu_{-\mathbf{k}} \\ \bar{\rho}_{-\mathbf{k}}\bar{\mu}_{\mathbf{k}} - \bar{\mu}_{-\mathbf{k}}\bar{\rho}_{\mathbf{k}} & |\bar{\rho}_{-\mathbf{k}}|^2 - |\bar{\mu}_{-\mathbf{k}}|^2 \end{pmatrix} \\ h_b &= \begin{pmatrix} |\mu_{\mathbf{k}}|^2 + |\bar{\rho}_{-\mathbf{k}}|^2 & \bar{\mu}_{\mathbf{k}}\rho_{\mathbf{k}} + \rho_{-\mathbf{k}}\bar{\mu}_{-\mathbf{k}} \\ \bar{\rho}_{\mathbf{k}}\mu_{\mathbf{k}} + \mu_{-\mathbf{k}}\bar{\rho}_{-\mathbf{k}} & |\rho_{\mathbf{k}}|^2 + |\bar{\mu}_{-\mathbf{k}}|^2 \end{pmatrix} \end{aligned} \tag{5.69}$$

with traces given by

$$\begin{aligned} tr\left(h_f\right) &= |\mu_{\mathbf{k}}|^2 + |\bar{\rho}_{-\mathbf{k}}|^2 - |\rho_{\mathbf{k}}|^2 - |\bar{\mu}_{-\mathbf{k}}|^2 \gtrless 0 \\ tr\left(h_b\right) &= |\mu_{\mathbf{k}}|^2 + |\rho_{\mathbf{k}}|^2 + |\bar{\rho}_{-\mathbf{k}}|^2 + |\bar{\mu}_{-\mathbf{k}}|^2 \geq 0 \end{aligned} \tag{5.70}$$

For the case where we set $\rho_{\pm\mathbf{k}} = 0$ corresponding to $U(1) \times U(1)$ symmetric model, we have

$$h_f = \begin{pmatrix} |\mu_{\mathbf{k}}|^2 & 0 \\ 0 & -|\bar{\mu}_{-\mathbf{k}}|^2 \end{pmatrix} \quad , \quad h_b = \begin{pmatrix} |\mu_{\mathbf{k}}|^2 & 0 \\ 0 & |\bar{\mu}_{-\mathbf{k}}|^2 \end{pmatrix} \tag{5.71}$$

If moreover, we demand time reversal symmetry, we must have $\bar{\mu}_{\mathbf{k}} = \mu_{-\mathbf{k}}$; this leads to the following matrices $h_f = |\mu_{\mathbf{k}}|^2 \sigma_z$ and $h_b = |\mu_{\mathbf{k}}|^2 \sigma_0$. For this model, an interesting function $\mu_{\mathbf{k}}$ is given by

$$\begin{array}{lll} \mu_{\mathbf{k}} & = & (M-\cos k)+i\sin k \\ |\mu_{\mathbf{k}}|^2 & = & (M-\cos k)^2+\sin^2 k \end{array} \tag{5.72}$$

showing the existence of four massless states at the fix points $k_* = 0, \pi$ and $M = \cos k_*$. For $|M| > 1$, we have massive states.

6. More on topological supermatter

In this section, we deepen the investigation of the topological properties of the super bands while illustrating these features on some classes of the AZ table [18,21].

6.1. Constraints from ORTIC and SUSY

Topological supermatter is given by ordinary matter constrained by orthosymmetric invariance or supersymmetry; as such it constitutes a subset of the AZ matter; but requires bosons. Here, we focus on spinless supermatter and develop a method for constructing a family of coupling matrices $\mathbf{q_k}$ by starting from known h_f's. This proposal of engineering $\mathbf{q_k}$'s out of h_f was first suggested in [49]; it will be given below an interpretation in terms of symmetries.

6.1.1. TPC symmetries of $q_{\mathbf{k}}$

In the AZ table, the topological classes are modelled by hamiltonians $h_f(\mathbf{k})$ constrained by TPC. One may also demand other discrete symmetries like crystalline symmetries [50,51]. For the spinless case, we have

$$\mathcal{T} = K, \qquad \mathcal{P} = XK, \qquad \mathcal{C} = X, \qquad X = \sigma_x \otimes I_{N\times N} \tag{6.1}$$

The actions of these symmetry generators on $h_f(\mathbf{k})$ are given by

$$\begin{array}{llll} \mathcal{T} & : & h_f(\mathbf{k})^* & = & +h_f(-\mathbf{k}) \\ \mathcal{P} & : & Xh_f(\mathbf{k})^* X & = & -h_f(-\mathbf{k}) \\ \mathcal{C} & : & Xh_f(\mathbf{k}) X & = & -h_f(\mathbf{k}) \end{array} \tag{6.2}$$

The topological indices $\mathcal{I}_m$ characterising the matter phases are nicely derived by considering involution hamiltonian matrices $\hat{h}_f$ on the Brillouin torus $\mathbb{T}^d$. These $\mathcal{I}_m$'s are integers that can be either in $\mathbb{Z}$, $2\mathbb{Z}$ or $\mathbb{Z}_2$; the expression of $\mathcal{I}_m$ in terms of $\hat{h}_f$ depends on the parity of the spatial dimensions d. For even $d = 2m$ for instance, the integer $\mathcal{I}_m$ is either given by the m-th Chern number Ch_m or the Fu-Kane index as follows,

$$Ch_m = \frac{1}{m!} \int\limits_{\mathbb{T}^{2m}} Tr\left(\frac{i\mathcal{F}}{2\pi}\right)^m \tag{6.3}$$

and

$$\begin{aligned} FK_m = {} & \frac{i^m}{(2\pi)^m\, m!} \int\limits_{\frac{1}{2}\mathbb{T}^{2m}} Tr\left(\mathcal{F}^m\right) \\ & - \frac{i^m}{(2\pi)^m\,(m-1)!} \int_{\partial(\frac{1}{2}\mathbb{T}^{2m})} \left[\int_0^1 dt Tr\left(\mathcal{A}\mathcal{F}_t^{m-1}\right)\right] \end{aligned} \tag{6.4}$$

In these relations, the Berry curvature $\mathcal{F}$ is given by the 2-form $d\mathcal{A} + \mathcal{A}\wedge\mathcal{A}$ and the parametric $\mathcal{F}_t$ by $td\mathcal{A} + t^2\mathcal{A}^2$.

In the super band theory, the hamiltonian matrix $h_f(\mathbf{k})$ is given by $\boldsymbol{q}_{\mathbf{k}}Z\boldsymbol{q}_{\mathbf{k}}^{\dagger}$ and the bosonic partner $h_b(\mathbf{k})$ by $\boldsymbol{q}_{\mathbf{k}}\boldsymbol{q}_{\mathbf{k}}^{\dagger}$.
Putting $h_f(\mathbf{k}) = \boldsymbol{q}_{\mathbf{k}}Z\boldsymbol{q}_{\mathbf{k}}^{\dagger}$ into (6.2), we get the following transformation of the coupling matrix $q(\mathbf{k})$ for spinless fermions

$$\begin{array}{lllll}\mathcal{T} & : & \boldsymbol{q}(\mathbf{k})^* & = & +\boldsymbol{q}(-\mathbf{k}) \\ \mathcal{P} & : & X\boldsymbol{q}(\mathbf{k})^* X & = & +\boldsymbol{q}(-\mathbf{k}) \\ \mathcal{C} & : & \mathcal{C}\boldsymbol{q}(\mathbf{k})\mathcal{C}^{-1} & = & +\boldsymbol{q}(\mathbf{k})\end{array} \tag{6.5}$$

Substituting these transformations into the bosonic $h_b(\mathbf{k}) = \boldsymbol{q}_{\mathbf{k}}^{\dagger}\boldsymbol{q}_{\mathbf{k}}$, we end up with the following transformations

$$\begin{array}{lllll}\mathcal{T} & : & h_b(\mathbf{k})^* & = & +h_b(-\mathbf{k}) \\ \mathcal{P} & : & Xh_b(\mathbf{k})^* X & = & +h_b(-\mathbf{k}) \\ \mathcal{C} & : & \mathcal{C}h_b(\mathbf{k})\mathcal{C}^{-1} & = & +h_b(\mathbf{k})\end{array} \tag{6.6}$$

from which we see that $\mathcal{P}$ and $\mathcal{C}$ have different actions on $h_f(\mathbf{k})$ and $h_b(\mathbf{k})$. Notice that also that $h_f(\mathbf{k})$ and $Zh_b(\mathbf{k})$ have similar transformations

$$\begin{array}{lllll}\mathcal{T} & : & Zh_b(\mathbf{k})^* & = & +Zh_b(-\mathbf{k}) \\ \mathcal{P} & : & X[Zh_b(\mathbf{k})^*]X & = & -Zh_b(-\mathbf{k}) \\ \mathcal{C} & : & \mathcal{C}[Zh_b(\mathbf{k})^*]\mathcal{C}^{-1} & = & -h_b(\mathbf{k})\end{array} \tag{6.7}$$

showing that the TPC symmetries agree with the fact that the matrix $Zh_b(\mathbf{k}) = Zq_{\mathbf{k}}^{\dagger}q_{\mathbf{k}}$ has the same spectrum as $h_f(\mathbf{k})$. This feature follows from (i) the fact that h_f and Zh_b factorise like AB and BA with A$=\boldsymbol{q}_{\mathbf{k}}$ and B$=Z\boldsymbol{q}_{\mathbf{k}}^{\dagger}$; and (ii) the factors AB and BA have the same spectrum. Below, we consider the super TBM family with $U(1)^N$ symmetry; they allow to extract straightforwardly information on the topological phases.

6.1.2. *Coupling* $\boldsymbol{q}(\mathbf{k})$ *of super models* $U(1)^N$

To engineer the $2N \times 2N$ coupling matrix $\boldsymbol{q}(\mathbf{k})$ for the super models $U(1)^N$, we start from an AZ hamiltonian $h_f(\mathbf{k})$ with a given TPC symmetry like in (6.2). Because h_f is given by $\boldsymbol{q}_{\mathbf{k}}Z\boldsymbol{q}_{\mathbf{k}}^{\dagger}$, the fermionic Hamiltonian to start with must be $2N \times 2N$.

• **Spectrum of $h_f(\mathbf{k})$**

To get the eigenstates and the eigenvalues for $U(1)^N$ super models, we use the even parity 2N to expand this $h_f(\mathbf{k})$ like

$$h_f(\mathbf{k}) = \sum_{\mu=x,y,z} \sigma^{\mu} d_{\mu}(\mathbf{k}) \tag{6.8}$$

where we have taken $h_f(\mathbf{k})$ traceless. Explicitly, we have

$$h_f = \begin{pmatrix} d_z & d_x - id_y \\ d_x + id_y & -d_z \end{pmatrix} \tag{6.9}$$

In this relation, the three functions $d_{\mu} = d_{\mu}(\mathbf{k})$ are hermitian $N \times N$ matrices valued in the $u(N)$ Lie algebra with N^2 dimensions. Denoting by $E_{IJ} = |I\rangle\langle J|$ the generators of $u(N)$, we can expand the d_{μ}s like $\sum_{I,J} d_{\mu}^{IJ} E_{IJ}$ that split as

$$d_{\mu} = \sum_{I=1}^{N} d_{\mu}^{I} E_{I}^{0} + \sum_{I<J}^{N} d_{\mu}^{IJ} E_{IJ}^{-} + \sum_{I>J=1}^{N} d_{\mu}^{IJ} E_{IJ}^{+} \tag{6.10}$$

with $E_I^0 = |I\rangle\langle I|$. According to the classification (5.43), we have the following constraints on the d_μ^{IJ}'s

symmetry	constraints	values of d_μ^{IJ}	parameters
$U(N)$	$\left[E_{IJ}, d_\mu\right]=0$	$d_\mu = \mathrm{d}_\mu^0 I_{N\times N}$	d_μ^0
$U(1)^N$	$\left[E_I^0, d_\mu\right]=0$	$d_\mu = \sum_{I=1}^N d_\mu^I E_I^0$	$d_\mu^1, ..., d_\mu^N$

(6.11)

Because of the abelian symmetry, the calculations for the $U(1)^N$ super models family are similar to the case of one factor $U(1)$; as such we will hide the Table l in the d_μ^Is seen that this omission does not affect the calculations.
The eigenvalues of h_f are $\varepsilon_\pm = \pm\varepsilon$ with $\varepsilon = \sqrt{d_x^2+d_y^2+d_z^2}$ with eigenvectors $V_\pm$ given by

$$V_+ = \begin{pmatrix} \frac{d_x - id_y}{d_x^2+d_y^2}\left[d_z+\varepsilon\right] \\ 1 \end{pmatrix} \quad , \quad V_- = \begin{pmatrix} -\frac{d_x - id_y}{d_x^2+d_y^2}\left[\varepsilon-d_z\right] \\ 1 \end{pmatrix} \tag{6.12}$$

By setting $e^{i\phi} = \left(d_x - id_y\right)/\sqrt{d_x^2+d_y^2}$, the eigenvectors $V_\pm$ are normalised as follows

$$V_+ = \frac{1}{\sqrt{2\varepsilon}}\begin{pmatrix} e^{-i\phi}\sqrt{d_z+\varepsilon} \\ \sqrt{\varepsilon-d_z} \end{pmatrix} \quad , \quad V_- = \frac{1}{\sqrt{2\varepsilon}}\begin{pmatrix} \sqrt{\varepsilon-d_z} \\ -e^{+i\phi}\sqrt{\varepsilon+d_z} \end{pmatrix} \tag{6.13}$$

- **Building $q_\mathbf{k}$ from** h_f

Given the above hermitian matrix h_f, we can diagonalise it by a unitary transformation $h_f = \boldsymbol{V}_\mathbf{k}\Delta_\mathbf{k}\boldsymbol{V}_\mathbf{k}^\dagger$ where the diagonal matrix $\Delta_\mathbf{k}$ is given by

$$\Delta_\mathbf{k} = \begin{pmatrix} \varepsilon_{\mathbf{k}(I)} & 0 \\ 0 & -\varepsilon_{-\mathbf{k}(I)} \end{pmatrix} \tag{6.14}$$

and

$$\varepsilon_{\pm\mathbf{k}(I)} = \begin{pmatrix} \varepsilon_{\pm\mathbf{k}1} & & \\ & \ddots & \\ & & \varepsilon_{\pm\mathbf{k}N} \end{pmatrix} \tag{6.15}$$

with the property $\varepsilon_{\pm\mathbf{k}i} \geq 0$ and the ordering $\varepsilon_{\pm\mathbf{k}(i+1)} \geq \varepsilon_{\pm\mathbf{k}i}$. For later use, it is interesting to introduce the two following matrices

$$D_\mathbf{k} = \begin{pmatrix} \varepsilon_{\mathbf{k}(I)} & 0 \\ 0 & \varepsilon_{-\mathbf{k}(I)} \end{pmatrix} \quad , \quad D_\mathbf{k}^{1/2} = \begin{pmatrix} \sqrt{\varepsilon_{\mathbf{k}(I)}} & 0 \\ 0 & \sqrt{\varepsilon_{-\mathbf{k}(I)}} \end{pmatrix} \tag{6.16}$$

which are related to (6.14) like $\Delta_\mathbf{k} = ZD_\mathbf{k} = D_\mathbf{k}Z$. Notice that the square root $D_\mathbf{k}^{1/2}$ is well defined because of the positivity of the eigenvalues $\varepsilon_{\pm\mathbf{k}i}$. Using the relation $h_f = \boldsymbol{V}_\mathbf{k}\Delta_\mathbf{k}\boldsymbol{V}_\mathbf{k}^\dagger$ and the factorisation $D_\mathbf{k} = D_\mathbf{k}^{1/2}D_\mathbf{k}^{1/2}$ as well as $ZD_\mathbf{k}^{1/2} = D_\mathbf{k}^{1/2}Z$, we can first express h_f like $\boldsymbol{V}_\mathbf{k}\left(ZD_\mathbf{k}\right)\boldsymbol{V}_\mathbf{k}^\dagger$ and then as follows

$$h_f = \left(\boldsymbol{V}_\mathbf{k}D_\mathbf{k}^{1/2}\right)Z\left(D_\mathbf{k}^{1/2}\boldsymbol{V}_\mathbf{k}^\dagger\right) \tag{6.17}$$

Equating with $h_f = q_\mathbf{k}Zq_\mathbf{k}^\dagger$, we end up with

$$q_{\mathbf{k}} = \boldsymbol{V}_{\mathbf{k}} D_{\mathbf{k}}^{1/2} \tag{6.18}$$

Putting this expression of $D_{\mathbf{k}}^{1/2}$ back into the value of the bosonic $h_b = q_{\mathbf{k}}^{\dagger} q_{\mathbf{k}}$, we obtain the bosonic hamiltonian $h_b = D_{\mathbf{k}}^{1/2} V_{\mathbf{k}}^{\dagger} V_{\mathbf{k}} D_{\mathbf{k}}^{1/2}$ reading as follows

$$h_b = D_{\mathbf{k}} \tag{6.19}$$

with no dependence into $\boldsymbol{V}_{\mathbf{k}}$ indicating that h_b is topologically trivial.

6.2. *Interpreting the $\boldsymbol{q}_{\mathbf{k}}$-coupling tensor*

Here, we give an algebraic interpretation of the 2N×2N Bose-Fermi coupling tensor $[\boldsymbol{q}_{\mathbf{k}}]_A^{\dot{C}}$ given by (5.13) and derive supersymmetric constraints relating the four N×N block matrices $\boldsymbol{q}_{1\mathbf{k}}$, $\boldsymbol{q}_{2\mathbf{k}}$, $\boldsymbol{q}_{3\mathbf{k}}$, $\boldsymbol{q}_{4\mathbf{k}}$ making $[\boldsymbol{q}_{\mathbf{k}}]_A^{\dot{C}}$. The basic idea behind this interpretation goes back to eqs. (4.32)-(4.33) that we discuss them further in this subsection. For this purpose, we first study special limits of $[\boldsymbol{q}_{\mathbf{k}}]_A^{\dot{C}}$ given by the diagonal $z_{\mathbf{k}}^A \delta_A^{\dot{C}}$ with the $z_{\mathbf{k}}^A$'s 2N complex numbers. Then, we turn to investigate the deformation away from $z_{\mathbf{k}}^A \delta_A^{\dot{C}}$.

6.2.1. *The coupling limit $\boldsymbol{q}_{\mathbf{k}} = z_{\mathbf{k}} I_{id}$*

We begin by the fermionic $\hat{c}_{\mathbf{k}}^{j}/\hat{c}_{\mathbf{k}j}^{\dagger}$ and the bosonic $\hat{b}_{\mathbf{k}}^{I}/\hat{b}_{\mathbf{k}I}^{\dagger}$ oscillator realisation of the ORTIC charge $Q_{\mathbf{k}}$ given by $\hat{\lambda}_{\mathbf{k}\dot{C}}[\boldsymbol{q}_{\mathbf{k}}]_A^{\dot{C}} \hat{\xi}_{\mathbf{k}}^A$ reading explicitly as follows

$$\begin{aligned} Q_{\mathbf{k}} \quad = \quad & \sum_{I,j} \hat{c}_{\mathbf{k}j}^{\dagger} \left(\boldsymbol{q}_{1\mathbf{k}}\right)_I^j \hat{b}_{\mathbf{k}}^I + \hat{c}_{\mathbf{k}j}^{\dagger} \hat{b}_{\mathbf{k}I}^{\dagger} \left(\boldsymbol{q}_{2\mathbf{k}}\right)^{jI} \\ & + \sum_{I,j} \left(\boldsymbol{q}_{3\mathbf{k}}\right)_{jI} \hat{c}_{\mathbf{k}}^j \hat{b}_{\mathbf{k}}^I + \hat{b}_{\mathbf{k}I}^{\dagger} \left(\boldsymbol{q}_{4\mathbf{k}}\right)_j^I \hat{c}_{\mathbf{k}}^j \end{aligned} \tag{6.20}$$

where the four blocks $(\boldsymbol{q}_{1\mathbf{k}})_I^j$, $(\boldsymbol{q}_{2\mathbf{k}})^{jI}$, $(\boldsymbol{q}_{3\mathbf{k}})_{jI}$ and $(\boldsymbol{q}_{4\mathbf{k}})_j^I$ are $N \times N$ coupling matrices as in (5.13); they are functions of the momentum $\mathbf{k}$. In the diagonal case where $[\boldsymbol{q}_{\mathbf{k}}]_A^{\dot{C}}$ is given by

$$[\boldsymbol{q}_{\mathbf{k}}]_A^{\dot{C}} = z_{\mathbf{k}}^A \delta_A^{\dot{C}}$$

we have $\left(\boldsymbol{q}_{1\mathbf{k}}\right)_I^j = u_{\mathbf{k}}^I \delta_I^j$ and $\left(\boldsymbol{q}_{4\mathbf{k}}\right)_j^I = v_{\mathbf{k}}^j \delta_j^I$ while $\boldsymbol{q}_{2\mathbf{k}} = \boldsymbol{q}_{3\mathbf{k}} = 0_{N\times N}$. By substituting, the above ORTIC charge $Q_{\mathbf{k}}$ becomes

$$Q_{\mathbf{k}} = \sum u_{\mathbf{k}}^I \hat{c}_{\mathbf{k}I}^{\dagger} \hat{b}_{\mathbf{k}}^I + \sum v_{\mathbf{k}}^i \hat{b}_{\mathbf{k}i}^{\dagger} \hat{c}_{\mathbf{k}}^i) \tag{6.21}$$

Here, the $z_{\mathbf{k}}^A = (u_{\mathbf{k}}^I; v_{\mathbf{k}}^i)$ are complex functions of momentum with $\left|z_{\mathbf{k}}^A\right|^2 = \omega_{\mathbf{k}}^A$ thought of in terms of real frequencies scaling as energy. In the very special limit where $[\boldsymbol{q}_{\mathbf{k}}]_A^{\dot{C}}$ is proportional to the identity $z_{\mathbf{k}} \delta_A^{\dot{C}}$, we have $u_{\mathbf{k}}^I = v_{\mathbf{k}}^i = z_{\mathbf{k}}$ and $\boldsymbol{q}_{1\mathbf{k}} = \boldsymbol{q}_{4\mathbf{k}} = z_{\mathbf{k}} I_{N\times N}$. In this particular situation, the above ORTIC charge $Q_{\mathbf{k}}$ reduces further to

$$Q_{\mathbf{k}} = z_{\mathbf{k}} \hat{c}_{\mathbf{k}I}^{\dagger} \hat{b}_{\mathbf{k}}^I + z_{\mathbf{k}} \hat{b}_{\mathbf{k}I}^{\dagger} \hat{c}_{\mathbf{k}}^I \tag{6.22}$$

Up to the scale factor $z_{\mathbf{k}}$, this supercharge is just the sum of two fermionic operators $\hat{c}_{\mathbf{k}I}^{\dagger} \hat{b}_{\mathbf{k}}^I$ and its adjoint conjugate $\hat{b}_{\mathbf{k}I}^{\dagger} \hat{c}_{\mathbf{k}}^I$ which are nothing but the F_-^+ and $\bar{F}_+^-$ generators of osp(2|2) within osp(2N|2N). So, for the choice $\boldsymbol{q}_{\mathbf{k}} = z_{\mathbf{k}} I_{id}$, the orthosymplectic Hamiltonian is given by

$$H_{\mathbf{k}}^{ortic} = \frac{|z_{\mathbf{k}}|^2}{2}\left(\hat{b}^{\dagger}_{\mathbf{k}\dot{I}}\hat{b}^{\dot{I}}_{\mathbf{k}} + \hat{b}^{\dot{I}}_{\mathbf{k}}\hat{b}^{\dagger}_{\mathbf{k}\dot{I}}\right) + \frac{|z_{\mathbf{k}}|^2}{2}\left(\hat{c}^{\dagger}_{\mathbf{k}\dot{I}}\hat{c}^{\dot{I}}_{\mathbf{k}} - \hat{c}^{\dot{I}}_{\mathbf{k}}\hat{c}^{\dagger}_{\mathbf{k}\dot{I}}\right) \tag{6.23}$$

with matrix representation in the basis $(\hat{b}^{\dot{I}}_{\mathbf{k}}, \hat{b}^{\dagger}_{\mathbf{k}\dot{I}}, \hat{c}^{\dot{I}}_{\mathbf{k}}, \hat{c}^{\dagger}_{\mathbf{k}\dot{I}})$ as follows

$$\frac{1}{2}\begin{pmatrix} +\omega_{\mathbf{k}} & & & \\ & +\omega_{\mathbf{k}} & & \\ & & +\omega_{\mathbf{k}} & \\ & & & -\omega_{\mathbf{k}} \end{pmatrix} \tag{6.24}$$

where we have set $|z_{\mathbf{k}}|^2 = \omega_{\mathbf{k}}$. From this particular limit, one may think about eqs (6.20) and (6.20) as follows. (**i**) Eq. (6.20) is given by the sum of two terms like $Q_{\mathbf{k}} = Q^+_{\mathbf{k}} + Q^-_{\mathbf{k}}$ with

$$\begin{aligned} Q^+_{\mathbf{k}} &= z_{\mathbf{k}}\hat{c}^{\dagger}_{\mathbf{k}I}\hat{b}^{I}_{\mathbf{k}} \\ Q^-_{\mathbf{k}} &= z_{\mathbf{k}}\hat{b}^{\dagger}_{\mathbf{k}I}\hat{c}^{I}_{\mathbf{k}} \end{aligned} \tag{6.25}$$

(**ii**) Eq. (6.20) is a deformation of the above (6.25); and its coupling $\boldsymbol{q}_{\mathbf{k}}$ as a deviation away from $z_{\mathbf{k}}I_{id}$. Before exploring this deformation, notice that the orthosymplectic eq. (6.23) corresponding to $\boldsymbol{q}_{\mathbf{k}} = z_{\mathbf{k}}I_{id}$ coincides with the supersymmetric Hamiltonian $H^{susy}_{\mathbf{k}}$. This is because the $Q^{\pm}_{\mathbf{k}}$ of (6.25) are nilpotent and the Hamiltonian eq. (6.23) commutes with $Q^{\pm}_{\mathbf{k}}$; that is

$$\left[H^{ortic}_{\mathbf{k}}, Q^{\pm}_{\mathbf{k}}\right] = 0 \quad , \quad \left(Q^-_{\mathbf{k}}\right)^2 = \left(Q^+_{\mathbf{k}}\right)^2 = 0 \tag{6.26}$$

Extending the fermionic charges (6.25) for couplings $[\boldsymbol{q}_{\mathbf{k}}]^{\dot{C}}_{A}$ beyond the diagonal $z_{\mathbf{k}}\delta^{\dot{C}}_{A}$, we can present the $Q^+_{\mathbf{k}}$ and the $Q^-_{\mathbf{k}}$ as well as their adjoint conjugates $\bar{Q}^-_{\mathbf{k}}$ and $\bar{Q}^+_{\mathbf{k}}$ as follows,

$$\begin{aligned} Q^+_{\mathbf{k}} &= \hat{c}^{\dagger}_{\mathbf{k}j}\left(\boldsymbol{q}_{1\mathbf{k}}\right)^{j}_{I}\hat{b}^{I}_{\mathbf{k}} + \hat{c}^{\dagger}_{\mathbf{k}j}\hat{b}^{\dagger}_{\mathbf{k}I}\left(\boldsymbol{q}_{2\mathbf{k}}\right)^{jI} \\ Q^-_{\mathbf{k}} &= \left(\boldsymbol{q}_{3\mathbf{k}}\right)_{jI}\hat{c}^{j}_{\mathbf{k}}\hat{b}^{I}_{\mathbf{k}} + \hat{b}^{\dagger}_{\mathbf{k}I}\left(\boldsymbol{q}_{4\mathbf{k}}\right)^{I}_{j}\hat{c}^{j}_{\mathbf{k}} \\ \bar{Q}^-_{\mathbf{k}} &= \hat{b}^{\dagger}_{\mathbf{k}I}\left(\boldsymbol{q}^{\dagger}_{1\mathbf{k}}\right)^{I}_{j}\hat{c}^{j}_{\mathbf{k}} + \left(\boldsymbol{q}^{\dagger}_{2\mathbf{k}}\right)_{Ij}\hat{b}^{I}_{\mathbf{k}}\hat{c}^{j}_{\mathbf{k}} \\ \bar{Q}^+_{\mathbf{k}} &= \hat{b}^{\dagger}_{I\mathbf{k}}\hat{c}^{\dagger}_{j\mathbf{k}}\left(\boldsymbol{q}^{\dagger}_{3\mathbf{k}}\right)^{Ij} + \hat{c}^{\dagger}_{j\mathbf{k}}\left(\boldsymbol{q}^{\dagger}_{4\mathbf{k}}\right)^{j}_{I}\hat{b}^{I}_{\mathbf{k}} \end{aligned} \tag{6.27}$$

By mimicking eqs. (4.32)-(4.33), we see that the above supercharges (6.27) can be handled in two ways as given here after:

- *First way:* The $Q^{\pm}_{\mathbf{k}}$ and $\bar{Q}^{\pm}_{\mathbf{k}}$ in (6.27) are imagined in a condensed form as follows

$$\begin{aligned} Q^+_{\mathbf{k}} &= \hat{c}^{\dagger}_{\mathbf{k}j}\hat{B}^{j}_{\mathbf{k}} \\ Q^-_{\mathbf{k}} &= \hat{D}^{\dagger}_{\mathbf{k}j}\hat{c}^{j}_{\mathbf{k}} \end{aligned} \quad , \quad \begin{aligned} \bar{Q}^-_{\mathbf{k}} &= \hat{B}^{\dagger}_{\mathbf{k}j}\hat{c}^{j}_{\mathbf{k}} \\ \bar{Q}^+_{\mathbf{k}} &= \hat{c}^{\dagger}_{\mathbf{k}j}\hat{D}^{j}_{\mathbf{k}} \end{aligned} \tag{6.28}$$

  where the $\hat{B}^{j}_{\mathbf{k}}$ and $\hat{D}^{j}_{\mathbf{k}}$ are linear functions of the bosonic $\hat{b}^{I}_{\mathbf{k}}$ and $\hat{b}^{\dagger}_{\mathbf{k}I}$.
- *Second way:* The $Q^{\pm}_{\mathbf{k}}$ and $\bar{Q}^{\pm}_{\mathbf{k}}$ are thought of like

$$\begin{aligned} Q^+_{\mathbf{k}} &= \hat{C}^{\dagger}_{\mathbf{k}j}\hat{b}^{j}_{\mathbf{k}} \\ Q^-_{\mathbf{k}} &= \hat{b}^{\dagger}_{\mathbf{k}j}\hat{E}^{j}_{\mathbf{k}} \end{aligned} \quad , \quad \begin{aligned} \bar{Q}^-_{\mathbf{k}} &= \hat{b}^{\dagger}_{\mathbf{k}j}\hat{C}^{j}_{\mathbf{k}} \\ \bar{Q}^+_{\mathbf{k}} &= \hat{E}^{\dagger}_{\mathbf{k}j}\hat{b}^{j}_{\mathbf{k}} \end{aligned} \tag{6.29}$$

  where now $\hat{C}^{j}_{\mathbf{k}}$ and $\hat{E}^{j}_{\mathbf{k}}$ are linear functions of the fermionic $\hat{c}^{I}_{\mathbf{k}}$ and $\hat{c}^{\dagger}_{\mathbf{k}I}$.

In what follows, we develop the picture given by eq. (6.28); a similar analysis can be done for (6.29).

6.2.2. Coupling $\boldsymbol{q}_{\mathbf{k}}$ as a deviation away from $\boldsymbol{q}_{\mathbf{k}} = z_{\mathbf{k}} I_{id}$

By singling out the fermionic operators $\hat{c}_{\mathbf{k}}^{j}$ and $\hat{c}_{\mathbf{k}j}^{\dagger}$, one can put the ORTIC charge (6.20) like $Q_{\mathbf{k}} = Q_{\mathbf{k}}^{+} + Q_{\mathbf{k}}^{-}$ with $Q_{\mathbf{k}}^{\pm}$ as in (6.28) and the new bosonic operators $\hat{B}_{\mathbf{k}}^{j}$ and $\hat{D}_{\mathbf{k}}^{j}$ given by

$$\begin{array}{lll} \hat{B}_{\mathbf{k}}^{j} & = & \left(\boldsymbol{q}_{1\mathbf{k}}\right)_{I}^{j} \hat{b}_{\mathbf{k}}^{I} + \hat{b}_{\mathbf{k}I}^{\dagger} \left(\boldsymbol{q}_{2\mathbf{k}}\right)^{Ij} \\ \hat{B}_{\mathbf{k}j}^{\dagger} & = & \left(\bar{\boldsymbol{q}}_{2\mathbf{k}}\right)_{jI} \hat{b}_{\mathbf{k}}^{I} + \hat{b}_{\mathbf{k}I}^{\dagger} \left(\bar{q}_{1\mathbf{k}}\right)_{j}^{I} \end{array} \quad , \quad \begin{array}{lll} \hat{D}_{\mathbf{k}j}^{\dagger} & = & \left(\boldsymbol{q}_{3\mathbf{k}}\right)_{jI} \hat{b}_{\mathbf{k}}^{I} + \hat{b}_{\mathbf{k}I}^{\dagger} \left(\boldsymbol{q}_{4\mathbf{k}}\right)_{j}^{I} \\ \hat{D}_{\mathbf{k}}^{j} & = & \hat{b}_{\mathbf{k}I}^{\dagger} \left(\bar{\boldsymbol{q}}_{3\mathbf{k}}\right)^{Ij} + \left(\bar{\boldsymbol{q}}_{4\mathbf{k}}\right)_{I}^{j} \hat{b}_{\mathbf{k}}^{I} \end{array} \tag{6.30}$$

For the case $\boldsymbol{q}_{1\mathbf{k}} = \boldsymbol{q}_{4\mathbf{k}} = z_{\mathbf{k}} I_{N\times N}$ and $\boldsymbol{q}_{2\mathbf{k}} = \boldsymbol{q}_{3\mathbf{k}} = 0_{N\times N}$, one has $\hat{B}_{\mathbf{k}}^{I} = z\hat{b}_{\mathbf{k}}^{I}$ and $\hat{D}_{\mathbf{k}}^{I} = \bar{z}\hat{b}_{\mathbf{k}}^{I}$.

A) Orthosymplectic hamiltonian

The ORTIC hamiltonian $H_{ortic} = \sum_{\mathbf{k}} H_{\mathbf{k}}^{ortic}$ is defined by the anticommutators $\{Q_{\mathbf{k}}, Q_{\mathbf{k}}^{\dagger}\} = H_{\mathbf{k}}^{ortic}$; it reads in terms of $Q_{\mathbf{k}}^{\pm}$ and their adjoint conjugates $\bar{Q}_{\mathbf{k}}^{\mp}$ as follows

$$H_{\mathbf{k}}^{ortic} = \left\{Q_{\mathbf{k}}^{+}, \bar{Q}_{\mathbf{k}}^{-}\right\} + \left\{Q_{\mathbf{k}}^{-}, \bar{Q}_{\mathbf{k}}^{+}\right\} + \left\{Q_{\mathbf{k}}^{-}, \bar{Q}_{\mathbf{k}}^{-}\right\} + \left\{Q_{\mathbf{k}}^{+}, \bar{Q}_{\mathbf{k}}^{+}\right\} \tag{6.31}$$

As for the ORTIC supercharge $Q_{\mathbf{k}}$, the bosonic $H_{\mathbf{k}}^{ortic}$ is valued in the osp(2N|2N) Lie superalgebra; and has four anticommutators blocks $H_{1\mathbf{k}} + H_{2\mathbf{k}} + Z_{1\mathbf{k}} + \bar{Z}_{1\mathbf{k}}$ given by

$$\begin{array}{lll} \left\{Q_{\mathbf{k}}^{+}, \bar{Q}_{\mathbf{k}}^{-}\right\} & = & H_{1\mathbf{k}} \\ \left\{Q_{\mathbf{k}}^{-}, \bar{Q}_{\mathbf{k}}^{+}\right\} & = & H_{2\mathbf{k}} \end{array} \quad , \quad \begin{array}{lll} \left\{Q_{\mathbf{k}}^{-}, \bar{Q}_{\mathbf{k}}^{-}\right\} & = & Z_{1\mathbf{k}} \\ \left\{Q_{\mathbf{k}}^{+}, \bar{Q}_{\mathbf{k}}^{+}\right\} & = & \bar{Z}_{1\mathbf{k}} \end{array} \tag{6.32}$$

where $H_{1\mathbf{k}}$ and $H_{2\mathbf{k}}$ are hermitian while $Z_{1\mathbf{k}}$ and $\bar{Z}_{1\mathbf{k}}$ are exchanged under adjoint conjugation. They read in terms of the oscillators $\hat{c}_{\mathbf{k}}/\hat{c}_{\mathbf{k}}^{\dagger}$ and the new $\hat{B}_{\mathbf{k}}/\hat{B}_{\mathbf{k}}^{\dagger}$ as well as $\hat{D}_{\mathbf{k}}/\hat{D}_{\mathbf{k}}^{\dagger}$ as follows

$$\begin{array}{lll} H_{1\mathbf{k}} & = & \hat{B}_{\mathbf{k}i}^{\dagger} \hat{B}_{\mathbf{k}}^{i} + \hat{c}_{\mathbf{k}i}^{\dagger} \mathbb{B}_{j}^{i} \hat{c}_{\mathbf{k}}^{j} \\ H_{2\mathbf{k}} & = & \hat{D}_{\mathbf{k}i}^{\dagger} \hat{D}_{\mathbf{k}}^{i} - \hat{c}_{\mathbf{k}i}^{\dagger} \mathbb{D}_{j}^{i} \hat{c}_{\mathbf{k}}^{j} \end{array} \quad , \quad \begin{array}{lll} Z_{1\mathbf{k}} & = & \hat{c}_{\mathbf{k}}^{i} \bar{\mathbb{F}}_{ij} \hat{c}_{\mathbf{k}}^{j} \\ \bar{Z}_{1\mathbf{k}} & = & \hat{c}_{\mathbf{k}i}^{\dagger} \mathbb{F}^{ij} \hat{c}_{\mathbf{k}j}^{\dagger} \end{array} \tag{6.33}$$

with matrices $\mathbb{B}_{j}^{i}$, $\mathbb{D}_{j}^{i}$ and so on given by the following commutators

$$\begin{array}{lll} \mathbb{B}_{j}^{i} & = & [\hat{B}_{\mathbf{k}}^{i}, \hat{B}_{\mathbf{k}j}^{\dagger}] \\ \mathbb{X}^{IJ} & = & [\hat{B}_{\mathbf{k}}^{I}, \hat{B}_{\mathbf{k}}^{J}] \\ \mathbb{F}^{ij} & = & [\hat{B}_{\mathbf{k}}^{i}, \hat{D}_{\mathbf{k}}^{j}] \\ \mathbb{G}_{i}^{j} & = & [\hat{B}_{\mathbf{k}}^{j}, \hat{D}_{\mathbf{k}i}^{\dagger}] \end{array} \quad , \quad \begin{array}{lll} \mathbb{D}_{j}^{i} & = & [\hat{D}_{\mathbf{k}}^{i}, \hat{D}_{\mathbf{k}j}^{\dagger}] \\ \mathbb{Y}^{IJ} & = & [\hat{D}_{\mathbf{k}}^{I}, \hat{D}_{\mathbf{k}}^{J}] \\ \bar{\mathbb{F}}_{ij} & = & [\hat{D}_{\mathbf{k}i}^{\dagger}, \hat{B}_{\mathbf{k}j}^{\dagger}] \\ \bar{\mathbb{G}}_{j}^{i} & = & [\hat{D}_{\mathbf{k}}^{i}, \hat{B}_{\mathbf{k}j}^{\dagger}] \end{array} \tag{6.34}$$

Using (6.30), these quantities read in terms of the coupling tensors as follows

$$\begin{array}{lll} \mathbb{B} & = & \boldsymbol{q}_{1\mathbf{k}}\boldsymbol{q}_{1\mathbf{k}}^{\dagger} - \boldsymbol{q}_{2\mathbf{k}}^{\dagger}\boldsymbol{q}_{2\mathbf{k}} \\ \mathbb{D} & = & \boldsymbol{q}_{4\mathbf{k}}^{\dagger}\boldsymbol{q}_{4\mathbf{k}} - \boldsymbol{q}_{3\mathbf{k}}\boldsymbol{q}_{3\mathbf{k}}^{\dagger} \\ \mathbb{X} & = & \boldsymbol{q}_{1\mathbf{k}}\boldsymbol{q}_{2\mathbf{k}} - \boldsymbol{q}_{1\mathbf{k}}\boldsymbol{q}_{2\mathbf{k}} \end{array} \quad , \quad \begin{array}{lll} \mathbb{F} & = & \boldsymbol{q}_{1\mathbf{k}}\boldsymbol{q}_{3\mathbf{k}}^{\dagger} - \boldsymbol{q}_{4\mathbf{k}}^{\dagger}\boldsymbol{q}_{2\mathbf{k}} \\ \mathbb{G} & = & \boldsymbol{q}_{1\mathbf{k}}\boldsymbol{q}_{4\mathbf{k}} - \boldsymbol{q}_{3\mathbf{k}}\boldsymbol{q}_{2\mathbf{k}} \\ \mathbb{Y} & = & \bar{\boldsymbol{q}}_{4\mathbf{k}}\bar{\boldsymbol{q}}_{3\mathbf{k}} - \bar{\boldsymbol{q}}_{4\mathbf{k}}\bar{\boldsymbol{q}}_{3\mathbf{k}} \end{array} \tag{6.35}$$

where the diagonal blocks are hermitian; that is $\mathbb{B}^{\dagger} = \mathbb{B}$ and $\mathbb{D}^{\dagger} = \mathbb{D}$. Putting these relations into (6.31), we end up with

$$\begin{array}{lll} H_{\mathbf{k}}^{ortic} & = & \hat{B}_{\mathbf{k}i}^{\dagger} \hat{B}_{\mathbf{k}}^{i} + \hat{D}_{\mathbf{k}i}^{\dagger} \hat{D}_{\mathbf{k}}^{i} + \hat{c}_{\mathbf{k}i}^{\dagger} \mathbb{M}_{j}^{i} \hat{c}_{\mathbf{k}}^{j} \\ & & + \bar{\mathbb{F}}_{ij} \hat{c}_{\mathbf{k}}^{i} \hat{c}_{\mathbf{k}}^{j} + \hat{c}_{\mathbf{k}i}^{\dagger} \hat{c}_{\mathbf{k}j}^{\dagger} \mathbb{F}^{ij} \end{array} \tag{6.36}$$

where $\mathbb{M} = \mathbb{B} - \mathbb{D}$. From this hamiltonian, we learn the bosonic $H_{\mathbf{k}}^{bose}$ and the fermionic $H_{\mathbf{k}}^{fermi}$ contributions namely

$$\begin{array}{lll} H_{\mathbf{k}}^{bose} & = & \hat{B}_{\mathbf{k}i}^{\dagger}\hat{B}_{\mathbf{k}}^{i} + \hat{D}_{\mathbf{k}i}^{\dagger}\hat{D}_{\mathbf{k}}^{i} \\ H_{\mathbf{k}}^{fermi} & = & \hat{c}_{\mathbf{k}i}^{\dagger}\mathbb{M}_{j}^{i}\hat{c}_{\mathbf{k}}^{j} + \bar{\mathbb{F}}_{ij}\hat{c}_{\mathbf{k}}^{i}\hat{c}_{\mathbf{k}}^{j} + \hat{c}_{\mathbf{k}i}^{\dagger}\hat{c}_{\mathbf{k}j}^{\dagger}\mathbb{F}^{ij} \end{array} \tag{6.37}$$

B) Supersymmetric Hamiltonian

In the absence of supersymmetric central charges, the hamiltonian $H_{\mathbf{k}}^{susy}$ is obtained from the ORTIC $H_{\mathbf{k}}^{ortic}$ by imposing the supersymmetric conditions required by the supersymmetric algebra on world line. First, we impose the nilpotency of the anticommutators $\{Q_{\mathbf{k}}^{+}, \bar{Q}_{\mathbf{k}}^{+}\}$ and $\{Q_{\mathbf{k}}^{-}, \bar{Q}_{\mathbf{k}}^{-}\}$; that is $Z_{1\mathbf{k}} = \bar{Z}_{1\mathbf{k}} = 0$. These nilpotencies reduce the ORTIC $H_{\mathbf{k}}^{ortic}$ to the sum of two terms namely $H_{\mathbf{k}}^{susy} = H_{1\mathbf{k}} + H_{2\mathbf{k}}$. Second, we require the following commutation relations to hold

$$[H_{\mathbf{k}}^{susy}, Q_{\mathbf{k}}^{+}] = H_{\mathbf{k}}^{susy}, Q_{\mathbf{k}}^{-}] = 0 \tag{6.38}$$

Clearly, this supersymmetric invariance put constraints on the coupling tensors $\boldsymbol{q}_{1\mathbf{k}}, \boldsymbol{q}_{2\mathbf{k}}, \boldsymbol{q}_{3\mathbf{k}}$ and $\boldsymbol{q}_{4\mathbf{k}}$; they are no longer free tensors; candidate of such $\boldsymbol{q}_{n\mathbf{k}}$'s can be determined by using (6.35). Here, we omit the details; but to fix the ideas, we describe below some steps of the calculations regarding the resolution of the constraints $\{Q_{\mathbf{k}}^{+}, \bar{Q}_{\mathbf{k}}^{+}\} = \{Q_{\mathbf{k}}^{-}, \bar{Q}_{\mathbf{k}}^{-}\} = 0$.

These constraints correspond to imposing $Z_{1\mathbf{k}} = 0$ in (6.31) and (6.36); as such they require $\hat{c}_{\mathbf{k}i}^{\dagger}\hat{c}_{\mathbf{k}j}^{\dagger}\mathbb{F}^{ij} = 0$ with tensor $\mathbb{F}^{ij}$ as in eq. (6.35). This condition gives a matrix relation between the four coupling tensors $\boldsymbol{q}_{1\mathbf{k}}, \boldsymbol{q}_{2\mathbf{k}}, \boldsymbol{q}_{3\mathbf{k}}$ and $\boldsymbol{q}_{4\mathbf{k}}$ namely

$$\left(\boldsymbol{q}_{1\mathbf{k}}\boldsymbol{q}_{3\mathbf{k}}^{\dagger}\right)^{ij} - \left(\boldsymbol{q}_{4\mathbf{k}}^{\dagger}\boldsymbol{q}_{2\mathbf{k}}\right)^{ij} = \eta_q \Sigma_{\mathbf{k}}^{ij} \tag{6.39}$$

where $\Sigma_{\mathbf{k}}^{ij}$ is a symmetric matrix ($\Sigma_{\mathbf{k}}^{ij} = \Sigma_{\mathbf{k}}^{ji}$) because $\hat{c}_{\mathbf{k}i}^{\dagger}\hat{c}_{\mathbf{k}j}^{\dagger}\Sigma_{\mathbf{k}}^{ij} = 0$. As far as the constraint (6.39) is concerned, let us give some comments regarding its solutions.

(**1**) The simplest solution is given by the diagonal coupling $\boldsymbol{q}_{\mathbf{k}} = z_{\mathbf{k}} I_{2N\times 2N}$ considered previously with sub-blocks $\boldsymbol{q}_{1\mathbf{k}} = \boldsymbol{q}_{4\mathbf{k}} = z_{\mathbf{k}} I_{N\times N}$ and $\boldsymbol{q}_{2\mathbf{k}} = \boldsymbol{q}_{3\mathbf{k}} = 0_{N\times N}$. It corresponds just to the trivial case $\eta_q = 0$. This requires

$$\boldsymbol{q}_{1\mathbf{k}}\boldsymbol{q}_{3\mathbf{k}}^{\dagger} = \boldsymbol{q}_{4\mathbf{k}}^{\dagger}\boldsymbol{q}_{2\mathbf{k}} \tag{6.40}$$

(**2**) A second set of solutions of (6.39) is given by the remarkable case where $\Sigma_{\mathbf{k}}^{ij} = \delta^{ij}$ with $\eta_q \neq 0$. In this case, a solution of (6.39) is given by

$$\boldsymbol{q}_{1\mathbf{k}}\boldsymbol{q}_{3\mathbf{k}}^{\dagger} = \varrho_{\mathbf{k}} I_{N\times N} \quad , \quad \boldsymbol{q}_{4\mathbf{k}}^{\dagger}\boldsymbol{q}_{2\mathbf{k}} = \tilde{\varrho}_{\mathbf{k}} I_{N\times N} \tag{6.41}$$

with the relation $\varrho_{\mathbf{k}} - \tilde{\varrho}_{\mathbf{k}} = \zeta_{\mathbf{k}}$.

(**3**) Other solutions of the constraint (6.39) can be also written down; for instance by taking $\boldsymbol{q}_{3\mathbf{k}}^{\dagger} = \boldsymbol{q}_{1\mathbf{k}}^{T}$ and $\boldsymbol{q}_{4\mathbf{k}}^{\dagger} = \boldsymbol{q}_{2\mathbf{k}}^{T}$.

6.3. Topological super model with two bands

Here, we consider a 2D Brillouin $\mathbb{T}^2$ parameterised by (k_x, k_y) with $0 \leq k_x, k_y < 2\pi$; and we apply the above construction to the hamiltonian (6.9). Because of the particle-hole symmetry, we must have $\sigma_x h_f(\mathbf{k})^* \sigma_x = -h_f(-\mathbf{k})$; thus requiring conditions on the $d_{x,y,z}$ functions namely

$$\begin{array}{lll} d_{x,y}(-\mathbf{k}) & = & -d_{x,y}(\mathbf{k}) \\ d_z(-\mathbf{k}) & = & d_z(\mathbf{k}) \end{array} \tag{6.42}$$

We solve these constraint relations as

$$d_x = t_z \sin k_x, \qquad d_y = t_t \sin k_y, \qquad d_z = M - \cos k_x - \cos k_y \tag{6.43}$$

6.3.1. Constructing the coupling matrix $\boldsymbol{q}_{\mathbf{k}}$

For this supersymmetric two bands model, the diagonal matrices $\mathcal{D}_{\mathbf{k}}$ and $\Delta_{\mathbf{k}}$ are given by

$$\mathcal{D}_{\mathbf{k}} = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \quad , \qquad \Delta_{\mathbf{k}} = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix} \tag{6.44}$$

where $\varepsilon = \sqrt{d_x^2 + d_y^2 + d_z^2}$ with the remarkable property $\mathcal{D}_{\mathbf{k}} = \varepsilon\sigma_0$. We also have $\Delta_{\mathbf{k}} = \varepsilon\sigma_z$ with gap energy 2ε. The above diagonal (6.44) can be compared to (6.24) with ε given by $\omega_{\mathbf{k}}$. For fermionic gapless states, the hamiltonians h_f and h_b have zero modes ($\det h_f = \det h_b = 0$).

• *Building the passage matrix $\boldsymbol{V}_{\mathbf{k}}$*

The unitary matrix $\boldsymbol{V}_{\mathbf{k}}$ involved in the construction of (6.18) is given by (V_+, V_-) where $V_\pm$ are the normalisation of the eigenvectors (6.13) of h_f. Substituting, we obtain

$$\boldsymbol{V}_{\mathbf{k}} = \frac{1}{\sqrt{2\varepsilon}} \begin{pmatrix} e^{-i\phi}\sqrt{\varepsilon + d_z} & \sqrt{\varepsilon - d_z} \\ \sqrt{\varepsilon - d_z} & -e^{+i\phi}\sqrt{\varepsilon + d_z} \end{pmatrix} \tag{6.45}$$

Putting this expression back into $\boldsymbol{q}_{\mathbf{k}} = \boldsymbol{V}_{\mathbf{k}} D_{\mathbf{k}}^{1/2}$, we end up with the coupling matrix

$$\boldsymbol{q}_{\mathbf{k}} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\phi}\sqrt{\varepsilon + d_z} & \sqrt{\varepsilon - d_z} \\ \sqrt{\varepsilon - d_z} & -e^{+i\phi}\sqrt{\varepsilon + d_z} \end{pmatrix} \tag{6.46}$$

with $\det \boldsymbol{q} = -\varepsilon$ and

$$e^{i\phi} = \frac{d_x + i d_y}{\sqrt{d_x^2 + d_y^2}} \tag{6.47}$$

Comparing (6.46) to (5.13), we learn the expressions of $\boldsymbol{q}_{1\mathbf{k}}$, $\boldsymbol{q}_{2\mathbf{k}}$, $\boldsymbol{q}_{3\mathbf{k}}$ and $\boldsymbol{q}_{4\mathbf{k}}$ namely

$$\boldsymbol{q}_{1\mathbf{k}} = \frac{\sqrt{\varepsilon + d_z}}{\sqrt{2}} e^{-i\phi}, \quad \boldsymbol{q}_{2\mathbf{k}} = \boldsymbol{q}_{3\mathbf{k}} = \frac{\sqrt{\varepsilon - d_z}}{\sqrt{2}}, \quad \boldsymbol{q}_{4\mathbf{k}} = -\frac{\sqrt{\varepsilon + d_z}}{\sqrt{2}} e^{+i\phi} \tag{6.48}$$

In the Fig. 7, we plot the real part $e^{i\phi}$ where a distortion lives at the high symmetry points $k = 0, \pi$.

• *Topological distortion*

The above coupling $\boldsymbol{q}_{\mathbf{k}}$ matrix is a function of (k_x, k_y); it obeys the PH symmetry (5.17). At the high symmetry point, the matrix coupling has distortion manifested by an ill-definiteness. The fix points of PH are given by the solution of the vanishing $d_{x,y}(\mathbf{k}) = 0$. For the model where $d_x = t_x \sin k_x$, $d_y = t_t \sin k_y$, the points are given by $\left(k_x^*, k_y^*\right) = \left(n_x\pi, n_y\pi\right)$ reading explicitly as

$$\left(k_x^*, k_y^*\right) = (0,0), \qquad (\pi, 0), \qquad (0, \pi), \qquad (\pi, \pi) \tag{6.49}$$

At these points $\mathbf{k}_*$, the phase $e^{i\phi}$ shows an obstruction; it behaves like $\frac{0}{0^+} + i\frac{0}{0^+}$ as it can be checked on eq. (6.47). This singularity survives in the limit $d_z \to 0$ where $\det q \to 0$ and where live fermionic gapless states and bosonic partners.

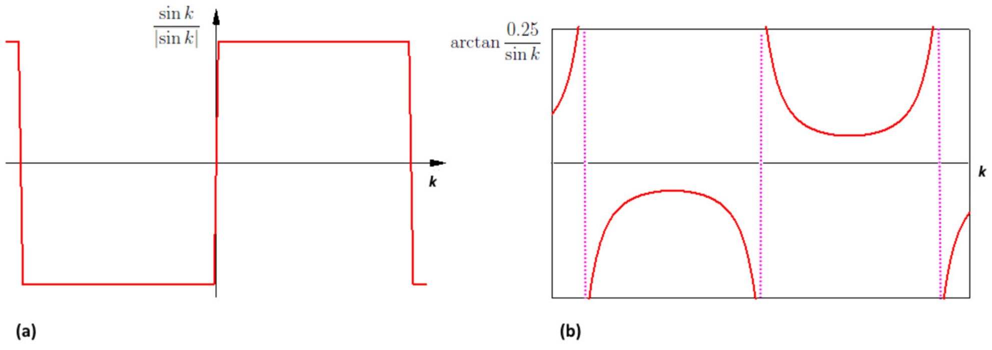

Fig. 7. Variation of factor the phase $\left(d_x + id_y\right)/\sqrt{d_x^2+d_y^2}$ with $d_i = \sin k_i$ for $\sin k_y = 0$ and $\sin k_y \neq 0$. **a**) The plot is given for $\sin k_y = 0$; thus reducing to $\cos\phi = \sin k/|\sin k|$ showing a discontinuity at $\sin k_x = 0$. **b**) $\sin k_y \neq 0$ taken as 0.25 having a discontinuity at $\sin k_x = 0$.

6.3.2. Time reversal symmetry

Under time reversal symmetry, the Hamiltonian for spinless fermions is constrained by $h_f(\mathbf{k})^* = h_f(-\mathbf{k})$. For the two band model with hamiltonian $h_f = \sum d_\mu(\mathbf{k})\sigma^\mu$ the three $d_\mu(\mathbf{k})$ functions are constrained like

$$\begin{array}{lll} d_{x,z}(\mathbf{k})^* & = & d_{x,z}(-\mathbf{k}) \\ d_y(-\mathbf{k}) & = & -d_y(-\mathbf{k}) \end{array} \tag{6.50}$$

We solve these conditions as

$$d_x = M - t\cos k_x, \qquad d_y = t\sin k_y, \qquad d_z = 0 \tag{6.51}$$

leading to

$$h_f = \begin{pmatrix} 0 & d_x - id_y \\ d_x + id_y & 0 \end{pmatrix} \tag{6.52}$$

Its eigenvalues $\varepsilon_\pm$ are given by $\pm\varepsilon = \pm\sqrt{d_x^2+d_y^2}$ with gap $E_g = 2\varepsilon$. The vanishing condition for this gap corresponds to $d_x = d_y = 0$. The normalised eigenvectors $V_\pm$ are given by

$$V_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\phi} \\ 1 \end{pmatrix} \quad , \quad V_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -e^{+i\phi} \end{pmatrix} \tag{6.53}$$

with $e^{-i\phi} = \left(d_x - id_y\right)/\sqrt{d_x^2+d_y^2}$. Using the above eigenvectors, we obtain the unitary matrix $\boldsymbol{V}_{\mathbf{k}}$ diagonalising h_f namely

$$\boldsymbol{V}_{\mathbf{k}} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\phi} & 1 \\ 1 & -e^{+i\phi} \end{pmatrix} \tag{6.54}$$

Putting this expression into $q_{\mathbf{k}} = \boldsymbol{V}_{\mathbf{k}} D_{\mathbf{k}}^{1/2}$, we end up with the coupling matrix

$$q_{\mathbf{k}} = \frac{\sqrt{\varepsilon}}{\sqrt{2}}\begin{pmatrix} e^{-i\phi} & 1 \\ 1 & -e^{+i\phi} \end{pmatrix} \tag{6.55}$$

with $\det q = -\varepsilon$ showing that $q_{\mathbf{k}}$ has a singularity for gapless states. Here also the bosonic h_b is given by $\varepsilon\sigma_0$ and $h_f = \varepsilon\sigma_z$.

7. Conclusion and Discussions

In this paper, we used the orthosymplectic structure of the quantized graded phase space of quantum super oscillators $\hat{c}^i/\hat{c}_i^\dagger$ and $\hat{b}^\alpha/\hat{b}_\alpha^\dagger$ to develop tight binding models for super bands and use this construction to investigate topological phases of supermatter. We distinguished two families of supermatter: supersymmetric and orthosymplectic.

(**1**) the supersymmetric family, termed as SUSY matter; is based on two fermionic charges $Q_{susy}^\pm = Q_1 \pm iQ_2$ constrained by the usual $\mathcal{N}=2$ *supersymmetric* algebra of QM. This is a four dimensional graded Lie algebra generated by the hermitian Q_1, Q_2, interpreted as $\mathcal{N}=2$ supercharge operators, and the bosonic $H_{susy}^{\mathcal{N}=2}$ defining the $\mathcal{N}=2$ SUSY Hamiltonian as well as a bosonic charge operator T_0 generating the SO(2) R-symmetry rotating the two Q_i's. Given $Q_{susy}^\pm$, we have the anticommutator $\{Q_{susy}^+, Q_{susy}^-\}$ allowing to construct $H_{susy}^{\mathcal{N}=2}$ and the $\{Q_{susy}^\pm, Q_{susy}^\pm\}=0$ as well as the $[Q_{susy}^\pm, H_{susy}^{\mathcal{N}=2}]=0$ giving the supersymmetric constraints. The nilpotencies $(Q_{susy}^\pm)^2=0$ lead to finite dimensional supermultiplets, and the commutativities $[Q_{susy}^\pm, H_{susy}^{\mathcal{N}=2}]=0$ give superstates with same energy. Notice that in absence of central charges, this $\mathcal{N}=2$ system is just the union of two isomorphic $\mathcal{N}=1$ SUSY systems; one generated by Q_1 and the other by Q_2. For the $\mathcal{N}=1$ SUSY family generated by Q_1, we have one anticommutator $\{Q_1, Q_1\}$ giving the SUSY Hamiltonian $H_{susy}^{\mathcal{N}=1}$ while the previous supersymmetric constraints get restricted to the commutativity $[Q_1, H_{susy}^{\mathcal{N}=1}]=0$.

(**2**) the orthosymplectic family, termed as ORTIC matter, is based on the *orthosymplectic* osp(2N|2M) Lie superalgebra with $M \geq N \geq 1$. For the particular case $M=N$, the osp(2N|2N) superalgebra is generated by (i) $4N^2$ bosonic operators given by the $N(2N-1)$ generators of so(2N) and the $N(2N+1)$ ones of sp(2N); and (ii) $4N^2$ fermionic operators realised as given by

$$F_i^\alpha = \hat{c}_i^\dagger \hat{b}^\alpha, \qquad \bar{F}_\alpha^i = \hat{b}_\alpha^\dagger \hat{c}^i, \qquad \bar{F}_{\alpha i} = \hat{c}_i^\dagger \hat{b}_\alpha^\dagger, \qquad F^{\alpha i} = \hat{b}^\alpha \hat{c}^i \tag{7.1}$$

Using this orthosymplectic structure, we can construct four hermitian fermionic operators Q_1, Q_2, Q_3, Q_4 that characterise ORTIC matter. These odd operators can be combined like $Q^\pm = Q_1 \pm iQ_2$ and $\tilde{Q}^\pm = Q_3 \pm iQ_4$, and are realised by linear combinations of the fermionic generators (7.1) line $Q^+ = \sum R_\alpha^i F_i^\alpha$ and so on. In terms of the oscillators, we have

$$Q^+ = \hat{c}_i^\dagger R_\alpha^i \hat{b}^\alpha, \qquad Q^- = \hat{b}_\alpha^\dagger \bar{R}_i^\alpha \hat{c}^i, \qquad \tilde{Q}^+ = \hat{c}_i^\dagger T^{i\alpha} \hat{b}_\alpha^\dagger, \qquad \tilde{Q}^- = \hat{b}^\alpha \bar{T}_{\alpha i} \hat{c}^i \tag{7.2}$$

where the R_α^i and $T^{i\alpha}$ are complex coupling tensors. With these four fermionic charges (7.2), we can engineer the basic observables for the tight binding modelling of ORTIC matter as follows.

(**i**) Two ORTIC charges $Q_{ortic}^\pm$ given by the most general linear combination of (7.2); the Q_{ortic}^+ is given by $z_+Q^+ + z_-Q^- + w_+\tilde{Q}^+ + w_-\tilde{Q}^-$ with complex $z_\pm$ and $w_\pm$; and its adjoint conjugate Q_{ortic}^- by $\bar{z}_-Q^+ + \bar{z}_+Q^- + \bar{w}_-\tilde{Q}^+ + \bar{w}_+\tilde{Q}^-$. Using (7.2), the $Q_{ortic}^\pm$ can be also presented like

$$\begin{aligned} Q_{ortic}^+ &= (\hat{c}_i^\dagger, \hat{c}^i) \begin{pmatrix} z_+R_\alpha^i & w_+T^{i\alpha} \\ w_-\bar{T}_{\alpha i} & z_-\bar{R}_i^\alpha \end{pmatrix} \begin{pmatrix} \hat{b}^\alpha \\ \hat{b}_\alpha^\dagger \end{pmatrix} \\ Q_{ortic}^- &= (\hat{b}_\alpha^\dagger, \hat{b}^\alpha) \begin{pmatrix} \bar{z}_+\bar{R}_i^\alpha & w_-T^{i\alpha} \\ \bar{w}_+\bar{T}_{\alpha i} & \bar{z}_-R_\alpha^i \end{pmatrix} \begin{pmatrix} \hat{c}^i \\ \hat{c}_i^\dagger \end{pmatrix} \end{aligned} \tag{7.3}$$

they have been denoted in the main text like $\hat{\lambda}_{\dot{A}}[\boldsymbol{q}_{\mathbf{k}}]_A^{\dot{A}}\hat{\xi}^A$ and $\hat{\xi}_A[\boldsymbol{q}_{\mathbf{k}}^\dagger]_{\dot{A}}^A\hat{\lambda}^{\dot{A}}$; see also (8.66) in appendix B. Notice that a hermitian ORTIC charge Q_{ortic} requires the identification $Q_{ortic}^+ = Q_{ortic}^-$ which is solved by setting $\bar{z}_\mp = z_\pm$ and $\bar{w}_\mp = w_\pm$.

(**ii**) Three anticommutators; the first $\{Q^{+}_{ortic}, Q^{-}_{ortic}\}$ defines the ORTIC hamiltonian H_{ortic}. The $\{Q^{+}_{ortic}, Q^{+}_{ortic}\}$ and its adjoint $\{Q^{-}_{ortic}, Q^{-}_{ortic}\}$ give bosonic charge operators Z^{++}_{ortic} and Z^{--}_{ortic}. The H_{ortic} and Z^{++}_{ortic} and Z^{--}_{ortic} are valued in the bosonic sector of osp(2N|2N). By imposing the constraints

$$[H_{ortic}, Q^{\pm}_{ortic}] = 0 \qquad , \qquad Z^{++}_{ortic} = Z^{--}_{ortic} = 0 \tag{7.4}$$

one obtains the SUSY matter. These conditions are non trivial as they are mapped into constraint relations on the coupling tensor $[\boldsymbol{q}_{\mathbf{k}}]^{\dot{A}}_{A}$ that has been studied in this investigation; see also appendix B for other details.

To deal with the super TBM and the coupling matrix $\boldsymbol{q}_{\mathbf{k}}$ of the supercharge $Q^{\pm}_{ortic}$ (5.12), we revisited properties of the quantum super oscillators in connection with: (**i**) The orthosymplectic structure given by the triplet (Ω, G, J) and described by the supergroup $OSP(2N|2M)$. (**ii**) The construction of models for super topological matter on hypercubic super lattice in relation with the periodic AZ table. For the triplet (Ω, G, J), we found that the three structures are remarkably accommodated in the $OSP(2N|2M)$ supergroup. For application of our study, we have considered two typical topological models from the AZ table; one having charge conjugation (particle-hole) symmetry and the other has time reversal invariance; but the method extends straightforwardly to other discrete symmetries like those of [18,21] and also to spinfull matter. As an important fact of supermatter resulting from this study, is that the topological obstructions is captured by h_f while h_b is some how topologically trivial in agreement with literature results. The general set up of the super TBM has been given in the core of the paper and in the appendices A and B. In the remainder of this discussion section, we want to give additional comments regarding the classification of the symmetries of the supercharge

$$Q^{+}_{\mathbf{k}} = \hat{\lambda}_{\mathbf{k}\dot{C}} \left[(\boldsymbol{q}_{\mathbf{k}})^{\dot{C}}_{A} \right] \hat{\xi}^{A}_{\mathbf{k}}, \qquad Q^{-}_{\mathbf{k}} = \hat{\xi}_{\mathbf{k}A} \left[\left(\boldsymbol{q}^{\dagger}_{\mathbf{k}} \right)^{A}_{\dot{C}} \right] \hat{\lambda}^{\dot{C}}_{\mathbf{k}} \tag{7.5}$$

We show below that the constraint eq. (5.19) giving $(\boldsymbol{q}_{\mathbf{k}})^{\dot{B}}_{A}$ has $2 \times P_{(N)}$ solutions with $P_{(N)}$ being the number of partitions of N. To that purpose, we start by recalling that our super TBM lives on hypercubic super lattice and is based on a supercharge $\boldsymbol{Q}$ with Fourier modes $Q^{\pm}_{\mathbf{k}}$ as in (7.5). From the view of the $\mathcal{O}_{(\mathrm{N,M})}$ observables of eqs. (3.17)-(3.19), this is the simplest fermionic operator living on the Brillouin torus $\mathbb{T}^{d}$ combining bosons $\hat{\xi}^{A}_{\mathbf{k}}$ and fermions $\hat{\lambda}_{\mathbf{k}\dot{C}}$. The $(q_{\mathbf{k}})^{\dot{B}}_{A}$ is a $2N \times 2N$ matrix coupling fermions and bosons; it plays a basic role in the super modelling. It belongs to the bi-fundamental of $Sp(2N) \times SO(2N)$ which is just the even part of the orthosymplectic $osp(2N|2N)$. The novelty of the super TBM is that the square of the supercharge generates a Hamiltonian operator H_{ortic} (resp. H_{susy}) which (i) splits as the sum $H^{ortic}_{f} + H^{ortic}_{b}$ (resp. $H^{susy}_{f} + H^{susy}_{b}$) describing fermionic and bosonic contributions (5.57)-(5.58); and (ii) hides a remarkable cancellation property (5.56) indicating that the effect of bosons in topological supermatter is not as trivial as one might think. The coupling matrices $h_f(\mathbf{k})$ and $h_b(\mathbf{k})$ associated with H_f and H_b are respectively given by $\boldsymbol{q}_{\mathbf{k}} Z \boldsymbol{q}^{\dagger}_{\mathbf{k}}$ and $h_b(\mathbf{k}) = \boldsymbol{q}^{\dagger}_{\mathbf{k}} \boldsymbol{q}_{\mathbf{k}}$ with $Z = \sigma_z \otimes I_N$. While $h_b(\mathbf{k})$ is a positive definite operator, the $h_f(\mathbf{k})$ has an indefinite sign because of the ± 1 eigenvalues of σ_z in agreement with valence and conduction bands. As additional comments on the results obtained in this study, we cite the following:

1) The set of topological supermatter is a subset of ordinary topological matter; but with the presence of bosonic matter described by $h_b(\mathbf{k})$. It is classified by the periodic AZ table with hamiltonian $h_f(\mathbf{k})$ factorised like

$$h_f(\mathbf{k}) = \boldsymbol{q}_{\mathbf{k}} Z \boldsymbol{q}_{\mathbf{k}}^{\dagger} \tag{7.6}$$

For this subset of AZ matter, "gapless states" are given by SUSY (ORTIC) multiplets having massless fermions and massless bosons. These massless super states constraint $h_f(\mathbf{k})$ and $h_b(\mathbf{k})$ to have zero modes; thus requiring $\det h_f(\mathbf{k}) = 0$ and $\det h_b(\mathbf{k}) = 0$. These two requirements are solved by $\det \boldsymbol{q}(\mathbf{k}) = 0$; and so is the condition for massless super states. Another consequence of SUSY (ORTIC) is that super bands come in representation multiplets. For instance, in the case of $\mathcal{N} = 2$ supersymmetry, the SUSY multiplets have four modes: 2 fermionic with energies $\pm\epsilon$ (conduction ane valence) and 2 bosonic positive energy $+\epsilon$. So, $\mathcal{N} = 2$ multiplets in super TBM (1.2), (5.1) have 3 states with positive $+\epsilon$ and one with negative $-\epsilon$. This feature is illustrated by the Fig. 5.

2) Besides TPC discrete symmetries (6.5) and crystalline ones like mirrors, a classification of the super TBM charges is given by global symmetries of $q_A^{\dot{C}}$. As investigated in the main text and as it will be described below, it turns out that the classes of $\boldsymbol{q}_A^{\dot{C}}$ are given by subgroups $Sp(2N) \times SO(2N)$ by help of the splitting (5.24)

$$\begin{aligned} SO(2N) &: S[U(2) \times U(N)] \\ Sp(2N) &: S\left[U(2)' \times SU(N)'\right] \end{aligned} \tag{7.7}$$

The invariance of the supercharge (5.12) requires the condition (5.19). By using (5.24) and (7.7), this condition reads as follows

$$q_{\alpha I}^{\dot{\gamma}\dot{K}} = \left[(U_1)_{\dot{\delta}}^{\dot{\gamma}} (U_2)_{\dot{L}}^{\dot{K}}\right] q_{\beta J}^{\dot{\delta}\dot{L}} \left[(U_1')_{\alpha}^{\beta} (U_2')_{I}^{J}\right] \tag{7.8}$$

where the 2×2 matrices U_1 and U_1' are elements of the $U(2) \times U(2)'$ in (7.7) and the $N \times N$ matrices U_2 and U_2' are elements of $U(N) \times U(N)'$. The condition (7.8) has several solutions classified by subgroups of (7.7). We comment here after on this classification while illustrating the constructions on particular examples.

a) $U(2) \times U(N)$ *symmetry:* The simplest solution of the condition (7.8) is given by the following one parameter family

$$q_{\beta J}^{\dot{\delta}\dot{L}}(\mathbf{k}) = \mu(\mathbf{k})\, \delta_{\beta}^{\dot{\delta}} \delta_{J}^{\dot{L}} \tag{7.9}$$

where the complex $\mu(\mathbf{k})$ lives on $\mathbb{T}^d$. This particular solution has a strong symmetry given by the $U(2) \times U(N)$ group. This invariance is derived as follows: First, substitute the above realisation in the condition; so the right hand side of eq. (7.8) becomes

$$q_{\alpha I}^{\dot{\gamma}\dot{K}} = \mu(\mathbf{k}) \left[(U_1' U_1)_{\alpha}^{\dot{\gamma}}\right] \left[(U_2' U_2)_{I}^{\dot{K}}\right] \tag{7.10}$$

To insure invariance, we must have $U_1' U_1 = I_2$ and $U_2' U_2 = I_N$ which are respectively solved by requiring $U_1' = U_1^{\dagger}$ and $U_2' = U_2^{\dagger}$ generating the group $U(2) \times U(N)$.

b) $U(1)^2 \times U(N)$ *symmetry:* A second solution of the constraint eq. (7.8) is given by

$$q_{\beta J}^{\dot{\delta}\dot{L}}(\mathbf{k}) = \begin{pmatrix} \mu(\mathbf{k})\, \delta_J^{\dot{L}} & 0 \\ 0 & \nu(\mathbf{k})\, \delta_J^{\dot{L}} \end{pmatrix} \tag{7.11}$$

It has two complex parameters $\mu(\mathbf{k})$ and $\nu(\mathbf{k})$ living on $\mathbb{T}^d$. This solution is invariant under $U(1)^2 \times U(N)$ which is a subgroup of $U(2) \times U(N)$. By setting $\mu(\mathbf{k}) = \nu(\mathbf{k})$, we recover the previous class.

c) *$U(N)$ symmetry:* A third solution of eq. (7.8) is given by

$$q_{\beta J}^{\dot{\delta}\dot{L}}(\mathbf{k}) = \begin{pmatrix} \mu(\mathbf{k}) & \rho(\mathbf{k}) \\ \varsigma(\mathbf{k}) & \nu(\mathbf{k}) \end{pmatrix} \otimes \delta_J^{\dot{L}} \tag{7.12}$$

It has four complex parameters $\mu(\mathbf{k})$, $\nu(\mathbf{k})$, $\rho(\mathbf{k})$ and $\varsigma(\mathbf{k})$ living on $\mathbb{T}^d$. This solution is invariant under $U(N)$ which is a subgroup of $U(2) \times U(N)$.

Following this method, one can construct several solutions of eq. (7.8); they are classified by the subgroups of $U(2) \times U(N)$; the classes follow from the factorisation

$$q_{\beta J}^{\dot{\delta}\dot{L}}(\mathbf{k}) = \mathcal{A}_{\beta}^{\dot{\delta}}(\mathbf{k}) \times \mathcal{B}_J^{\dot{L}}(\mathbf{k}) \tag{7.13}$$

where in general $\mathcal{A}_{\beta}^{\dot{\delta}}(\mathbf{k})$ has four complex functions and $\mathcal{B}_J^{\dot{L}}(\mathbf{k})$ has N^2 complex functions. In this regard, we recall that unitary subgroups of $U(N)$ are given by

$$\prod_{i=1}^{l} U(n_i) \qquad , \qquad \sum_{i=1}^{l} n_i = N \tag{7.14}$$

including the maximal abelian $U(1)^N$ corresponding to the choice $\mathcal{B}_J^{\dot{L}} = \mathcal{B}_J \delta_J^{\dot{L}}$ where $\mathcal{B}_J$ are N complex functions. Notice that the partition $P(2) = 2$ because there are two ways to decompose the integer 2: either just as 2 of like $1 + 1$. Notice also that the number of possibilities of decomposing a positive integer like $\sum_{i=1}^{l} n_i = N$ is given by the partition $P(N)$. This number is obtained by expanding the Mac-Mahon partition function in a series as follows [52],

$$\prod_{l=1}^{\infty} \left(1 - X^l\right)^{-1} = \sum_{n=1}^{\infty} P_{(n)} X^n \tag{7.15}$$

So the number of classes of is given by $P_{(2)} \times P_{(N)} = 2P_{(N)}$.

8. Two appendices

In this section, we give two appendices A and B where we collect useful ingredients and complete some results given in the main text. In appendix A, we give useful tools on (1) the supersymmetric algebras in 2D and 1D having n supersymmetric charges ($n \leq 4$); (2) their embedding in the orthosymplectic osp(2|2) and (3) the embedding of osp(2|2) in the $\mathcal{N} = 2$ super conformal invariance in the Neveu-Schwaz (NS) sector. In appendix B, we revisit basic algebraic aspects of the orthosymplectic osp(2N|2M) underlying results obtained in this paper and also in the study of the supersymmetric fivefold ways of [20]. We also complete partial results given in section 4 and subsection 6.2 of our investigation.

8.1. Appendix A: fermions' algebra in 2D and 1D

We begin by recalling useful features on fermions in 2D world sheet and in 1D world line. The two worlds are somehow related due to the abelian property of the SO(2) rotation group of the real plane $\mathbb{R}^2$ (SO(1,1) for $\mathbb{R}^{1,1}$). Indeed 2D fermions are described by $SO(2)$ spinors $\boldsymbol{\psi}_\alpha$ having two components $(\psi_{+1/2}, \psi_{-1/2})$. They can be either hermitian with $\psi_{\pm 1/2}$ real (Majorana) or complex (Dirac). Because of the abelian property of $SO(2)$, these fermionic components $\psi_{\pm 1/2}$ can be handled separately (as Weyl or Majorana-Weyl spinors). So; one real fermion $\boldsymbol{\psi}_\alpha$ in 2D may be imagined in terms of two real 1D fermions λ_1 and λ_2 respectively associated with $\psi_{+1/2}$ and $\psi_{-1/2}$. A complex $\boldsymbol{\psi}_\alpha$ in 2D splitting as $\boldsymbol{\xi}_\alpha + i\boldsymbol{\chi}_\alpha$ can thought of in terms of four real fermions in 1D denoted as λ_1, λ_2, λ_3, λ_4 and respectively given by $\xi_{+1/2}$, $\xi_{-1/2}$, $\chi_{+1/2}$, $\chi_{-1/2}$. In sum, we have the following 2D/1D correspondence

$$n \text{ fermions } \boldsymbol{\psi}^i_\alpha \text{ in 2D} \qquad \rightarrow \qquad 2n \text{ fermions } \lambda_A \text{ in 2D} \tag{8.1}$$

This correspondence applies also to the conserved supersymmetric charges in 2D and 1D (super QM). In 2D $\mathcal{N} = 1$ supersymmetric theory, the fermionic generator Q_α is a Majorana spinor; it has two conserved super charges: a left supercharge $Q_{-1/2}$ and a right one $Q_{+1/2}$ that can realised independently due to the reducibility of SO(2) representations. This 2D $\mathcal{N} = 1$ theory can be put in correspondence with $\mathcal{N} = 2$ supersymmetry in 1D. So, by using (8.1), we have the following dictionary:

(i) $\mathcal{N} = 1$ supersymmetry in 2D generated by a Majorana spinor operator $Q_\alpha = Q_{\pm 1/2}$ is generally termed as 2D $\mathcal{N} = (1, 1)$. In this case, we have two real supersymmetric charges: $Q_{+1/2}$ and $Q_{-1/2}$. From the 1D view, the super QM has two supercharges given by Q_1 and Q_2.

(ii) $\mathcal{N} = 2$ supersymmetry in 2D has two Majorana supercharges Q^1_α, Q^2_α often denoted like $Q^{\pm}_{\pm 1/2}$ with the upper $\pm$ charges referring to an extra $SO(2)_R$ symmetry rotating Q^1_α and Q^2_α. This structure is generally termed as 2D $\mathcal{N} = (2, 2)$. Here, we have four real supersymmetric charges given by the following Majorana-Weyl fermions,

$$Q^1_{+1/2}, \qquad Q^1_{-1/2}, \qquad Q^2_{+1/2}, \qquad Q^2_{-1/2} \tag{8.2}$$

From the 1D view, the associated theory has four supercharges Q_1, Q_2, Q_3 and Q_4.

(iii) In the case of $n + m$ Majorana-Weyl spinors; we use the terminology $\mathcal{N} = (n, m)$ supersymmetry having n right charges $Q^1_{+1/2}, ...Q^n_{+1/2}$; and m left charges $Q^1_{-1/2}, ...Q^m_{-1/2}$. In this case, we have $n + m$ real supersymmetric charges. So, for $n \neq m$, the some how apparent classical left-right symmetry is violated.

We end this intro hat by noticing that our interest in going to 1D $\mathcal{N} = 2$ super QM through 2D theory is motivated by its embedding in osp(2|2).

8.1.1. Supersymmetry in 2D and 1D

Focusing on the interesting case of a supersymmetric theory with two real supercharges Q_1 and Q_2 combined into a complex $Q = Q_1 + iQ_2$ and its adjoint $\bar{Q} = Q_1 - iQ_2$, the underlying Lie superalgebra is, generally speaking, defined by

$$\begin{array}{lll} QQ^\dagger + Q^\dagger Q & = & 2P \\ \{Q, Q\} & = & 2Z \\ \left\{Q^\dagger, Q^\dagger\right\} & = & 2Z^\dagger \end{array} \tag{8.3}$$

In these relations, the P is a bosonic hermitian operator (the Hamiltonian in $\mathcal{N} = 2$ super QM) and the complex Z is the central charge of the superalgebra. They obey the commutations

$$\begin{aligned} &[P, Q] = \left[P, Q^\dagger\right] = 0 \\ &[Z, Q] = \left[Z, Q^\dagger\right] = 0 \\ &[P, Z] = 0 \end{aligned} \tag{8.4}$$

As far as this 1D $\mathcal{N} = 2$ superalgebra is concerned, notice that the four following features:

(**1**) the central charge Z scales as energy, the same as P; it plays an important role in the study of

BPS states. These states are beyond the scope of the present paper; and so we will disregard Z here. So, the relation $\{Q, Q\} = Z$ becomes a nilpotency condition of the fermionic charge; that is

$$Q^2 = \bar{Q}^2 = 0 \tag{8.5}$$

(**2**) The highest weight representations $\mathcal{R}_{susy}$ of the superalgebra (8.3)-(8.4) are two complex dimensional (even complex dimensional in general) due to $Q^2 = 0$; they contain as complex bosonic state degrees $|b\rangle$ as complex fermionic ones $|f\rangle$; and they have the same energy ε. For the example of the 1D $\mathcal{N} = 2$ supersymmetric scalar representation with ground state as

$$Q\,|b\rangle = 0 \qquad , \qquad P\,|b\rangle = \varepsilon\,|b\rangle \tag{8.6}$$

the fermionic partner state is given by $|f\rangle = \bar{Q}\,|b\rangle$.
(**3**) The energy of this state is determined by computing $P\,|f\rangle = P\,\bar{Q}\,|b\rangle$; it equals to $\varepsilon\,|f\rangle$; thanks to the commutativity $P\,\bar{Q} = \bar{Q}\,P$. The feature $Q^2 = \bar{Q}^2 = 0$ and $\left[P, \bar{Q}\right] = 0$ are present in supersymmetric algebra (8.3)-(8.4); but are violated for osp(2|2) as its supercharges Q_{ortic} have the typical property

$$[P_{ortic}, Q_{ortic}] \neq 0 \tag{8.7}$$

This orthosymplectic feature will be described in the next sub-subsection.
(**4**) In the case where the supercharge is hermitian $Q^\dagger = Q$, the eqs. (8.3)-(8.4) reduce to

$$Q^2 = 2P \qquad , \qquad [P, Q] = 0 \tag{8.8}$$

They define the 1D $\mathcal{N} = 1$ supersymmetric algebra underlying the $\mathcal{N} = 1$ super QM.

8.1.2. The orthosymplectic osp(2|2)

The osp(2|2) is eight dimensional; it has a bosonic sector $\boldsymbol{g}_{\bar{0}}$ given by $so(2, \mathbb{R}) \oplus sp(2, \mathbb{R})$; and a fermionic sector $\boldsymbol{g}_{\bar{1}}$ given by a 4-dimensional module of $\boldsymbol{g}_{\bar{0}}$. Their graded commutations are as follows:

*I. **The bosonic sector** $so(2) \oplus sp(2)$:* The even part $\boldsymbol{g}_{\bar{0}}$ has four bosonic generators; the hermitian J_0 generating the abelian $so(2)$; and the three $S_{0,\pm}$ generating the $sp(2)$ obeying $S_0^\dagger = S_0$ and $S_\pm^\dagger = S_\mp$. The commutation relations defining $\boldsymbol{g}_{\bar{0}}$ are given by

$$\begin{array}{lcl} \left[J_0, S_{0,\pm}\right] & = & 0 \\ \left[S_0, S_\pm\right] & = & \pm S_\pm \\ \left[S_-, S_+\right] & = & 2S_0 \end{array} \tag{8.9}$$

As far as these relations are concerned, notice the following interesting features.
(**1**) the two hermitian J_0 and S_0 are the commuting generators of $so(2) \oplus sp(2)$; they can be diagonalised simultaneously in the same basis. Their quantum charges (q, p) label the irreducible representations $\mathcal{R}_{(q,p)}^{ortic}$ of osp(2|2). The quantum number $p = s_z$, it is the usual spin projection ranging as $-s \leq s_z \leq s$.
(**2**) The two J_0 and S_0 appear in the graded commutations of the osp(2|2) superalgebra through the linear combinations $S_0 \pm J_0$; see eq. (8.14). They can be imagined as candidates for the H_{susy} of eq. (8.3) and a charge operator generating a $\mathrm{U}(1)_R$ symmetry; see below for further details. In terms of the supersymmetric oscillators $\hat{b}/\hat{b}^\dagger$ and $\hat{c}/\hat{c}^\dagger$, we have the following oscillator realisation of the bosonic operators J_0, S_0 and $S_\pm$,

$$J_0 = \frac{1}{4}\left(\hat{c}^\dagger \hat{c} - \hat{c}\hat{c}^\dagger\right) \tag{8.10}$$

$$S_0 = \frac{1}{4}\left(\hat{b}^\dagger \hat{b} + \hat{b}\hat{b}^\dagger\right), \qquad S_- = \frac{1}{2}\hat{b}\hat{b}, \qquad S_+ = \frac{1}{2}\hat{b}^\dagger \hat{b}^\dagger \tag{8.11}$$

From this realisation, we can check that we have

$$\begin{array}{lllllll} \left[J_0, \hat{c}\right] & = & -\frac{1}{2}\hat{c} & , & \left[J_0, \hat{c}^\dagger\right] & = & \frac{1}{2}\hat{c}^\dagger \\ \left[S_0, \hat{b}\right] & = & -\frac{1}{2}\hat{b} & , & \left[S_0, \hat{b}^\dagger\right] & = & \frac{1}{2}\hat{b}^\dagger \end{array} \tag{8.12}$$

indicating that (**i**) the fermionic $\hat{c}/\hat{c}^\dagger$ carry a half charge of $so(2)$ ($q = \pm 1/2$) but no charge of S_0 ($p = 0$); (**ii**) the bosonic $\hat{b}/\hat{b}^\dagger$ carry a half charge of $sp(2)$ ($p = \pm 1/2$) but no $so(2)$ charge. Notice that in the $\mathcal{R}_{(q,p)}$ representation language, we can think of $\hat{c}/\hat{c}^\dagger$ in terms of an so(2) doublet $\hat{c}^q = (\hat{c}, \hat{c}^\dagger)$ and about $\hat{b}/\hat{b}^\dagger$ in terms of an sp(2) doublet $\hat{b}^p = (\hat{b}, \hat{b}^\dagger)$.

*II. **The fermionic sector:*** The four fermionic operators generating $\boldsymbol{g}_{\bar{1}}$ are denoted like $\boldsymbol{F}_p^q$ with $q = \pm$ (short of $\pm 1/2$) labelling the charges of $so(2)$ and $p = \pm$ indexing the charges of $sp(2)$. They behave as doublets under $so(2)$ and under $sp(2)$. These $\boldsymbol{F}_p^q$'s are realised in terms of the super oscillators as follows

$$\begin{array}{lll} \boldsymbol{F}_-^- = \hat{b}\hat{c} & , & \boldsymbol{F}_+^+ = \hat{b}^\dagger \hat{c}^\dagger \\ \boldsymbol{F}_-^+ = \hat{b}\hat{c}^\dagger & , & \boldsymbol{F}_+^- = \hat{b}^\dagger \hat{c} \end{array} \tag{8.13}$$

with (**i**) the adjoint conjugations $\left(\boldsymbol{F}_-^-\right)^\dagger = \boldsymbol{F}_+^+$ and $\left(\boldsymbol{F}_-^+\right)^\dagger = \boldsymbol{F}_+^-$; and (**ii**) the nilpotency $\left(\boldsymbol{F}_p^q\right)^2 = 0$. By using the notation $\hat{c}^q$ and $\hat{b}_p$, the above fermionic generators can be presented collectively like $F_p^q = \hat{c}^q \hat{b}_p$. The graded commutation relations between these fermionic operators are given by

$$\begin{array}{lllllll} \left\{\boldsymbol{F}_-^-, \boldsymbol{F}_+^+\right\} & = & 2S_0 - 2J_0 & & \left\{\boldsymbol{F}_-^+, \boldsymbol{F}_+^-\right\} & = & 2S_0 + 2J_0 \\ \left\{\boldsymbol{F}_-^-, \boldsymbol{F}_-^+\right\} & = & 2S_- & , & \left\{\boldsymbol{F}_+^-, \boldsymbol{F}_+^+\right\} & = & 2S_+ \\ \left\{\boldsymbol{F}_-^-, \boldsymbol{F}_+^-\right\} & = & 0 & & \left\{\boldsymbol{F}_-^+, \boldsymbol{F}_+^+\right\} & = & 0 \end{array} \tag{8.14}$$

and

$$\begin{array}{lllllll} \left[J_0, \boldsymbol{F}_\pm^-\right] & = & -\frac{1}{2}\boldsymbol{F}_\pm^- & , & \left[S_0, \boldsymbol{F}_-^\pm\right] & = & -\frac{1}{2}\boldsymbol{F}_-^\pm \\ \left[J_0, \boldsymbol{F}_\pm^+\right] & = & +\frac{1}{2}\boldsymbol{F}_\pm^+ & & \left[S_0, \boldsymbol{F}_+^\pm\right] & = & +\frac{1}{2}\boldsymbol{F}_+^\pm \end{array} \tag{8.15}$$

as well as

$$\begin{array}{lllllll} \left[S_-, \boldsymbol{F}_-^\pm\right] & = & 0 & , & \left[S_+, \boldsymbol{F}_-^\pm\right] & = & -\boldsymbol{F}_+^\pm \\ \left[S_+, \boldsymbol{F}_+^\pm\right] & = & 0 & & \left[S_-, \boldsymbol{F}_+^\pm\right] & = & +\boldsymbol{F}_-^+ \end{array} \tag{8.16}$$

The quadratic Casimir C_2 commuting with the generators of osp(2|2) and characterising the osp(2|2) representations is given by [53–56]

$$C_2 = S_0\,(2S_0 + 1) - J_0\,(2J_0 + 1) + 2S_+S_- + 2\boldsymbol{F}_-^-\boldsymbol{F}_-^+ - 2\boldsymbol{F}_+^+\boldsymbol{F}_+^- \tag{8.17}$$

8.1.3. Embedding $\mathcal{N} = 2$ SUSY in osp(2|2) and in $\mathcal{N} = 2$ CFT$_2$

From the above graded commutation relations of osp(2|2), we can deduce a set of interesting properties; those useful features for us are as listed below:

*I. **Two complex generators $\boldsymbol{F}_{-}^{-}$ and $\boldsymbol{F}_{+}^{-}$:*** The osp(2|2) has two complex supercharges $\boldsymbol{F}_{-}^{-}$ and $\boldsymbol{F}_{+}^{-}$ (four real ones F_1, F_2, F_3, F_4). Using eqs. (8.12)-(8.16), we draw the following relations

$$\begin{array}{lll} \left[2S_0 - 2J_0, \boldsymbol{F}_p^q\right] & = & (p-q)\, \boldsymbol{F}_p^q \\ \left[2S_0 + 2J_0, \boldsymbol{F}_p^q\right] & = & (p+q)\, \boldsymbol{F}_p^q \end{array} \tag{8.18}$$

So for $p = +q$, we have the following

$$\begin{array}{lll} \left[2S_0 - 2J_0, \boldsymbol{F}_{-}^{-}\right] & = & 0 \\ \left[2S_0 - 2J_0, \boldsymbol{F}_{+}^{+}\right] & = & 0 \end{array} \quad , \qquad \begin{array}{lll} \left[2S_0 - 2J_0, \boldsymbol{F}_{+}^{-}\right] & = & +2\boldsymbol{F}_{+}^{-} \\ \left[2S_0 - 2J_0, \boldsymbol{F}_{-}^{+}\right] & = & -2\boldsymbol{F}_{-}^{+} \end{array} \tag{8.19}$$

and for $p = -q$, we have

$$\begin{array}{lll} \left[2S_0 + 2J_0, \boldsymbol{F}_{-}^{-}\right] & = & -2\boldsymbol{F}_{-}^{-} \\ \left[2S_0 + 2J_0, \boldsymbol{F}_{+}^{+}\right] & = & +2\boldsymbol{F}_{+}^{+} \end{array} \quad , \qquad \begin{array}{lll} \left[2S_0 + 2J_0, \boldsymbol{F}_{+}^{-}\right] & = & 0 \\ \left[2S_0 + 2J_0, \boldsymbol{F}_{-}^{+}\right] & = & 0 \end{array} \tag{8.20}$$

*II. $\mathcal{N} = 2$ **SUSY as a subalgebra of osp(2|2):*** If restricting the four bosonic generators of osp(2|2) down to the combination $P = \omega\,(S_0 + J_0)$ and $T = \upsilon\,(S_0 - J_0)$ with no $S_\pm$; and the four fermionic ones down to the two $\boldsymbol{F}_{-}^{+} \sim Q/\sqrt{\omega}$ and $\boldsymbol{F}_{+}^{-} \sim \bar{Q}/\sqrt{\omega}$ with no $\boldsymbol{F}_{+}^{+}$ nor $\boldsymbol{F}_{-}^{-}$, the above graded commutations reduce to

$$\begin{array}{lll} \left\{Q, \bar{Q}\right\} & = & P \\ \left[P, Q\right] & = & 0 \end{array} \quad , \qquad \begin{array}{lll} \left\{Q, Q\right\} & = & 0 \\ \left\{\bar{Q}, \bar{Q}\right\} & = & 0 \end{array} \tag{8.21}$$

These relations should be compared with eqs. (8.3); they define a supersymmetric algebra with two supercharges Q and $\bar{Q}$. Notice the two following: (**i**) in terms of the harmonic oscillators, the supercharges are realised as $Q = \sqrt{\omega}\hat{c}^{\dagger}\hat{b}$ and $\bar{Q} = \sqrt{\omega}\hat{b}^{\dagger}\hat{c}$ while the bosonic operator P is given by

$$P = \omega(\hat{b}^{\dagger}\hat{b} + \hat{c}^{\dagger}\hat{c}) \tag{8.22}$$

(**ii**) Because of supersymmetry, the $\mathcal{N} = 2$ superalgebra has another bosonic charge namely the charge operator which we denoted as T. It generates the $U(1)_R$ symmetry of the $\mathcal{N} = 2$ $U(1)$ superalgebra and it acts as $[T, Q] = Q$. Its oscillator realisation is given by $(\hat{c}^{\dagger}\hat{c} - \hat{b}^{\dagger}\hat{b})/2$.

*III. **OSp(2|2) as subinvariance of** $\mathcal{N} = 2$ **CFT$_2$:*** From the view of the 2D $\mathcal{N} = 2$ superconformal field theory (CFT$_2$), the osp(2|2) is a subalgebra of the $\mathcal{N} = 2$ super CFT$_2$ algebra in the NS sector. In terms of the generators of the $\mathcal{N} = 2$ super Virasoro algebra [54] given by: (**i**) the usual bosonic L_n and J_n with n integer, respectively referring the Virasoro and the $U(1)$ Kac-Moody generators, and (**ii**) the fermionic partners $G_{n+1/2}^{+}$ and $G_{n+1/2}^{-}$, we have

$$\begin{array}{ll} L_0 = -S_0 \quad , & G_{\pm 1/2}^{-} = F_{\pm}^{-} \\ L_{\pm} = \pm S_{\pm} \quad , & G_{\pm 1/2}^{+} = F_{\pm}^{+} \end{array} \tag{8.23}$$

Recall that the osp(2|2) Lie superalgebra corresponds just to the anomaly free sub- superalgebra of the $\mathcal{N} = 2$ super Virasoro in NS sector defined as

$$
\begin{array}{lll}
[L_m, L_n] & = & (m-n)\, L_{m+n} + \frac{c}{12}\left(m^3-m\right)\delta_{m+n} \\
[J_m, J_n] & = & \frac{c}{3} m \delta_{m+n} \\
\left\{G_r^+, G_s^-\right\} & = & 2L_{r+s} + (r-s)\, J_{r+s} + \frac{c}{12}\left(4r^2-1\right)\delta_{r+s} \\
\left\{G_r^\pm, G_s^\pm\right\} & = & 0 \\
\left[L_m, G_r^\pm\right] & = & \left(\frac{m}{2}-r\right) G_{m+r}^\pm \\
\left[J_m, G_r^\pm\right] & = & \pm G_{m+r}^\pm
\end{array}
\tag{8.24}
$$

with $r, s \in \mathbb{Z}+1/2$. The anomalous terms $\frac{c}{12}\left(m^3-m\right)\delta_{m+n}$ and $\frac{c}{12}\left(4r^2-1\right)\delta_{r+s}$ disappear for $m = 0, \pm 1$ and $r = \pm 1/2$. This anomaly free condition reduces the infinite set $L_m, J_m, G_r^\pm$ down to the four bosonic $L_0, L_\pm, J_0$ and the four fermionic $G^+_{\pm 1/2}, G^-_{\pm 1/2}$ as in (8.23).

8.2. Appendix B: From osp(2N|2M) to SUSY

In this appendix, we use the tools developed in the present study to shed some light on the graded algebraic structure underlying the section 4 and subsection 6.2 as well as the supersymmetric fivefold ways of [20].

8.2.1. General on oscillator realisation of fermionic charges

For simplicity of the presentation, we use short cuts to reach some results of this construction while keeping quite similar notations as in [20]. For the details regarding other interesting results in particular the intrinsic aspects of the fivefold way classes and the physical applications, we report the reader to the above mentioned reference. To make an idea on the types of fermionic charges we will consider in this appendix, we anticipate this construction by noticing that we will consider three families (I, II, III) of fermionic charges as summarised in the following table

families	supercharges	realisations	couplings	number of Q's
I	Q^+	$\hat{c}_i^\dagger R_\alpha^i \hat{b}^\alpha$	R_α^i	1 complex
II	Q_1^+	$\hat{c}_i^\dagger R_\alpha^i \hat{b}^\alpha$	R_α^i	2 complex
	Q_2^+	$\hat{c}_i^\dagger \hat{b}_\alpha^\dagger T^{\alpha i}$	$T^{\alpha i}$	
III	Q_1^+	$\hat{c}_i^\dagger R_\alpha^i \hat{b}^\alpha$	R_α^i	4 complex
	Q_2^+	$\hat{c}_i^\dagger \hat{b}_\alpha^\dagger T^{\alpha i}$	$T^{\alpha i}$	
	Q_3^-	$S_{\alpha i} \hat{c}^i \hat{b}^\alpha$	$S_{\alpha i}$	
	Q_4^-	$\hat{b}_\alpha^\dagger W_i^\alpha \hat{c}^i$	W_i^α	

(8.25)

From these supercharges one can construct several anticommutators (observables). For example, from the Q^+ in the first row of the above table (family I) and its adjoint $(Q^+)^\dagger = Q^-$, one can build $\{Q^+, Q^-\} = H$ and $\{Q^+, Q^+\} = Z^{++}$ as well as $\{Q^-, Q^-\} = Z^{--}$. From the Q_1^+, Q_2^+ of the family II and their adjoints Q_1^-, Q_2^-, we can build 10 anticommutators. Special supersymmetric Hamiltonians H_{susy} using fermionic charges as in the two first rows of this table (families I and II) were considered in [20]. General ones will be given here.

8.2.2. Fermionic charge operators: examples

Following [20], the fermionic charge operator Q can be realised in terms of tensor products of N+M supersymmetric quantum oscillators with the following features; see also subsection 4.2 of our present study.

- N free fermionic quantum oscillators $\hat{c}^i$ and their adjoint $\hat{c}_i^\dagger$ labelled by $i = 1, ..., N$ and obeying the usual anticommutation relations. These $\hat{c}^i$ and $\hat{c}_i^\dagger$ transform in the fundamental representations of $U(N)$. This symmetry group $U(N)$ is the maximal unitary part of the SO(2N) orthogonal group rotating the underlying 2N Majorana fermions $\hat{\gamma}_{2l-1}$ and $\hat{\gamma}_{2l}$ making $\hat{c}^l = (\hat{\gamma}_{2l-1} + i\hat{\gamma}_{2l-1})/2$ and $\hat{c}_l^\dagger = (\hat{\gamma}_{2l-1} + i\hat{\gamma}_{2l-1})/2$. The N^2 generators of $U(N)$ are given by
$$\mathcal{O}_i^j = \frac{1}{4}\left(\hat{c}_i^\dagger \hat{c}^j - \hat{c}^j \hat{c}_i^\dagger\right) \tag{8.26}$$
including the commuting Cartan charge operators as $J_i = \mathcal{O}_i^i$ namely
$$J_i = \frac{1}{4}\left(\hat{c}_i^\dagger \hat{c}^i - \hat{c}^i \hat{c}_i^\dagger\right), \qquad i = 1, ..., N \tag{8.27}$$
From these abelian J_i's; the previous J_0 corresponds to $\sum_i J_i$. We also have $\mathcal{O}^{[ij]} = \hat{c}^i \hat{c}^j/2$ and $\mathcal{O}_{[ij]} = \hat{c}_i^\dagger \hat{c}_j^\dagger/2$ transforming under the antisymmetric representations of $U(N)$. In the tight binding modelling, the above fermionic oscillator operators are fibred over the Brillouin zone as $\hat{c}_{\mathbf{k}}^i$ and $\hat{c}_{\mathbf{k}i}^\dagger$ with momentum $\mathbf{k}$.
- M free bosonic operators $\hat{b}^\alpha$ and their adjoint $\hat{b}_\alpha^\dagger$ with label $i = 1, ..., M$. The $\hat{b}^\alpha$ and $\hat{b}_\alpha^\dagger$ transform in the fundamental representations of $U(M)$. This unitary $U(M)$ is the maximal unitary subgroup within the usual $Sp(2M)$ phase space symplectic symmetry. The M^2 generators of $U(M)$ are given by
$$\mathcal{S}_\alpha^\beta = \frac{1}{4}\left(\hat{b}_\alpha^\dagger \hat{b}^\beta + \hat{b}^\beta \hat{b}_\alpha^\dagger\right) \tag{8.28}$$
including the commuting Cartan charge operators
$$S_\alpha = \frac{1}{4}\left(\hat{b}_\alpha^\dagger \hat{b}^\alpha + \hat{b}^\beta \hat{b}_\alpha^\dagger\right), \qquad \alpha = 1, ..., M \tag{8.29}$$
with $S_0 = \sum_\alpha S_\alpha$. We also have $\mathcal{S}^{(\alpha\beta)} = \hat{b}^\alpha \hat{b}^\beta/2$ and $\mathcal{S}_{(\alpha\beta)} = \hat{b}_\alpha^\dagger \hat{b}_\beta^\dagger/2$ transforming under the symmetric representations of $U(N)$.
Here also the tight binding model operators are fibred over the Brillouin zone as $\hat{b}_{\mathbf{k}}^\alpha$ and $\hat{b}_{\mathbf{k}\alpha}^\dagger$.

I. Oscillator realisation of Q_{susy}: In terms of the fermionic and bosonic quantum oscillators, a particular realisation of the supersymmetric charge Q reads as follows
$$Q = \sum_{i,\alpha} \hat{c}_i^\dagger R_\alpha^i \hat{b}^\alpha, \qquad Q^\dagger = \sum_{j,\beta} \hat{b}_\beta^\dagger \bar{R}_j^\beta \hat{c}^j \tag{8.30}$$
with coupling tensors given by the $N \times M$ rectangular matrix R_α^i and its adjoint conjugate $\left(R^\dagger\right)_j^\beta \equiv \bar{R}_j^\beta$. This R_α^i carries NM complex degrees of freedom. A far this complex supercharge is concerned, notice the following features.

(1) fermionic generators of osp(2N|2M)

The Lie superalgebras has 4NM fermionic generators ($F^{i\alpha}$, $\bar{F}_{i\alpha}$, G_i^α, $\bar{G}_\alpha^i$); they extend the ones given by (8.13) and are realised in terms of the fermionic $\hat{c}_i^\dagger/\hat{c}^i$ and bosonic $\hat{b}_\alpha^\dagger/\hat{b}^\alpha$ operators as follows

$$\begin{array}{lll} \boldsymbol{F}_{i\alpha} = \hat{c}_i^\dagger \hat{b}_\alpha^\dagger & , & \bar{\boldsymbol{F}}^{\alpha i} = \hat{b}^\alpha \hat{c}^i \\ \boldsymbol{G}_i^\alpha = \hat{c}_i^\dagger \hat{b}^\alpha & , & \bar{\boldsymbol{G}}_\alpha^i = \hat{b}_\alpha^\dagger \hat{c}^i \end{array} \tag{8.31}$$

obeying amongst others the following anticommutations

$$\begin{array}{lllllll} \left\{\boldsymbol{F}^{i\alpha}, \bar{\boldsymbol{F}}_{j\beta}\right\} & = & 2\delta_j^i \mathcal{S}_\alpha^\beta - 2\delta_\beta^\alpha \mathcal{O}_i^j & , & \left\{\boldsymbol{F}^{i\alpha}, \boldsymbol{F}^{j\beta}\right\} & = & 0 \\ \left\{\boldsymbol{G}_i^\alpha, \bar{\boldsymbol{G}}_\beta^j\right\} & = & 2\delta_j^i \mathcal{S}_\alpha^\beta + 2\delta_\beta^\alpha \mathcal{O}_i^j & & \left\{\boldsymbol{G}_i^\alpha, \boldsymbol{G}_j^\beta\right\} & = & 0 \end{array} \tag{8.32}$$

and

$$\begin{array}{lllllll} \left\{\bar{\boldsymbol{F}}^{\alpha i}, \boldsymbol{G}_j^\beta\right\} & = & 2\delta_j^i \mathcal{S}^{(\alpha\beta)} & , & \{\boldsymbol{F}_{i\alpha}, \boldsymbol{G}_j^\beta\} & = & 2\delta_\beta^\alpha \overline{\mathcal{O}}_{[ij]} \\ \left\{\bar{\boldsymbol{F}}^{\alpha i}, \bar{\boldsymbol{G}}_\beta^j\right\} & = & 2\delta_\beta^\alpha \mathcal{O}^{[ij]} & , & \{\boldsymbol{F}_{i\alpha}, \bar{\boldsymbol{G}}_\beta^j\} & = & 2\delta_i^j \bar{\mathcal{S}}_{(\alpha\beta)} \end{array} \tag{8.33}$$

In terms of these fermionic operators, we learn that the supercharge (8.30) and its adjoint conjugate $Q^\dagger$ are given by the linear combinations

$$Q = \sum_{i,\alpha} R_\alpha^i \boldsymbol{G}_i^\alpha \qquad , \qquad Q^\dagger = \sum_{i,\alpha} \bar{R}_i^\alpha \bar{\boldsymbol{G}}_\alpha^i \tag{8.34}$$

These are complex fermionic charges carrying charges $q = \pm$ of the orthogonal J_0 and charges $p = \pm$ under the symplectic S_0; they correspond to the Q and its adjoint $\bar{Q}$ in eqs. (8.3)-(8.4). Notice that Q is valued in the bifundamental $\left(\bar{N}, M\right)$ of the group $U(N) \times U(M)$; this feature will have important consequences on supersymmetry.

(2) the superalgebra of Q and $Q^\dagger$ of (8.34)

Here we show that eqs. (8.34) generate a supersymmetric algebra with two fermionic charges Q and $\bar{Q}$. To that purpose, we first check that $Q^2 = 0$; then we calculate the anticommutator $QQ^\dagger + Q^\dagger Q = H_{susy}$ and after that we verify that we have $\left[H_{susy}, Q\right] = 0$.

a) Calculating Q^2

By substituting (8.34) into Q^2; we find that it reads as $\mathcal{O}_{[ij]} T_{\alpha\beta}^{ij} \mathcal{S}^{(\alpha\beta)}$ with complex coupling $T_{\alpha\beta}^{ij}$ quadratic into R_α^i namely

$$T_{\alpha\beta}^{ij} = 4 R_\alpha^i R_\beta^j \tag{8.35}$$

and $\mathcal{O}_{[ij]} = \hat{c}_i^\dagger \hat{c}_j^\dagger/2$ as well as $\mathcal{S}^{(\alpha\beta)} = \hat{b}^\alpha \hat{b}^\beta/2$. Clearly the contraction of this tensor $T_{\alpha\beta}^{ij}$ vanishes identically due to $T_{\alpha\beta}^{ij} = T_{\beta\alpha}^{ji}$ and because of the properties $\mathcal{O}_{[ij]} = -\mathcal{O}_{[ji]}$ and $\mathcal{S}^{(\alpha\beta)} = \mathcal{S}^{(\beta\alpha)}$. This nilpotency feature can be explicitly exhibited. First, by using the symmetry of $\mathcal{S}^{(\alpha\beta)}$, we have

$$Q^2 = 2\mathcal{O}_{ij} \left(R_\alpha^i R_\beta^j + R_\alpha^j R_\beta^i\right) \mathcal{S}^{(\alpha\beta)} \tag{8.36}$$

then using the antisymmetry $\mathcal{O}_{[ij]}$, we end up with

$$Q^2 = \mathcal{O}_{ij} \left(R_\alpha^i R_\beta^j - R_\alpha^j R_\beta^i + R_\alpha^j R_\beta^i - R_\alpha^i R_\beta^j\right) \mathcal{S}^{(\alpha\beta)} \tag{8.37}$$

indicating that Q^2 vanishes identically with no constraint on R_α^i.

b) Computing H_{susy}

For the calculation of the anticommutator $QQ^\dagger + Q^\dagger Q = H_{susy}$, we substitute $Q = \hat{c}_i^\dagger R_\alpha^i \hat{b}^\alpha$ and $Q^\dagger = \hat{b}_\beta^\dagger \bar{R}_j^\beta \hat{c}^j$, we obtain

$$H_{susy} = \hat{c}_i^\dagger [h_f]_j^i \hat{c}^j + \hat{b}_\alpha^\dagger [h_b]_\beta^\alpha \hat{b}^\beta \tag{8.38}$$

with hermitian coupling tensors $[h_f]_j^i$ and $[h_b]_\beta^\alpha$ respectively given by the $N \times N$ square matrix $[RR^\dagger]_j^j$ and the $M \times M$ square matrix $[R^\dagger R]_\beta^\alpha$. So, we have

$$h_f = RR^\dagger, \qquad h_b = R^\dagger R \tag{8.39}$$

with $H_b^{susy} = \hat{b}_\alpha^\dagger [h_b]_\beta^\alpha \hat{b}^\beta$ and $H_f^{susy} = \hat{c}_i^\dagger [h_f]_j^i \hat{c}^j$ reading also as

$$\begin{array}{lll} H_b^{susy} & = & 2[h_b]_\beta^\alpha \mathcal{S}_\alpha^\beta - \frac{1}{2} tr\,(h_b) \\ H_f^{susy} & = & 2[h_f]_j^i \mathcal{O}_j^i + \frac{1}{2} tr\left(h_f\right) \end{array} \tag{8.40}$$

indicating that $H^{susy} = 2[h_b]_\beta^\alpha \mathcal{S}_\alpha^\beta + 2[h_f]_j^i \mathcal{O}_j^i$ is valued into $u\,(N) \oplus u\,(M)$. Notice that we have

$$h_f R - R h_b = 0 \tag{8.41}$$

c) Checking the commutativity $H_{susy}, Q = QH_{susy}$

Regarding the calculation of $\left[H_{susy}, Q\right]$, we compute the commutators $[H_b^{susy}, Q]$ and $[H_f^{susy}, Q]$; we find

$$\begin{array}{lll} \left[H_b^{susy}, \hat{c}_i^\dagger R_\gamma^i \hat{b}^\gamma\right] & = & -\hat{c}_i^\dagger [RR^\dagger R]_\beta^i \hat{b}^\beta \\ \left[H_f^{susy}, \hat{c}_i^\dagger R_\gamma^i \hat{b}^\gamma\right] & = & +\hat{c}_i^\dagger [RR^\dagger R]_\beta^i \hat{b}^\beta \end{array} \tag{8.42}$$

whose sum vanishes identically. So, eqs. ((8.30)-(8.34)) realise a SUSY charge.

II. Beyond the realisation ((8.30)-(8.34)): Clearly, eq. (8.34) is a special realisation of the fermionic charge Q; a general oscillator realisation can be written down; it involves the coupling tensors $T^{i\alpha}$ and $\bar{T}_{i\alpha}$ in addition R_α^i and $\bar{R}_i^\alpha$. We refer to these orthosymplectic supercharges like $Q_{ortic} \equiv Q^+$ and $\bar{Q}_{ortic} \equiv Q^-$ with realisations given by

$$\begin{array}{lll} Q^+ & = & \hat{c}_i^\dagger R_\alpha^i \hat{b}^\alpha + \hat{c}_i^\dagger \hat{b}_\alpha^\dagger T^{\alpha i} \\ Q^- & = & \hat{b}_\alpha^\dagger \bar{R}_i^\alpha \hat{c}^i + \bar{T}_{i\alpha} \hat{b}^\alpha \hat{c}^i \end{array} \tag{8.43}$$

These supercharges goes beyond (8.30)-(8.34) corresponding to $T^{\alpha i} = 0$; the above $Q^\pm$ have some specific properties as listed below:

(1) Chiral odd spaces

The supercharges $Q^\pm$ belong to two adjoint conjugate subspaces in the odd sector of the osp(2N|2M). Indeed, using the fermionic operators generating the odd sector of osp(2N|2M) namely $\boldsymbol{G}_i^\alpha$, $\boldsymbol{F}_{i\alpha}$, $\bar{\boldsymbol{G}}_\alpha^i$, $\bar{\boldsymbol{F}}^{\alpha i}$ given eq(8.31), then, we have

$$\begin{array}{lll} Q^+ & = & R_\alpha^i \boldsymbol{G}_i^\alpha + T^{\alpha i} \boldsymbol{F}_{i\alpha} \\ Q^- & = & \bar{R}_i^\alpha \bar{\boldsymbol{G}}_\alpha^i + \bar{T}_{i\alpha} \bar{\boldsymbol{F}}^{\alpha i} \end{array} \tag{8.44}$$

with Q^+ sitting in the complex directions $\boldsymbol{G}_i^\alpha, \boldsymbol{F}_{i\alpha}$; and the $Q^- = (Q^+)^\dagger$ in their images $\bar{\boldsymbol{G}}_\alpha^i, \bar{\boldsymbol{F}}^{\alpha i}$ under conjugation. From these $Q^\pm$, one can also construct a hermitian supercharge $\mathfrak{Q}$ and an antihermitian $\mathfrak{P}$ as usual like

$$\mathfrak{Q} = \frac{Q^+ + Q^-}{\sqrt{2}} \qquad , \qquad i\mathfrak{P} = \frac{Q^+ - Q^-}{\sqrt{2}} \tag{8.45}$$

Notice that the two following features of eq. (8.44). First, it carries $2NM$ complex degrees of freedom; NM coming from R_α^i and NM from $T^{i\alpha}$. Second, the $Q^\pm$ are valued in reducible representations of $U(N) \times U(M)$. This is because, the $\boldsymbol{G}_i^\alpha$ sits in the bifundamental $\left(\bar{N}, M\right)$ while the $\boldsymbol{F}_{i\alpha}$ sits in the antisymmetric $\bar{N} \wedge \bar{M}$. This feature deforms the expressions of the supersymmetric constraints satisfied by $\{Q^\pm, Q^\pm\} = 0$ and $[H, Q^\pm] = 0$.

(2) Anticommutators $\left\{Q^\pm, Q^\pm\right\}$

From the orthosymplectic supercharges (8.44), we can define three anticommutators: (i) The anticommutator $\{Q^+, Q^-\}$ defining the orthosymplectic Hamiltonian H_{ortic} which is hermitian. (ii) The anticommutator $\{Q^+, Q^+\}$, which in general is non vanishing, defines a complex operator Z_{ortic} scaling as H_{ortic}. The third anticommutator is given by $Z_{ortic}^\dagger = \{Q^-, Q^-\}$; it is the adjoint conjugate of Z_{ortic}.

These three bosonic operators H_{ortic}, Z_{ortic} and $Z_{ortic}^\dagger$ are valued in the bosonic subalgebra $so(2N) \oplus sp(2M)$ generated by $\mathcal{O}_i^j$, $\mathcal{O}^{[ij]}$, $\overline{\mathcal{O}}_{[ij]}$ (for so_{2N}) and $\mathcal{S}_\beta^\alpha$, $\mathcal{S}^{(\alpha\beta)}$, $\bar{\mathcal{S}}_{(\alpha\beta)}$ (for sp_{2M}). To exhibit this feature, we calculate these anticommutators.

a) the Hamiltonian H_{ortic}

Substituting (8.44) into $\{Q^+, Q^-\}$, we get

$$\begin{array}{rcl} H_{ortic} & = & R_\alpha^i \bar{R}_j^\beta \{\boldsymbol{G}_i^\alpha, \bar{\boldsymbol{G}}_\beta^j\} + T^{\alpha i} \bar{R}_j^\beta \{\boldsymbol{F}_{i\alpha}, \bar{\boldsymbol{G}}_\beta^j\} \\ & & R_\alpha^i \bar{T}_{j\beta} \{\boldsymbol{G}_i^\alpha, \bar{\boldsymbol{F}}^{\beta j}\} + T^{\alpha i} \bar{T}_{j\beta} \{\boldsymbol{F}_{i\alpha}, \bar{\boldsymbol{F}}^{\beta j}\} \end{array} \tag{8.46}$$

By using (8.32)-(8.33), this H_{ortic} reads as a linear combination of the above mentioned generators namely

$$\begin{array}{rcl} H_{ortic} & = & 2\left(R_\alpha^i \bar{R}_i^\beta + T^{\alpha i} \bar{T}_{i\beta}\right) \mathcal{S}_\beta^\alpha + \\ & & 2\left(R_\alpha^i \bar{R}_j^\alpha - \bar{T}_{j\alpha} T^{\alpha i}\right) \mathcal{O}_i^j + \\ & & 2\left(R_\alpha^i \bar{T}_{i\beta}\right) \mathcal{S}^{(\alpha\beta)} + 2(T^{\alpha i} \bar{R}_i^\beta) \bar{\mathcal{S}}_{(\alpha\beta)} \end{array} \tag{8.47}$$

As it has no $\mathcal{O}^{[ij]}$, $\overline{\mathcal{O}}_{[ij]}$, itt is valued in $u_N \oplus sp_{2M}$. Interesting families of such Hamiltonians are given by the two following constraints on the coupling tensors: (**i**) The R_α^i and $T^{\alpha i}$ couplings are constrained like

$$\begin{array}{rcl} R_\alpha^i \bar{T}_{i\beta} & = & \eta_{RT} \Omega_{\alpha\beta} \\ T^{\beta i} \bar{R}_i^\alpha & = & \bar{\eta}_{RT} \Omega^{\beta\alpha} \end{array} \quad , \quad \begin{array}{rcl} \Omega_{\alpha\beta} & = & -\Omega_{\beta\alpha} \\ \Omega^{\beta\alpha} & = & -\Omega^{\alpha\beta} \end{array} \tag{8.48}$$

where η_{RT} scaling as energy. For simple calculations, we will set $\eta_{RT} = 0$. For this choice, the $\mathcal{S}^{(\alpha\beta)}$ and $\bar{\mathcal{S}}_{(\alpha\beta)}$ terms disappear and the Hamiltonian (8.47) reduces to a form similar to (8.40); it reads as follows

$$\begin{array}{rcl} H_{ortic} & = & 2\left(\bar{R}_i^\beta R_\alpha^i + T^{\beta i} \bar{T}_{i\alpha}\right) \mathcal{S}_\beta^\alpha + \\ & & 2\left(R_\alpha^i \bar{R}_j^\alpha - \bar{T}_{j\alpha} T^{\alpha i}\right) \mathcal{O}_i^j \end{array} \tag{8.49}$$

it is valued into $u(N) \oplus u(M)$. In this case, we have the following bosonic and fermionic contributions

$$\begin{array}{lll} (h_b)^{\beta}_{\gamma} & = & \bar{R}^{\beta}_{l} R^{l}_{\gamma} + T^{\beta l} \bar{T}_{l\gamma} \\ \left(h_f\right)^{i}_{j} & = & R^{i}_{\alpha} \bar{R}^{\alpha}_{j} - \bar{T}_{j\alpha} T^{\alpha i} \end{array} \tag{8.50}$$

(**ii**) If in addition to (8.48) with $\eta_{RT} = 0$, the R^i_α and $T^{\alpha i}$ couplings are restricted to the following diagonal choices

$$\begin{array}{lllclll} R^{i}_{\alpha} \bar{R}^{\beta}_{i} & = & \lambda^{\alpha}_{R} \delta^{\beta}_{\alpha} & , & R^{i}_{\alpha} \bar{R}^{\alpha}_{j} & = & \mu^{i}_{R} \delta^{i}_{j} \\ T^{\alpha i} \bar{T}_{i\beta} & = & \lambda^{\alpha}_{T} \delta^{\beta}_{\alpha} & , & T^{\alpha i} \bar{T}_{j\alpha} & = & \mu^{i}_{T} \delta^{i}_{j} \end{array} \tag{8.51}$$

where λ^{α}_{R}, λ^{α}_{T} and μ^{i}_{R}, μ^{i}_{T} are $M + N$ real numbers, the Hamiltonian takes an interesting form. For this choice, the resulting H_{ortic} is valued in the Cartan subalgebra of $so(2N) \oplus sp(2M)$ as given here after

$$H_{ortic} = 2 \sum_{\alpha=1}^{M} \left(\lambda^{\alpha}_{R} + \lambda^{\alpha}_{T}\right) \mathcal{S}_{\alpha} + 2 \sum_{i=1}^{N} \left(\mu^{i}_{R} - \mu^{i}_{T}\right) \mathcal{J}_{i} \tag{8.52}$$

b) the anticommutator $\{Q^+, Q^+\}$

This anticommutator defines the observable Z_{ortic}; we find after substituting $Q^+ = R^i_\alpha \boldsymbol{G}^\alpha_i + T^{\alpha i} \boldsymbol{F}_{i\alpha}$ and using $\{\boldsymbol{G}^\alpha_i, \boldsymbol{G}^\beta_j\} = \{\boldsymbol{F}_{i\alpha}, \boldsymbol{F}_{j\beta}\} = 0$, the following expression

$$Z_{ortic} = R^{i}_{\alpha} T^{\beta j} \left\{\boldsymbol{G}^{\alpha}_{i}, \boldsymbol{F}_{j\beta}\right\} + T^{\beta j} R^{i}_{\alpha} \left\{\boldsymbol{F}_{j\beta}, \boldsymbol{G}^{\alpha}_{i}\right\} \tag{8.53}$$

Using (8.32)-(8.33), we can put this relation as follows

$$Z_{ortic} = -4 \left(R^{i}_{\alpha} T^{\alpha j}\right) \overline{\mathcal{O}}_{[ij]} \tag{8.54}$$

showing that Z_{ortic} is valued in the antisymmetric representation of the $u(N)$ subalgebra of so$(2N)$. It vanishes for the case

$$R^{i}_{\alpha} T^{\alpha j} = \zeta_{RT} G^{ij} \qquad , \qquad G^{ij} = G^{ji} \tag{8.55}$$

c) the commutator $[H, Q^+]$

Substituting the bosonic $H_{ortic} = (h_b)^{\beta}_{\alpha} \mathcal{S}^{\alpha}_{\beta} + \left(h_f\right)^{i}_{j} \mathcal{O}^{j}_{i}$ and the fermionic $Q^+ = R^i_\alpha \boldsymbol{G}^\alpha_i + T^{\alpha i} \boldsymbol{F}_{i\alpha}$, then using

$$\begin{array}{lllclll} \left[\mathcal{S}^{\alpha}_{\beta}, \boldsymbol{G}^{\gamma}_{l}\right] & = & -\frac{1}{2} \delta^{\gamma}_{\beta} \boldsymbol{G}^{\alpha}_{l} & , & \left[\mathcal{O}^{j}_{i}, \boldsymbol{G}^{\gamma}_{l}\right] & = & \frac{1}{2} \delta^{j}_{l} \boldsymbol{G}^{\gamma}_{i} \\ \left[\mathcal{S}^{\alpha}_{\beta}, \boldsymbol{F}_{l\gamma}\right] & = & +\frac{1}{2} \delta^{\alpha}_{\gamma} \boldsymbol{F}_{l\beta} & & \left[\mathcal{O}^{j}_{i}, \boldsymbol{F}_{l\gamma}\right] & = & \frac{1}{2} \delta^{j}_{l} \boldsymbol{F}_{i\gamma} \end{array} \tag{8.56}$$

we find after some algebra, the following

$$\begin{array}{lll} [H, Q^+] & = & \frac{1}{2} \{\left(h_f\right)^{i}_{j} R^{j}_{\gamma} - (h_b)^{\beta}_{\gamma} R^{i}_{\beta}\} \boldsymbol{G}^{\gamma}_{i} + \\ & & \frac{1}{2} \{T^{\gamma j} \left(h_f\right)^{i}_{j} + (h_b)^{\gamma}_{\beta} T^{\beta i}\} \boldsymbol{F}_{i\gamma} \end{array} \tag{8.57}$$

The vanishing condition of this commutator requires

$$\begin{array}{lll} \left(h_f\right)^{i}_{j} R^{j}_{\gamma} - (h_b)^{\beta}_{\gamma} R^{i}_{\beta} & = & 0 \\ T^{\gamma j} \left(h_f\right)^{i}_{j} + (h_b)^{\gamma}_{\beta} T^{\beta i} & = & 0 \end{array} \tag{8.58}$$

By replacing $\left(h_f\right)^i_j$ and $(h_b)^\beta_\gamma$ by eq. (8.50), we obtain

$$\begin{array}{lll}(h_b)^\beta_\gamma R^i_\beta & = & R^i_\beta \bar{R}^\beta_l R^l_\gamma + R^i_\beta T^{\beta l}\bar{T}_{l\gamma}, \\ \left(h_f\right)^i_j R^j_\gamma & = & R^i_\beta \bar{R}^\beta_l R^l_\gamma - R^j_\gamma \bar{T}_{j\alpha} T^{\alpha i}, \\ (h_b)^\beta_\gamma T^{\gamma i} & = & \bar{R}^\beta_l R^l_\gamma T^{\gamma i} + T^{\beta l}\bar{T}_{l\gamma} T^{\gamma i}, \\ \left(h_f\right)^i_j T^{\beta j} & = & T^{\beta j}\bar{R}^\gamma_j R^i_\gamma - T^{\beta l}\bar{T}_{l\gamma} T^{\gamma i}\end{array} \tag{8.59}$$

which by substituting into (8.58), we get the following constraint relations

$$\begin{array}{lll}R^j_\gamma \bar{T}_{j\alpha} T^{\alpha i} + R^i_\alpha T^{\alpha j}\bar{T}_{j\gamma} & = & 0 \\ \bar{R}^\beta_j R^j_\gamma T^{\gamma i} + T^{\beta j}\bar{R}^\gamma_j R^i_\gamma & = & 0\end{array} \tag{8.60}$$

These constraints are naturally solved by the orthogonalites $R^i_\alpha T^{\alpha j} = R^i_\alpha \bar{T}_{i\beta} = 0$. If substituting $R^i_\alpha T^{\alpha j} = \zeta_{RT} G^{ij}$ and $R^i_\alpha \bar{T}_{i\beta} = \eta_{RT}\Omega_{\alpha\beta}$, these constraints read as

$$\begin{array}{lll}\eta_{RT}\Omega_{\gamma\alpha} T^{\alpha i} + \zeta_{RT} G^{ij}\bar{T}_{j\gamma} & = & 0 \\ \zeta_{RT} G^{ji}\bar{R}^\beta_j + \bar{\eta}_{RT}\Omega^{\beta\gamma} R^i_\gamma & = & 0\end{array} \tag{8.61}$$

III. Fermionic charges beyond (8.44) In eq. (8.44) the two fermionic charge Q^+ and Q^- are related by adjoint conjugation; the combination $\sqrt{2}\mathfrak{Q} = Q^+ + Q^-$ is hermitian and reads like

$$\sqrt{2}\mathfrak{Q} = R^i_\alpha \boldsymbol{G}^\alpha_i + T^{\alpha i}\boldsymbol{F}_{i\alpha} + \bar{R}^\alpha_i \bar{\boldsymbol{G}}^i_\alpha + \bar{T}_{i\alpha}\bar{\boldsymbol{F}}^{\alpha i} \tag{8.62}$$

having two complex coupling tensors R^i_α and $T^{\alpha i}$. However, this is still a particular case because one may relax this hermiticity constraint by thinking about the above fermionic charge $\sqrt{2}\mathfrak{Q}$ as follows

$$\mathfrak{Q}^+ = R^i_\alpha \boldsymbol{G}^\alpha_i + T^{\alpha i}\boldsymbol{F}_{i\alpha} + W^\alpha_i \bar{\boldsymbol{G}}^i_\alpha + S_{i\alpha}\bar{\boldsymbol{F}}^{\alpha i} \tag{8.63}$$

where now we have four complex coupling tensors namely the old complex R^i_α and $T^{\alpha i}$ and the two new W^α_i and $S_{i\alpha}$. This $\mathfrak{Q}^+$ involves $4NM$ complex degrees of freedom; it is the general form of the fermionic generator one can build out of the fermionic $\hat{c}^\dagger_i/\hat{c}^i$ and bosonic $\hat{b}^\dagger_\alpha/\hat{b}^\alpha$ operators. The $\mathfrak{Q}^+$ is valued in the complexified osp(2N|2M) and can be viewed as the given by sum $\mathfrak{Q}^+ = Q^+_1 + Q^+_2$ with

$$\begin{array}{lll}Q^+_1 & = & R^i_\alpha \boldsymbol{G}^\alpha_i + T^{\alpha i}\boldsymbol{F}_{i\alpha} \\ Q^+_2 & = & W^\alpha_i \bar{\boldsymbol{G}}^i_\alpha + S_{i\alpha}\bar{\boldsymbol{F}}^{\alpha i}\end{array} \tag{8.64}$$

with Q^+_1 involving the R^i_α, $T^{\alpha i}$ and the Q^+_2 using the new coupling tensors $S_{i\alpha}$ and W^α_i. The adjoint conjugate of the $\mathfrak{Q}^+$ is given by $\mathfrak{Q}^- = Q^-_1 + Q^-_2$ with

$$\begin{array}{lll}Q^-_1 & = & \bar{R}^\alpha_i \bar{\boldsymbol{G}}^i_\alpha + \bar{T}_{i\alpha}\bar{\boldsymbol{F}}^{\alpha i} \\ Q^-_2 & = & \bar{W}^i_\alpha \boldsymbol{G}^\alpha_i + \bar{S}^{\alpha i}\boldsymbol{F}_{i\alpha}\end{array} \tag{8.65}$$

An interesting way to deal with these $\mathfrak{Q}^\pm$ is to use the notations $\hat{\boldsymbol{\lambda}}^\dagger_{\dot{A}} = (\hat{\boldsymbol{c}}^\dagger_I, \hat{\boldsymbol{c}}^I)$ and $\hat{\xi}^\dagger_A = (\hat{\boldsymbol{b}}^\dagger_I, \hat{\boldsymbol{b}}^I)$ as well as $\hat{\boldsymbol{\lambda}}^{\dot{A}} = (\hat{\boldsymbol{c}}^I, \hat{\boldsymbol{c}}^\dagger_I)^T$ and $\hat{\xi}^A = (\hat{\boldsymbol{b}}^I, \hat{\boldsymbol{b}}^\dagger_I)$ in terms of which the $\mathfrak{Q}^+$ and $\mathfrak{Q}^-$ read in a condensed way as follows

$$\mathfrak{Q}^{-}=\hat{\boldsymbol{\lambda}}_{\dot{A}}^{\dagger}\left[\boldsymbol{q}\right]_{B}^{\dot{A}}\hat{\xi}^{B}\qquad,\qquad\mathfrak{Q}^{+}=\hat{\xi}_{A}^{\dagger}\left[\boldsymbol{q}^{\dagger}\right]_{\dot{B}}^{A}\hat{\boldsymbol{\lambda}}^{\dot{B}}\tag{8.66}$$

Here, the $\left[\boldsymbol{q}\right]_{B}^{\dot{A}}$ coupling tensor and adjoint $\left[\boldsymbol{q}^{\dagger}\right]_{\dot{B}}^{A}$ are respectively given by $2N\times 2M$ and $2M\times 2N$ rectangular matrices reading in $N\times M$ and $M\times N$ sub-block matrices as follows

$$\mathfrak{Q}^{-}=\left(\hat{\boldsymbol{c}}^{\dagger},\hat{\boldsymbol{c}}\right)\begin{pmatrix}R & T\\ S & W\end{pmatrix}\begin{pmatrix}\hat{\boldsymbol{b}}\\ \hat{\boldsymbol{b}}^{\dagger}\end{pmatrix}\quad,\quad\mathfrak{Q}^{+}=\left(\hat{\boldsymbol{b}}^{\dagger},\hat{\boldsymbol{b}}\right)\begin{pmatrix}R^{\dagger} & S^{\dagger}\\ T^{\dagger} & W^{\dagger}\end{pmatrix}\begin{pmatrix}\hat{\boldsymbol{c}}\\ \hat{\boldsymbol{c}}^{\dagger}\end{pmatrix}\tag{8.67}$$

From these fermionic charges, one can calculate three anticommutators: the hermitian $\{\mathfrak{Q}^{-},\mathfrak{Q}^{+}\}$ and the complex $\{\mathfrak{Q}^{+},\mathfrak{Q}^{+}\}$ and the adjoint conjugate $\{\mathfrak{Q}^{-},\mathfrak{Q}^{-}\}$. The calculations of these anticommutators go in the same manner as we have done before. The novelty is that in this general case the graded canonical commutation relations are some how unusual: (**i**) For the case of fermions, the anticommutation relations have the form

$$\{\hat{\boldsymbol{\lambda}}_{\dot{A}}^{\dagger},\hat{\boldsymbol{\lambda}}^{\dot{B}}\}=\delta_{\dot{A}}^{\dot{B}},\quad\{\hat{\boldsymbol{\lambda}}_{\dot{A}}^{\dagger},\hat{\boldsymbol{\lambda}}_{\dot{B}}^{\dagger}\}=\dot{\Sigma}_{\dot{A}\dot{B}}^{x},\quad\{\hat{\boldsymbol{\lambda}}^{\dot{A}},\hat{\boldsymbol{\lambda}}^{\dot{B}}\}=\dot{\Sigma}_{x}^{\dot{A}\dot{B}}\tag{8.68}$$

where $\{\hat{\boldsymbol{\lambda}}_{\dot{A}}^{\dagger},\hat{\boldsymbol{\lambda}}_{\dot{B}}^{\dagger}\}$ and $\{\hat{\boldsymbol{\lambda}}^{\dot{A}},\hat{\boldsymbol{\lambda}}^{\dot{B}}\}$ are non vanishing. This is because

$$\{\hat{\boldsymbol{\lambda}}_{\dot{A}}^{\dagger},\hat{\boldsymbol{\lambda}}_{\dot{B}}^{\dagger}\}=\begin{pmatrix}\{\hat{\boldsymbol{c}}_{\dot{I}}^{\dagger},\hat{\boldsymbol{c}}^{\dot{J}}\} & \{\hat{\boldsymbol{c}}_{\dot{I}}^{\dagger},\hat{\boldsymbol{c}}_{\dot{J}}^{\dagger}\}\\ \{\hat{\boldsymbol{c}}^{\dot{I}},\hat{\boldsymbol{c}}^{\dot{J}}\} & \{\hat{\boldsymbol{c}}^{\dot{I}},\hat{\boldsymbol{c}}_{\dot{J}}^{\dagger}\}\end{pmatrix}\tag{8.69}$$

In these relations, the $\dot{\Sigma}_{\dot{A}\dot{B}}^{x}$ and $\dot{\Sigma}_{x}^{\dot{A}\dot{B}}$ are given by

$$\delta_{\dot{A}}^{\dot{B}}=\begin{pmatrix}\delta_{\dot{I}}^{\dot{J}} & 0\\ 0 & \delta_{\dot{J}}^{\dot{I}}\end{pmatrix},\quad\dot{\Sigma}_{\dot{A}\dot{B}}^{x}=\begin{pmatrix}0 & \delta_{\dot{I}}^{\dot{J}}\\ \delta_{\dot{J}}^{\dot{I}} & 0\end{pmatrix},\quad\dot{\Sigma}_{x}^{\dot{A}\dot{B}}=\begin{pmatrix}0 & \delta_{\dot{J}}^{\dot{I}}\\ \delta_{\dot{I}}^{\dot{J}} & 0\end{pmatrix}\tag{8.70}$$

(**ii**) For the bosons, we also have

$$\left[\hat{\xi}^{A},\hat{\xi}_{B}^{\dagger}\right]=\Sigma_{B}^{zA},\quad\left[\hat{\xi}^{A},\hat{\xi}^{B}\right]=i\,\Sigma_{y}^{AB},\quad\left[\hat{\xi}_{A}^{\dagger},\hat{\xi}_{B}^{\dagger}\right]=-i\,\Sigma_{AB}^{y}\tag{8.71}$$

where here also the $[\hat{\xi}^{A},\hat{\xi}^{B}]$ and $[\hat{\xi}_{A}^{\dagger},\hat{\xi}_{B}^{\dagger}]$ are non vanishing. In these relations, the Σ_{B}^{zA} and Σ_{y}^{AB} are given by

$$\left(\Sigma^{z}\right)_{B}^{A}=\begin{pmatrix}\delta_{J}^{I} & 0\\ 0 & -\delta_{J}^{I}\end{pmatrix},\quad i\,\Sigma_{y}^{AB}=\begin{pmatrix}0 & \delta_{J}^{I}\\ -\delta_{I}^{J} & 0\end{pmatrix},\quad -i\,\Sigma_{AB}^{y}=\begin{pmatrix}0 & -\delta_{I}^{J}\\ \delta_{J}^{I} & 0\end{pmatrix}\tag{8.72}$$

Declaration of competing interest

The authors declare that there is no competing financial interests or personal relationships that could influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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