Singular supertranslations and Chern-Simons theory on the black hole horizon

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We construct the standard and dual supertranslation charges on the future horizon of the Schwarzschild black hole, using the first-order formulation of gravity with the Holst action. The Dirac bracket algebra of standard and dual supertranslation charges is shown to exhibit a central term in the presence of singularities in the two-sphere function associated with supertranslation. We show that one can cancel this anomalous term and restore the asymptotic symmetry algebra by introducing a gravitational Chern-Simons theory on the horizon. This demonstrates that consistency of the asymptotic symmetry algebra requires a new structure on the horizon.

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I. INTRODUCTION

Black hole physics provides us with a paradox that has been around now for almost fifty years [1]. The information paradox illustrates an apparent conflict between classical and semiclassical general relativity and the fundamental tenets of quantum theory. The classical black hole uniqueness theorems appear to indicate that the Kerr-Newman family of black holes, characterized by their mass, angular momentum and electric charge, is sufficient to describe all black hole stationary states [2]. If this were true in a complete quantum theory, then it would not be possible to distinguish between a black hole formed from matter and one formed form antimatter. It appears that to any observer outside a black hole that has reached a stationary state, the black hole is independent of the details of its formation. In particular, the black hole has no memory of the quantum state of the material that formed it. The trouble comes when black holes evaporate. Hawking showed that the outgoing radiation is thermal and so has a large von Neumann entropy. Suppose that the matter forming the black hole was in a pure quantum state. In quantum mechanics, the von Neumann entropy is constant because the time evolution operator is unitary but that is inconsistent with the picture outlined above. Identifying what is wrong with this picture has been a huge challenge and, despite much hard labor, has not yet yielded any clear solution.

Recently, it has been realized that black holes can have soft hair [3,4]. Soft hair are extra degrees of freedom that a black hole can have. The geometry remains that of the Kerr-Newman sequence with the soft hair being described by a particular class of gauge transformations. Suppose we look at an asymptotically flat spacetime that does not contain a black hole. Bondi-Metzner-Sachs (BMS) transformations acting on the gravitational field at both past and future null infinity generalize the Poincaré symmetry group familiar from nongravitational settings [5,6]. The Poincaré group acts on Minkowski space with large gauge transformations generating translations, rotations or boosts. Each of these large gauge transformations is associated with a charge namely the momentum, angular momentum and the boost

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charge. Similarly, the BMS transformations are associated with a charge conjugate to large gauge transformations. These charges distinguish the infinite number of distinct vacua of the gravitational field. In electromagnetism, one is familiar with integrating a current over a spacelike three-surface to describe the charge passing through the surface. Gauss' theorem then guarantees that this integral can be turned into a surface integral that measures the charge inside that surface. Exactly the same thing happens for the BMS charges so they can be described by surface integrals on sections of past or future null infinity. These charges can change as the result of incoming matter or gravitational waves passing through past null infinity or outgoing matter or gravitational waves passing through future null infinity. Gravitational memory gives a method of observing the changes in these charges. We refer interested readers to [7–20] for some earlier literature on this development. See [21] for a review.

In black hole spacetimes, the horizon is a boundary of what can be observed form the exterior. The integrals of currents may then have two boundary components, one at null infinity and the other on the horizon. As a consequence black holes will also carry soft charges in much the same way as they can be found at null infinity. This paper is concerned with some of the consequences of this observation. Our main aim here is to consider pure gravity without matter. However, in the interests of clarity and simplicity, we will also provide an outline discussion of the case of electromagnetism as a model of the more complicated case of gravitation. The main results of this paper have been outline in the letter [22], and in this paper we provide details of the calculation.

We examine in detail the physics of horizon (standard and dual) BMS charges for the Schwarzschild black hole. The horizon charges are computed using the machinery presented by [23,24] in the first-order formalism of gravity. The algebra of the charges is expected to reflect the algebra of the vector fields that generate the corresponding symmetry. We find an anomaly in the algebra of charges. To preserve the symmetry of the theory, we need to introduce some degrees of freedom to cancel the anomaly [25]. We show that this can be done by the introduction of a (holographic) gravitational Chern-Simons theory on the horizon. It would be satisfying to show that the states of this Chern-Simons theory reproduce the correct black hole entropy and thereby describe the states of the black hole itself. Such a goal is currently beyond us but a subject of current investigation [26]. We find the Chern-Simons theory for electromagnetism to have gauge group $U(1) \otimes U(1)$ and for gravitation to be $SL(2,\mathbb{C})$ [27]. Thus from this point of view, consistency requires the introduction of a new structure at the horizon.

In Sec. II, we describe the analog of standard and dual BMS transformations on the horizon in the Bondi gauge. The use of the Bondi gauge on the horizon makes computations particularly simple for the case of the Schwarzschild metric. It is noteworthy that the algebra of vector fields that generate supertranslations and superrotations is identical to that found for the BMS group at null infinity. However, to establish this result, we had to revisit some earlier work of Barnich and Troessaert where a modified Lie bracket was introduced; the rationale and description is also discussed in Sec. II. In Sec. III, we introduce the charges associated to the diffeomorphism symmetries. In parallel, we also discuss the dual (magnetic) counterpart of the diffeomorphism symmetries. We restrict ourselves here to use of smooth vector fields to generate the symmetries. In Sec. IV, we continue the discussion of charges but allow for the possibility that there could be singularities in the supertranslations. We examine in detail the case of the supertranslation generator having a pole when expressed in the usual complex coordinates on the S^2 of the horizon. In Sec. V, we show that the algebra of electric and magnetic supertranslation charges is anomalous and discover the nature of a central charge. In Sec. VI, we give an alternative derivation of the same result. In Sec. VII, we examine electromagnetic soft hair and show that a singularity lead to an anomaly in the charge algebra when one has both electric and magnetic transformations. We show that this anomaly can be canceled by supposing that the horizon has a Chern-Simons theory living on it. It is fortunate that the Chern-Simons is a topological theory as it is metric independent. There are two nice properties that follow. The first is that since the horizon is a null surface, the metric is degenerate there and one cannot invert the metric. Had the theory been metric-dependent, as most are, it would have been impossible to formulate a theory that is restricted to the null surface. The second also follows from being metric-independent. The energy-momentum tensor of a theory is given by varying the action with respect to the metric. Therefore, in the Chern-Simons case, the energy-momentum tensor vanishes and the holographic theory does not disturb the black hole geometry. In Sec. VIII, we repeat this analysis for the gravitational case. Finally, there is a brief discussion of our results in Sec. IX. In addition, there are three appendices that deal with some technical matters involved in our computations [28].

II. HORIZON BMS TRANSFORMATIONS IN THE BONDI GAUGE

We briefly review the BMS supertranslations and superrotations on the future horizon of a Schwarzschild black hole [4]. Throughout our paper, we work in the Bondi gauge,

$$g_{rr} = g_{rA} = 0, \qquad \partial_r \det\left(\frac{g_{AB}}{r^2}\right) = 0. \tag{1}$$

In terms of the ingoing Eddington-Finkelstein coordinates, the Schwarzschild metric is given by

$$ds^2 = -\Lambda dv^2 + 2dvdr + r^2\gamma_{AB}d\Theta^A d\Theta^B,$$
$$\Lambda \equiv 1 - \frac{2M}{r}, \tag{2}$$

where γ_{AB} is the metric on the unit 2-sphere. A diffeomorphism ξ that preserves these conditions should satisfy

$$\mathcal{L}_\xi g_{rr} = \mathcal{L}_\xi g_{rA} = 0, \qquad \gamma^{AB}\mathcal{L}_\xi g_{AB} = 0. \tag{3}$$

Such diffeomorphisms can be parametrized as [4]

$$\xi = X\partial_v - \frac{1}{2}(rD_A X^A + D^2 X)\partial_r + \left(X^A + \frac{1}{r}D^A X\right)\partial_A, \tag{4}$$

where $X^A = X^A(v, \Theta)$ is an arbitrary vector field and $X = X(v, \Theta)$ is an arbitrary scalar field on the future horizon $\mathcal{H}^+$. Here D_A denotes the covariant derivative on the unit 2-sphere and so $D^A = \gamma^{AB}D_B$ and also $D^2 \equiv D^A D_A = \gamma^{AB}D_A D_B$.

A supertranslation is given by

$$X = f(\Theta), \qquad X^A = 0, \tag{5}$$

where f is a smooth function on the 2-sphere. In later sections, we relax the smoothness condition to allow f to have poles.

A superrotation is given by

$$X = \frac{v}{2}D_A Y^A, \qquad X^A = Y^A(\Theta), \tag{6}$$

where Y^A is a smooth vector field on the 2-sphere.

Since supertranslations and superrotations are metric-dependent, the diffeomorphisms (4) do not form a closed algebra under the Lie bracket of vector fields. To see why, consider a transformation of the metric generated by ξ_1^a. Under such a transformation $g_{ab} \to g_{ab} + h_{ab}$ with

$$h_{ab} = \mathcal{L}_{\xi_1} g_{ab} = \xi_1^c \partial_c g_{ab} + g_{ac}\partial_b \xi_1^c + g_{cb}\partial_a \xi_1^c. \tag{7}$$

Now a second transformation generated by ξ_2^a will produce the second order variation of the metric but will also produce a variation of ξ_1^a. The variation of ξ_1^a needs to be removed in order to isolate the second order variation of the metric. The Lie bracket $[\xi_1, \xi_2]$ of two vector fields ξ_1^a and ξ_2^a is conventionally defined by

$$\mathcal{L}_{\xi_1}\mathcal{L}_{\xi_2} - \mathcal{L}_{\xi_2}\mathcal{L}_{\xi_1} = \mathcal{L}_{[\xi_1,\xi_2]} \tag{8}$$

so that

$$[\xi_1, \xi_2]^a = \xi_1^b \partial_b \xi_2^a - \xi_2^b \partial_b \xi_1^a. \tag{9}$$

The Lie bracket needs to be modified in order to isolate just the second order variation of the metric. An appropriately modified Lie bracket of vector fields was introduced by Barnich and Troessaert [29] of vector fields and is

$$[\xi_1, \xi_2]_M^a = [\xi_1, \xi_2]^a - \delta_{\xi_1}\xi_2^a + \delta_{\xi_2}\xi_1^a, \tag{10}$$

where $\delta_{\xi_1}\xi_2^a$ denotes the change in the vector component ξ_2^a induced by the diffeomorphism ξ_1. Supertranslations and superrotations acting on the metric then form a closed algebra under the modified bracket.

For example, given a pair of vector fields ξ_i $(i = 1, 2)$ that generate a supertranslation f_i and a superrotation Y_i, one can show that

$$[\xi_1, \xi_2]_M = \xi_3, \tag{11}$$

where ξ_3 is a vector field that generates both a supertranslation $\hat{f}$ and a superrotation $\hat{Y}$ given by

$$\hat{f} = \frac{1}{2}f_1 D_A Y_2^A - \frac{1}{2}f_2 D_A Y_1^A + Y_1^A D_A f_2 - Y_2^A D_A f_1, \tag{12}$$

$$\hat{Y}^A = Y_1^B D_B Y_2^A - Y_2^B D_B Y_1^A. \tag{13}$$

A derivation of the above result is given in Appendix A. We note that this is the same as for the BMS_4 algebra at null infinity [29].

Another important ingredient that plays a central role in this work is dual supertranslation, which is a new set of asymptotic symmetries of gravity that has recently been uncovered [30]. Interestingly, dual supertranslations are not diffeomorphisms of any kind [31], and they have a natural interpretation as the magnetic dual of the standard BMS supertranslation [30–33]. In electromagnetism, magnetic large gauge symmetry is tied to the complexification of the large gauge transformation charge (see [34] for instance). Similarly in gravity, the appearance of dual supertranslation can be understood as the complexification of the BMS charge. Just like the BMS supertranslation charge $Q_f^{\mathcal{I}^+}$ can be written as the real part of a complex Weyl scalar,

$$Q_f^{\mathcal{I}^+} = \frac{1}{4\pi}\int_{\mathcal{I}^+_-} d^2z\sqrt{\gamma}f(z,\bar{z})\mathrm{Re}[\Psi_2^0(u,z,\bar{z})], \tag{14}$$

the dual supertranslation charge $\tilde{Q}_f^{\mathcal{I}^+}$ is associated to its imaginary part [30,31],

$$\tilde{Q}_f^{\mathcal{I}^+} = \frac{1}{4\pi}\int_{\mathcal{I}^+_-} d^2z\sqrt{\gamma}f(z,\bar{z})\mathrm{Im}[\Psi_2^0(u,z,\bar{z})]. \tag{15}$$

A prime example of a spacetime with a nontrivial global dual supertranslation charge is the Taub-NUT spacetime [35,36], which has been studied in detail in the context of dual supertranslation in [31]. There are also examples of asymptotically flat spacetimes with bulk dust configurations that lead to a nontrivial dual supertranslation at the null infinity, see Sec. III.D of [37].

More recently, it has been demonstrated by [23,24] that dual supertranslation charges (or dual diffeomorphism charges in general) can be computed using covariant phase space formalism in first-order formalism of gravity with the Holst action [38]. In the next section, we employ this method to compute the dual supertranslation charge on the future Schwarzschild horizon. This dual charge is then used along with the standard horizon supertranslation charge to compute the Dirac bracket algebra of horizon charges.

III. HORIZON CHARGES

We will now construct the supertranslation charges on the future horizon $\mathcal{H}^+$ assuming smoothness of the supertranslation parameter f.

Following [4], let us define Σ to be a spacelike hypersurface extending from a section of $\mathcal{I}^+$ to a section of the horizon $\mathcal{H}^+$. A charge Q^Σ associated with Σ breaks into two parts, one being on the horizon and the other on null infinity. These two parts of Q^Σ correspond to the two components of the boundary of Σ, $\partial\Sigma$.

$$Q^\Sigma = Q^{\mathcal{H}^+} + Q^{\mathcal{I}^+}. \tag{16}$$

In [23,24], the authors provide a formula for the variation of electric and magnetic charges associated with a vector field ξ [39]. The metric is varied inducing a variation of the connection 1-form $\omega^{\alpha\beta}$ of $\delta\omega^{\alpha\beta}$

$$\not{\delta} Q_E^\Sigma = \frac{1}{16\pi}\epsilon_{\alpha\beta\gamma\delta}\int_{\partial\Sigma}(i_\xi E^\gamma)\delta\omega^{\alpha\beta}\wedge E^\delta, \tag{17}$$

$$\not{\delta} Q_M^\Sigma = \frac{1}{8\pi}\int_{\partial\Sigma}(i_\xi E^\alpha)\delta\omega_{\alpha\beta}\wedge E^\beta, \tag{18}$$

where E^α is the vielbein, see Appendix B for details. Each of these break into two contributions $\not{\delta} Q^{\mathcal{H}^+}$ and $\not{\delta} Q^{\mathcal{I}^+}$. On the horizon, there is an advanced time coordinate v and the horizon contributions at time v_0 take the form

$$\not{\delta} Q_E^{\mathcal{H}^+} = \frac{1}{16\pi}\epsilon_{\alpha\beta\gamma\delta}\int_{\mathcal{H}^+_{v_0}}(i_\xi E^\gamma)\delta\omega^{\alpha\beta}\wedge E^\delta, \tag{19}$$

$$\not{\delta} Q_M^{\mathcal{H}^+} = \frac{1}{8\pi}\int_{\mathcal{H}^+_{v_0}}(i_\xi E^\alpha)\delta\omega_{\alpha\beta}\wedge E^\beta. \tag{20}$$

Throughout this paper, we will take the viewpoint that the black hole ultimately evaporates. Therefore, although there is a future boundary $\mathcal{H}^+$ to the horizon, we assume that there is no contribution to the charge there. If an horizon has a future endpoint, in classical general relativity it must be singular. We presume, in conformity with common practice, that this is not an issue and that quantum phenomena will take care of matters. We therefore take $\partial\mathcal{H}^+ \equiv \mathcal{H}^+_-$, the past endpoint of the horizon, and ignore all possible contributions of $\mathcal{H}^+_+$, the future endpoint of the horizon. [41].

Expressions for the horizon contributions in Bondi coordinates are derived in Appendix B. Taking ξ to be the supertranslation vector field

$$\xi = f\partial_v - \frac{1}{2}D^2 f\partial_r + \frac{1}{r}D^A f\partial_A, \tag{21}$$

we obtain the horizon supertranslation charge $\not{\delta} Q_f^{\mathcal{H}^+}$ from (B48) and the dual supertranslation charge $\not{\delta}\tilde{Q}_f^{\mathcal{H}^+}$ from (B68) to be

$$\not{\delta} Q_f^{\mathcal{H}^+} = \frac{M}{8\pi}\int_{\mathcal{H}^+_{v_0}} d^2\Theta\sqrt{\gamma}\left[D^A\left(\frac{f}{M}h_{vA} + (D_A f)h_{vr}\right)\right. \tag{22}$$

$$\left. -(D^A f)\partial_r h_{vA} + 2f h_{vv} + (D^2 f)h_{vr}\right], \tag{23}$$

$$\not{\delta}\tilde{Q}_f^{\mathcal{H}^+} = -\frac{1}{32\pi M}\int_{\mathcal{H}^+_{v_0}} d^2\Theta\sqrt{\gamma}(D^B f)\epsilon_A{}^C D^A h_{BC}. \tag{24}$$

ϵ^{AB} is the alternating tensor on the unit 2-sphere and take $\epsilon^{\theta\phi} = \frac{1}{\sin\theta}$.

For smooth functions everywhere, we can discard total derivatives in the integrand, and the supertranslation charge is then in exact agreement with that of [4]. After residual gauge fixing and using a combination of the constraints on $\mathcal{H}^+$, the supertranslation charge simplifies to the expression

$$\not{\delta} Q_f^{\mathcal{H}^+} = \frac{1}{16\pi M}\int_{\mathcal{H}^+} dv\, d^2\Theta\sqrt{\gamma}f(\Theta)D^A D^B\sigma_{AB}, \tag{25}$$

where $\sigma_{AB} = \frac{1}{2}\partial_v h_{AB}$ is the conjugate momentum of h_{AB}. The integral over the advanced time parameter v is taken from $\mathcal{H}^+_-$ to v_0. The phase space of the horizon $\mathcal{H}^+$ has the Dirac bracket [4],

$$\begin{aligned}&\{\sigma_{AB}(v,\Omega), h_{CD}(v',\Omega')\}_D \\ &\quad = 32\pi M^2\gamma_{ABCD}\delta(v-v')\delta(\Omega-\Omega'),\end{aligned} \tag{26}$$

where $\gamma_{ABCD} \equiv \gamma_{AC}\gamma_{BC} + \gamma_{AD}\gamma_{BC} - \gamma_{AB}\gamma_{CD}$ is proportional to the DeWitt metric [47].

Since we can integrate by parts freely without having to worry about boundary terms, we can move all covariant derivatives to act on f. As such, we can now identify the integrable horizon supertranslation charge $\delta Q_f^{\mathcal{H}^+}$ and dual supertranslation charge $\delta\tilde{Q}_f^{\mathcal{H}^+}$ as

$$\delta Q_f^{\mathcal{H}^+} \equiv \frac{1}{16\pi M}\int_{\mathcal{H}^+} dv\, d^2\Theta\sqrt{\gamma}(D^B D^A f)\sigma_{AB}, \tag{27}$$

$$\delta\tilde{Q}_f^{\mathcal{H}^+} \equiv -\frac{1}{32\pi M}\int_{\mathcal{H}^+_\pm} d^2\Theta\sqrt{\gamma}(D^B D^A f)\epsilon_A{}^C h_{BC}. \tag{28}$$

Notice that in this form, the dual supertranslation charge is related to supertranslation charge by the twisting procedure $h_{AB} \to \epsilon_A{}^C h_{CB}$ proposed in [32,33]. When we have smooth functions everywhere, $\not\delta Q_f^{\mathcal{H}^+} = \delta Q_f^{\mathcal{H}^+}$ and $\not\delta\tilde{Q}_f^{\mathcal{H}^+} = \delta\tilde{Q}_f^{\mathcal{H}^+}$, i.e. the charges are integrable.

IV. SUPERTRANSLATION CHARGE WITH POLES ON THE COMPLEX PLANE

We now extend the construction of previous section to allow for the possibility that the supertranslation parameters, f, have simple poles.

The easiest way to explore this possibility is to use complex stereographic coordinates $(z, \bar{z})$, defined as

$$z = e^{i\phi}\tan\frac{\theta}{2}, \qquad \bar{z} = e^{-i\phi}\tan\frac{\theta}{2}, \tag{29}$$

where θ and ϕ are the standard spherical coordinates on a unit sphere. The metric on the unit sphere in these coordinates is $\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$, $\gamma_{zz} = \gamma_{\bar{z}\bar{z}} = 0$. The integration measure on the sphere is

$$d^2\Theta\sqrt{\gamma} = d^2z\sqrt{\gamma}, \quad \text{with} \quad d^2z \equiv i dz\wedge d\bar{z}, \quad \text{and} \quad \sqrt{\gamma} = \gamma_{z\bar{z}}. \tag{30}$$

The notation has been organized such that d^2z is real. The alternating tensor is defined such that $\epsilon_{z\bar{z}} = i\sqrt{\gamma}$. The only nonvanishing Christoffel symbols are ${}^{(2)}\Gamma^z_{zz} = \frac{-2\bar{z}}{1+z\bar{z}}$ and ${}^{(2)}\Gamma^{\bar{z}}_{\bar{z}\bar{z}} = \frac{-2z}{1+z\bar{z}}$.

Let us compute the supertranslation charge $\not\delta Q_f^{\mathcal{H}^+}$ when $f(z, \bar{z})$ has a pole at some complex coordinate w, that is, $f = \frac{1}{z-w}$. After fully fixing the residual gauge freedom on $\mathcal{H}^+$, as in [4], we have

$$\begin{aligned} h_{vv} &= h_{vA} = 0, \\ h_{vr} &= \frac{1}{4M^2}[D^2-1]^{-1}D^B D^C h_{BC}, \\ \partial_r h_{vA} &= -\frac{1}{4M^2}D_A[D^2-1]^{-1}D^B D^C h_{BC} \\ &\quad + \frac{1}{4M^2}D^B h_{AB}, \end{aligned} \tag{31}$$

and the supertranslation charge (23) takes the form

$$\begin{aligned} \not\delta Q_f^{\mathcal{H}^+} = \frac{M}{8\pi}\int_{\partial\mathcal{H}^+} d^2z\sqrt{\gamma}\Big(&-(D^A f)\frac{1}{4M^2}D^B h_{AB} \\ &+ 2D_A(D^A f h_{vr})\Big). \end{aligned} \tag{32}$$

In obtaining this we have used (31) for $D_A h_{vr} + \partial_r h_{vA}$. Now consider the total derivative term $D_A(D^A f h_{vr})$. For $f = \frac{1}{z-w}$ we have,

$$\begin{aligned} &\int d^2z\sqrt{\gamma}D^A((D_A f)h_{vr}) \\ &\quad = i\int dz\wedge d\bar{z}(\partial_{\bar{z}}(h_{vr}\partial_z f) + \partial_z(h_{vr}\partial_{\bar{z}} f)) \end{aligned} \tag{33}$$

$$= -i\oint_w dz\, h_{vr}\partial_z f + i\oint_w d\bar{z}\, h_{vr}\partial_{\bar{z}} f \tag{34}$$

$$= -2\pi\partial_z h_{vr}|_{z=w}. \tag{35}$$

In the second line, the contour is a small circle taken counter-clockwise around $z = w$. The second term on the right-hand side (rhs) of the second line vanishes because $f = \frac{1}{z-w}$ satisfies the identity [48]

$$\partial_{\bar{z}} f = 2\pi\delta^2(z-w). \tag{36}$$

The contour of $\oint_w d\bar{z}$ is a small circle around $z = w$ and so does not pick up any contribution from the delta-function. In the first term of the second line since $\partial_z f = \frac{-1}{(z-w)^2}$, there is a contribution proportional to $\partial_z h_{vr}$ evaluated at w, which is the result (35). Substituting in the expression (31) for h_{vr} we find

$$\begin{aligned} \not\delta Q_f^{\mathcal{H}^+} &= -\frac{1}{16\pi M}\int_{\mathcal{H}^+} dv\, d^2z\sqrt{\gamma}(D^A f)D^B\sigma_{AB} \\ &\quad - \frac{1}{4M}\int_{-\infty}^{\infty} dv D_z[D^2-1]^{-1}D^B D^A\sigma_{AB}\Big|_{z=w}. \end{aligned} \tag{37}$$

Partial integration of the first term gives

$$\begin{aligned} \not\delta Q_f^{\mathcal{H}^+} &= \frac{1}{16\pi M}\int_{\mathcal{H}^+} dv\, d^2z\sqrt{\gamma}(D^B D^A f)\sigma_{AB} \\ &\quad - \frac{1}{4M}\int_{-\infty}^{\infty} dv D_z[D^2-1]^{-1}D^B D^A\sigma_{AB}\Big|_{z=w}. \end{aligned} \tag{38}$$

In (38), the first term vanishes since

$$\begin{aligned} &\int d^2z\sqrt{\gamma}D^B(\sigma_{AB}D^A f) \\ &\quad = \int d^2z(\partial_{\bar{z}}(\sigma_{zz}D^z f) + \partial_z(\sigma_{\bar{z}\bar{z}}D^{\bar{z}} f)) \end{aligned} \tag{39}$$

$$= -i\oint_w dz\gamma^{z\bar{z}}\sigma_{zz}\partial_{\bar{z}} f + i\oint_w d\bar{z}\gamma^{z\bar{z}}\sigma_{\bar{z}\bar{z}}\partial_z f \tag{40}$$

$$= 0. \tag{41}$$

In obtaining (41) we have again used $\partial_{\bar z} f = 2\pi\delta^2(z-w)$, the contour of $\oint_w dz$ is a circle around w, and $\sigma_{\bar z\bar z}\partial_z f = -\frac{1}{2}(z-w)^{-2}\partial_v h_{\bar z\bar z}$ does not have poles in $\bar z$.

We recognize the first term in (38) for general f to be the integrable supertranslation charge $\delta Q_f^{\mathcal{H}^+}$ (27). Thus, we find that a pole in f leads $\delta Q_f^{\mathcal{H}^+}$ to acquire a nonintegrable part $\mathcal{N}_f^{\mathcal{H}^+}$,

$$\not{\delta} Q_f^{\mathcal{H}^+} = \delta Q_f^{\mathcal{H}^+} + \mathcal{N}_f^{\mathcal{H}^+}, \tag{42}$$

where $\delta Q_f^{\mathcal{H}^+}$ is given by (27), and

$$\mathcal{N}_f^{\mathcal{H}^+} = -\frac{1}{4M}\int_{-\infty}^{\infty} dv D_z[D^2-1]^{-1}D^B D^A \sigma_{AB}\Big|_{z=w}. \tag{43}$$

This splitting into integrable and nonintegrable parts is, of course, not unique (see for instance [24]). Our choice is justified as first $\delta Q_f^{\mathcal{H}^+}$ is the horizon supertranslation charge in the absence of poles in f, and second $\mathcal{N}_f^{\mathcal{H}^+}$ has zero Dirac bracket with both $\delta Q_g^{\mathcal{H}^+}$ and $\delta\tilde{Q}_g^{\mathcal{H}^+}$ and so carries no degrees of freedom. We encountered the first observation at the end of Sec. III and we will demonstrate second in Appendix C.

V. DIRAC BRACKET BETWEEN CHARGES

We now compute the Dirac bracket $\{\delta Q_f^{\mathcal{H}^+}, \delta\tilde{Q}_g^{\mathcal{H}^+}\}_D$, where $f = \frac{1}{z-w}$ and g is assumed to be smooth. This bracket probes central terms of the algebra of charges. To see this, note that the charges have the expansions,

$$Q_f^{\mathcal{H}^+} = Q_f^{(h=0)} + \delta Q_f^{\mathcal{H}^+} + O(h^2), \tag{44}$$

$$\tilde{Q}_g^{\mathcal{H}^+} = \tilde{Q}_g^{(h=0)} + \delta\tilde{Q}_g^{\mathcal{H}^+} + O(h^2), \tag{45}$$

where $Q_f^{(h=0)}$ and $\tilde{Q}_g^{(h=0)}$ are the constant charges of the background metric and hence do not carry degrees of freedom. This gives,

$$\{Q_f^{\mathcal{H}^+}, \tilde{Q}_g^{\mathcal{H}^+}\}_D = \underbrace{\{\delta Q_f^{\mathcal{H}^+}, \delta\tilde{Q}_g^{\mathcal{H}^+}\}_D}_{\text{constant}} + O(h). \tag{46}$$

The constant term corresponds to the central charge of the charge algebra. This method using linearized charges is conceptually useful due to the presence of a nonintegrable term $\mathcal{N}_f^{\mathcal{H}^+}$, since its Dirac bracket with any linearized charge vanishes. We check the final result using another method in the next section.

Now let us compute $\{\delta Q_f^{\mathcal{H}^+}, \delta\tilde{Q}_g^{\mathcal{H}^+}\}_D$, with $f = \frac{1}{z-w}$ and g smooth. Using the expressions (27) and (28) and applying (26), we obtain

$$\begin{aligned}\{\delta Q_f^{\mathcal{H}^+}, \delta\tilde{Q}_g^{\mathcal{H}^+}\}_D &= \frac{-1}{2(16\pi M)^2}\left\{\int_{\mathcal{H}^+} dv\, d^2z\sqrt{\gamma}(D^B D^A f)\sigma_{AB}, \int_{\mathcal{H}^+_-} d^2z\sqrt{\gamma}(D^E D^C g)\epsilon_E{}^D h_{CD}\right\}_D \\ &= -\frac{1}{16\pi}\int_{\mathcal{H}^+_-} d^2z\sqrt{\gamma}(D^B D^A f)(D^E D^C g)\epsilon_E{}^D\gamma_{ABCD}.\end{aligned} \tag{47}$$

Rearranging D^B in (47) results in

$$\{\delta Q_f^{\mathcal{H}^+}, \delta\tilde{Q}_g^{\mathcal{H}^+}\}_D = -\frac{1}{16\pi}\int_{\mathcal{H}^+_-} d^2z\sqrt{\gamma}(D^B((D^A f)(D^E D^C g)\epsilon_E{}^D\gamma_{ABCD}) - (D^A f)(D^B D^E D^C g)\epsilon_E{}^D\gamma_{ABCD}). \tag{48}$$

Substituting in the expressions for $\epsilon_A{}^B$ and γ_{ABCD}, we can see that the first term is zero,

$$\begin{aligned}\int_{\mathcal{H}^+_-} d^2z\sqrt{\gamma}D^B((D^A f)(D^E D^C g)\epsilon_E{}^D\gamma_{ABCD}) &= \int_{\mathcal{H}^+_-} d^2z\partial_{\bar z}((D^z f)(D^{\bar z} D^{\bar z} g)\epsilon_{\bar z}{}^{\bar z}\gamma_{z\bar z z\bar z\bar z}) + \int_{\mathcal{H}^+_-} d^2z\partial_z((D^{\bar z} f)(D^z D^z g)\epsilon_z{}^z\gamma_{\bar z\bar z zz}) \\ &= -2\oint_w dz(\partial_{\bar z} f)(D^{\bar z} D^{\bar z} g)\gamma_{z\bar z} - 2\oint_w d\bar z(\partial_z f)(D^z D^z g)\gamma_{z\bar z} \\ &= 0.\end{aligned} \tag{49}$$

In obtaining this result, we have used the fact that the $\oint_w dz$ integral vanishes since its contour is a circle around w and does not intersect the singularity of the delta function $\partial_{\bar z} f = 2\pi\delta^2(z-w)$, and the $\oint_w d\bar z$ integral vanishes since $(\partial_z f)(D^z D^z g)\gamma_{z\bar z}$ does not have a pole in $\bar z$. We obtain

$$\{\delta Q_f^{\mathcal{H}^+},\delta\tilde{Q}_g^{\mathcal{H}^+}\}_D=\frac{1}{8\pi}\int_{\mathcal{H}^+_-}d^2z\sqrt{\gamma}\gamma_{z\bar{z}}{}^2((D^z f)(D^zD^{\bar{z}}D^{\bar{z}}g)\epsilon_{\bar{z}}{}^{\bar{z}}+(D^{\bar{z}}f)(D^{\bar{z}}D^zD^zg)\epsilon_z{}^z) \tag{50}$$

$$=\frac{-i}{8\pi}\int_{\mathcal{H}^+_-}d^2z((\partial_{\bar{z}}f)D^zD_{\bar{z}}^2g-(\partial_z f)D^{\bar{z}}D_{\bar{z}}^2g). \tag{51}$$

$$=\frac{-i}{8\pi}\int_{\mathcal{H}^+_-}d^2z\gamma^{z\bar{z}}((\partial_{\bar{z}}f)[D_{\bar{z}},D_z]D_zg+(\partial_{\bar{z}}f)D_zD_{\bar{z}}D_zg-(\partial_z f)[D_z,D_{\bar{z}}]D_{\bar{z}}g-(\partial_z f)D_{\bar{z}}D_zD_{\bar{z}}g). \tag{52}$$

The commutators are $[D_{\bar{z}},D_z]D_zg=\gamma_{z\bar{z}}D_zg$ and $[D_z,D_{\bar{z}}]D_{\bar{z}}g=\gamma_{z\bar{z}}D_{\bar{z}}g$. Thus we have

$$\{\delta Q_f^{\mathcal{H}^+},\delta\tilde{Q}_g^{\mathcal{H}^+}\}_D=\frac{-i}{8\pi}\int d^2z((\partial_{\bar{z}}f)D_zg-(\partial_z f)D_{\bar{z}}g+(\partial_{\bar{z}}f)D_zD_{\bar{z}}D^{\bar{z}}g-(\partial_z f)D_{\bar{z}}D_zD^zg). \tag{53}$$

For the last two terms in the parentheses, we have used $\gamma^{z\bar{z}}$ to purposely raise the index of the first derivative acting on g. This allows us to write the third covariant derivatives acting on g as partial derivatives,

$$\{\delta Q_f^{\mathcal{H}^+},\delta\tilde{Q}_g^{\mathcal{H}^+}\}_D=\frac{-i}{8\pi}\int d^2z((\partial_{\bar{z}}f)D_zg-(\partial_z f)D_{\bar{z}}g+(\partial_{\bar{z}}f)\partial_zD_{\bar{z}}D^{\bar{z}}g-(\partial_z f)\partial_{\bar{z}}D_zD^zg). \tag{54}$$

Now we partial integrate all $\partial_A f$'s inside the parentheses. Only the boundary terms survive since partial derivatives commute and

$$D_{\bar{z}}D^{\bar{z}}g-D_zD^zg=\gamma^{z\bar{z}}(\partial_{\bar{z}}\partial_zg-\partial_z\partial_{\bar{z}})g=0. \tag{55}$$

Therefore, we have via Stokes' theorem,

$$\{\delta Q_f^{\mathcal{H}^+},\delta\tilde{Q}_g^{\mathcal{H}^+}\}_D=\frac{-i}{8\pi}\int d^2z(\partial_{\bar{z}}(fD_zg)-\partial_z(fD_{\bar{z}}g)+\partial_{\bar{z}}(f\partial_zD_{\bar{z}}D^{\bar{z}}g)-\partial_z(f\partial_{\bar{z}}D_zD^zg)) \tag{56}$$

$$=-\frac{1}{8\pi}\oint_w\left(dz\frac{(D_zg+\partial_zD_{\bar{z}}D^{\bar{z}}g)}{z-w}+d\bar{z}\frac{(D_{\bar{z}}g+\partial_{\bar{z}}D_zD^zg)}{z-w}\right). \tag{57}$$

The $\oint_w d\bar{z}$ integral vanishes due to the absence of $\bar{z}$-poles. Now observe that we can use $[D_{\bar{z}},D_z]D_zg=\gamma_{z\bar{z}}D_zg$ to simplify

$$D_zg+\partial_zD_{\bar{z}}D^{\bar{z}}g=D_zg+D_zD_{\bar{z}}D^{\bar{z}}g \tag{58}$$

$$=D_zg+\gamma^{z\bar{z}}D_zD_{\bar{z}}D_zg \tag{59}$$

$$=D_zg+\gamma^{z\bar{z}}[D_z,D_{\bar{z}}]D_zg+\gamma^{z\bar{z}}D_{\bar{z}}D_zD_zg \tag{60}$$

$$=D^zD_zD_zg \tag{61}$$

and write

$$\{\delta Q_f^{\mathcal{H}^+},\delta\tilde{Q}_g^{\mathcal{H}^+}\}_D=-\frac{1}{8\pi}\oint_w dz\frac{D^zD_zD_zg}{z-w}. \tag{62}$$

The residue theorem then gives

$$\{\delta Q_f^{\mathcal{H}^+},\delta\tilde{Q}_g^{\mathcal{H}^+}\}_D=-\frac{i}{4}D^zD_z^2g\Big|_{z=w}. \tag{63}$$

VI. ANOTHER APPROACH TO THE COMPUTATION OF THE CENTRAL TERM

The result for the central term is new and has important implications. We will now reproduce the central term of the previous section using a completely different method.

We start from our expression (28) for the integrable variation $\delta\tilde{Q}_f^{\mathcal{H}^+}$ of dual supertranslation charge, which reads

$$\delta\tilde{Q}_g^{\mathcal{H}^+}=-\frac{1}{32\pi M}\int_{\mathcal{H}^+_-}d^2z\sqrt{\gamma}(D^BD^Ag)\epsilon_A{}^Ch_{BC}, \tag{64}$$

and invoke Eq. (3.4) in the work of Barnich and Troessaert [29],

$$\{Q_f^{\mathcal{H}^+},\tilde{Q}_g^{\mathcal{H}^+}\}_D=\delta_f\tilde{Q}_g^{\mathcal{H}^+}, \tag{65}$$

where $\delta_f \tilde{Q}_g$ denotes taking the expression (64) for $\delta\tilde{Q}_g^{\mathcal{H}^+}$ and replacing h_{AB} with a diffeomorphism constructed from f with f being only dependent on z and $\bar{z}$. A general diffeomorphism is of the form $h_{ab} = \nabla_a \xi_b + \nabla_b \xi_a$. Let $\xi_a = \partial_a f$. Then restricting h_{ab} to the sphere gives

$$h_{BC} \to 2M(2D_B D_C f - \gamma_{BC} D^2 f). \tag{66}$$

This leads to the expression

$$\{Q_f^{\mathcal{H}^+}, \tilde{Q}_g^{\mathcal{H}^+}\}_D = -\frac{1}{16\pi}\int d^2z\sqrt{\gamma}(D^B D^A g)\epsilon_A{}^C(2D_B D_C f - \gamma_{BC} D^2 f) \tag{67}$$

$$= -\frac{1}{8\pi}\int d^2z\sqrt{\gamma}(D^B D^A g)\epsilon_A{}^C D_B D_C f + \frac{1}{16\pi}\int d^2z\sqrt{\gamma}(D^B D^A g)\epsilon_{AB} D^2 f. \tag{68}$$

The second term on the rhs is zero, since $D^B D^A g$ is symmetric and ϵ_{AB} is antisymmetric. We are just left with the first term,

$$\{Q_f^{\mathcal{H}^+}, \tilde{Q}_g^{\mathcal{H}^+}\}_D = -\frac{1}{8\pi}\int d^2z\sqrt{\gamma}(D^B D^A g)\epsilon_A{}^C D_B D_C f. \tag{69}$$

Rewrite this as the sum of two terms

$$\{Q_f^{\mathcal{H}^+}, \tilde{Q}_g^{\mathcal{H}^+}\}_D = -\frac{1}{8\pi}(X + Y), \tag{70}$$

with

$$X \equiv \int d^2z\sqrt{\gamma} D_B((D^B D^A g)\epsilon_A{}^C D_C f), \tag{71}$$

$$Y \equiv -\int d^2z\sqrt{\gamma}(D^2 D^A g)\epsilon_A{}^C D_C f. \tag{72}$$

X is of the form of an integral over the sphere of the divergence of a vector field V^A on the sphere. So,

$$\int d^2z\sqrt{\gamma} D_B V^B = i\int dz \wedge d\bar{z}\gamma_{z\bar{z}}(D_z V^z + D_{\bar{z}} V^{\bar{z}}) \tag{73}$$

$$= i\int dz \wedge d\bar{z}(\partial_z V_{\bar{z}} + \partial_{\bar{z}} V_z) \tag{74}$$

$$= i\oint_w d\bar{z} V_{\bar{z}} - i\oint_w dz V_z. \tag{75}$$

In the second line, we have used the fact that the only nonvanishing Christoffel symbols are Γ^z_{zz} and $\Gamma^{\bar{z}}_{\bar{z}\bar{z}}$ to write $D_z V_{\bar{z}} = \partial_z V_{\bar{z}}$ and $D_{\bar{z}} V_z = \partial_{\bar{z}} V_z$. Finally we use Stokes' theorem to write X as

$$X = i\oint_w d\bar{z}(D_{\bar{z}} D^A g)\epsilon_A{}^C D_C f - i\oint dz(D_z D^A g)\epsilon_A{}^C D_C f. \tag{76}$$

Everything is smooth except for $f = \frac{1}{z-w}$, so the first term with $\oint d\bar{z}$ never sees a pole in $\bar{z}$ and therefore vanishes. Writing out the second term while noting that the only nonvanishing components of $\epsilon_A{}^B$ are $\epsilon_z{}^z = -\epsilon_{\bar{z}}{}^{\bar{z}} = i$, we obtain

$$X = \oint dz(D_z D^z g)\partial_z f - \oint dz(D_z D^{\bar{z}} g)\partial_{\bar{z}} f. \tag{77}$$

The second term vanishes since it has $\partial_{\bar{z}} f = 2\pi\delta^2(z - w)$ and the contour never meets w. We can partial integrate the first term and using the residue theorem and $f = \frac{1}{z-w}$ obtain

$$X = -\oint dz(\partial_z D_z D^z g) f \tag{78}$$

$$= -\oint dz \frac{\partial_z D_z D^z g}{z - w} \tag{79}$$

$$= -2\pi i \partial_z D_z D^z g|_{z=w}. \tag{80}$$

Now we turn to Y in (70), which reads

$$Y = -\int d^2z\sqrt{\gamma}(D^2 D^A g)\epsilon_A{}^C D_C f \tag{81}$$

$$= -\int d^2z\sqrt{\gamma} D_C((D^2 D^A g)\epsilon_A{}^C f) + \int d^2z\sqrt{\gamma}(D_C D^2 D^A g)\epsilon_A{}^C f. \tag{82}$$

One quickly see that the second term vanishes as

$$\epsilon_A{}^C D_C D^2 D^A g = \epsilon^{AC} D_C D^2 D_A g \tag{83}$$

$$= \epsilon^{AC} D_C [D^2, D_A] g + \epsilon^{AC} D_C D_A D^2 g \tag{84}$$

$$= \epsilon^{AC} D_C D_A g + \epsilon^{AC} D_C D_A D^2 g \tag{85}$$

$$= 0, \tag{86}$$

since both $D_C D_A g$ and $D_C D_A D^2 g$ are symmetric in A and C and $[D^2, D_A] g = D_A g$. So we are left with just

$$Y = -\int d^2z\sqrt{\gamma} D_C((D^2 D^A g)\epsilon_A{}^C f), \tag{87}$$

which again is of the form (75), so we can writing the explicit form of f as $\frac{1}{z-w}$, and $\epsilon_{z\bar{z}} = -\epsilon_{\bar{z}z} = i\gamma_{z\bar{z}}$ we find

$$Y = -i \oint_w d\bar{z}(D^2 D^z g)\epsilon_{z\bar{z}} f + i \oint_w dz (D^2 D^{\bar{z}} g)\epsilon_{\bar{z}z} f \tag{88}$$

$$= \oint_w d\bar{z} \frac{(D^2 D_{\bar{z}} g)}{z-w} + \oint_w dz \frac{(D^2 D_z g)}{z-w}. \tag{89}$$

Explicitly writing f as $\frac{1}{z-w}$, and $\epsilon_{z\bar{z}} = -\epsilon_{\bar{z}z} = i\gamma_{z\bar{z}}$. The first term is zero since there are no poles in $\bar{z}$, and the second term yields the residue at $z = w$,

$$Y = 2\pi D^2 D_z g|_{z=w}. \tag{90}$$

Collecting the results (80) and (90) and plugging them into (70), we obtain

$$\{Q_f^{\mathcal{H}^+}, \tilde{Q}_g^{\mathcal{H}^+}\}_D = -\frac{1}{8\pi}(X+Y) \tag{91}$$

$$= -\frac{i}{4}(-\partial_z D_z D^z g + D^2 D_z g)\Big|_{z=w}. \tag{92}$$

Simplifying

$$-\partial_z D_z D^z g + D^2 D_z g$$
$$= -\partial_z D^z D_z g + D^2 D_z g \tag{93}$$

$$= -D_z D^z D_z g + D_z D^z D_z g + D_{\bar{z}} D^{\bar{z}} D_z g \tag{94}$$

$$= D_{\bar{z}} D^{\bar{z}} D_z g \tag{95}$$

$$= D^z D_z D_z g, \tag{96}$$

where in the second line we have used $\partial_z D^z D_z g = D_z D_{\bar{z}} D^{\bar{z}} g = D_z D^z D_z g$. This finally leads to

$$\{Q_f^{\mathcal{H}^+}, \tilde{Q}_g^{\mathcal{H}^+}\}_D = -\frac{i}{4} D^z D_z^2 g\Big|_{z=w}. \tag{97}$$

This is in complete agreement with our earlier result (63) for the infinitesimal bracket $\{\delta Q_f^{\mathcal{H}^+}, \delta\tilde{Q}_g^{\mathcal{H}^+}\}_D$.

What are the implications of this central term? It is usually understood that this is indicative of an anomaly in the theory which must be canceled in order for the theory to make sense. In order to understand how to remove the central term in the supertranslation algebra we will first take a look at the simpler case of the electromagnetic charges of large gauge transformation which is discussed in the next section.

VII. ELECTROMAGNETISM

Consider now electromagnetic soft charges on the Schwarzschild horizon. Our discussion is parallel to the case of future null infinity $\mathcal{I}^+$ since both $\mathcal{H}^+$ and $\mathcal{I}^+$ are null hypersurfaces. We refer the reader to [42,43] for a treatment of the electromagnetic case on $\mathcal{I}^+$.

Just like the BMS charges, the electromagnetic charges split into the $\mathcal{H}^+$ and $\mathcal{I}^+$ contributions (16). Horizon contributions to the (soft) electric and magnetic charges are given by

$$\mathcal{Q}_\lambda^{\mathcal{H}^+} = \int_{\mathcal{H}^+} d\lambda \wedge *F, \tag{98}$$

$$\tilde{\mathcal{Q}}_\lambda^{\mathcal{H}^+} = \int_{\mathcal{H}^+} d\lambda \wedge F, \tag{99}$$

where λ is an arbitrary function on the sphere. We use the curly letter $\mathcal{Q}$ to distinguish these charges from the diffeomorphism charges.

We can write these charges as integrals over the null surface $\mathcal{H}^+$ subject to the same boundary conditions as described in Sec. III. In the complex coordinates (29)

$$\mathcal{Q}_\lambda^{\mathcal{H}^+} = -i\int_{\mathcal{H}^+} dv\, d^2z(\partial_{\bar{z}}\lambda(*F)_{vz} - \partial_z\lambda(*F)_{v\bar{z}}) \tag{100}$$

$$= -\int_{\mathcal{H}^+} dv\, d^2z(F_{vz}\partial_{\bar{z}}\lambda + F_{v\bar{z}}\partial_z\lambda), \tag{101}$$

$$\tilde{\mathcal{Q}}_\lambda^{\mathcal{H}^+} = -i\int_{\mathcal{H}^+} dv\, d^2z(F_{vz}\partial_{\bar{z}}\lambda - F_{v\bar{z}}\partial_z\lambda). \tag{102}$$

Alternatively we can write the charges as integrals over section of the horizon at some instant of advanced time v. In the temporal gauge $A_v = 0$, we have $F_{vz} = \partial_v A_z$ and

$$\tilde{\mathcal{Q}}_\lambda^{\mathcal{H}^+} = i\int_{\mathcal{H}^+_-} d^2z(A_z\partial_{\bar{z}}\lambda - A_{\bar{z}}\partial_z\lambda). \tag{103}$$

The relevant Dirac bracket is [43] (see [12,21] for details on the symplectic structure),

$$\{\mathcal{Q}_\lambda^{\mathcal{H}^+}, A_z\}_D = -\partial_z\lambda \tag{104}$$

using which we obtain

$$\{\mathcal{Q}_\lambda^{\mathcal{H}^+}, \tilde{\mathcal{Q}}_\sigma^{\mathcal{H}^+}\}_D = \int_{\mathcal{H}^+_-} d^2z\sqrt{\gamma}\epsilon^{AB}\partial_A\lambda\partial_B\sigma \tag{105}$$

$$= \int_{S^2} d\lambda \wedge d\sigma. \tag{106}$$

For λ with singularities in z, this gives rise to a central term in the algebra, just as in the case of gravity.

To get rid of the central term in the algebra, one may imagine that there exists a boundary theory on $\mathcal{H}^+$ whose purpose is to cancel the anomalous contribution suggested by the central charge discussed above. For this purpose, let us consider a $U(1)\times U(1)$ Chern-Simons theory with two independent 1-form fields a and $\tilde{a}$ on a null surface Σ,

$$S = \frac{k}{4\pi}\int_{\Sigma} a \wedge d\tilde{a}. \tag{107}$$

Under an electric large gauge transformation a and $\tilde{a}$ transform as

$$a \to a + d\phi, \tag{108}$$

$$\tilde{a} \to \tilde{a}, \tag{109}$$

and under a magnetic large gauge transformation they transform as

$$a \to a, \tag{110}$$

$$\tilde{a} \to \tilde{a} + d\chi. \tag{111}$$

From the action we find the equations of motion to be $da = 0$ and $d\tilde{a} = 0$. Variation of the action yields

$$\delta S = \frac{k}{4\pi}\int_{\Sigma}(\delta a \wedge d\tilde{a} + a \wedge d\delta\tilde{a}) \tag{112}$$

$$= \frac{k}{4\pi}\int_{\Sigma}(\delta a \wedge d\tilde{a} - da \wedge \delta\tilde{a}) + \frac{k}{4\pi}\int_{\partial\Sigma} a \wedge \delta\tilde{a}, \tag{113}$$

from which we obtain the symplectic potential as,

$$\theta(a, \tilde{a}, \delta a, \delta\tilde{a}) = \frac{k}{4\pi} a \wedge \delta\tilde{a}. \tag{114}$$

Accordingly, the symplectic current density is

$$\omega(a, \tilde{a}, \delta_1 a, \delta_1\tilde{a}, \delta_2 a, \delta_2\tilde{a}) = \frac{k}{4\pi}(\delta_1 a \wedge \delta_2\tilde{a} - \delta_2 a \wedge \delta_1\tilde{a}). \tag{115}$$

Since there are two types of large gauge transformations, we have two integrable charge variations. One is the electric charge,

$$\delta\mathcal{Q}_\phi = \int_{\partial\Sigma}\omega(a, \tilde{a}, \delta a, \delta\tilde{a}, d\phi, 0) \tag{116}$$

$$= -\frac{k}{4\pi}\int_{\partial\Sigma} d\phi \wedge \delta\tilde{a}, \tag{117}$$

the other is the magnetic charge,

$$\delta\tilde{\mathcal{Q}}_\chi = \int_{\partial\Sigma}\omega(a, \tilde{a}, \delta a, \delta\tilde{a}, 0, d\chi) \tag{118}$$

$$= \frac{k}{4\pi}\int_{\partial\Sigma}\delta a \wedge d\chi. \tag{119}$$

We can compute the algebra using either one of the variations,

$$\{\mathcal{Q}_\phi, \tilde{\mathcal{Q}}_\chi\}_D = \delta_\phi\tilde{\mathcal{Q}}_\chi = -\delta_\chi\mathcal{Q}_\phi, \tag{120}$$

and one can see that we get the same answer for both cases,

$$\{\mathcal{Q}_\phi, \tilde{\mathcal{Q}}_\chi\}_D = -\frac{k}{4\pi}\int_{\partial\Sigma} d\phi \wedge d\chi. \tag{121}$$

The electric-electric and magnetic-magnetic brackets vanish regardless of the presence of poles,

$$\{\mathcal{Q}_\phi, \mathcal{Q}_{\phi'}\}_D = 0, \tag{122}$$

$$\{\tilde{\mathcal{Q}}_\chi, \tilde{\mathcal{Q}}_{\chi'}\}_D = 0. \tag{123}$$

Therefore, one finds the algebra to be exactly parallel to that of standard and dual large gauge transformation charges on the horizon. The algebra (121), (122) and (123) tells us that putting a $U(1) \times U(1)$ Chern-Simons theory with the proper choice of the level k on the horizon, we can get rid of the central term obtained earlier in the standard and dual large gauge transformation algebra.

Chern-Simons theory is a topological theory and as such is independent of the metric. This is how it is possible to have a holographic theory defined on the null surface forming the horizon. There is no obstacle to theory being defined on a surface with a degenerate metric. A further benefit is that, being independent of the metric, the theory has vanishing energy-momentum tensor and so does not affect the spacetime geometry.

VIII. GRAVITATIONAL CHERN-SIMONS THEORY

Gravity in three dimensions is a topological theory. Suppose one starts from the Einstein action, with or without a cosmological term, and count up the number of physical degrees of freedom at each point in spacetime. In d-dimensions the metric has $\frac{1}{2}d(d+1)$ components. Diffeomorphisms are related to first class constraints generated by a vector fields that subtract out $2d$ degrees of freedom. The total number of physical degrees of freedom is therefore $\frac{1}{2}d(d-3)$. So in $d = 3$ there are no local degrees of freedom. We should therefore expect to find a topological gravitational theory that is independent of any metric. The Einstein action is not such a construct. However, Witten [27] found a Chern-Simons theory that is equivalent to the Einstein theory provided some of its fields are identified with the metric. The Chern-Simons theory is independent of any metric and can therefore be formulated consistently on null surfaces where the spacetime metric is degenerate. In more conventional theories the necessity of using an inverse metric prevents their formulation on null surfaces.

A. Chern-Simons actions

We now briefly describe Witten's gravitational Chern-Simons theory. The ingredients are a basis of one-forms $e^a = e^a_i dx^i$, a connection one-form $\omega^{ab} = \omega_i{}^{ab} dx^i$ and a dimensionful real parameter λ that is in many way analogous to a cosmological constant. We will see in Sec. VIII D that the equations of motion fixes $-\lambda$ to be the curvature of a section on the horizon. The indices $i, j\ldots$ are spacetime indices whereas $a, b\ldots$ are tangent space indices. Spacetime indices never need to be raised or lowered, however we do need as an extra piece of spacetime structure, the alternating symbol ϵ^{ijk}. By contrast, tangent space indices are raised or lowered using the Lorentz metric η_{ab}. To construct a three-dimensional spacetime, we construct its metric g_{ij} using $e^a_i e^b_j \eta_{ab}$ where $\eta_{ab} = \mathrm{diag}(-{+}{+})$. In Witten's approach to gravity in three spacetime dimensions, this identification is used to conclude the equivalence with the Einstein theory. A Chern-Simons theory needs a gauge group $\mathbf{G}$, and in the case $\lambda = 0$, $\mathbf{G}$ is chosen to be $ISO(2,1)$. If $\lambda < 0$, $\mathbf{G}$ is $SO(3,1)$ and if $\lambda > 0$, $\mathbf{G}$ is $SO(2,2)$. Note that in last case the gauge group can be factorized as $SO(2,2) \equiv SL(2,\mathbb{R}) \otimes SL(2,\mathbb{R})$. The case of $SO(3,1)$ cannot be factorized, but it can be regarded as a complex group $SL(2,\mathbb{C})$.

One can write an all encompassing gauge field $A_i = e^a_i P_a + \omega^a_i J_a$ where $\omega^a_i = \frac{1}{2}\epsilon^{abc}\omega_{i\,bc}$ and P_a and J_a are the generators of the gauge group. They have the commutation relations

$$[J_a, J_b] = \epsilon_{abc} J^c, \quad [J_a, P_b] = \epsilon_{abc} P^c, \quad [P_a, P_b] = \lambda \epsilon_{abc} J^c \tag{124}$$

For arbitrary λ, the Killing form is given by

$$\langle J_a, J_b \rangle = 0, \quad \langle J_a, P_b \rangle = \eta_{ab}, \quad \langle P_a, P_b \rangle = 0. \tag{125}$$

However, when $\lambda \neq 0$, $\mathbf{G}$ factorizes, a second Killing form exists

$$\langle J_a, J_b \rangle = \eta_{ab}, \quad \langle J_a, P_b \rangle = 0, \quad \langle P_a, P_b \rangle = \lambda \eta_{ab}. \tag{126}$$

If $\lambda = 0$, the second Killing form is degenerate and not particularly useful.

From these relations we can construct Chern-Simons theory from the general expression

$$I_{\mathrm{CS}} = \frac{k}{4\pi} \int tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \tag{127}$$

If $\lambda \neq 0$, we can construct two different actions using the two different Killing forms. For any value of λ, we can construct an "electric" theory using the Killing form of (125). In terms of the differential forms e^a and ω^a we get

$$I_{\mathrm{electric}} = \frac{k}{2\pi} \int 2e^a \wedge d\omega_a + \epsilon_{abc} e^a \wedge \omega^a \wedge \omega^c + \frac{1}{3}\lambda \epsilon_{abc} e^a \wedge e^b \wedge e^c. \tag{128}$$

or, perhaps more conveniently for some of the following calculations, in terms of components

$$I_{\mathrm{electric}} = \frac{k}{2\pi} \int d^3x\, \epsilon^{ijk}\, e^a_i \times \left(2\partial_j \omega_{ka} + \epsilon_{abc} \omega^b_j \omega^c_k + \frac{1}{3}\lambda \epsilon_{abc} e^b_j e^c_k \right). \tag{129}$$

When $\lambda \neq 0$, we can use the alternative Killing form (126) to construct a different action, the "magnetic" action

$$I_{\mathrm{magnetic}} = \frac{\tilde{k}}{\pi} \int \omega^a \wedge d\omega_a + \frac{1}{3}\epsilon_{abc} \omega^a \wedge \omega^b \wedge \omega^c + \lambda e^a \wedge de_a + \lambda \epsilon_{abc} \omega^a \wedge e^b \wedge e^c. \tag{130}$$

This too can be more conveniently for practical calculations be written in terms of components as

$$I_{\mathrm{magnetic}} = \frac{\tilde{k}}{\pi} \int d^3x\, \epsilon^{ijk} \left(\omega^a_i \left(\partial_j \omega_{ka} + \frac{1}{3}\epsilon_{abc} \omega^b_j \omega^c_k \right) + \lambda e^a_i \partial_j e_{ka} + \lambda \epsilon_{abc} \omega^a_i e^b_j e^k_c \right). \tag{131}$$

Both the electric and the magnetic action have the same equations of motion and the same gauge invariance. The equation of motion from variation e^a in the electric action is

$$d\omega^a + \frac{1}{2}\epsilon^{abc} \omega_b \wedge \omega_c + \frac{1}{2}\lambda \epsilon_{abc} e^b \wedge e^c = 0. \tag{132}$$

It is the analog of the Einstein equation and specifies the curvature of the connection ω^a. Variation of ω^a in the electric action gives

$$de^a + \epsilon^{abc} \omega_b \wedge e_c = 0 \tag{133}$$

which shows that the connection is torsion-free. For the magnetic action, it is the variation of e^a that specifies the curvature of the connection and the variation of ω^a that tells us that it is torsion-free. It is in this sense that these two actions are dual to each other.

The gauge transformations are of two types. The first is labeled by a tangent-space vector ρ^a. The gauge variations of e^a and ω^a are

$$\delta e^a_i = -\partial_i \rho^a - \epsilon^{abc} \omega_{ib} \wedge \rho_c \tag{134}$$

and

$$\delta\omega_i^a = -\lambda\epsilon^{abc}e_{ib}\rho_c. \quad (135)$$

The second gauge transformation is generated by a second vector τ^a. The resulting gauge variations are

$$\delta e_i^a = -\epsilon^{abc}e_{ib} \wedge \tau_c \quad (136)$$

and

$$\delta\omega_i^a = -\partial_i\tau^a - \epsilon^{abc}\omega_{ib} \wedge \tau_c. \quad (137)$$

After recalling that one has dualized the spin connection, one observes that the τ-transformations are just local Lorentz rotations.

The nature of diffeomorphisms is not quite so straightforward. Suppose that one has a diffeomorphism generated by an infinitesimal vector field v^i. The the variation of the components of the basis of 1-forms is

$$\delta e_i^a = -v^k(\partial_k e_i^a - \partial_i e_k^a) - \partial_i(v^k e_k^a) \quad (138)$$

Similarly, the variation of the spin connection is

$$\delta\omega_i^a = -v^k(\partial_k\omega_i^a - \partial_i\omega_k^a) - \partial_i(v^k\omega_k^a). \quad (139)$$

We now see how to find a diffeomorphism in terms of ρ^a and τ^a. Setting

$$\rho^a = v^k e_k^a \quad \text{and} \quad \tau^a = v^k\omega_k^a \quad (140)$$

reproduces what is expected for the transformations of both e^a and ω^a under a diffeomorphism.

B. The charges

We now need to find the soft charges resulting from this pair of actions. The calculation is routine in the covariant phase space formalism. First, one performs a variation of the action in terms of the variation of the fields δe^a and $\delta\omega^a$. The bulk term then gives the usual equations of motion which we have already described. However, there is also a boundary term, the symplectic potential θ. For our actions we find for the electric case

$$\theta_{\text{electric}} = -\frac{k}{\pi}\omega^a \wedge \delta\omega_a \quad (141)$$

and for the magnetic case

$$\theta_{\text{magnetic}} = -\frac{\tilde{k}}{\pi}(\omega^a \wedge \delta\omega_a + \lambda e^a \wedge \delta e_a). \quad (142)$$

Given a symplectic potential, one finds the symplectic form Ω by carrying out a second variation in θ of the fields, $\delta' e^a$ and $\delta'\omega^a$, antisymmetrizing over the two variations and integrating the resultant 2-form over a spacelike surface Σ. For the electric action we find

$$\Omega_{\text{electric}} = -\frac{k}{\pi}\int_\Sigma \delta e^a \wedge \delta'\omega_a + \delta\omega^a \wedge \delta' e_a \quad (143)$$

and for the magnetic case

$$\Omega_{\text{magnetic}} = -\frac{2\tilde{k}}{\pi}\int_\Sigma \delta'\omega^a \wedge \delta\omega_a + \lambda\delta' e^a \wedge \delta e_a. \quad (144)$$

The charges are now found by setting the second variation $\delta' e^a$ and $\delta'\omega^a$ to be pure gauge transformations determined by ρ' and τ'. Now substituting these variations into the symplectic form and using the equations of motion, one finds that the integral for Ω collapse into boundary terms giving the variation of the charges conjugate to ρ' and τ' on $\partial\Sigma$ under the variation of the fields δe^a and $\delta\omega^a$.

For the electric case, we find

$$\delta Q^E_{\rho,\tau} = -\frac{k}{\pi}\int_{\partial\Sigma}\tau_a\delta e^a + \rho_a\delta\omega^a \quad (145)$$

and for the magnetic case

$$\delta Q^M_{\rho,\tau} = -\frac{2\tilde{k}}{\pi}\int_{\partial\Sigma}\tau_a\delta\omega^a + \lambda\rho_a\delta e^a. \quad (146)$$

Both of these charges are integrable, and so we will define the charges to be

$$Q^E_{\rho,\tau} = -\frac{k}{\pi}\int_{\partial\Sigma}\tau_a e^a + \rho_a\omega^a \quad (147)$$

for the electric case and

$$Q^M_{\rho,\tau} = -\frac{2\tilde{k}}{\pi}\int_{\partial\Sigma}\tau_a\omega^a + \lambda\rho_a e^a \quad (148)$$

for the magnetic case.

A knowledge of the symplectic form allows one to compute the Dirac bracket of various quantities of importance in the theory. For reasons that will be explained later, we do this now for just the electric theory. On the sphere coordinatized by the complex coordinate z, the (electric) symplectic form becomes

$$\Omega_{\text{electric}} = \frac{ik}{\pi}\int d^2z(\delta e_z^a\delta'\omega_{\bar{z}a} - \delta e_{\bar{z}}^a\delta'\omega_{za} - (\delta \leftrightarrow \delta')). \quad (149)$$

From this it follows that only nontrivial Dirac brackets are

$$\{e_z^a(z,\bar{z}), \omega_{\bar{z}}^b\} = -\frac{i\pi}{k}\eta^{ab}\delta^2(z-z') \quad (150)$$

and its complex conjugate.

We can use these expressions to compute the bracket of the charges with the field variables. Modulo the equations of motion, these brackets should reproduce the gauge transformations of the fields. Explicit calculation reveals that

$$\{Q^E, e_i^a\} = -\partial_i \rho^a - \epsilon^{abc} e_{ib} \tau_c - \epsilon^{abc} \omega_{ib} \rho_c,$$
$$\{Q^E, \omega_i^a\} = -\partial_i \tau^a - \epsilon^{abc} \omega_{ib} \tau_c - \lambda \epsilon^{abc} e_{ib} \rho_c, \quad (151)$$

as expected. Similarly, the brackets of the magnetic charges with e^a and ω^a are

$$\{Q^M, e_i^a\} = 2\frac{\tilde{k}}{k}(-\partial_i \tau^a - \epsilon^{abc} \omega_{ib} \tau_c - \lambda \epsilon^{abc} e_{ib} \rho_c),$$
$$\{Q^M, \omega_i^a\} = 2\lambda \frac{\tilde{k}}{k}(-\partial_i \rho^a - \epsilon^{abc} \omega_{ib} \rho_c - \epsilon^{abc} e_{ib} \tau_c). \quad (152)$$

Again, these are gauge transformations but with the role of τ and ω interchanged and rescaled.

C. Charge algebra

We now compute the charge algebra. From here on we are going to work exclusively with the electric theory. One might then wonder what the point of introducing the magnetic theory is. The answer is that allows us to find the magnetic charges in a straightforward fashion. Had we not done so, finding the magnetic charges would have been an involved, convoluted and obscure process. The magnetic charges still exist in the electric theory just as electric charges exist in the magnetic theory. However, one needs to make a choice of symplectic form at some point and we choose the electric picture.

1. Electric-electric bracket

The bracket between two electric charges is

$$\{Q^E_{\tau,\rho}, Q^E_{\tau',\rho'}\} = -\frac{k}{\pi}\int_{\partial\Sigma} (\epsilon^{abc}(\tau'_b \tau_c + \lambda \rho'_b \rho_c) e_a + \epsilon^{abc}(\tau'_b \rho_c - \tau_b \rho'_c)\omega_a) - \frac{k}{\pi}\int_{\partial\Sigma} (\rho^a d\tau'_a + \tau^a d\rho'_a). \quad (153)$$

Recall from (147) that the integrated electric charge takes the form

$$Q^E_{\tau,\rho} = -\frac{k}{\pi}\int_{\partial\Sigma} (\tau_a e^a + \rho_a \omega^a). \quad (154)$$

Comparing this with the result for the bracket, we observe that

$$\{Q^E_{\tau,\rho}, Q^E_{\tau',\rho'}\} = Q^E_{\tau'',\rho''} - \frac{k}{\pi}\int_{\partial\Sigma} (\rho^a d\tau'_a + \tau^a d\rho'_a), \quad (155)$$

where

$$\tau''^a = \epsilon^{abc}(\tau'_b \tau_c + \lambda \rho'_b \rho_c), \quad (156)$$

$$\rho''^a = \epsilon^{abc}(\tau'_b \rho_c - \tau_b \rho'_c). \quad (157)$$

With $\rho^a = v^i e_i^a$, the central term is zero whenever $\tau^a = 0$ or $\tau^a = v^i \omega_i^a$.

2. Electric-magnetic bracket

The bracket between electric and magnetic charges can be obtained in two distinct ways since

$$\{Q^E_{\tau,\rho}, Q^M_{\tau',\rho'}\} = \delta^E_{\tau,\rho} Q^M_{\tau',\rho'} = -\delta^M_{\tau',\rho'} Q^E_{\tau,\rho}, \quad (158)$$

where here $\delta^E_{\tau,\rho}$ denotes the gauge transformation generated by $Q^E_{\rho,\tau}$ given in (151) and $\delta^M_{\tau',\rho'}$ denotes the gauge transformation generated by $Q^M_{\rho,\tau}$ given in (152). These two results must agree. Let us first compute

$$\delta^E_{\tau,\rho} Q^M_{\tau',\rho'} = -\frac{2\tilde{k}}{\pi}\int_{\partial\Sigma} (\epsilon^{abc}(\tau'_b \tau_c + \lambda \rho'_b \rho_c)\omega_a + \lambda \epsilon^{abc}(\tau'_b \rho_c - \tau_b \rho'_c) e_a) - \frac{2\tilde{k}}{\pi}\int_{\partial\Sigma} (\tau_a d\tau'^a + \lambda \rho_a d\rho'^a). \quad (159)$$

Comparing this to the magnetic charge (148), we can see that

$$\delta^E_{\tau,\rho} Q^M_{\tau',\rho'} = \tilde{Q}^M_{\tau'',\rho''} - \frac{2\tilde{k}}{\pi}\int_{\partial\Sigma} (\tau_a d\tau'^a + \lambda \rho_a d\rho'^a) \quad (160)$$

where τ'' and ρ'' are defined in (156) and (157).

Let us next use (152) to compute $-\delta^M_{\tau',\rho'} Q^E_{\tau,\rho}$. We obtain

$$-\delta^M_{\tau',\rho'} Q^E_{\tau,\rho} = -\frac{2\tilde{k}}{\pi}\int_{\partial\Sigma} (\epsilon^{abc}(\tau'_b \tau_c + \lambda \rho'_b \rho_c)\omega_a + \lambda \epsilon^{abc}(\tau'_b \rho_c - \tau_b \rho'_c) e_a) - \frac{2\tilde{k}}{\pi}\int_{\partial\Sigma} (\tau_a d\tau'^a + \lambda \rho_a d\rho'^a). \quad (161)$$

Observe that this is exactly the same as the expression for $\delta^E_{\tau,\rho} Q^M_{\tau',\rho'}$. This is a nice consistency check.

Therefore, we conclude that the electric-magnetic charge bracket is

$$\{Q^E_{\tau,\rho}, Q^M_{\tau',\rho'}\} = \tilde{Q}^M_{\tau'',\rho''} - \frac{2\tilde{k}}{\pi}\int_{\partial\Sigma} (\tau_a d\tau'^a + \lambda \rho_a d\rho'^a), \quad (162)$$

with τ'' and ρ'' given in (156) and (157).

3. Magnetic-magnetic bracket

The bracket between two magnetic charges is

$$\{Q^M_{\tau,\rho}, Q^M_{\tau',\rho'}\} = \delta^M_{\tau,\rho} Q^M_{\tau',\rho'}. \tag{163}$$

Using (146) and (152), we obtain

$$\begin{aligned}\{Q^M_{\tau,\rho}, Q^M_{\tau',\rho'}\} = &-\frac{4\lambda\tilde{k}^2}{\pi k}\int_{\partial\Sigma}(\epsilon^{abc}(\tau'_b\tau_c + \lambda\rho'_b\rho_c)e_a \\ &+ \epsilon^{abc}(\tau'_b\rho_c - \tau_b\rho'_c)\omega_a) \\ &-\frac{4\lambda\tilde{k}^2}{\pi k}\int_{\partial\Sigma}(\rho^a d\tau'_a + \tau^a d\rho'_a).\end{aligned} \tag{164}$$

Comparing this to the electric charge (147), we conclude that

$$\{Q^M_{\tau,\rho}, Q^M_{\tau',\rho'}\} = 4\lambda\frac{\tilde{k}^2}{k^2}Q^E_{\tau'',\rho''} - 4\lambda\frac{\tilde{k}^2}{\pi k}\int_{\partial\Sigma}(\rho^a d\tau'_a + \tau^a d\rho'_a). \tag{165}$$

Again, τ'' and ρ'' are given in (156) and (157). The central term is the same (up to a constant) as that of $\{Q^E, Q^E\}$, so it vanishes for supertranslations.

It may be worth noting that there is a relation

$$\{Q^M_{\tau,\rho}, Q^M_{\tau',\rho'}\} = 4\lambda\frac{\tilde{k}^2}{k^2}\{Q^E_{\tau,\rho}, Q^E_{\tau',\rho'}\}. \tag{166}$$

The factor of 4λ seems to be just an artifact for a less than optimal choice of scale for Q^M (and preceding that for I_{magnetic}). For instance, if we started from $2I_{\text{magnetic}}$ we would have $2Q^M$ in place of Q^M and this would have led to having 16λ in place of the factor 4λ.

D. e^a and ω^a on the horizon

In this section, we consider putting a gravitational Chern-Simons theory on the future Schwarzschild horizon $\mathcal{H}^+$, and find the solutions of the equations of motion for e^a and ω^a. We observe that the "cosmological constant" λ is fixed by the equations of motion.

In the context of our work, g_{ij} is the pullback of the four-dimensional metric in advanced Eddington-Finkelstein coordinates to the future Schwarzschild horizon,

$$g_{ij} = 4M^2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{2}{(1+z\bar{z})^2} \\ 0 & \frac{2}{(1+z\bar{z})^2} & 0 \end{pmatrix}, \tag{167}$$

where i, j span $(v, z, \bar{z})$. The "flat metric" η_{ab} is the Cartan metric $\eta_{ab} = \mathrm{diag}(-1, 1, 1)$. They are connected by the "triad"

$$e_i{}^a = 2M\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{1+z\bar{z}} & \frac{i}{1+z\bar{z}} \\ 0 & \frac{1}{1+z\bar{z}} & \frac{-i}{1+z\bar{z}} \end{pmatrix} \tag{168}$$

that satisfies

$$e_i{}^a e_j{}^b \eta_{ab} = g_{ij}. \tag{169}$$

We do not have the inverse relation $g^{ij}e_i{}^a e_j{}^b = \eta^{ab}$ because g_{ij} is not invertible. We can write the above matrix form as collection of one-forms,

$$e^0 = 0, \tag{170}$$

$$e^1 = \frac{2M}{1+z\bar{z}}dz + \frac{2M}{1+z\bar{z}}d\bar{z}, \tag{171}$$

$$e^2 = \frac{2iM}{1+z\bar{z}}dz - \frac{2iM}{1+z\bar{z}}d\bar{z}, \tag{172}$$

from which we obtain

$$dz = \frac{1}{4M}(1+z\bar{z})(e^1 - ie^2), \tag{173}$$

$$d\bar{z} = \frac{1}{4M}(1+z\bar{z})(e^1 + ie^2), \tag{174}$$

$$dz\wedge d\bar{z} = \frac{i}{8M^2}(1+z\bar{z})^2 e^1\wedge e^2. \tag{175}$$

The spin connection can be obtained using the equations of motion and the anholonomy coefficients

$$de^a = -\omega^a{}_b\wedge e^b = \frac{1}{2}c^a{}_{bc}e^b\wedge e^c, \tag{176}$$

$$\omega_{ab} = \frac{1}{2}(c_{abc} - c_{bac} - c_{cab})e^c, \tag{177}$$

where we keep in mind that $\omega^{bc} = -\omega^a\epsilon_a{}^{bc}$. The exterior derivative of e^a yields

$$de^0 = 0 \tag{178}$$

$$de^1 = 2M\frac{(z-\bar{z})}{(1+z\bar{z})^2}dz\wedge d\bar{z} = \frac{i}{4M}(z-\bar{z})e^1\wedge e^2, \tag{179}$$

$$de^2 = 2iM\frac{(z+\bar{z})}{(1+z\bar{z})^2}dz\wedge d\bar{z} = -\frac{1}{4M}(z+\bar{z})e^1\wedge e^2, \tag{180}$$

from which we read off

$$c^1{}_{12} = c_{112} = \frac{i}{4M}(z - \bar{z}),$$
$$c^2{}_{12} = c_{212} = -\frac{1}{4M}(z + \bar{z}), \tag{181}$$

with all other coefficients vanishing. Accordingly, the only nonvanishing component of the spin connection is

$$\omega_{12} = -\frac{i}{4M}(z - \bar{z})e^1 + \frac{1}{4M}(z + \bar{z})e^2 \tag{182}$$

$$= \frac{i\bar{z}}{1 + z\bar{z}}dz - \frac{iz}{1 + z\bar{z}}d\bar{z}. \tag{183}$$

The only nonvanishing component of the dual $\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{bc}$ is thus

$$\omega^0 = -\omega_{12} = -\frac{i\bar{z}}{1 + z\bar{z}}dz + \frac{iz}{1 + z\bar{z}}d\bar{z} \tag{184}$$

since $\epsilon^{012} = -1$.

Let us see if this satisfies the other set of equations of motion

$$d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b \wedge \omega_c + \frac{\lambda}{2}\epsilon^{abc}e_b \wedge e_c = 0. \tag{185}$$

Since only ω^0 is nonzero, we have $\epsilon^{abc}\omega_b \wedge \omega_c = 0$. The only nonvanishing component of $d\omega^a$ is

$$d\omega^0 = \frac{2i}{(1 + z\bar{z})^2}dz \wedge d\bar{z}, \tag{186}$$

and the only nonvanishing term of $\frac{\lambda}{2}\epsilon^{abc}e_b \wedge e_c$ is

$$\frac{\lambda}{2}\epsilon^{0bc}e_b \wedge e_c = -\lambda e^1 \wedge e^2 = \frac{8iM^2\lambda}{(1 + z\bar{z})^2}dz \wedge d\bar{z}. \tag{187}$$

Therefore, the above equations of motion boils down to fixing λ,

$$\lambda = -\frac{1}{4M^2}. \tag{188}$$

1. Compensating τ-transformation for central term

We have seen that the electric and magnetic charges satisfy the algebra (155), (162) and (165), which reads

$$\{Q^E_{\tau,\rho}, Q^E_{\tau',\rho'}\} = Q^E_{\tau'',\rho''} - \frac{k}{\pi}\int_{\partial\Sigma}(\rho^a d\tau'_a + \tau^a d\rho'_a), \tag{189}$$

$$\{Q^E_{\tau,\rho}, Q^M_{\tau',\rho'}\} = Q^M_{\tau'',\rho''} - \frac{2\tilde{k}}{\pi}\int_{\partial\Sigma}(\tau_a d\tau'^a + \lambda\rho_a d\rho'^a), \tag{190}$$

$$\{Q^M_{\tau,\rho}, Q^M_{\tau',\rho'}\} = 4\lambda\frac{\tilde{k}^2}{k^2}Q^E_{\tau'',\rho''} - 4\lambda\frac{\tilde{k}^2}{\pi k}\int_{\partial\Sigma}(\rho^a d\tau'_a + \tau^a d\rho'_a), \tag{191}$$

with the composition τ'' and ρ'' given by (156) and (157). We want the central terms of this algebra to cancel the central term of supertranslation algebra on the Schwarzschild horizon. Recall that the ρ transformation is related to a diffeomorphism v^i by

$$v^i = \left(f, \frac{1}{2M}D^z f, \frac{1}{2M}D^{\bar{z}} f\right), \tag{192}$$

$$\rho^a = v^i e^a_i \tag{193}$$

$$= \frac{1}{1 + z\bar{z}}(0, D^z f + D^{\bar{z}} f, i(D^z f - D^{\bar{z}} f)). \tag{194}$$

We demand that, in this Chern-Simons theory, supertranslation is accompanied a compensating Lorentz transformation (τ-transformation) given by

$$\tau^a = \left(\frac{1}{8\tilde{k}^{1/2}}(D^2 + 2)f, i\sqrt{\lambda}\rho^2, -i\sqrt{\lambda}\rho^1\right). \tag{195}$$

This leads to the algebra

$$\{Q^E_{\tau,\rho}, Q^E_{\tau',\rho'}\} = Q^E_{\tau'',\rho''}, \tag{196}$$

$$\{Q^E_{\tau,\rho}, Q^M_{\tau',\rho'}\} = Q^M_{\tau'',\rho''} + \frac{i}{4}(D^z D^2_{\bar{z}} f')\Big|_{z=w}, \tag{197}$$

$$\{Q^M_{\tau,\rho}, Q^M_{\tau',\rho'}\} = 4\lambda\frac{\tilde{k}^2}{k^2}Q^E_{\tau'',\rho''}. \tag{198}$$

Observe that the standard supertranslations commute by themselves, the dual supertranslations commute by themselves, but the standard and dual charges have the correct form of central term. Thus, we see that exactly the form of the central term obtained in Secs. V and VI is reproduced. The Chern-Simons theory can then be used to cancel the anomalous behavior of the supertranslation charge algebra in the case that the supertranslation parameter f has a pole.

Finally, we note that the constant $\tilde{k}$ can also be fixed in terms of k by demanding that the complexified charge algebra is closed up to the central terms. One may readily check that the complexified charge $\mathbf{Q}_{\tau,\rho} = Q^E_{\tau,\rho} + iQ^M_{\tau,\rho}$ satisfies the bracket

$$\{\mathbf{Q}_{\tau,\rho}, \mathbf{Q}_{\tau',\rho'}\} = \left[1 - 4\lambda\frac{\tilde{k}^2}{k^2}\right]Q^E_{\tau'',\rho''} + 2iQ^M_{\tau'',\rho''} - \frac{1}{2}(D^z D^2_{\bar{z}} f')\Big|_{z=w}. \tag{199}$$

For this to close up to the central term, we demand that the coefficient of $Q^E_{\tau'',\rho''}$ be 2, which fixes $\tilde{k}^2 = -\frac{k^2}{4\lambda} = k^2 M^2$. Then, we obtain

$$\{\mathbf{Q}_{\tau,\rho}, \mathbf{Q}_{\tau',\rho'}\} = \mathbf{Q}_{2\tau'',2\rho''} - \frac{1}{2}(D^z D_z^2 f')\Big|_{z=w}. \tag{200}$$

IX. DISCUSSION

We have constructed standard and dual supertranslation charges on the future horizon of the Schwarzschild black hole using the first-order formalism of [23,24]. Then, we have explored the consequences of allowing for singularities in the parameter function of supertranslations. Singular supertranslations arise naturally in the extended phase space associated with the BMS algebra [29]. Also, in electrodynamics singular large gauge transformations are closely related to Dirac string configurations in the bulk [43], and singular supertranslations can be considered as their gravitational analog. Using a simple pole as an example, we have demonstrated that singularities lead to the presence of a central term in the Dirac bracket charge algebra, implying that the symmetry algebra becomes anomalous. In order to remove such a term, we have introduced a gravitational Chern-Simons theory [27] with gauge group $SL(2,\mathbb{C})$ on the horizon. Being a topological theory, this theory is suitable to live on the horizon which is a null surface, and in addition does not contribute a stress-energy tensor which may perturb the gravitational field. We have shown that the large gauge transformation of this boundary theory can be organized such that its charge algebra cancels the anomalous central term of the bulk gravity theory.

Some comments are in order. In this paper, we have shown that an $SL(2,\mathbb{C})$ Chern-Simons theory on the horizon can cancel the central term, but what we have not shown is that this theory is unique in being capable of this job. Whether there exist other topological field theories that can cancel the central term is an interesting question, as the properties shared by the set of such theories will teach us more about the fundamental nature of the structure on the black hole horizon. Also, we have considered one singular supertranslation with smooth dual supertranslation, as it is the simplest case that yields a nontrivial result. Considering more complicated cases is an interesting problem, and may lead to further physical insight. We leave such directions to future investigations.

Since the standard and dual supertranslation algebra on the horizon is an asymptotic symmetry algebra and hence is not gauged, one may observe the anomalous central term and decide that we extend the symmetry algebra to incorporate such a term instead of removing it. As an example of this viewpoint, central extension of classical asymptotic symmetry algebra is also present in the literature such as [49]. It would be very interesting to explore this direction, as the work of Brown and Henneaux is intimately related to the existence of a dual two-dimensional holographic boundary CFT. We leave this for future investigation.

In electromagnetism, there are specific examples of configurations that are associated with singular gauge transformations [43]. Then, one may ask whether there are well-known gravitational configurations associated with singular supertranslations. It has been shown by Strominger and Zhiboedov [50] that finite superrotations at the null infinity map asymptotically flat spacetimes to spacetimes with isolated defects, which are interpreted as cosmic strings. It is not clear whether singular supertranslations can have similar effects. It would be very interesting to see find such an example associated with singular supertranslations.

Finally, the structure of null infinity is very similar to the future Schwarzschild horizon, and thus we expect a similar structure to be present at the null infinity as well. It would be interesting to explore how such a structure could affect scattering amplitudes.

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APPENDIX A: MODIFIED LIE BRACKET

In this appendix we describe, in detail, the construction of the modified Lie bracket [29] on the Schwarzschild horizon $\mathcal{H}^+$.

The vector field ξ that generates a supertranslation $f(\Theta)$ and a superrotation $Y(\Theta)$ is

$$\xi = \left(f + \frac{v}{2}\psi\right)\partial_v - \frac{1}{2}\left(D^2\left(f + \frac{v}{2}\psi\right) + r\psi\right)\partial_r + \left(\frac{1}{r}D^A\left(f + \frac{v}{2}\psi\right) + Y^A\right)\partial_A, \tag{A1}$$

where $\psi \equiv D_A Y^A$. Let

$$F(v,\Theta) \equiv f(\Theta) + \frac{v}{2}\psi(\Theta), \tag{A2}$$

such that $\partial_v F = \frac{1}{2}\psi$. Then,

$$\xi = F\partial_v - \frac{1}{2}D^2F\partial_r + \frac{1}{r}D^AF\partial_A - \frac{r}{2}\psi\partial_r + Y^A\partial_A,$$
$$\xi_v = -\Lambda F - \frac{1}{2}D^2F - \frac{r}{2}\psi, \qquad \xi_r = F,$$
$$\xi_A = rD_AF + r^2Y_A, \tag{A3}$$

where $Y_A = \gamma_{AB}Y^B$. In this form, ξ is like a v-dependent supertranslation F but with "corrections" $-\frac{r}{2}\psi\partial_r + Y^A\partial_A$. Since we are only interested in terms linear in ξ, we can compute the contributions of F and the remainders separately.

For the unperturbed Schwarzschild spacetime, the nonvanishing Christoffel symbols $\bar{\Gamma}^a_{bc}$ are

$$\bar{\Gamma}^v_{vv} = \frac{M}{r^2}, \quad \bar{\Gamma}^v_{AB} = -r\gamma_{AB}, \quad \bar{\Gamma}^r_{vv} = \frac{M\Lambda}{r^2}, \quad \bar{\Gamma}^r_{vr} = -\frac{M}{r^2},$$
$$\bar{\Gamma}^r_{AB} = -r\Lambda\gamma_{AB}, \quad \bar{\Gamma}^A_{rB} = \frac{1}{r}\delta^A_B, \quad \bar{\Gamma}^A_{BC} = {}^{(2)}\Gamma^A_{BC}. \tag{A4}$$

Now, the metric perturbations $\bar{\delta}g_{ab}$ generated by ξ are $\delta\bar{g}_{ab} \equiv \mathcal{L}_\xi \bar{g}_{ab}$ and so we find

$$\delta\bar{g}_{vv} = \frac{M}{r^2}D^2F - \psi + \frac{3M}{r}\psi - \frac{1}{2}D^2\psi, \tag{A5}$$

$$\delta\bar{g}_{vr} = 0, \tag{A6}$$

$$\delta\bar{g}_{vA} = -D_A\left(\Lambda F + \frac{1}{2}D^2F\right), \tag{A7}$$

$$\delta\bar{g}_{AB} = 2rD_AD_BF - r\gamma_{AB}D^2F + r^2(D_AY_B + D_BY_A - \gamma_{AB}\psi). \tag{A8}$$

The perturbed metric is then

$$ds^2 = -\left(\Lambda - \frac{M}{r^2}D^2F + \psi - \frac{3M}{r}\psi + \frac{1}{2}D^2\psi\right)dv^2 + 2dvdr - D_A(2\Lambda F + D^2F)dvd\Theta^A + [r^2\gamma_{AB} + 2rD_AD_BF - r\gamma_{AB}D^2F + r^2(D_AY_B + D_BY_A - \gamma_{AB}\psi)]d\Theta^A d\Theta^B. \tag{A9}$$

Using this and the relation

$$\Gamma^a_{bc} = \bar{\Gamma}^a_{bc} + \frac{1}{2}\bar{g}^{ad}(\overline{\nabla}_b\delta\bar{g}_{dc} + \overline{\nabla}_c\delta\bar{g}_{db} - \overline{\nabla}_d\delta\bar{g}_{bc}) + O(\delta\bar{g}^2), \tag{A10}$$

we compute some of the perturbed Christoffel symbols to linear order in ξ,

$$\Gamma^v_{rr} = \Gamma^r_{rr} = \Gamma^A_{rr} = 0, \tag{A11}$$

$$\Gamma^v_{rA} = 0, \tag{A12}$$

$$\Gamma^r_{rA} = \frac{1}{r}D_AF - \frac{3M}{r^2}D_AF + \frac{1}{2r}D_AD^2F, \tag{A13}$$

$$\Gamma^B_{rA} = \frac{1}{r}\delta^B_A - \frac{1}{2r^2}(2D^BD_AF - \delta^B_AD^2F), \tag{A14}$$

which turn out to be exactly the same as the components of supertranslated metric with just $f \to F$. Also

$$\gamma^{AB}\Gamma^v_{AB} = -2r, \tag{A15}$$

$$\gamma^{AB}\Gamma^r_{AB} = -2r\Lambda - D^2F + \frac{4M}{r}D^2F - 2r\psi + 6M\psi - rD^2\psi - \frac{1}{2}D^2D^2F, \tag{A16}$$

$$\gamma^{AB}\Gamma^C_{AB} = \gamma^{AB(2)}\Gamma^C_{AB} + \frac{4M}{r^2}D^CF + Y^C + D^2Y^C. \tag{A17}$$

Using the above, we can write for any vector field ζ_a

$$\nabla_r\zeta_r = \partial_r\zeta_r, \tag{A18}$$

$$\nabla_r\zeta_A + \nabla_A\zeta_r = \partial_r\zeta_A + D_A\zeta_r - \zeta_r\left(\frac{2}{r}D_AF - \frac{6M}{r^2}D_AF + \frac{1}{r}D_AD^2F\right) - \frac{2}{r}\zeta_A + \frac{1}{r^2}\zeta_B(2D^BD_AF - \delta^B_AD^2F), \tag{A19}$$

$$\gamma^{AB}\nabla_A\zeta_B = \zeta_r\left(D^2F - \frac{4M}{r}D^2F + 2r\psi - 6M\psi + rD^2\psi + \frac{1}{2}D^2D^2F + 2r\Lambda\right) + D^A\zeta_A + 2r\zeta_v - \zeta_C\left(\frac{4M}{r^2}D^CF + Y^C + D^2Y^C\right). \tag{A20}$$

Now we can relate the components of any contravariant vector field ζ^a to the components of a covaraint vector field ζ_a using the perturbed metric,

$$\zeta_v = -\Lambda\zeta^v + \zeta^v\left(\frac{M}{r^2}D^2F - \psi + \frac{3M}{r}\psi - \frac{1}{2}D^2\psi\right) + \zeta^r - \zeta^AD_A\left(\Lambda F + \frac{1}{2}D^2F\right), \tag{A21}$$

$$\zeta_r = \zeta^v, \tag{A22}$$

$$\zeta_A = -\zeta^v D_A\left(\Lambda F + \frac{1}{2}D^2 F\right) + r^2\gamma_{AB}\zeta^B + \zeta^B(2rD_A D_B F - r\gamma_{AB}D^2 F + r^2(D_A Y_B + D_B Y_A - \gamma_{AB}\psi)). \tag{A23}$$

Now let ζ^a to be a new Schwarzschild supertranslation plus superrotation vector field parametrized by $g(\Theta)$ and $Z^A(\Theta)$. Employing the shorthand $\phi \equiv D_A Z^A$ and $G \equiv g + \frac{v}{2}\phi$,

$$\zeta^v = G + \delta\zeta^v, \qquad \zeta^r = -\frac{1}{2}D^2 G - \frac{r}{2}\phi + \delta\zeta^r,$$
$$\zeta^A = \frac{1}{r}D^A G + Z^A + \delta\zeta^A, \tag{A24}$$

where $\delta\zeta^a$ is the change in ζ^a due to the original diffeomorphism ξ^a. To first order in the perturbation,

$$\zeta_v = -\Lambda\delta\zeta^v + \delta\zeta^r + G\left(-\Lambda + \frac{M}{r^2}D^2F - \psi + \frac{3M}{r}\psi - \frac{1}{2}D^2\psi\right) - \frac{1}{2}D^2G - \frac{r}{2}\phi - \left(\frac{1}{r}D^A G + Z^A\right)\left(\Lambda D_A F + \frac{1}{2}D_A D^2 F\right), \tag{A25}$$

$$\zeta_r = G + \delta\zeta^v, \tag{A26}$$

$$\zeta_A = \left(\frac{1}{r}D^B G + Z^B\right)(2rD_A D_B F - r\gamma_{AB}D^2 F + r^2(D_A Y_B + D_B Y_A - \gamma_{AB}\psi)) - G\left(\Lambda D_A F + \frac{1}{2}D_A D^2 F\right) + rD_A G + r^2 Z_A + r^2\gamma_{AB}\delta\zeta^B. \tag{A27}$$

Plugging back in and demanding that $\nabla_r\zeta_r = \nabla_A\zeta_r + \nabla_r\zeta_A = \gamma^{AB}\nabla_A\zeta_B = 0$, we obtain

$$0 = \partial_r\delta\zeta^v, \tag{A28}$$

$$0 = r^2\gamma_{AB}\partial_r\delta\zeta^B + D_A\delta\zeta^v - \frac{2}{r}(D^B G)D_A D_B F + \frac{1}{r}(D_A G)D^2 F - (D^B G)(D_A Y_B + D_B Y_A) + (D_A G)\psi, \tag{A29}$$

$$0 = -\Lambda(D^A G)D_A F + 2(D^A D^B G)D_A D_B F + \frac{r^2}{2}(D^A Z^B + D^B Z^A)(D_A Y_B + D_B Y_A) - (D^2 G)D^2 F - r^2\phi\psi - r(D^2 G)\psi - r(D^2 F)\phi + 2r(D^A D^B G)D_A Y_B - \frac{1}{2}(D^A G)D_A D^2 F + 2r(D^A Z^B)D_A D_B F + r^2 D_A\delta\zeta^A + 2r\delta\zeta^r. \tag{A30}$$

Solving for $\delta\zeta$, we obtain

$$\delta\zeta^v = 0, \tag{A31}$$

$$\delta\zeta^r = \frac{1}{2r}\left(\Lambda(D^A G)D_A F - (D^A D^B G)D_A D_B F + \frac{1}{2}(D^2 G)D^2 F\right) - rD^{(A}Z^{B)}D_{(A}Y_{B)} + \frac{1}{2r}(D^A G)D^2 D_A F - (D^A Z^B)D_A D_B F + \frac{1}{2}(D^2 F)\phi + \frac{1}{2}(D_B G)D^2 Y^B + \frac{1}{2}(D_B G)Y^B + \frac{r}{2}\phi\psi, \tag{A32}$$

$$\delta\zeta^A = -\frac{1}{r^2}(D^B G)D^A D_B F + \frac{1}{2r^2}(D^A G)D^2 F - \frac{1}{r}(D_B G)(D^A Y^B + D^B Y^A) + \frac{1}{r}(D^A G)\psi. \tag{A33}$$

We need to remind ourselves here that these $\delta\zeta^a$ are the changes in ζ^a due to the transformation ξ^a. Due to this nature of $\delta\zeta^a$, we will change our notation to $\delta\zeta^a \to \delta_\xi\zeta^a$. The changes in ξ^a due to ζ^a can be obtained by exchanging $\xi \leftrightarrow \zeta$, and we will denote this as $\delta_\zeta\xi^a$.

The regular Lie bracket $[\xi,\zeta]^a = \xi^b\partial_b\zeta^a - \zeta^b\partial_b\xi^a$ of two vector fields can be computed straightforwardly from (A1),

$$[\xi,\zeta]^v = \frac{1}{2}F\phi - \frac{1}{2}G\psi + Y^A D_A G - Z^A D_A F, \tag{A34}$$

$$[\xi,\zeta]^r = -\frac{1}{4}FD^2\phi + \frac{1}{4}GD^2\psi + \frac{1}{4}\phi D^2F - \frac{1}{4}\psi D^2G - \frac{1}{2r}(D^AF)D_AD^2G + \frac{1}{2r}(D^AG)D_AD^2F$$
$$-\frac{1}{2}(D^AF)D_A\phi + \frac{1}{2}(D^AG)D_A\psi - \frac{1}{2}Y^AD_AD^2G + \frac{1}{2}Z^AD_AD^2F - \frac{r}{2}Y^AD_A\phi + \frac{r}{2}Z^AD_A\psi, \quad \text{(A35)}$$

$$[\xi,\zeta]^A = \frac{1}{2r}FD^A\phi - \frac{1}{2r}GD^A\psi + \frac{1}{2r^2}(D^2F)D^AG - \frac{1}{2r^2}(D^2G)D^AF + \frac{1}{2r}\psi D^AG - \frac{1}{2r}\phi D^AF + \frac{1}{r^2}(D^BF)D_BD^AG$$
$$-\frac{1}{r^2}(D^BG)D_BD^AF + \frac{1}{r}Y^BD_BD^AG - \frac{1}{r}Z^BD_BD^AF + \frac{1}{r}(D^BF)D_BZ^A$$
$$-\frac{1}{r}(D^BG)D_BY^A + Y^BD_BZ^A - Z^BD_BY^A. \quad \text{(A36)}$$

We define the modified bracket by correcting this by $\delta_\xi\zeta^a$ and $\delta_\zeta\xi^a$,

$$[\xi,\zeta]^a_M = [\xi,\zeta]^a - \delta_\xi\zeta^a + \delta_\zeta\xi^a. \quad \text{(A37)}$$

Using the expressions for $\delta_\xi\zeta^a$ that we have computed earlier, we obtain

$$[\xi,\zeta]^v_M = \frac{1}{2}F\phi + Y^AD_AG - (\xi\leftrightarrow\zeta), \quad \text{(A38)}$$

$$[\xi,\zeta]^r_M = -\frac{1}{4}FD^2\phi - \frac{1}{4}(D^2F)\phi - \frac{1}{2}(D^AF)D_A\phi$$
$$-\frac{1}{2}Y^AD^2D_AG - (D^AY^B)D_AD_BG$$
$$-\frac{1}{2}(D_BG)D^2Y^B - \frac{r}{2}Y^AD_A\phi - (\xi\leftrightarrow\zeta), \quad \text{(A39)}$$

$$[\xi,\zeta]^A_M = \frac{1}{2r}FD^A\phi - \frac{1}{2r}\psi D^AG + \frac{1}{r}Y^BD_BD^AG + Y^BD_BZ^A$$
$$+\frac{1}{r}(D_BG)D^AY^B - (\xi\leftrightarrow\zeta). \quad \text{(A40)}$$

The v-component can be reorganized as

$$[\xi,\zeta]^v = \frac{1}{2}f\phi - \frac{1}{2}g\psi + Y^AD_Ag - Z^AD_Af$$
$$+\frac{v}{2}D_A(Y^BD_BZ^A - Z^BD_BY^A). \quad \text{(A41)}$$

Let us define

$$\hat{f} = \frac{1}{2}f\phi - \frac{1}{2}g\psi + Y^AD_Ag - Z^AD_Af, \quad \text{(A42)}$$

$$\hat{Y}^A = Y^BD_BZ^A - Z^BD_BY^A. \quad \text{(A43)}$$

Then, define $\hat{\psi} \equiv D_A\hat{Y}^A$, and take $\hat{F} \equiv \hat{f} + \frac{v}{2}\hat{\psi}$ so that we have $[\xi,\zeta]^v = \hat{F}$,

$$\hat{F} = \frac{1}{2}F\phi + Y^AD_AG - (\xi\leftrightarrow\zeta). \quad \text{(A44)}$$

With this definition, observe that we have exactly the modified bracket components

$$-\frac{1}{2}D^2\hat{F} - \frac{r}{2}\hat{\psi} = -\frac{1}{4}FD^2\phi - \frac{1}{4}(D^2F)\phi - \frac{1}{2}(D^AF)D_A\phi$$
$$-\frac{1}{2}Y^AD^2D_AG - (D^AY^B)D_AD_BG$$
$$-\frac{1}{2}(D_AG)D^2Y^A - \frac{r}{2}Y^AD_A\phi - (\xi\leftrightarrow\zeta) \quad \text{(A45)}$$

$$= [\xi,\zeta]^r_M, \quad \text{(A46)}$$

and

$$\frac{1}{r}D^A\hat{F} + \hat{Y}^A = \frac{1}{2r}FD^A\phi - \frac{1}{2r}\psi D^AG + \frac{1}{r}Y^BD_BD^AG$$
$$+ Y^BD_BZ^A + \frac{1}{r}(D_BG)D^AY^B$$
$$-(\xi\leftrightarrow\zeta) \quad \text{(A47)}$$

$$= [\xi,\zeta]^A_M. \quad \text{(A48)}$$

This implies that

$$[\xi,\zeta]_M = \left(\hat{f} + \frac{v}{2}\hat{\psi}\right)\partial_v - \frac{1}{2}\left(D^2\left(\hat{f} + \frac{v}{2}\hat{\psi}\right) + r\hat{\psi}\right)\partial_r$$
$$+\left(\frac{1}{r}D^A\left(\hat{f} + \frac{v}{2}\hat{\psi}\right) + \hat{Y}^A\right)\partial_A. \quad \text{(A49)}$$

Comparing the right-hand side (rhs) to the expression (A1), we can see that it is another supertranslation $\hat{f}$ together with superrotation $\hat{Y}^A$.

We conclude that given two pairs $(f_1, Y_1), (f_2, Y_2)$ of supertranslation and superrotation, the modified bracket has the algebra

$$[(f_1, Y_1), (f_2, Y_2)]_M = (\hat{f}, \hat{Y}), \tag{A50}$$

with the product being another supertranslation together with a superrotation parametrized by

$$\hat{f} = \frac{1}{2} f_1 D_A Y_2^A - \frac{1}{2} f_2 D_A Y_1^A + Y_1^A D_A f_2 - Y_2^A D_A f_1, \tag{A51}$$

$$\hat{Y}^A = Y_1^B D_B Y_2^A - Y_2^B D_B Y_1^A, \tag{A52}$$

which is equivalent to the BMS algebra at the null infinity [29].

APPENDIX B: DERIVATION OF HORIZON CHARGES

In this section, we will give a derivation of the supertranslation and dual supertranslation charges using the formula of [23,24],

$$\not\delta Q_E^{\mathcal{H}^+} = \frac{1}{16\pi} \epsilon_{\alpha\beta\gamma\delta} \int_{\partial\mathcal{H}^+} (i_\xi E^\gamma) \delta\omega^{\alpha\beta} \wedge E^\delta, \tag{B1}$$

$$\not\delta Q_M^{\mathcal{H}^+} = \frac{1}{8\pi} \int_{\partial\mathcal{H}^+} (i_\xi E^\alpha) \delta\omega_{\alpha\beta} \wedge E^\beta. \tag{B2}$$

Here $\omega_{\alpha\beta}$ is the (torsion-free) spin connection 1-form, and $\delta\omega$ is the change in ω induced by the variation $\delta g_{ab} = h_{ab}$ of the metric.

In order to incorporate the variation of the metric, we will parametrize a generic metric in Bondi gauge by

$$g_{ab} = \begin{pmatrix} V + W_A W^A & U & W_B \\ U & 0 & 0 \\ W_A & 0 & g_{AB} \end{pmatrix}, \tag{B3}$$

where V, U, W_A are real functions of v, r, Θ^A. The inverse metric is

$$g^{ab} = \begin{pmatrix} 0 & U^{-1} & 0 \\ U^{-1} & -VU^{-2} & -U^{-1}W^B \\ 0 & -U^{-1}W^A & g^{AB} \end{pmatrix}, \tag{B4}$$

where g^{AB} is the inverse of the two-dimensional metric g_{AB}, and $W^A = g^{AB} W_B$ (not $\gamma^{AB} W_B$). Since this metric may deviate from that of Schwarzschild, the two-dimensional curved indices $A, B, C, \ldots$ in this section, and only in this section, are raised and lowered using g^{AB} and g_{AB} rather than γ^{AB} and γ_{AB}, the metric on the unit 2-sphere metric.

We will employ the following set of vielbein $E^\alpha = E^\alpha{}_a dx^a$,

$$E^1 = \frac{V}{2} dv + U dr, \tag{B5}$$

$$E^2 = -dv, \tag{B6}$$

$$E^3 = W_A \mu^A dv + \mu_A d\Theta^A, \tag{B7}$$

$$E^4 = W_A \bar{\mu}^A dv + \bar{\mu}_A d\Theta^A, \tag{B8}$$

where μ_A, $\bar{\mu}_A$ are complex functions of v, r, Θ^A, and $\mu^A = g^{AB}\mu_B$, $\bar{\mu}^A = g^{AB}\bar{\mu}_B$ (bar denotes complex conjugation, so $\bar{\mu}_A$ is the complex conjugate of μ_A and hence $E^3 = \overline{E^4}$). They satisfy the conditions

$$\mu_A \bar{\mu}_B + \mu_B \bar{\mu}_A = g_{AB}, \quad \mu^A \bar{\mu}_A = 1, \quad \mu^A \mu_A = \bar{\mu}^A \bar{\mu}_A = 0. \tag{B9}$$

The tangent space metric and its inverse are

$$\eta_{\alpha\beta} = \eta^{\alpha\beta} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{B10}$$

and the inverse vielbeins $E_\alpha = E_\alpha{}^a \partial_a$ are

$$E_1 = U^{-1} \partial_r, \tag{B11}$$

$$E_2 = -\partial_v + \frac{V}{2U} \partial_r + W^A \partial_A, \tag{B12}$$

$$E_3 = \bar{\mu}^A \partial_A, \tag{B13}$$

$$E_4 = \mu^A \partial_A. \tag{B14}$$

One can readily check that

$$E_\alpha{}^a E^\alpha{}_b = \delta^a{}_b, \qquad E_\alpha{}^a E^\beta{}_a = \delta_\alpha{}^\beta, \tag{B15}$$

$$E^\alpha{}_a E^\beta{}_b \eta_{\alpha\beta} = g_{ab}, \qquad E_\alpha{}^a E_\beta{}^b \eta^{\alpha\beta} = g^{ab}. \tag{B16}$$

The spin connection 1-form $\omega_{\alpha\beta}$ is defined as

$$dE^\alpha = -\omega^\alpha{}_\beta \wedge E^\beta = \frac{1}{2} c^\alpha{}_{\beta\gamma} E^\beta \wedge E^\gamma, \tag{B17}$$

$$\omega_{\alpha\beta} = \frac{1}{2} (c_{\alpha\beta\gamma} - c_{\beta\alpha\gamma} - c_{\gamma\alpha\beta}) E^\gamma, \tag{B18}$$

where $c_{\alpha\beta\gamma}$ are the anholonomy coefficients. Explicit expressions for the coefficients read

$$c_{1\beta\gamma} = 0, \tag{B19}$$

$$c_{212} = \frac{1}{U} \left(\frac{1}{2} V' - \dot{U} + W^A \partial_A U \right), \tag{B20}$$

$$c_{213} = \frac{\bar{\mu}^A \partial_A U}{U}, \tag{B21}$$

$$c_{223} = -\frac{1}{2}\bar{\mu}^A\partial_A V + \frac{\bar{\mu}^A\partial_A U}{2U}V, \tag{B22}$$

$$c_{234} = 0, \tag{B23}$$

$$c_{312} = -\frac{W^{A\prime}\bar{\mu}_A}{U}, \tag{B24}$$

$$c_{313} = \frac{\bar{\mu}^A\bar{\mu}A'}{U}, \tag{B25}$$

$$c_{314} = \frac{\mu^A\bar{\mu}'_A}{U}, \tag{B26}$$

$$c_{323} = \bar{\mu}^A\partial_A(W\cdot\bar{\mu}) - \bar{\mu}^A\dot{\bar{\mu}}_A + \frac{\bar{\mu}^A\bar{\mu}'_A}{2U}V + W^A\bar{\mu}^B(\partial_A\bar{\mu}_B - \partial_B\bar{\mu}_A), \tag{B27}$$

$$c_{324} = \mu^A\partial_A(W\cdot\bar{\mu}) - \mu^A\dot{\bar{\mu}}_A + \frac{\mu^A\bar{\mu}'_A}{2U}V + W^A\mu^B(\partial_A\bar{\mu}_B - \partial_B\bar{\mu}_A), \tag{B28}$$

$$c_{334} = (\mu^A\bar{\mu}^B - \bar{\mu}^A\mu^B)\partial_B\bar{\mu}_A. \tag{B29}$$

The remaining coefficients can be obtained using the antisymmetry $c_{\alpha\beta\gamma} = -c_{\alpha\gamma\beta}$ and the fact $E^3 = \overline{E^4}$ implies switching indices $3 \leftrightarrow 4$ corresponds to complex conjugation, for instance $c_{213} = \overline{c_{214}}$ and $c_{434} = \overline{c_{343}} = -\overline{c_{334}}$. Using this to compute $\omega_{\alpha\beta}$, we obtain

$$\omega_{12} = \frac{1}{U}\left(-\frac{1}{2}V' + \dot{U} - W^A\partial_A U\right)E^2 + \frac{1}{2U}(-\bar{\mu}^A\partial_A U + W^{A\prime}\bar{\mu}_A)E^3 + \frac{1}{2U}(-\mu^A\partial_A U + W^{A\prime}\mu_A)E^4, \tag{B30}$$

$$\omega_{13} = \frac{1}{2U}(W^{A\prime}\bar{\mu}_A - \bar{\mu}^A\partial_A U)E^2 - \frac{\bar{\mu}^A\bar{\mu}'_A}{U}E^3 - \frac{1}{2U}(\mu^A\bar{\mu}'_A + \bar{\mu}^A\mu'_A)E^4, \tag{B31}$$

$$\begin{aligned}\omega_{23} = {}& \frac{1}{2U}(-\bar{\mu}^A\partial_A U - W^{A\prime}\bar{\mu}_A)E^1 + \frac{1}{2}\left(\bar{\mu}^A\partial_A V - \frac{\bar{\mu}^A\partial_A U}{U}V\right)E^2\\ &- \left(\bar{\mu}^A\partial_A(W\cdot\bar{\mu}) - \bar{\mu}^A\dot{\bar{\mu}}_A + \frac{\bar{\mu}^A\bar{\mu}'_A}{2U}V + W^A\bar{\mu}^B(\partial_A\bar{\mu}_B - \partial_B\bar{\mu}_A)\right)E^3\\ &- \frac{1}{2}\left(\mu^A\partial_A(W\cdot\bar{\mu}) - \mu^A\dot{\bar{\mu}}_A + \frac{\mu^A\bar{\mu}'_A}{2U}V + W^A\mu^B(\partial_A\bar{\mu}_B - \partial_B\bar{\mu}_A) + \text{c.c.}\right)E^4,\end{aligned} \tag{B32}$$

$$\begin{aligned}\omega_{34} = {}& \frac{1}{2U}(-\mu^A\bar{\mu}'_A + \bar{\mu}^A\mu'_A)E^1 + \frac{1}{2}\left(\bar{\mu}^A\partial_A(W\cdot\mu) - \bar{\mu}^A\dot{\mu}_A + \frac{\bar{\mu}^A\mu'_A}{2U}V + W^A\bar{\mu}^B(\partial_A\mu_B - \partial_B\mu_A) - \text{c.c.}\right)E^2\\ &- (\mu^A\bar{\mu}^B - \bar{\mu}^A\mu^B)\partial_B\bar{\mu}_A E^3 - (\mu^A\bar{\mu}^B - \bar{\mu}^A\mu^B)\partial_B\mu_A E^4.\end{aligned} \tag{B33}$$

We keep in mind that $E^3 = \overline{E^4}$. The remaining components can be obtained by antisymmetry and complex conjugation, for instance $\omega_{42} = \overline{\omega_{32}} = -\overline{\omega_{23}}$.

1. Supertranslation charge

The conserved electric charge involves the differential form

$$\frac{1}{16\pi}\epsilon_{\alpha\beta\gamma\delta}(i_\xi E^\gamma)\delta\omega^{\alpha\beta} \wedge E^\delta. \tag{B34}$$

We are interested in integrating

$$\begin{aligned}\frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}(i_\xi E^\gamma)\delta\omega^{\alpha\beta}\wedge E^\delta = {}& \epsilon_{1234}i_\xi E^3\delta\omega^{12}\wedge E^4 + \epsilon_{1324}i_\xi E^2\delta\omega^{13}\wedge E^4 + \epsilon_{2314}i_\xi E^1\delta\omega^{23}\wedge E^4 + \epsilon_{1243}i_\xi E^4\delta\omega^{12}\wedge E^3\\ &+ \epsilon_{1423}i_\xi E^2\delta\omega^{14}\wedge E^3 + \epsilon_{2413}i_\xi E^1\delta\omega^{24}\wedge E^3 + \cdots\end{aligned} \tag{B35}$$

over S^2. Observe that the alternating tensor $\epsilon_{\alpha\beta\gamma\delta}$ is purely imaginary,

$$\overline{\epsilon_{1234}} = \epsilon_{1243} = -\epsilon_{1234}. \tag{B36}$$

By explicit computation, one finds that

$$\epsilon_{1234} = E_1{}^a E_2{}^b E_3{}^c E_4{}^d \epsilon_{abcd} = -i, \tag{B37}$$

where ϵ_{abcd} is the alternating tensor in the curved coordinates with $\epsilon_{vr\theta\phi} = \sqrt{-\det g} = Ur^2 \sin\theta$. Using this and rearranging the indices, we obtain

$$\begin{aligned}
&\frac{i}{2}\epsilon_{\alpha\beta\gamma\delta}(i_\xi E^\gamma)\delta\omega^{\alpha\beta} \wedge E^\delta \\
&= -i_\xi E^3 \delta\omega_{12} \wedge E^4 + i_\xi E^2 \delta\omega_{24} \wedge E^4 - i_\xi E^1 \delta\omega_{14} \wedge E^4 \\
&\quad + i_\xi E^4 \delta\omega_{12} \wedge E^3 - i_\xi E^2 \delta\omega_{23} \wedge E^3 + i_\xi E^1 \delta\omega_{13} \wedge E^3 \\
&\quad + \cdots .
\end{aligned} \tag{B38}$$

Let us look at this expression term by term. We are interested only in coefficients of $E^3 \wedge E^4$ as we are integrating a the two-sphere on the horizon. The first and fourth terms combine to yield

$$\begin{aligned}
&- i_\xi E^3 \delta\omega_{12} \wedge E^4 + i_\xi E^4 \delta\omega_{12} \wedge E^3 \\
&= \frac{1}{2}\xi^A \left(\partial_A h_{vr} + \frac{2}{r} h_{vA} - \partial_r h_{vA}\right) E^3 \wedge E^4 \\
&\quad + \cdots .
\end{aligned} \tag{B39}$$

For the second term we have

$$\begin{aligned}
&i_\xi E^2 \delta\omega_{24} \wedge E^4 \\
&= \frac{\xi^v}{2}\left(\frac{1}{r} h_{vv} + \partial_A h^A{}_v + h^A{}_v (\bar\mu^B \partial_A \mu_B + \mu^B \partial_A \bar\mu_B)\right) E^3 \wedge E^4 \\
&\quad + \cdots ,
\end{aligned} \tag{B40}$$

where we have used $\delta(\bar\mu^A \dot\mu_A) = \delta(\mu^A \dot{\bar\mu}_A) = 0$. It turns out that

$$\begin{aligned}
\partial_A h^A{}_v + h^A{}_v (\bar\mu^B \partial_A \mu_B + \mu^B \partial_A \bar\mu_B) &= g^{AB} D_A h_{vB} \\
&= \frac{1}{r^2}\gamma^{AB} D_A h_{vB},
\end{aligned} \tag{B41}$$

where D_A denotes covariant derivative on the unit 2-sphere (that is, compatible with γ_{AB}, not g_{AB}). Thus, we can write

$$i_\xi E^2 \delta\omega_{24} \wedge E^4 = \frac{\xi^v}{2}\left(\frac{1}{r} h_{vv} + \frac{1}{r^2}\gamma^{AB} D_A h_{vB}\right) E^3 \wedge E^4 + \cdots . \tag{B42}$$

The coefficient of $E^3 \wedge E^4$ is real,

$$\begin{aligned}
&i_\xi E^2 \delta\omega_{24} \wedge E^4 - i_\xi E^2 \delta\omega_{23} \wedge E^3 \\
&\quad = \xi^v \left(\frac{1}{r} h_{vv} + \frac{1}{r^2}\gamma^{AB} D_A h_{vB}\right) E^3 \wedge E^4 + \cdots .
\end{aligned} \tag{B43}$$

We also have

$$\begin{aligned}
-i_\xi E^1 \delta\omega_{14} \wedge E^4 &= \xi^r \delta\left[\frac{1}{2U}(\bar\mu^A \mu'_A + \mu^A \bar\mu'_A)\right] E^3 \wedge E^4 \\
&\quad + \frac{\xi^r}{2}(\bar\mu^A \mu'_A + \mu^A \bar\mu'_A)\delta E^3 \wedge E^4 \\
&\quad + \cdots ,
\end{aligned} \tag{B44}$$

$$\begin{aligned}
i_\xi E^1 \delta\omega_{13} \wedge E^3 &= \xi^r \delta\left[\frac{1}{2U}(\mu^A \bar\mu'_A + \bar\mu^A \mu'_A)\right] E^3 \wedge E^4 \\
&\quad + \frac{\xi^r}{2}(\mu^A \bar\mu'_A + \bar\mu^A \mu'_A) E^3 \wedge \delta E^4 \\
&\quad + \cdots .
\end{aligned} \tag{B45}$$

Together we have

$$\begin{aligned}
&- i_\xi E^1 \delta\omega_{14} \wedge E^4 + i_\xi E^1 \delta\omega_{13} \wedge E^3 \\
&\quad = -\frac{2\xi^r}{r} h_{vr} E^3 \wedge E^4 + \frac{\xi^r}{r}\delta(E^3 \wedge E^4) + \cdots ,
\end{aligned} \tag{B46}$$

where we have used $\delta(\mu^A \bar\mu'_A) = \delta(\bar\mu^A \mu'_A) = 0$. With $\delta r = 0$, we also have $\delta(E^3 \wedge E^4) = 0$ due to the Bondi gauge condition $\gamma^{AB} h_{AB} = 0$.

Collecting the results, we obtain

$$\begin{aligned}
\frac{i}{2}\epsilon_{\alpha\beta\gamma\delta}(i_\xi E^\gamma)\delta\omega^{\alpha\beta} \wedge E^\delta = &\left[\frac{1}{2}\xi^A \left(\partial_A h_{vr} + \frac{2}{r} h_{vA} - \partial_r h_{vA}\right)\right. \\
&+ \xi^v \left(\frac{1}{r} h_{vv} + \frac{1}{r^2}\gamma^{AB} D_A h_{vB}\right) \\
&\left. - \frac{2\xi^r}{r} h_{vr}\right] E^3 \wedge E^4 + \cdots .
\end{aligned} \tag{B47}$$

Plugging this into (B1), we obtain the electric diffeomorphism charge associated with vector field ξ on the Schwarzschild horizon $r = 2M$ to be

$$\begin{aligned}
\not{\delta} Q_E^{\mathcal{H}^+} = &\frac{M^2}{4\pi}\int d^2\Theta \sqrt{\gamma}\left[\xi^A \left(\partial_A h_{vr} + \frac{1}{M} h_{vA} - \partial_r h_{vA}\right)\right. \\
&\left. + \frac{1}{M}\xi^v \left(h_{vv} + \frac{1}{2M}\gamma^{AB} D^A h_{vB}\right) - \frac{2\xi^r}{M} h_{vr}\right].
\end{aligned} \tag{B48}$$

For a smooth function $f(\Theta)$ and the horizon supertranslation vector field (21), this formula is in exact agreement with the horizon supertranslation charge derived in [4], as anticipated.

2. Dual supertranslation charge

The magnetic diffeomorphism charge associated with a vector field ξ takes the form

$$\delta\!\!\!/ Q_M^{\mathcal{H}^+} = \frac{i}{8\pi}\int_{\partial\mathcal{H}^+} (i_\xi E^\alpha)\delta\omega_{\alpha\beta}\wedge E^\beta. \tag{B49}$$

Again, we only need to compute the $E^3\wedge E^4$ component of the two-form

$$(i_\xi E^\alpha)\delta\omega_{\alpha\beta}\wedge E^\beta. \tag{B50}$$

The only part of the expression relevant to the S^2 integral is

$$\begin{aligned}(i_\xi E^\alpha)\delta\omega_{\alpha\beta}\wedge E^\beta = {}& (i_\xi E^1)(\delta\omega_{13}\wedge E^3 + \delta\omega_{14}\wedge E^4)\\ &+ (i_\xi E^2)(\delta\omega_{23}\wedge E^3 + \delta\omega_{24}\wedge E^4)\\ &+ (i_\xi E^3)\delta\omega_{34}\wedge E^4 + (i_\xi E^4)\delta\omega_{43}\wedge E^3\\ &+\cdots,\end{aligned} \tag{B51}$$

where $\cdots$ contains all the irrelevant components. Using the expression (B32) for the spin connection, we can write

$$\begin{aligned}&(\delta\omega_{13}\wedge E^3 + \delta\omega_{14}\wedge E^4)|_{d\Theta^A\wedge d\Theta^B}\\ &= -\delta\left[\frac{1}{2U}(\bar\mu^A\mu'_A + \mu^A\bar\mu'_A)\right](E^3\wedge E^4 + E^4\wedge E^3)\\ &\quad -\frac{1}{2}(\bar\mu^A\mu'_A + \mu^A\bar\mu'_A)(\delta E^3\wedge E^4 + \delta E^4\wedge E^3)\\ &\quad -(\bar\mu^A\bar\mu'_A)\delta E^3\wedge E^3 - (\mu^A\mu'_A)\delta E^4\wedge E^4.\end{aligned} \tag{B52}$$

The first line on the rhs is clearly zero since $E^3\wedge E^4 + E^4\wedge E^3 = 0$. The third line is also zero since

$$\bar\mu^A\bar\mu'_A = \frac{1}{r}\bar\mu^A\bar\mu_A = 0, \qquad \mu^A\mu'_A = \frac{1}{r}\mu^A\mu_A = 0. \tag{B53}$$

In the second line, we have

$$\begin{aligned}&\delta E^3\wedge E^4 + \delta E^4\wedge E^3\\ &\quad = (\delta\mu_A\bar\mu_B + \delta\bar\mu_A\mu_B)d\Theta^A\wedge d\Theta^B.\end{aligned} \tag{B54}$$

One can show that the expression in parentheses on the rhs is $\frac{1}{2}h_{AB}$ and is therefore symmetric,

$$h_{AB} = \delta(\mu_A\bar\mu_B + \bar\mu_A\mu_B) = 2(\delta\mu_A\bar\mu_B + \delta\bar\mu_A\mu_B), \tag{B55}$$

which implies $\delta E^3\wedge E^4 + \delta E^4\wedge E^3 = 0$. Therefore we have

$$(\delta\omega_{13}\wedge E^3 + \delta\omega_{14}\wedge E^4)|_{d\Theta^A\wedge d\Theta^B} = 0. \tag{B56}$$

The expression for $\delta\omega_{23}\wedge E^3 + \delta\omega_{24}\wedge E^4$ is similar but with just more complicated coefficients. To see this, first observe that the E^3 and E^4 components of ω_{23} and ω_{24} have the form

$$\omega_{23} = \cdots - AE^3 - BE^4, \qquad \omega_{24} = \cdots - BE^3 - \bar A E^4, \tag{B57}$$

where A is complex and B is real,

$$\begin{aligned}A = {}& \bar\mu^A\partial_A(W\cdot\bar\mu) - \bar\mu^A\dot{\bar\mu}_A + \frac{\bar\mu^A\bar\mu'_A}{2U}(V - W^2)\\ &+ W^A\bar\mu^B(\partial_A\bar\mu_B - \partial_B\bar\mu_A),\end{aligned} \tag{B58}$$

$$\begin{aligned}B = {}& \frac{1}{2}\Big(\mu^A\partial_A(W\cdot\bar\mu) - \mu^A\dot{\bar\mu}_A + \frac{\mu^A\bar\mu'_A}{2U}(V - W^2)\\ &+ W^A\mu^B(\partial_A\bar\mu_B - \partial_B\bar\mu_A) + \text{c.c.}\Big).\end{aligned} \tag{B59}$$

Note that $A = B = 0$ on Schwarzschild; it is only the variations δA and δB that do not necessarily vanish. Thus, we have

$$\begin{aligned}&(\delta\omega_{23}\wedge E^3 + \delta\omega_{24}\wedge E^4)|_{d\Theta^A\wedge d\Theta^B}\\ &= -(\delta B)(E^3\wedge E^4 + E^4\wedge E^3) - B(\delta E^3\wedge E^4 + \delta E^4\wedge E^3)\\ &\quad - A\delta E^3\wedge E^3 - \bar A\delta E^4\wedge E^4\\ &= 0,\end{aligned} \tag{B60}$$

where the second line vanishes since $A = B = 0$, and the first line vanishes due to $E^3\wedge E^4 + E^4\wedge E^3 = 0$.

At this point we are left with the two terms,

$$(i_\xi E^3)\delta\omega_{34}\wedge E^4 + (i_\xi E^4)\delta\omega_{43}\wedge E^3. \tag{B61}$$

We first note that the E^3 and E^4 components of $\omega_{34} = -\omega_{43}$ can be written compactly using $\mu^A\bar\mu^B - \bar\mu^A\mu^B = i\epsilon^{AB}$ as

$$\omega_{34} = \cdots + i\epsilon^{AB}(\partial_A\bar\mu_B E^3 + \partial_A\mu_B E^4). \tag{B62}$$

The variation $\delta\epsilon^{AB}$ is proportional to the trace $\gamma^{AB}h_{AB}$ and therefore vanishes in Bondi gauge. Therefore if we vary ω_{34}, the variation only acts on the expression inside the parentheses,

$$\begin{aligned}\delta\omega_{34} &= \cdots + i\epsilon^{AB}\delta(\partial_A\bar\mu_B E^3 + \partial_A\mu_B E^4)\\ &= \cdots + i\epsilon^{AB}(\partial_A\delta\bar\mu_B E^3 + \partial_A\delta\mu_B E^4\\ &\qquad + \partial_A\bar\mu_B\delta E^3 + \partial_A\mu_B\delta E^4).\end{aligned} \tag{B63}$$

Plugging this in and using $i_\xi E^3 = \xi^A\mu_A$ and $i_\xi E^4 = \xi^A\bar\mu_A$, we obtain

$$(i_\xi E^3)\delta\omega_{34}\wedge E^4+(i_\xi E^4)\delta\omega_{43}\wedge E^3 = i\epsilon^{AB}\xi^C\mu_C(\partial_A\delta\bar{\mu}_B E^3+\partial_A\bar{\mu}_B\delta E^3+\partial_A\mu_B\delta E^4)\wedge E^4 \\ -i\epsilon^{AB}\xi^C\bar{\mu}_C(\partial_A\delta\mu_B E^4+\partial_A\bar{\mu}_B\delta E^3+\partial_A\mu_B\delta E^4)\wedge E^3 \\ =\xi^C X_C, \tag{B64}$$

where X_C takes the form

$$X_C = i\epsilon^{AB}\mu_C(\partial_A\delta\bar{\mu}_B E^3+\partial_A\bar{\mu}_B\delta E^3+\partial_A\mu_B\delta E^4)\wedge E^4 - i\epsilon^{AB}\bar{\mu}_C(\partial_A\delta\mu_B E^4+\partial_A\bar{\mu}_B\delta E^3+\partial_A\mu_B\delta E^4)\wedge E^3 \\ = i\epsilon^{AB}[\mu_C(\partial_A\delta\bar{\mu}_B)\mu_D\bar{\mu}_E+\mu_C(\partial_A\bar{\mu}_B)\delta\mu_D\bar{\mu}_E+\mu_C(\partial_A\mu_B)\delta\bar{\mu}_D\bar{\mu}_E+\bar{\mu}_C(\partial_A\delta\mu_B)\mu_D\bar{\mu}_E \\ +\bar{\mu}_C(\partial_A\bar{\mu}_B)\mu_D\delta\mu_E+\bar{\mu}_C(\partial_A\mu_B)\mu_D\delta\bar{\mu}_E]d\Theta^D\wedge d\Theta^E. \tag{B65}$$

One finds that this expression is

$$X_C=\frac{1}{2}\left(\partial_\theta\frac{h_{\phi\theta}}{\sin\theta}+\frac{2\cos\theta}{\sin^2\theta}h_{\phi\theta}-\frac{\partial_\phi h_{\theta\theta}}{\sin\theta},\sin\theta\partial_\theta\frac{h_{\phi\phi}}{\sin^2\theta}+\frac{2\cos\theta}{\sin^2\theta}h_{\phi\phi}-\partial_\phi\frac{h_{\theta\phi}}{\sin\theta}\right)d\Omega=-\frac{r^2}{2}\epsilon^{AB}D_A h_{BC}d\Omega, \tag{B66}$$

where $d\Omega=\sin\theta d\theta\wedge d\phi$, and D_A denotes the unit 2-sphere covariant derivative compatible with γ_{AB}. Notice that ϵ^{AB} here is the Levi-Civita tensor for the metric g_{AB}, which contains the r^2 factor. If we write $\bar{\epsilon}^{AB}$ for the Levi-Civita tensor corresponding to the S^2 metric γ_{AB}, we have the relation $\bar{\epsilon}^{AB}=r^2\epsilon^{AB}$ and

$$X_C=-\frac{1}{2}\bar{\epsilon}^{AB}D_A h_{BC}d\Omega. \tag{B67}$$

Collecting the results, we obtain the magnetic diffeomorphism charge associated with a vector field ξ to be

$$\delta\!\!\!/ Q_M^{\mathcal{H}^+}=\frac{1}{8\pi}\int_{\partial\mathcal{H}^+}\xi^C X_C \\ =-\frac{1}{16\pi}\int_{\partial\mathcal{H}^+}d^2\Theta\sqrt{\gamma}\xi^C\bar{\epsilon}^{AB}D_A h_{BC}. \tag{B68}$$

APPENDIX C: DIRAC BRACKET OF NONINTEGRABLE PIECE

We can rewrite $\mathcal{N}_f^{\mathcal{H}^+}$ in terms of the delta function $\partial_{\bar{z}}f=2\pi\delta^2(z-w)$. Doing so and taking note that the covariant derivative D_z is acting on a scalar and is therefore a plain partial derivative, we obtain

$$\mathcal{N}_f^{\mathcal{H}^+}=-\frac{1}{8\pi M}\int_{\mathcal{H}^+}dv\,d^2z(\partial_{\bar{z}}f)\partial_z[D^2-1]^{-1}D^B D^A\sigma_{AB} \tag{C1}$$

Partial integration in the second term by $\bar{z}$ yields

$$\mathcal{N}_f^{\mathcal{H}^+}=\frac{1}{8\pi M}\int_{\mathcal{H}^+}dv\,d^2z(\partial_z\partial_{\bar{z}}f)[D^2-1]^{-1}D^B D^A\sigma_{AB}. \tag{C2}$$

The boundary term arising from this vanishes, since $\partial_{\bar{z}}f=2\pi\delta^2(z-w)$ and the contour does not cross w. To treat $[D^2-1]^{-1}$ explicitly, let us consider its Green's function $\Delta(z,z')$ of D^2-1, [51]

$$(D^2-1)\Delta(z,z')=\frac{1}{\gamma_{z\bar{z}}}\delta^2(z-z'), \tag{C3}$$

which is derived in Appendix C 1 to be,

$$\Delta(z,z')=\frac{1}{4\sin(\pi\lambda)}P_\lambda(-\mathbf{n}_z\cdot\mathbf{n}_{z'}), \tag{C4}$$

where $\lambda=\frac{1}{2}(-1+i\sqrt{3})$, P_λ is the Legendre function, and

$$\mathbf{n}_z=\left(\frac{z+\bar{z}}{1+z\bar{z}},\frac{i(\bar{z}-z)}{1+z\bar{z}},\frac{1-z\bar{z}}{1+z\bar{z}}\right) \tag{C5}$$

is the Cartesian coordinates of a unit vector on the sphere characterized by $(z,\bar{z})$. The quantity $\mathbf{n}_z\cdot\mathbf{n}_{z'}$ reduces to $\cos\theta$ when $(z',\bar{z}')$ is set to the north pole, as it should. Using Δ, we can write (43) as

$$\mathcal{N}_f^{\mathcal{H}^+}=\frac{1}{8\pi M}\int_{\mathcal{H}^+}dv\,d^2z(\partial_z\partial_{\bar{z}}f) \\ \times\int d^2z'\sqrt{\gamma'}\Delta(z,z')D^{B'}D^{A'}\sigma_{A'B'}. \tag{C6}$$

In the second term on the rhs, let us partial integrate the two covariant derivatives on $\sigma_{A'B'}$ to Δ. This gives rise to two boundary terms, but one can use (C4) to show that they vanish, see Appendix C 2 for details,

$$\mathcal{N}_f^{\mathcal{H}^+} = \frac{1}{8\pi M}\int_{\mathcal{H}^+} d^2z(\partial_z\partial_{\bar z}f)\int d^2z'\sqrt{\gamma'}(D^{A'}D^{B'}\Delta(z,z'))\sigma_{A'B'}. \tag{C7}$$

First, let us compute the Dirac bracket $\{\mathcal{N}_f^{\mathcal{H}^+}, \delta Q_g^{\mathcal{H}^+}\}_D$. This is zero, since it is proportional to the expression

$$\left\{\int_{\mathcal{H}^+} dv\, d^2z(\partial_z\partial_{\bar z}f)\int d^2z'\sqrt{\gamma'}(D^{A'}D^{B'}\Delta(z,z'))\sigma_{A'B'}, \int_{\mathcal{H}^+} dv\, d^2z''\sqrt{\gamma''}(D^{E''}D^{C''}g)\sigma_{D''C''}\right\}_D$$

that vanishes. Next, we compute $\{\mathcal{N}_f^{\mathcal{H}^+}, \delta\tilde{Q}_g^{\mathcal{H}^+}\}_D$. It is proportional to the quantity

$$\begin{aligned}&\left\{\int_{\mathcal{H}^+} dv\, d^2z(\partial_z\partial_{\bar z}f)\int d^2z'\sqrt{\gamma'}(D^{A'}D^{B'}\Delta(z,z'))\sigma_{A'B'}, \int_{\mathcal{H}^+_-} d^2z''\sqrt{\gamma''}(D^{E''}D^{C''}g)\epsilon_{E''}{}^{D''}h_{D''C''}\right\}_D\\ &= 32\pi M^2\int_{\mathcal{H}^+_-} d^2z(\partial_z\partial_{\bar z}f)\int d^2z'\sqrt{\gamma'}(D^{A'}D^{B'}\Delta(z,z'))(D^{E'}D^{C'}g)\epsilon_{E'}{}^{D'}\gamma_{A'B'D'C'},\end{aligned} \tag{C8}$$

where we have used (26), with $\gamma_{ABCD} = \gamma_{AC}\gamma_{BD} + \gamma_{AD}\gamma_{BC} - \gamma_{AB}\gamma_{CD}$. Partial integrating the two covariant derivatives on Δ to g while noting that $D_A\epsilon_{BC} = 0$ and $D_A\gamma_{BCDE} = 0$, we obtain

$$\begin{aligned}&\left\{\int_{\mathcal{H}^+} dv\, d^2z(\partial_z\partial_{\bar z}f)\int d^2z'\sqrt{\gamma'}(D^{A'}D^{B'}\Delta(z,z'))\sigma_{A'B'}, \int_{\mathcal{H}^+_-} d^2z''\sqrt{\gamma''}(D^{E''}D^{C''}g)\epsilon_{E''}{}^{D''}h_{D''C''}\right\}_D\\ &= 32\pi M^2\int d^2z(\partial_z\partial_{\bar z}f)\int d^2z'\sqrt{\gamma'}\Delta(z,z')(D^{B'}D^{A'}D^{E'}D^{C'}g)\epsilon_{E'}{}^{D'}\gamma_{A'B'D'C'}\\ &= 64\pi iM^2\int d^2z(\partial_z\partial_{\bar z}f)\int d^2z'\sqrt{\gamma'}\Delta(z,z')(D^{\bar z'}D^{\bar z'}D^{z'}D^{z'}g - D^{z'}D^{z'}D^{\bar z'}D^{\bar z'}g)(\gamma_{z'\bar z'})^2\\ &= 64\pi iM^2\int d^2z(\partial_z\partial_{\bar z}f)\int d^2z'\sqrt{\gamma'}\Delta(z,z')(\gamma^{z'\bar z'})^2[D^2_{z'}, D^2_{\bar z'}]g.\end{aligned} \tag{C9}$$

The boundary term arising from the partial integration is similar to that discussed in Appendix C 2 and vanish for the same reason. [52] In the second equation, we have used the fact that the only nonvanishing components of $\epsilon_A{}^B$ and γ_{ABCD} are $\epsilon_z{}^z = -\epsilon_{\bar z}{}^{\bar z} = i$ and $\gamma_{zz\bar z\bar z} = \gamma_{\bar z\bar z zz} = \frac{8}{(1+z\bar z)^2} = 2\gamma_{z\bar z}{}^2$ respectively. One can readily check that $[D^2_z, D^2_{\bar z}]g = 0$.

We conclude that $\mathcal{N}_f^{\mathcal{H}^+}$ has zero bracket with both charges,

$$\{\mathcal{N}_f^{\mathcal{H}^+}, \delta Q_g^{\mathcal{H}^+}\}_D = 0, \qquad \{\mathcal{N}_f^{\mathcal{H}^+}, \delta\tilde{Q}_g^{\mathcal{H}^+}\}_D = 0, \tag{C10}$$

and therefore we do not be concerned about this term when computing Dirac brackets.

1. Green's function for $D^2 - 1$

In this section, we present a derivation of the Green's function for the negative-definite operator $D^2 - 1$ on the unit sphere using standard textbook techniques. As operators of this form are of interest in various areas of physics, their Green's functions can be found in many places in the literature, see for example [53] and references therein.

The Green's function $\Delta(\Omega, \Omega')$ for $D^2 - 1$ is a solution to the equation

$$\begin{aligned}(D^2 - 1)\Delta(\Omega, \Omega') &= \delta(\Omega - \Omega')\\ &\equiv \frac{1}{\sin\theta}\delta(\theta - \theta')\delta(\phi - \phi'),\end{aligned} \tag{C11}$$

where Ω and Ω' represent points on the unit sphere, and the differential operator acts on Ω. Due to spherical symmetry, the Green's function will only depend on the geodesic distance between Ω and Ω'. Without any loss of generality, we can assign the coordinates on the sphere such that Ω' sits at the north pole. Then, the geodesic distance between Ω and Ω' is given by θ. By spherical symmetry, this solution must be the same as when Ω' is not necessarily at the north pole but instead $\phi = \phi'$, in which case the geodesic distance is $|\theta - \theta'|$. Thus, we will solve the following equation first,

$$(D^2 - 1)\Delta(|\theta - \theta'|) = \frac{1}{2\pi\sin\theta}\delta(\theta - \theta'), \tag{C12}$$

and restore the ϕ-dependence later. The operator D^2 in spherical coordinates reads

$$D^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}, \tag{C13}$$

so by changing variables to $t = \cos\theta$, we can write (C12) as

$$\left(\frac{d}{dt}(1-t^2)\frac{d}{dt} - 1\right)\Delta(t,t') = \frac{1}{2\pi}\delta(t-t'). \tag{C14}$$

We can obtain the Green's function Δ by solving this equation for $t < t'$ and $t > t'$, and then stitching the two solutions together at $t = t'$.

The differential equation (C14) states that a second-order differential operator acting on Δ yields a delta function. This implies that Δ is continuous at $t = t'$; otherwise the discontinuity can locally be written in terms of the Heaviside step function, and $\frac{d^2}{dt^2}$ acting on it will yield a derivative of the delta function, which is not present in (C14). So, we have

$$\lim_{\epsilon\to 0^+}\Delta(t'-\epsilon,t') = \lim_{\epsilon\to 0^+}\Delta(t'+\epsilon,t'). \tag{C15}$$

On the other hand, $\frac{d\Delta}{dt}$ is discontinuous, which can be seen by integrating (C14) around an infinitesimal region around $t = t'$,

$$\lim_{\epsilon\to 0^+}(1-t'^2)\left(\frac{d\Delta}{dt}\bigg|_{t=t'+\epsilon} - \frac{d\Delta}{dt}\bigg|_{t=t'-\epsilon}\right) = \frac{1}{2\pi}. \tag{C16}$$

With the stitching conditions (C15) and (C16) in mind, let us solve (C14) for $t \neq t'$. Equation (C14) for $t \neq t'$ takes the form of a Legendre equation,

$$\left(\frac{d}{dt}(1-t^2)\frac{d}{dt} + \lambda(\lambda+1)\right)\Delta(t,t') = 0, \tag{C17}$$

with $\lambda = \frac{-1\pm i\sqrt{3}}{2}$ (such that $\lambda(\lambda+1) = -1$). Being a second-order ordinary differential equation, this has two linearly independent solutions, the Legendre functions $P_\lambda(t)$ and $Q_\lambda(t)$ of the first and second kind. When $\lambda = n$ where n is an integer, $P_n(t)$ is a Legendre polynomial. Legendre polynomials have a definite parity, so for instance $P_n(t)$ and $P_n(-t) = (-1)^n P_n(t)$ are not linearly independent. However, for noninteger λ, $P_\lambda(t)$ is linearly independent to $P_\lambda(-t)$, (Eqs. 8.2.3 and 8.3.1 of [54])

$$P_\lambda(-t) = \cos(\lambda\pi)P_\lambda(t) - \frac{2}{\pi}\sin(\pi\lambda)Q_\lambda(t). \tag{C18}$$

This relation implies that for noninteger λ, we can use $P_\lambda(t)$ and $P_\lambda(-t)$ [instead of the standard pair $P_\lambda(t)$ and $Q_\lambda(t)$] as a basis of solutions to (C17). Thus, we can write

$$\Delta(t,t') = \begin{cases} a_1 P_\lambda(t) + a_2 P_\lambda(-t) & \text{for } t < t', \\ b_1 P_\lambda(t) + b_2 P_\lambda(-t) & \text{for } t > t', \end{cases} \tag{C19}$$

where a_1, a_2, b_1 and b_2 are functions of t' only. We demand that the Green's function $\Delta(t,t')$ is well-defined everywhere but $t = t'$. Taking note that $P_\lambda(1) = 1$ and $P_\lambda(-1) = \infty$, one can see that this fixes $a_1 = b_2 = 0$,

$$\Delta(t,t') = \begin{cases} a_2 P_\lambda(-t) & \text{for } t < t', \\ b_1 P_\lambda(t) & \text{for } t > t'. \end{cases} \tag{C20}$$

The remaining coefficients a_2 and b_1 are fixed by the stitching conditions (C15) and (C16), which read

$$a_2 P_\lambda(-t') = b_1 P_\lambda(t'), \tag{C21}$$

$$b_1 P'_\lambda(t') + a_2 P'_\lambda(-t') = \frac{1}{2\pi(1-t'^2)}. \tag{C22}$$

These can equivalently be written as

$$\begin{pmatrix} P_\lambda(-t') & -P_\lambda(t') \\ P'_\lambda(-t') & P'_\lambda(t') \end{pmatrix}\begin{pmatrix} a_2 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2\pi(1-t'^2)} \end{pmatrix}. \tag{C23}$$

Solving for a_2 and b_1, we obtain

$$\begin{aligned}\begin{pmatrix} a_2 \\ b_1 \end{pmatrix} &= \frac{1}{(P_\lambda(-t')P'_\lambda(t') + P_\lambda(t')P'_\lambda(-t'))} \\ &\quad\times\begin{pmatrix} P'_\lambda(t') & P_\lambda(t') \\ -P'_\lambda(-t') & P_\lambda(-t') \end{pmatrix}\begin{pmatrix} 0 \\ \frac{1}{2\pi(1-t'^2)} \end{pmatrix} \\ &= \frac{-1}{2\pi(1-t'^2)\mathcal{W}\{P_\lambda(t),P_\lambda(-t)\}|_{t=t'}}\begin{pmatrix} P_\lambda(t') \\ P_\lambda(-t') \end{pmatrix},\end{aligned} \tag{C24}$$

where $\mathcal{W}\{\cdot,\cdot\}|_{t=t'}$ is the Wronskian,

$$\begin{aligned}\mathcal{W}\{P_\lambda(t),P_\lambda(-t)\} &= \begin{vmatrix} P_\lambda(t) & P_\lambda(-t) \\ \frac{d}{dt}P_\lambda(t) & \frac{d}{dt}P_\lambda(-t) \end{vmatrix} \\ &= \begin{vmatrix} P_\lambda(t) & P_\lambda(-t) \\ P'_\lambda(t) & -P'_\lambda(-t) \end{vmatrix} \\ &= -(P_\lambda(t)P'_\lambda(-t) \\ &\quad + P_\lambda(-t)P'_\lambda(t)),\end{aligned} \tag{C25}$$

evaluated at $t = t'$. To compute the Wronskian of $P_\lambda(t)$ and $P_\lambda(-t)$, we first note that the Wronskian of $P_\lambda(t)$ and $Q_\lambda(t)$ is (Eq. 8.1.9 of [54])

$$\mathcal{W}\{P_\lambda(t),Q_\lambda(t)\} = \frac{1}{1-t^2}. \tag{C26}$$

Then, we use the relation (C18) to obtain

$$\begin{aligned}\mathcal{W}\{P_\lambda(t), P_\lambda(-t)\} &= \cos(\lambda\pi)\mathcal{W}\{P_\lambda(t), P_\lambda(t)\} \\ &\quad - \frac{2}{\pi}\sin(\pi\lambda)\mathcal{W}\{P_\lambda(t), Q_\lambda(t)\} \\ &= \frac{-2\sin(\pi\lambda)}{\pi(1-t^2)}, \end{aligned} \tag{C27}$$

since $\mathcal{W}\{P_\lambda(t), P_\lambda(t)\} = 0$. This with (C24) implies that a_2 and b_1 are

$$\begin{pmatrix} a_2 \\ b_1 \end{pmatrix} = \frac{1}{4\sin(\pi\lambda)} \begin{pmatrix} P_\lambda(t') \\ P_\lambda(-t') \end{pmatrix}. \tag{C28}$$

Plugging these into (C20), we obtain the Green's function

$$\Delta(t,t') = \frac{1}{4\sin(\pi\lambda)} \begin{cases} P_\lambda(t')P_\lambda(-t) & \text{for } t < t', \\ P_\lambda(-t')P_\lambda(t) & \text{for } t > t'. \end{cases} \tag{C29}$$

Putting Ω' back at the north pole (and hence $\theta' = 0$ and $t' = 1$) and recalling that $\lambda = \frac{-1+i\sqrt{3}}{2}$, we obtain

$$\Delta(\theta) = \frac{1}{4\sin(\pi\lambda)} P_{\frac{-1+i\sqrt{3}}{2}}(-\cos\theta). \tag{C30}$$

So, this is the Green's function when Ω' is the north pole. For a generic point Ω' on the sphere, spherical symmetry demands that Δ only depend on the geodesic distance γ between Ω and Ω', which is given as

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'), \tag{C31}$$

and we have

$$\Delta(\Omega,\Omega') = \frac{1}{4\sin(\pi\lambda)} P_{\frac{-1+i\sqrt{3}}{2}}(-\cos\gamma), \tag{C32}$$

as a solution to the Eq. (C11). We note that it does not matter which of the two orders $\lambda = \frac{-1\pm i\sqrt{3}}{2}$ we choose, since $P_\lambda(t) = P_{\lambda^*}(t)$; we have just chosen a plus sign for definiteness.

2. Treatment of boundary term

In this section, we show that the boundary terms arising from partial integrating the rhs of (C6) vanish.

One can see that this partial integration involves

$$\begin{aligned}&\int d^2z'\sqrt{\gamma'}\Delta(z,z')D^{B'}D^{A'}\sigma_{A'B'} \\ &\quad = \int d^2z'\sqrt{\gamma'}D^{B'}(\Delta(z,z')D^{A'}\sigma_{A'B'}) \\ &\qquad - \int d^2z'\sqrt{\gamma'}D^{A'}(\sigma_{A'B'}D^{B'}\Delta(z,z')) \\ &\qquad + \int d^2z'\sqrt{\gamma'}(D^{A'}D^{B'}\Delta(z,z'))\sigma_{A'B'}, \end{aligned} \tag{C33}$$

so the boundary term arising from this procedure is proportional to the quantity

$$\begin{aligned}&\int d^2z'\sqrt{\gamma'}D^{B'}(\Delta(z,z')D^{A'}\sigma_{A'B'}) \\ &\quad - \int d^2z'\sqrt{\gamma'}D^{A'}(\sigma_{A'B'}D^{B'}\Delta(z,z')) \\ &= -i\oint_z dz'\gamma^{z'\bar{z}'}(\Delta(z,z')\partial_{\bar{z}'}\sigma_{z'z'} - \sigma_{z'z'}\partial_{\bar{z}'}\Delta(z,z')) \\ &\quad + i\oint_z d\bar{z}'\gamma^{z'\bar{z}'}(\Delta(z,z')\partial_{z'}\sigma_{\bar{z}'\bar{z}'} - \sigma_{\bar{z}'\bar{z}'}\partial_{z'}\Delta(z,z')), \end{aligned} \tag{C34}$$

where we have used Stokes' theorem. This vanishes if (a) Δ and $\partial_{\bar{z}'}\Delta$ do not have z'-poles at $z' = z$ and (b) Δ and $\partial_{z'}\Delta$ do not have $\bar{z}'$-poles at $z' = z$.

To show that both (a) and (b) are true, we start from the Green's function $\Delta(z,z')$ given in (C4). For the moment, let us put $z', \bar{z}' = 0$ (the north pole) and restore them later. This gives

$$\Delta(z,0) = \frac{1}{4\sin(\lambda\pi)} P_\lambda\left(\frac{z\bar{z}-1}{z\bar{z}+1}\right). \tag{C35}$$

Only the asymptotic behavior of $\Delta(z,0)$ near $z, \bar{z} = 0$ is relevant for the boundary contribution (C34), and for this we need the asymptotic behavior of $P_\lambda(t)$ near $t = -1$. This can be derived via the asymptotic behaviors of $P_\lambda(t)$ and $Q_\lambda(t)$ near $t = 1$, which read [55]

$$P_\lambda(t) \sim 1, \qquad Q_\lambda(t) \sim \frac{1}{2}\ln\left(\frac{2}{1-t}\right), \quad \text{as } t \to 1, \tag{C36}$$

and using the relation (C18), which yields

$$P_\lambda(t) \sim \frac{1}{\pi}\sin(\pi\lambda)\ln(1+t), \quad \text{as } t \to -1. \tag{C37}$$

Applying this to the Green's function (C35) with $t = (z\bar{z}-1)/(z\bar{z}+1)$, we obtain

$$\Delta(z,0) \sim \frac{1}{4}\ln(z\bar{z}), \quad \text{as } z, \bar{z} \to 0. \tag{C38}$$

Restoring the reference point z', the asymptotic form of the Green's function near $z = z'$ is [56]

$$\Delta(z,z') \sim \frac{1}{4}\ln|z-z'|^2, \quad \text{as } (z,\bar{z}) \to (z',\bar{z}'). \tag{C39}$$

One immediately sees that Δ has a logarithmic singularity at $z = z'$ and therefore has no poles there. Also, $\partial_{z'}\Delta = \frac{1}{4(z'-z)}$ has no $\bar{z}'$-pole at $z' = z$, and $\partial_{\bar{z}'}\Delta = \frac{1}{4(\bar{z}'-\bar{z})}$ has no z'-pole at $z = z'$. Therefore, the boundary term (C34) receives no residues and vanishes.

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[52] The boundary term arising from the partial integration is proportional to the expressions

$$\int d^2z'\sqrt{\gamma'}[D^{A'}(D^{B'}\Delta(z,z')D^{E'}D^{C'}g) - D^{B'}(\Delta(z,z')D^{A'}D^{E'}D^{C'}g)]\epsilon_{E'}{}^{D'}\gamma_{A'B'D'C'}$$
$$= 2i\oint_z dz'\gamma^{z'\bar{z}'}((D_z^2 g)\partial_{\bar{z}'}\Delta(z,z') - \Delta(z,z')D_{\bar{z}'}D_z^2 g)$$
$$+ 2i\oint_z d\bar{z}'\gamma^{z'\bar{z}'}((D_{\bar{z}'}^2 g)\partial_{z'}\Delta(z,z') - \Delta(z,z')D_{z'}D_{\bar{z}'^2} g).$$

It is shown in Appendix C 1 that $\Delta \sim \frac{1}{4}\log|z-z'|^2$ as $z \to z'$, so the above expression vanishes due to lack of appropriate poles.

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$$1-\cos\gamma = 1-\mathbf{n}_z\cdot\mathbf{n}_{z'} = \frac{2(z'-z)(\bar{z}'-\bar{z})}{(1+z\bar{z})(1+z'\bar{z}')}. \quad \text{(C40)}$$

Then, taking $z = z' + re^{i\phi}$ and expanding around $r=0$ leads to

$$1-\cos\gamma = \frac{2r^2}{(1+z'\bar{z}')^2} + O(r^3), \quad \text{(C41)}$$

which plugged into (C37) for $P_\lambda(-\cos\gamma)$ and then into (C4) leads to $\Delta(z,z') \sim \frac{1}{4}\ln r^2 = \frac{1}{4}\ln(z-z')(\bar{z}-\bar{z}')$ for $r\to 0$, in agreement with (C39).