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A statistical mechanical model for non-perturbative regimes

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Abstract

Thanks to Feynman graphons, as mathematical tools in dealing with Dyson–Schwinger equations, we formulate a new statistical mechanical model for the study of equilibrium states and observables associated with the solution space of quantum motions in a strongly coupled gauge field theory underlying the evolution of running coupling constants, time and temperature.

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1. Introduction

The most difficult challenge of strongly coupled interacting gauge field theories is dealing with non-perturbative aspects of low energy levels (= long distance or time) while the space-time background should be quantized the same as other fields. The confinement of color-charged particles together with the dynamical chiral symmetry breaking, as an effective mechanism for mass generating, is the fundamental emergent phenomena in the universe which is considered in Quantum Chromodynamics (QCD) sector of Standard Model. QCD is the gauge field theory of the strong interaction with SU(3) gauge invariance such that the interaction carried by gluons acting on quarks and gluons. It considers the physics of elementary particles at the length scales on the order of 10^{-15} meter (= proton radius) or smaller with the energy levels around $\Lambda_{\mathrm{QCD}} \sim$ 0.2 GeV or greater while perturbation platform works only at energies greater than $10\Lambda_{\mathrm{QCD}}$ (= short distance or time). [15]

Thanks to the Connes–Kreimer renormalization Hopf algebraic approach to gauge field theories [12,19,43] and the theory of graphons in infinite combinatorics [5,6,8,23], the theory of Feynman graphons [31,32,37,38] provided the basic elements of a new constructive approach to non-perturbative aspects in terms of the geometry of the separable Banach manifold $\mathcal{S}^{\Phi,g}_{\approx}$ of weakly isomorphic classes of large Feynman diagrams which contribute to solutions of combinatorial Dyson–Schwinger equations (DSEs) in a strongly coupled gauge field theory Φ with the bare coupling constant g. In fact, $\mathcal{S}^{\Phi,g}_{\approx}$ is the solution space of quantum motions in Φ. Solutions of combinatorial DSEs under the evolution of strong running coupling constants c_g, which are interpreted by infinite direct sums of stretched Feynman graphons, encode intermediate phases of non-perturbative sector of Φ [35,36]. Geodesics in $\mathcal{S}^{\Phi,g}_{\approx}$ describe phase transitions where homomorphism densities of stretched Feynman graphons associated with combinatorial DSEs characterize non-perturbative phases [36]. Replacing space-time regions with cut-distance topological regions of Feynman diagrams provided a new constructive approach to formulate non-perturbative gauge field theory [32,34,37,38] while the background cut-distance metric space has a well-defined quantization program [39]. In addition, the space $\mathcal{S}^{\Omega,\Phi}_{\mathrm{graphon}}(\mathbb{R})$ of stretched Feynman graphons topologically completes the space of Feynman diagrams of Φ such that the resulting topologically enriched renormalization Hopf algebra $H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi)$ provides a non-perturbative renormalization program for the solution space of quantum motions [33,36].

1.1. Physical motivation

A quantum system of particles with finite degrees of freedom is defined by the complex Hilbert space $\mathbb{C}^n$ of states, the algebra of observables $M_n(\mathbb{C})$, the positive selfadjoint Hamiltonian operator H and a finite temperature T. The interaction of this system with a heat source is studied in terms of exchanging energy to achieve a final equilibrium state with a definite temperature at a feasible time. The Kubo–Martin–Schwinger (KMS) condition characterizes thermal equilibrium states at positive temperatures. Under a discrete setting with the canonical partition function $Z = \mathrm{Tr}(e^{-H/T})$, we have equilibrium states $\phi_{1/T}\big((m_{ij})_{n\times n}\big) = \frac{1}{Z}\mathrm{Tr}\,\big((m_{ij})_{n\times n}e^{-H/T}\big)$ with respect to the time evolution given by the continuous group homomorphism $\mathbb{R} \to \mathrm{Aut}(M_n(\mathbb{C}))$, $t \mapsto e^{itH}(m_{ij})_{n\times n}e^{-itH}$. [21]

A quantum system of particles with infinite degrees of freedom is considered under a continuum setting where the static behavior of the system is formulated on the basis of the partition function $Z = \frac{1}{\hbar}\int \langle q, p|e^{-H/T}|q, p\rangle dqdp$ with respect to the canonical position and momentum parameters p, q. The KMS condition is generalized under an infinite dimensional setting to determine equilibrium states of C^*-algebras of observables associated with conformal and perturbative theories [40,42]. Infinite limits address phase transitions. The phase transitions in interacting gauge field theories at high temperatures are studied in terms of statistical models depended on the time parameter. These statistical models are useful to describe the thermodynamics of the early universe where an equilibrium state is expected. In addition, they describe the thermodynamics of quark–gluon plasmas at high temperatures where the interaction between quarks and gluons is relatively weak and we expect the deconfined–confined phase transition. [1,2,13,20,24,25]

At 10^{-15} meter, the relative strengths of the strong interaction, the electromagnetic and the weak interaction are the scales 1, 10^{-2} and 10^{-6}, respectively. The QCD running coupling strength is depended on the energy scale. The confinement happens at low energy where running couplings are strong and they make impossible to extract a full description of a single quark or gluon annihilation operator such that there is no explicit formulation of a second quantized vacuum for QCD. Thanks to the quark–hadron duality, which provides a bridge between theoretical aspects (in terms of quarks and gluons) and experimental quantities (in terms of hadrons), the state with zero hadrons is considered as the ground state of the non-perturbative strong interaction Hamiltonian where creation and annihilation operators for such states are well-defined on a space-time lattice. The method of Dyson–Schwinger equation (DSE) has provided rigorous progresses in decoding non-perturbative quantum motions in terms of various mathematical disciplines. In this regard, algebraic quantum field theory, which applies master DSE to compute Feynman integrals in Lorentz signature, constructively analyzes non-perturbative aspects of a particular class of physical theories in space-time dimensions lower than 3+1. Lattice models, which consider DSEs in discrete space-time background, apply continuum limit to extract some non-perturbative information on the basis of large-N limit of multi-matrix models. The AdS/CFT correspondence provides some theoretical computations in the gravity approximation of a certain string theory in dealing with the strong limit of a corresponding dual conformal field theory with vanishing beta function. [14–16,22,27–29,41,44]

While at zero temperature we expect the confined phase, the lattice models at non-zero temperature T deal with deconfined–confined phase transitions. For the strong bare coupling constant $g \geq 1$, the non-perturbative behavior of QCD is investigated beyond the length scales $(c_g^2 T)^{-1}$ with respect to the running coupling constants c_g at relatively low energy levels. At short distances in enough high energy levels, QCD is asymptotically free and it can be well described by the quark–gluon perturbation theory. Therefore phase transitions between perturbation and non-perturbation regimes are controlled by the variation of running coupling constants. [10,11,26–28]

The C^*-dynamical system associated with a local quantum field theory is built in terms of (i) a system of C^*-algebras such that $\mathcal{O}_1 \subset \mathcal{O}_2\ : \mathcal{U}(\mathcal{O}_1) \subset \mathcal{U}(\mathcal{O}_2)$ labeled by open bounded space-time regions in $\mathbb{R}^d$ which acts on the Hilbert space $\mathcal{H}$ of states of the physical theory, (ii) a semi-group $\{\sigma_t\}_{t\in\mathbb{R}_+}$ of strongly continuous automorphisms which encode time translation and its local action on $\mathcal{U}(\mathcal{O})$ given by $\sigma_t(\mathcal{U}(\mathcal{O})) = \mathcal{U}(\mathcal{O} + t.n)$, (iii) a family $\Theta_{\beta,\mathcal{O}} : \mathcal{U}(\mathcal{O}) \to \mathcal{H}$, $\Theta_{\beta,\mathcal{O}}(A) = e^{-\beta H} A\Omega$ of nuclear maps for any $\beta = \frac{1}{T} > 0$ and the vacuum Ω [3,4]. Theory of Feynman graphons enables us to describe a gauge field theory on the space-time background in terms of replacing open regions of space-time with cut-distance open regions of Feynman diagrams [34–36]. We will apply this setting as a motivation to build a new family of C^*-dynamical systems which encode equilibrium states and observables of non-perturbative regimes. These systems are associated with non-local regions of space-time.

1.2. Mathematical framework

Types of elementary particles and types of their interactions in a (strongly coupled) gauge field theory Φ are encapsulated by Green's functions. Feynman diagrams are useful to formulate the combinatorial version of Green's functions in terms of formal expansions

$$G^{e_i}(c_g) = \mathbb{I} - \sum_{\mathrm{res}(\Gamma)=e_i} c_g^{|\Gamma|} \frac{\Gamma}{\mathrm{Sym}(\Gamma)}\ , \quad G^{v_j}(c_g) = \mathbb{I} + \sum_{\mathrm{res}(\Gamma)=v_j} c_g^{|\Gamma|} \frac{\Gamma}{\mathrm{Sym}(\Gamma)}\ , \tag{1.1}$$

which encode all possible interactions in Φ under different running coupling constants c_g generated by some regularization techniques. Sub-divergences of Feynman integrals are represented in terms of nested loops in their corresponding Feynman diagrams. Superficial degree of divergence is a parameter to recognize type of divergence or convergence of each loop. Quantum motions are determined in terms of fixed point equations of these Green's functions. The resulting integral equations are studied in the context of DSEs as quantized versions of the classical Euler–Lagrange equations of motion. [17–19,28]

The Connes–Kreimer renormalization Hopf algebra $H_{\mathrm{FG}}(\Phi) = \bigoplus_{n\geq 0} H_{(n)}$ of Feynman diagrams is a graded connected free commutative non-cocommutative Hopf algebra generated by 1PI Feynman diagrams. It is graded by the loop number parameter such that for each $n \geq 1$, $H_{(n)}$ is the vector space generated by 1PI Feynman diagrams with the loop number n or products of 1PI Feynman diagrams with the overall loop number n. This Hopf algebra, which encodes the BPHZ perturbative renormalization, is applied to combinatorially reformulate DSEs in terms of systems of coupled recursive Hochschild equations. The combinatorics of $H_{\mathrm{FG}}(\Phi)$ is discussed in terms of the Connes–Kreimer Hopf algebra of non-planar rooted trees and the universal Hopf algebra of renormalization where Feynman diagrams have representations in terms of decorated trees or forests and Hall sets of words. 1PI primitive Feynman diagrams in $H_{\mathrm{FG}}(\Phi)$ are applied as the decoration collection for non-planar rooted trees. Each vertex in a tree t_Γ is a symbol for a nested loop in its corresponding Feynman diagram Γ such that positions of nested loops are encoded in terms of the existence of edges between vertices in t_Γ. If Γ has an overlapped nested loop, then t_Γ is a linear expansion of decorated rooted trees. [9,17,19,30,43]

Graphons are tools in infinite combinatorics for the study of graph limits of sequences of finite weighted graphs in the context of homomorphism densities. A sequence $\{G_n\}_{n\geq 1}$ of simple graphs with increasing vertex numbers is convergent iff there exists a graphon $W : [0,1] \times [0,1] \to [0,1]$ such that the sequence $\{t(K, G_n)\}_{n\geq 1}$ of homomorphism densities is convergent to $t(K, W)$ for any simple graph K. The space of finite graphs is topologically completed by the space of graphons in terms of the cut-norm. $\{G_n\}_{n\geq 1}$ with $|G_n| \to \infty$ is convergent iff it is a Cauchy sequence with respect to the cut-norm. In other words, a graphon W is the graph limit of $\{G_n\}_{n\geq 1}$ iff the sequence $\{W_{G_n}\}_{n\geq 1}$ converges to W with respect to the cut-norm. Measure theoretic tools provide suitable non-trivial graphons such as reweighted or stretched versions of the canonical graphons for the description of graph limits of Cauchy sequences of sparse graphs. Theory of graphons addresses a new random process approach for the analysis of dense or extremely large graphs [5,6,8,23]. Graphons have also applications in quantum physics where Feynman graphons interpret graph limits of sequences of higher orders Feynman diagrams with respect to the cut-norm. In this setting, Feynman graphons provide a topological enrichment of the Connes–Kreimer renormalization Hopf algebra to formulate a non-perturbative generalization of the BPHZ renormalization program for the space of solutions of quantum motions in Φ. [31–38]

Definition 1.

- For any Feynman diagram Γ of loop order n with its tree representation t_Γ, the pixel picture presentation of the adjacency matrix of t_Γ is a symmetric Lebesgue measurable function $P^{\sigma}_{t_\Gamma}$: $[0,1]\times[0,1]\to[0,1]$ such that $\sigma := (I_1, ..., I_n)$ is a partition of $[0,1]$ with $\sum_{i=1}^{n} m(I_i) \leq 1$. The size of each subinterval $I_i \in \sigma$ is determined in terms of the weight of the vertex $v_i \in t_\Gamma$. If $m(I_i) = \frac{1}{n}$ for each $1 \leq i \leq n$, then it is called the canonical Feynman graphon associated with Γ.
- For the pixel picture presentation $P^{\sigma}_{t_\Gamma}$ associated with Γ and any invertible Lebesgue measure-preserving transformation $\rho : [0,1] \to [0,1]$, the graph function $W^{\rho} : [0,1] \times [0,1] \to [0,1]$ is called a labeled Feynman graphon associated with Γ iff $W^{\rho} = P^{\sigma}_{t_\Gamma}$ almost everywhere with respect to the Lebesgue measure such that $W^{\rho}(x,y) := W(\rho(x),\rho(y))$.
- Labeled Feynman graphons W_1, W_2 associated with Γ are called weakly isomorphic, iff there exist Lebesgue measure-preserving transformations τ_1, τ_2 such that $W_1^{\tau_1} = P^{\sigma}_{t_\Gamma} = W_2^{\tau_2}$ almost everywhere. Define $[W_\Gamma]_{\approx}$ as the equivalence class of weakly isomorphic labeled Feynman graphons corresponding to the pixel picture presentation $P^{\sigma}_{t_\Gamma}$. It is the unique unlabeled Feynman graphon class associated with Γ.

Definition 2.

- Feynman diagrams Γ_1, Γ_2 are called weakly isomorphic iff $[W_{\Gamma_1}]_{\approx} = [W_{\Gamma_2}]_{\approx}$. The cut-norm of any Feynman diagram Γ is given by

$$||W_\Gamma||_{\text{cut}} := \inf_\rho \sup_{A,B\subset[0,1]} \left| \int_{A\times B} W_\Gamma(\rho(x),\rho(y))dxdy \right| , \tag{1.2}$$

  such that the infimum is taken over all Lebesgue measure-preserving transformations on $[0,1]$ and the supremum is taken over non-empty Lebesgue measurable subsets A, B of $[0,1]$.
- For $1 \leq p < \infty$, W_Γ is called a L^p-stretched Feynman graphon if

$$||W_\Gamma||_{p,\text{cut}} := \inf_\rho \left(\int_{[0,1]\times[0,1]} |W^{\rho}_\Gamma(x,y)|^p dxdy \right)^{1/p} < \infty . \tag{1.3}$$

The space of (stretched) Feynman graphons topologically completes the space of Feynman diagrams in Φ where we can determine a new family of infinite graphs, called large Feynman diagrams, which contribute to the structure of complete 1PI Green's functions. These large Feynman diagrams are applied to interpret infinite power series generated by solutions of DSEs where a non-perturbative generalization of the BPHZ renormalization on the space of quantum motions is achieved. L^1-measurable functionals on the topological space of Feynman diagrams are applied to formulate some new generalizations of the Johnson–Lapidus Dyson series to characterize the evolution of quantum motions in terms of the behavior of partial sums of solutions of DSEs. The beta function generated by the non-perturbative BPHZ renormalization, which governs the running of strong coupling constants, is formulated in the language of Feynman graphons. [31–34,36]

The Connes–Kreimer theory [17,19] provided the basic elements of an alternative approach to quantum motions of gauge field theories in terms of combinatorial Dyson–Schwinger equations, noncommutative geometry and measure theoretic tools [18,32,34,37,38]. For a given ground measure space (Ω, μ), the space of (stretched) Feynman graphons $\mathcal{S}^{\Phi,\Omega}_{\text{graphon}}(\mathbb{R})$ is applied to formulate a separable Banach space $\mathcal{S}^{\Phi,g}_{\approx}$ of weakly isomorphic equivalence classes of large Feynman diagrams associated with solutions of quantum motions in a (strongly coupled) gauge field theory Φ [35,36]. Here we relate some new mathematical structures to $\mathcal{S}^{\Phi,g}_{\approx}$ to build a quantum statistical model for the study of equilibrium states and observables associated with the solution space of quantum motions underlying the evolution of running coupling constants during a continuum of time. These equilibrium states extract some data from transitions of intermediate phases in non-perturbative sector of Φ.

1.3. Original achievements

- The Banach $*$-algebra $\mathcal{B}(\mathcal{S}^{\Phi,g}_{\approx})$ of bounded operators on $\mathcal{S}^{\Phi,g}_{\approx}$ encodes phase transitions in Φ. The GNS representations of $\mathcal{B}(\mathcal{S}^{\Phi,g}_{\approx})$ determine the universal C^*-algebra $E^{\Phi,g}$. Applying strongly continuous one-parameter semigroups on $E^{\Phi,g}$, which encode the evolution of running coupling constants, lead us to formulate a new family of C^*-dynamical systems for the study of intermediate phase transitions in non-perturbative sector of Φ in the context of KMS states.
- Up to the commensurable equivalence relation, the quotient space of finite dimensional $\mathbb{Q}$-lattices in $\mathcal{S}^{\Phi,g}_{\approx}$ generated by any strongly continuous one-parameter semigroup $\rho = \{\rho_\lambda\}_\lambda$ is considered to build a new C^*-dynamical system $(\mathcal{A}^{\Phi,g}_{1,\rho}, \{\sigma_t\}_t)$ with respect to the time evolution $\{\sigma_t\}_t$. These systems encode observables of non-perturbative sector of Φ such that their KMS states enable us to identify equilibrium states associated with the solution space of quantum motions under the evolution of running coupling constants in Φ during a continuum of time.
- Intermediate phase transitions in non-perturbative sector of Φ are encoded in terms of a dynamical system $\left(\{\mathcal{A}^{\Phi,g}_{1,\rho,\mathcal{V}_l}, \Psi_{\mathcal{V}_l,T}\}_{l=1}^{\infty}, \{\sigma_t\}_t\right)$ of nested C^*-algebras associated with open regions in $\mathcal{S}^{\Phi,g}_{\approx}$ with respect to the time evolution. This system, which provides a generalization of systems associated with local physical theories [3,4], enjoys the nonlinearity condition with respect to the temperature parameter.

2. Limiting to non-perturbative quantum motions in the context of randomness

Consider a (strongly coupled) gauge field theory Φ with the collection $\mathcal{A}_\Phi = \{e_i, v_j\}$ of types of particles and interactions. Thanks to the addressed concepts in Subsection 1.2, it is possible to formulate the notion of convergence and graph limits for the space of Feynman diagrams in Φ in terms of stretched or rescaled versions of canonical Feynman graphons.

2.1. Solutions of quantum motions via Feynman graphons

Feynman graphons, which topologically complete the space of 1PI Green's functions of Φ, together with the Connes–Kreimer Hopf algebraic setting provide a random process for the computation of non-perturbative parameters generated by the BPHZ renormalization of Dyson–Schwinger equations. [35–38]

Definition 3. Consider a σ-finite measure space (Ω, μ).

- For any Feynman diagram Γ, a labeled stretched Feynman graphon associated with Γ is a symmetric bounded μ-measurable function $W : \Omega \times \Omega \to \mathbb{R}$ which is weakly isomorphic to the pixel picture presentation $P^{\nu}_{t_\Gamma}$ such that $\nu := (I_1, ..., I_n)$ is a partition of Ω with $\sum_{i=1}^{n} \mu(I_i) \leq \mu(\Omega)$.
- Set $[W_\Gamma]_{\approx}$ as the unique equivalence class of labeled stretched Feynman graphons which are weakly isomorphic to $P^{\nu}_{t_\Gamma}$ up to the μ-measure preserving transformations on Ω. It is called an unlabeled stretched Feynman graphon associated with Γ.

Lemma 2.1. *There exists a unique unlabeled Feynman graphon class associated with any Feynman diagram Γ in Φ with nested overlapped loops.*

Proof. We work on the Lebesgue measure space $([0, 1], m)$. Thanks to [9,17], $t_\Gamma = \alpha_1 s_1 + ... + \alpha_n s_n$ is a linear combination of non-planar rooted trees $s_1, ..., s_n$ such that $\alpha_1, ..., \alpha_n \in \mathbb{Q}$ or $\mathbb{R}$.

If $\sum_{i=1}^{n} |\alpha_i| < 1$, then consider n subintervals I_i of $[0, 1]$ such that $m(I_i) = |\alpha_i|$, $I_i \cap I_j = \emptyset$, $i \neq j$. Define the direct sum

$$W_\Gamma = \alpha_1 W_{s_1} + ... + \alpha_n W_{s_n} \tag{2.1}$$

such that for $1 \leq i \leq n$, $\pm W_{s_i} : I_i \times I_i \to \mathbb{R}$ is the stretched Feynman graphon associated with s_i which is weakly isomorphic to the Feynman graphon $W_{\alpha_i s_i} : [0, 1] \times [0, 1] \to \mathbb{R}$. Therefore $W_\Gamma : \bigsqcup_{i=1}^{n} I_i \times \bigsqcup_{i=1}^{n} I_i \to \mathbb{R}$ is the stretched Feynman graphon associated with t_Γ.

If $\sum_{j=1}^{n} |\alpha_j| \geq 1$, then we apply suitable affine monotone maps on $\mathbb{R}$, as compositions of linear transformations and translations, such as Lebesgue measure preserving transformations to define $W_\Gamma : \bigsqcup_{i=1}^{n} \tilde{I}_i \times \bigsqcup_{i=1}^{n} \tilde{I}_i \to \mathbb{R}$, with $m(\tilde{I}_i) = \frac{|\alpha_i|}{\sum_{j=1}^{n} |\alpha_j|}$, as the direct sum of stretched Feynman graphons $\pm \tilde{W}_{s_i} : \tilde{I}_i \times \tilde{I}_i \to \mathbb{R}$. □

Theorem 2.2. *Consider the linear space $\mathcal{S}^{\Phi,\Omega}_{\text{graphon}}(\mathbb{R})$ of unlabeled stretched Feynman graphon classes corresponding to 1PI Feynman diagrams in Φ.*

- $\mathcal{S}^{\Phi,[0,1]}_{\text{graphon}}([0, 1])$ *topologically completes the space of Feynman diagrams in Φ with respect to the cut-norm (1.2) and the L^p-metrics (1.3) for $1 \leq p < \infty$.*
- $\mathcal{S}^{\Phi,[a,b]}_{\text{graphon}}(\mathbb{R})$ *is complete and Hausdorff with respect to the cut-distance topology.*
- *Any complete 1PI Green's function $G^r(c_g)$, $r \in \mathcal{A}_\Phi$ under a running coupling constant c_g is presentable by a random process.*
- $\mathcal{S}^{\Phi,[0,1]}_{\text{graphon}}([0, 1])$ *is compact and Hausdorff.*

Proof. (i) and (ii). Thanks to Definitions 1, 3 and Lemma 2.1, the cut-distance between Feynman diagrams is given by

$$d_{\text{cut}}(\Gamma_1, \Gamma_2) := d_{\text{cut}}(W_{\Gamma_1}, W_{\Gamma_2}) = ||W_{\Gamma_1} - W_{\Gamma_2}||_{\text{cut}} \tag{2.2}$$
$$= \inf_{\tau_1,\tau_2} \sup_{A,B\subset\Omega}\left| \int\limits_{A\times B} \Big(W_{\Gamma_1}(\tau_1(x), \tau_1(y)) - W_{\Gamma_2}(\tau_2(x), \tau_2(y))\Big) d\mu(x)d\mu(y)\right| ,$$

and the L^p-distance is given by

$$d_{p,\text{cut}}(\Gamma_1, \Gamma_2) := d_{p,\text{cut}}(W_{\Gamma_1}, W_{\Gamma_2}) = \inf_{\tau_1,\tau_2} ||W^{\tau_1}_{\Gamma_1} - W^{\tau_2}_{\Gamma_2}||_{p,\text{cut}} \tag{2.3}$$
$$= \inf_{\tau_1,\tau_2} \left(\int\limits_{\Omega\times\Omega} \left|W_{\Gamma_1}(\tau_1(x), \tau_1(y)) - W_{\Gamma_2}(\tau_2(x), \tau_2(y))\right|^p d\mu(x)d\mu(y)\right)^{1/p}$$

where the infimum is taken over all μ-measure preserving transformations on Ω and the supremum is taken over all non-trivial μ-measurable subsets of Ω. Feynman diagrams Γ_1, Γ_2 are weakly isomorphic iff $d_{\text{cut}}(\Gamma_1, \Gamma_2) = 0$. They are L^p-weakly isomorphic iff $d_{p,\text{cut}}(\Gamma_1, \Gamma_2) = 0$.

When n tends to infinity, a sequence $\{\Gamma_n\}_{n\geq 1}$ of higher loop order Feynman diagrams is convergent iff the sequence $\{\frac{W_{\Gamma_n}}{||W_{\Gamma_n}||_{\text{cut}}}\}_{n\geq 1}$ of normalized stretched Feynman graphons is convergent with respect to the metric (2.2). It is called L^p-convergent iff the sequence $\{\frac{W_{\Gamma_n}}{||W_{\Gamma_n}||_{p,\text{cut}}}\}_{n\geq 1}$ of normalized L^p-stretched Feynman graphons is convergent with respect to the metric (2.3).

The space $\mathcal{G}^{[0,1]}([0, 1])$ of $[0, 1]$-valued graphons on the Lebesgue measure space $[0, 1]$ is a compact Hausdorff separable metric space with respect to the cut-distance topology [23]. $\mathcal{S}^{\Phi,[0,1]}_{\text{graphon}}([0, 1])$ is a closed subspace of $\mathcal{G}^{[0,1]}([0, 1])$. In addition, the space of $\mathbb{R}_+$-valued stretched graphons is completed with respect to the L^1-norm which enables us to complete this space with respect to some modified versions of the cut-distance metric. Therefore thanks to [5,6,8,35,36], up to the weakly isomorphic relation and L^p-weakly isomorphic relation, d_{cut} and $d_{p,\text{cut}}$ determine Hausdorff separable metrics on the space of Feynman diagrams in Φ which is completed by $\mathcal{S}^{\Phi,\Omega}_{\text{graphon}}(\mathbb{R})$.

(iii) For each loop order n, consider partial expansions

$$G^r_n(c_g) = \mathbb{I} \pm \sum_{\text{res}(\Gamma)=r\ ,\ |\Gamma|=n} c_g^{|\Gamma|} \frac{\Gamma}{\text{Sym}(\Gamma)} . \tag{2.4}$$

Its corresponding stretched Feynman graphon is given by the direct sum of stretched Feynman graphons associated with Feynman diagrams $\Gamma \in H_{(n)}$ with the loop order n given by

$$W_{G^r_n(c_g)} = +_{\text{res}(\Gamma)=r,|\Gamma|=n} \frac{c_g^{|\Gamma|}}{\text{Sym}(\Gamma)} W_\Gamma : A^{r,c_g}_n \times A^{r,c_g}_n \to \mathbb{R} , \tag{2.5}$$

such that for each $n \geq 1$,

$$A^{r,c_g}_n = \bigsqcup_{\text{res}(\Gamma)=r,|\Gamma|=n} I_\Gamma\ ,\ \mu(I_\Gamma) = \frac{c_g^{|\Gamma|}}{\text{Sym}(\Gamma)} , \tag{2.6}$$

and $W_\Gamma : I_\Gamma \times I_\Gamma \to \mathbb{R}$ is the stretched Feynman graphon associated with Γ. When n tends to infinity, the sequence $\{\frac{W_{G^r_n(c_g)}}{||W_{G^r_n(c_g)}||_{\text{cut}}}\}_{n\geq 1}$ converges to a non-zero normalized stretched Feynman graphon

$$W_{G^r(c_g)} : A^{r,c_g} \times A^{r,c_g} \to [0,1] \ , \ A^{r,c_g} := \bigsqcup_{n=1}^{\infty} A_n^{r,c_g} \tag{2.7}$$

with respect to the cut-distance topology.

Define an infinite random graph R_n^{r,c_g} on A_n^{r,c_g} such that there exists an edge between $x, y \in A_n^{r,c_g}$ with the probability $W_{G_n^r(c_g)}(x,y)$. When n tends to infinity, the sequence $\{R_n^{r,c_g}\}_{n\geq 1}$ converges to an infinite random graph R^{r,c_g} on the interval A^{r,c_g} with respect to the cut-distance topology such that with the probability $W_{G^r(c_g)}(u,v)$, there exists an edge between $u, v \in A^{r,c_g}$.

(iv) Thanks to (iii), $\mathcal{S}^{\Phi,[0,1]}_{\text{graphon}}([0,1])$, as a vector space, is generated by Feynman graphons $W_{G^r(1)}$, $r \in \mathcal{A}_\Phi$. The compactness is a direct result of (i), (ii) and [23]. □

Definition 4. For any family $\{\gamma_n\}_{n\geq 1}$ of (1PI) primitive Feynman diagrams in the Hopf algebra $H_{\text{FG}}(\Phi)$ with the corresponding family $\{B^+_{\gamma_n}\}_{n\geq 1}$ of Hochschild one cocycles with respect to the renormalization coproduct, the recursive equation

$$X = \mathbb{I} + \sum_{n\geq 1} c_g^n \omega_n B^+_{\gamma_n}(X^{n+1}) \ , \ \omega_n \in \mathbb{R} \tag{2.8}$$

is called a combinatorial Dyson–Schwinger equation under the running coupling constant c_g.

Feynman rules of Φ are a functional which associates an iterated integral together with subdivergences to each Feynman diagram. Feynman rules are encoded by some elements of the complex Lie group $\mathbb{G}_\Phi(A_{\text{dr}}) = \text{Hom}(H_{\text{FG}}(\Phi), A_{\text{dr}})$ such that A_{dr} is the dimensional regularization algebra of Laurent series with finite pole parts. Feynman rules characters are applied to translate any equation DSE to its original integral equation. The solution of the equation (2.8) is given by

$$X_{\text{DSE}(c_g)} = \sum_{n\geq 0} c_g^n X_n \ , \ X_n = \sum_{j=1}^{n} \omega_j B^+_{\gamma_j}\Big(\sum_{k_1+\ldots+k_{j+1}=n-j, \ k_i\geq 0} X_{k_1}\ldots X_{k_{j+1}} \Big) \tag{2.9}$$

such that $X_0 = \mathbb{I}$ is the empty graph. For $c_g < 1$, $X_{\text{DSE}(c_g)}$ is a convergent series. For $c_g \geq 1$, $X_{\text{DSE}(c_g)}$, as an object in the completion of $H_{\text{FG}}(\Phi)[[c_g]]$ with respect to the n-adic topology, is an infinite series of increasing powers of c_g^n together with Feynman diagrams of orders n such that n goes to infinity [7,18,19]. For any temperature T, the large Feynman diagram $X_{\text{DSE}(c_g)}$ encodes interactions of elementary particles which contribute to $\text{DSE}(c_g) =< \{\gamma_n\}_{n\geq 1} >$ at the length scale $(c_g^2 T)^{-1}$.

Lemma 2.3. *The space of Feynman graphons encodes the solution space of combinatorial Dyson–Schwinger equations under strong running coupling constants.*

Proof. Consider an equation DSE under the running coupling constant $c_g \geq 1$ with the solution X and the Lebesgue measure space $[0,\infty)$. For each $m \geq 1$, the stretched Feynman graphon associated with the partial sum $Y_m = \sum_{i=1}^m c_g^i X_i$ of X is given by the direct sum

$$\Big(||\tilde{W}_{X_1} + \ldots + \tilde{W}_{X_m}||_{\text{cut}} \Big) \tilde{W}_{Y_m} = \tilde{W}_{X_1} + \ldots + \tilde{W}_{X_m} \tag{2.10}$$

such that for each $1 \leq i \leq m$, $\tilde{W}_{X_i} : \tilde{I}_i \times \tilde{I}_i \to \mathbb{R}$ is the stretched Feynman graphon corresponding to X_i with $m(\tilde{I}_i) = c_g^i$ and $\tilde{I}_i \cap \tilde{I}_j = \emptyset$ for $i \neq j$. The large Feynman diagram X is the graph limit of the sequence $\{Y_m\}_{m\geq 1}$ with respect to the cut-distance topology [31]. In other words, the sequence $\{\tilde{W}_{Y_m}\}_{m\geq 1}$ is cut-distance convergent to the stretched Feynman graphon $\tilde{W}_X : \bigsqcup_{i=1}^{\infty} \tilde{I}_i \times \bigsqcup_{i=1}^{\infty} \tilde{I}_i \to \mathbb{R}$ given by the infinite direct sum

$$\left(||\tilde{W}_{X_1} + ... + \tilde{W}_{X_m} + ...||_{\text{cut}}\right)\tilde{W}_X = \tilde{W}_{X_1} + ... + \tilde{W}_{X_m} + ... , \tag{2.11}$$

such that $\tilde{W}_{X_i} : \tilde{I}_i \times \tilde{I}_i \to \mathbb{R}$.

Applying affine monotone maps enables us to project subintervals $\tilde{I}_i \subseteq [0, \infty)$ onto separate non-empty subintervals $I_i \subseteq [0, 1)$ such that $I_i \cap I_j = \emptyset$ for $i \neq j$. Then we project $\tilde{W}_X$ onto its corresponding canonical Feynman graphon $W_X : [0, 1) \times [0, 1) \to [0, 1]$. Thanks to the completion of the space of Feynman diagrams via the space of stretched Feynman graphons (i.e. Theorem 2.2), W_X is well-defined. [35–37] □

2.2. Randomness

Thanks to Lemma 2.3, it is now possible to recognize the difference between infinite series $X_{\text{DSE}(c_{g,1})}$ and $X_{\text{DSE}(c_{g,2})}$, as large Feynman diagrams, with the corresponding Feynman graphon representations $W_{\text{DSE}(c_{g,1})}$ and $W_{\text{DSE}(c_{g,2})}$ under different running coupling constants $c_{g,1}, c_{g,2} \geq 1$ [37,38]. The space of Feynman graphons led us to formulate a non-perturbative renormalization program for these large Feynman diagrams [33,35,36].

Modern gauge field theories are formulated on the basis of Standard Model. They contain the electroweak sector as a combination of Coulomb and Higgs phases, the confined phase of QCD and the deconfined–confined transition at non-zero temperatures. At high temperatures $T >> \Lambda_{\text{QCD}}$, the running coupling constants c_g, as functions of the temperature T, are small where quantum motions are studied in the context of perturbative combinatorial DSEs. At relatively low temperatures $T < \Lambda_{\text{QCD}}$, the running coupling constants c_g are 1 or larger than 1 where quantum motions have non-perturbative behavior [14,27,28]. Combinatorial DSEs under strongly coupled running coupling constants $c_g \geq 1$ allow us to recognize different intermediate phases in the confined phase of QCD. Thanks to the theory of Feynman graphons, these intermediate non-perturbative phases are studied in terms of the geometry of the Banach manifold $\mathcal{S}_{\approx}^{\Phi,g}$ of weakly isomorphic equivalence classes of large Feynman diagrams associated with combinatorial DSEs. [36]

Theorem 2.4. *There exists a category of random graphs which encodes intermediate phases in non-perturbative sector of* Φ *under different running coupling constants.*

Proof. Consider an equation DSE with the general form (2.8) under the running coupling constant $c_g \geq 1$ with the solution X given by (2.9), the sequence $\{Y_m\}_{m\geq 1}$ of partial sums with the corresponding sequence $\{W_{Y_m}\}_{m\geq 1}$ of Feynman graphon representations and the stretched Feynman graphon W_{DSE} defined on the Lebesgue measure space $[0, 1)$. For each $m \geq 1$, there exists an order structure on vertices of the rooted forest representation $s_{Y_m} = t_{X_1} + ... + t_{X_m}$ of the partial sum Y_m to project them onto separate nodes $x_1, ..., x_{|s_{Y_m}|}$ in $(0, 1)$ in terms of an embedding ρ_m. Define a random graph R_m on these nodes in such a way that with the probability $W_{Y_m}(x_i, x_j)$, there exists an edge between x_i, x_j.

The cut-distance topological completeness of the space of stretched Feynman graphons is applied to show that the sequence $\{R_m\}_{m\geq 1}$ converges to W_{DSE}. When m tends to infinity, we build a new infinite random graph R_∞ in terms of selecting infinite countable nodes $z_1, z_2, \ldots$ in $(0,1)$ as the projections of vertices of $s_{X_{\rm DSE}} = t_{X_1} + \ldots + t_{X_m} + \ldots$ such that with the probability $W_{\rm DSE}(z_k, z_l)$, there exists an edge $z_k z_l$ in R_∞.

The collection $\{X_n\}_{n\geq 0}$ provides the generators of a graded free commutative Hopf subalgebra $H_{\rm DSE}$ of the renormalization Hopf algebra $H_{\rm FG}(\Phi)$. Consider the category $\mathcal{C}_\Phi^{\rm Hopf}$ of pairs $(H_{\rm DSE}, L)$ with respect to Hochschild one-cocycles $L: H_{\rm DSE} \to H_{\rm DSE}$ which satisfy the equation

$$\mathbf{b}(L)(\Gamma) = ({\rm id} \otimes L)\Delta(\Gamma) + \sum_{k=1}^{n} (-1)^k \Delta_k(L(\Gamma)) + (-1)^{n+1} L(\Gamma) \otimes \mathbb{I} = 0 \tag{2.12}$$

for any Feynman diagram $\Gamma \in H_{\rm DSE}$ such that

$$\Delta(X_n) = \sum_{k=0}^{n} P_k^n \otimes X_k \ , \ P_{k+1}^{n+1} = \sum_{l=0}^{n-k} P_0^l P_k^{n-l} \ , \ P_0^{n+1} = X_{n+1} \ , \ l \leq n \ . \tag{2.13}$$

A morphism $f: H_{{\rm DSE}_1} \to H_{{\rm DSE}_2}$ in $\mathcal{C}_\Phi^{\rm Hopf}$ is a homomorphism of Hopf algebras such that $L_2 \circ f = f \circ L_1$. We extend objects of $\mathcal{C}_\Phi^{\rm Hopf}$ to cut-distance topological Hopf algebras to define a new category $\mathcal{C}_\Phi^{\rm Hopf,cut}$ such that $X_{\rm DSE} \in H_{\rm DSE}^{\rm cut}$. For any $f \in {\rm mor}(\mathcal{C}_\Phi^{\rm Hopf})$, a morphism $\tilde{f}: H_{{\rm DSE}_1}^{\rm cut} \to H_{{\rm DSE}_2}^{\rm cut}$ is a cut-distance continuous map which is a homomorphism of Hopf algebras such that $L_2 \circ \tilde{f} = \tilde{f} \circ L_1$.

Consider $H_{\rm graphon}^{\Phi}$ as the renormalization Hopf algebra of Feynman graphons which is a free commutative non-cocommutative Hopf algebra generated by unlabeled Feynman graphon classes corresponding to 1PI Feynman diagrams. The loop number is applied to define a graduation parameter on this Hopf algebra such that for each $n \geq 1$, $H_{\rm graphon}^{\Phi,(n)}$ is the vector space generated by unlabeled Feynman graphon classes corresponding to 1PI Feynman diagrams with the loop number n or products of unlabeled Feynman graphon classes corresponding to 1PI Feynman diagrams with the overall loop number n. The coproduct is given by

$$\Delta([W_\Gamma]_\approx) = [W_{\mathbb{I}}]_\approx \otimes [W_\Gamma]_\approx + [W_\Gamma]_\approx \otimes [W_{\mathbb{I}}]_\approx + \sum_\gamma [W_\gamma]_\approx \otimes [W_{\Gamma/\gamma}]_\approx \tag{2.14}$$

where

$$\Delta(\Gamma) = \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \sum_\gamma \gamma \otimes \Gamma/\gamma \tag{2.15}$$

such that γ is any disjoint union of non-trivial 1PI superficially divergent subgraphs of Γ.

The relation $\Gamma \mapsto [W_\Gamma]_\approx$ defines an injective homomorphism of topological Hopf algebras from $H_{\rm DSE}^{\rm cut}$ to $H_{\rm graphon}^{\Phi,{\rm cut}}$. This leads us to define a new category $\mathcal{C}_\Phi^{\rm Random}$ of random graphs $R_\infty^{\rm DSE}$ generated by Feynman graphons associated with solutions $X_{\rm DSE}$ of equations DSE. Any morphism $\tilde{f}: H_{{\rm DSE}_1}^{\rm cut} \to H_{{\rm DSE}_2}^{\rm cut} \in \mathcal{C}_\Phi^{\rm Hopf,cut}$ defines a graph homomorphism $\hat{f}: R_\infty^{{\rm DSE}_1} \to R_\infty^{{\rm DSE}_2}$ of random graphs. Suppose $R_\infty^{{\rm DSE}_1}$ has vertices $u_1, u_2, \ldots \in (0,1)$ such that with the probability $W_{{\rm DSE}_1}(u_i, u_j)$, there exists an edge between u_i and u_j. Then $R_\infty^{{\rm DSE}_2}$ has vertices $\hat{f}(u_1), \hat{f}(u_2), \ldots \in (0,1)$ such that with the probability $W_{{\rm DSE}_2}(\hat{f}(u_i), \hat{f}(u_j))$, there exists an edge between $\hat{f}(u_i)$ and $\hat{f}(u_j)$.

For any equation DSE with the corresponding Feynman graphon $W_{\rm DSE}$ and the sequence $\{R_m^{{\rm DSE}_1}\}_{m\geq 1}$ of finite random graphs which converges to $R_\infty^{{\rm DSE}_1} \in \mathcal{C}_\Phi^{\rm Random}$, define

$$t(R_m^{\mathrm{DSE}_1}, W_{\mathrm{DSE}}) = \int_{[0,1]^{n_m}} \prod_{x_i x_j \in R_m^{\mathrm{DSE}_1}} W_{\mathrm{DSE}}(x_i, x_j) \prod_{x_i x_j \notin R_m^{\mathrm{DSE}_1}} \Big(1 - W_{\mathrm{DSE}}(x_i, x_j)\Big) dx_1 ... dx_{n_m} \tag{2.16}$$

such that $n_m = |R_m^{\mathrm{DSE}_1}|$. The limit of the sequence $\Big\{t(R_m^{\mathrm{DSE}_1}, W_{\mathrm{DSE}})\Big\}_{m \geq 1}$ determines the probability of having the infinite random graph $R_\infty^{\mathrm{DSE}_1}$ as a subgraph in W_{DSE}.

Non-perturbative phases of Φ are classified in terms of homomorphism densities of Feynman graphons corresponding to solutions of quantum motions in Φ [36]. The equations $\mathrm{DSE}_1, \mathrm{DSE}_2$ generate the same phase iff for each $\epsilon > 0$, there exists an order M_ϵ such that for any $m > M_\epsilon$,

$$\Big| t(R_m^{\mathrm{DSE}_1}, W_{\mathrm{DSE}_2}) - t(R_m^{\mathrm{DSE}_2}, W_{\mathrm{DSE}_1}) \Big| < \epsilon \ . \tag{2.17}$$

Therefore, for a pair $(R_\infty^{\mathrm{DSE}_1}, R_\infty^{\mathrm{DSE}_2})$ of objects in $\mathcal{C}_\Phi^{\mathrm{Random}}$, the equations $\mathrm{DSE}_1, \mathrm{DSE}_2$ generate the same phase iff there exist morphisms $f_{12} : R_\infty^{\mathrm{DSE}_1} \to R_\infty^{\mathrm{DSE}_2}$ and $f_{21} : R_\infty^{\mathrm{DSE}_2} \to R_\infty^{\mathrm{DSE}_1}$ in $\mathcal{C}_\Phi^{\mathrm{Random}}$ such that $f_{12} \circ f_{21} = \mathrm{id}_{R_\infty^{\mathrm{DSE}_2}}$ and $f_{21} \circ f_{12} = \mathrm{id}_{R_\infty^{\mathrm{DSE}_1}}$. □

3. Equilibrium states associated with the solution space of quantum motions

Consider a strongly coupled gauge field theory Φ with the bare coupling constant g and a σ-finite measure space (Ω, μ). Set $\mathcal{S}_{\approx}^{\Phi,g}$ as the linear space generated by large Feynman diagrams X_{DSE} (up to the weakly isomorphic relation) as solutions of combinatorial Dyson–Schwinger equations DSE with the general form (2.8) in Φ. It is shown that the space $\mathcal{S}_{\mathrm{graphon}}^{\Phi,\Omega}(\mathbb{R})$ of stretched Feynman graphons provides a separable Banach structure on $\mathcal{S}_{\approx}^{\Phi,g}$ [36]. In this section, strongly continuous one-parameter semigroups on the Banach $*$-algebra $\mathcal{B}(\mathcal{S}_{\approx}^{\Phi,g})$ are applied to build a class of C^*-dynamical systems which characterize equilibrium states associated with the solution space of DSEs underlying the evolution of running coupling constants.

3.1. Intermediate phases

In this part, we provide a mathematical interpretation of the concept of "intermediate phase" and "phase transition" in non-perturbative sector of a gauge field theory.

Definition 5.

- Thanks to Definition 1 and Theorems 2.2 and 2.4, the cut-norm of any combinatorial Dyson–Schwinger equation DSE is given by

$$||X_{\mathrm{DSE}}||_{\mathrm{cut}} := ||W_{\mathrm{DSE}}||_{\mathrm{cut}} \ . \tag{3.1}$$

Define the equivalence class

$$[W_{\mathrm{DSE}}]_{\approx} := \{W_{\mathrm{DSE}}^{\tau} \ : \ \tau \ : \ \mu - \text{measure preserving transformation on } \Omega\} \ . \tag{3.2}$$

- Combinatorial Dyson–Schwinger equations $\mathrm{DSE}_1, \mathrm{DSE}_2$ are weakly isomorphic, iff their corresponding Feynman graphons are weakly isomorphic. In other words,

$$\mathrm{DSE}_1 \approx \mathrm{DSE}_2 \Leftrightarrow X_{\mathrm{DSE}_1} \approx X_{\mathrm{DSE}_2} \Leftrightarrow [W_{\mathrm{DSE}_1}]_{\approx} = [W_{\mathrm{DSE}_2}]_{\approx} \ . \tag{3.3}$$

Define the equivalence class

$$[\mathrm{DSE}]_{\approx} := \{\mathrm{DSE}' \ : \ \mathrm{DSE} \approx \mathrm{DSE}'\} \Leftrightarrow [X_{\mathrm{DSE}}]_{\approx} := \{X'_{\mathrm{DSE}} \ : \ X_{\mathrm{DSE}} \approx X'_{\mathrm{DSE}}\} \ . \tag{3.4}$$

- Up to the weakly isomorphic relation, the distance between equations $\mathrm{DSE}_1, \mathrm{DSE}_2$ is given by

$$d_{\mathrm{cut}}([\mathrm{DSE}_1]_{\approx}, [\mathrm{DSE}_2]_{\approx}) = d_{\mathrm{cut}}([W_{\mathrm{DSE}_1}]_{\approx}, [W_{\mathrm{DSE}_2}]_{\approx}) = d_{\mathrm{cut}}([X_{\mathrm{DSE}_1}]_{\approx}, [X_{\mathrm{DSE}_2}]_{\approx}) \ . \tag{3.5}$$

It defines a separable Hausdorff metric structure on the space of weakly isomorphic classes $[\mathrm{DSE}]_{\approx}$ corresponding to combinatorial Dyson–Schwinger equations in Φ.
- Thanks to Theorems 2.2 and 2.4,

$$d_{\mathrm{cut}}([\mathrm{DSE}_1]_{\approx}, [\mathrm{DSE}_2]_{\approx}) := d_{\mathrm{cut}}([X_{\mathrm{DSE}_1}]_{\approx}, [X_{\mathrm{DSE}_2}]_{\approx}) = \lim_{m \to \infty} d_{\mathrm{cut}}([W_{Y^1_m}]_{\approx}, [W_{Y^2_m}]_{\approx}) \ . \tag{3.6}$$

We use the notations DSE, X_{DSE}, W_{DSE} to simplify the presentations of weakly isomorphic classes $[\mathrm{DSE}]_{\approx}$, $[X_{\mathrm{DSE}}]_{\approx}$, $[W_{\mathrm{DSE}}]_{\approx}$.

Remark 1. For any positive real valued running coupling constant c_g, consider the sequence $\{r_n\}_{n\geq 1}$ of positive rational numbers which converges to c_g. For any combinatorial Dyson–Schwinger equation $\mathrm{DSE}(c_g)$ with the solution $X_{\mathrm{DSE}}(c_g)$, the sequence $\{X_{\mathrm{DSE}}(r_n)\}_{n\geq 1}$ is cut-distance convergent to $X_{\mathrm{DSE}}(c_g)$ when n tends to infinity. Therefore, up to the weakly isomorphic relation and thanks to (1.1), the collection

$$\{G^{e_i}(1), G^{v_j}(1) \ : \ e_i, v_j \in \mathcal{A}_{\Phi}\} \tag{3.7}$$

encodes the generators of the Banach space $\mathcal{S}^{\Phi,g}_{\approx}$.

Homomorphism densities, as continuous functionals on $\mathcal{S}^{\Phi,g}_{\approx}$, are applied for the analytic study of solutions of quantum motions in terms of some new geometric tools. [32,34–36]

Corollary 3.1. *The homomorphism densities of Feynman graphons characterize deconfined-confined phase transitions.*

Proof. This is a direct result of Theorems 2.2, 2.4 and the metric (3.6) together with [35,36].

Consider an equation DSE with the solution X_{DSE} in a deconfined high energy phase where the running coupling constant c_g tends to zero. Thanks to Feynman graphon representation W_{DSE}, for each order $m \geq 1$, the cut-norm of the remainder after $m+1$ terms of the infinite direct sum

$$W_{\mathrm{DSE}} = c_g W_{X_1} + c_g^2 W_{X_2} + ... + c_g^n W_{X_n} + ... \tag{3.8}$$

is much smaller than the cut-norm of the last retained term while c_g tends to zero. In other words,

$$\lim_{c_g \to 0} \ c_g^{-m} ||X_{\mathrm{DSE}} - Y_m||_{\mathrm{cut}} = 0 \ , \ Y_m = \sum_{n=1}^{m} c_g^n X_n \ , \tag{3.9}$$

which means that for each $\epsilon > 0$ there exists some neighborhood I_ϵ around 0 such that for each $c_g \in I_\epsilon$,

$$c_g^{-m}||\sum_{n=m+1}^{\infty} c_g^n X_n||_{\text{cut}} < \epsilon \Leftrightarrow c_g^{-m}||c_g^{m+1} W_{X_{m+1}} + c_g^{m+2} W_{X_{m+2}} + ...||_{\text{cut}} < \epsilon . \tag{3.10}$$

However, in a confined low energy phase where the running coupling constant c_g tends to values equal or larger than 1, the sequence $\{Y_m\}_{m\geq 1}$ of partial sums cut-distance converges to the stretched Feynman graphon $\tilde{W}_{\text{DSE}} : [0, \infty) \times [0, \infty) \to \mathbb{R}$ with $||\tilde{W}_{\text{DSE}}||_{\text{cut}} < \infty$ or its normalized version $W_{\text{DSE}} : [0, 1) \times [0, 1) \to [0, 1]$.

For each $m \geq 1$, the homomorphism density of the partial sum Y_m with respect to W_{DSE} is given by

$$t(Y_m, W_{\text{DSE}}) = \int_{[0,1]^{|t_{Y_m}|}} \prod_{(i,j)\in E(t_{Y_m})} W_{\text{DSE}}(x_i, x_j) \prod_{(i,j)\notin E(t_{Y_m})} \Big(1 - W_{\text{DSE}}(x_i, x_j)\Big) dx_1 ... dx_{|t_{Y_m}|} , \tag{3.11}$$

such that t_{Y_m} is the rooted forest representation of Y_m. In a deconfined phase with vanishing running coupling constants, we have

$$\lim_{c_g^m \to 0} t(Y_m, W_{\text{DSE}}) = 1 , \tag{3.12}$$

while in a confined phase with strong running coupling constants $c_g \geq 1$, we have

$$\lim_{m\to\infty} t(Y_m, W_{\text{DSE}}) = 1 . \quad \square \tag{3.13}$$

Consider the unital algebra $\mathcal{B}(\mathcal{S}_{\approx}^{\Phi,g})$ of all bounded linear operators $f : \mathcal{S}_{\approx}^{\Phi,g} \to \mathcal{S}_{\approx}^{\Phi,g}$ with respect to the composition. Thanks to (3.6), it is equipped with the norm

$$||f||_{\text{op,cut}} := \inf \Big\{ c \geq 0 : ||f(X_{\text{DSE}})||_{\text{cut}} \leq c||X_{\text{DSE}}||_{\text{cut}} \Big\} . \tag{3.14}$$

For any densely defined operator $f \in \mathcal{B}(\mathcal{S}_{\approx}^{\Phi,g})$, its adjoint operator $f^* : \text{Dom}(f^*) \subseteq (\mathcal{S}_{\approx}^{\Phi,g})^* \to (\mathcal{S}_{\approx}^{\Phi,g})^*$ is an operator with the domain

$$\text{Dom}(f^*) := \Big\{ G \in (\mathcal{S}_{\approx}^{\Phi,g})^* : \exists c \geq 0 : \forall X_{\text{DSE}} \in \text{Dom}(f) : \Big| G(f(X_{\text{DSE}})) \Big| \leq c||X_{\text{DSE}}||_{\text{cut}} \Big\}. \tag{3.15}$$

Therefore $\Big(\mathcal{B}(\mathcal{S}_{\approx}^{\Phi,g}), ||.||_{\text{op,cut}}, .^*\Big)$ is a unital Banach $*$-algebra.

For a fixed $f \in \mathcal{B}(\mathcal{S}_{\approx}^{\Phi,g})$ and any $G \in (\mathcal{S}_{\approx}^{\Phi,g})^*$, define

$$F_G : \text{Dom}(f) \to \mathbb{R}, \quad X_{\text{DSE}} \mapsto G(f(X_{\text{DSE}})) . \tag{3.16}$$

The Hahn–Banach Theorem is applied to obtain an extension $\tilde{F}_G$ of F_G on $\mathcal{S}_{\approx}^{\Phi,g}$ given by

$$G \mapsto f^*(G) = \tilde{F}_G , \ \forall X_{\text{DSE}} \in \text{Dom}(f) \ : \ G(f(X_{\text{DSE}})) = (f^*G)(X_{\text{DSE}}) . \tag{3.17}$$

Corollary 3.2. *The Banach $*$-algebra $\mathcal{B}(\mathcal{S}_{\approx}^{\Phi,g})$ encodes intermediate phase transitions in non-perturbative region.*

Proof. Any operator $f : \mathcal{S}^{\Phi,g}_{\approx} \to \mathcal{S}^{\Phi,g}_{\approx}, X_{\mathrm{DSE}} \mapsto f(X_{\mathrm{DSE}})$ in $\mathcal{B}(\mathcal{S}^{\Phi,g}_{\approx})$ transfers the non-perturbative phase p_0 corresponding to X_{DSE} to a new non-perturbative phase p_1 corresponding to $f(X_{\mathrm{DSE}})$. For any $n \geq 1$, the operator $f^n = f \circ ... \circ f$ determines a hierarchy of intermediate (non-perturbative) phases $p_1, p_2, ..., p_{n-1}$ between the initial phase p_0 and the final phase p_n. □

Lemma 3.3. *Intermediate phase transitions of non-perturbative sector of Φ and the deconfined–confined phase transition under running coupling constants are encapsulated in terms of strongly continuous one-parameter semi-groups.*

Proof. For $c_g \geq 1$, the rescaling $c_g \mapsto e^{-\lambda} c_g$ is applied to define a new functional

$$\rho : \mathbb{R}_+ \to \mathcal{B}(\mathcal{S}^{\Phi,g}_{\approx}) \ , \ \lambda \mapsto \rho_\lambda(X_{\mathrm{DSE}}(c_g)) = X_{\mathrm{DSE}}(e^{-\lambda} c_g) \ , \ \rho_0 = \mathrm{Id}_{\mathcal{S}^{\Phi,g}_{\approx}} \ , \tag{3.18}$$

which encodes intermediate phases generated by any combinatorial Dyson–Schwinger equation $\mathrm{DSE}(c_g)$ under the initial running coupling constant c_g with the solution $X_{\mathrm{DSE}}(c_g) = \sum_{n\geq 0} c_g^n X_n$.

We have

$$\rho_{\lambda_1+\lambda_2}(X_{\mathrm{DSE}}(c_g)) = \rho_{\lambda_1+\lambda_2}(\sum_{n\geq 0} c_g^n X_n) =$$

$$\sum_{n\geq 0}(e^{-(\lambda_1+\lambda_2)} c_g)^n X_n = \sum_{n\geq 0} e^{-\lambda_1 n} e^{-\lambda_2 n} c_g^n X_n = \rho_{\lambda_1} \circ \rho_{\lambda_2}(X_{\mathrm{DSE}}(c_g)) \ . \tag{3.19}$$

$$\lim_{\lambda\to 0} ||\rho_\lambda(X_{\mathrm{DSE}}(c_g)) - X_{\mathrm{DSE}}(c_g)||_{\mathrm{cut}} = \lim_{\lambda\to 0} || \sum_{n\geq 0} e^{-\lambda n} c_g^n X_n - \sum_{n\geq 0} c_g^n X_n ||_{\mathrm{cut}}$$

$$= \lim_{\lambda\to 0} || \sum_{n\geq 0} (e^{-\lambda n} c_g^n - c_g^n) X_n ||_{\mathrm{cut}} = 0 \ . \tag{3.20}$$

The functional ρ is continuous with respect to the strong operator topology. This means that any sequence $\{\rho_{\lambda_n}\}_{n\geq 1} \subset \{\rho_\lambda\}_{\lambda\in\mathbb{R}_+}$ of operators in $\mathcal{B}(\mathcal{S}^{\Phi,g}_{\approx})$ converges to $U \in \mathcal{B}(\mathcal{S}^{\Phi,g}_{\approx})$ with respect to the strong topology operator iff for any combinatorial Dyson–Schwinger equation DSE, the sequence

$$\Big\{ ||\rho_{\lambda_n}(X_{\mathrm{DSE}}(c_g)) - U(X_{\mathrm{DSE}}(c_g))||_{\mathrm{cut}} \Big\}_{n\geq 1} \tag{3.21}$$

is convergent to zero. The functional ρ is applied to define the new notion of Hamiltonian flow on observables associated with solutions of combinatorial Dyson–Schwinger equations under the variation of running coupling constants.

For $c_g \geq 1$ and any combinatorial Dyson–Schwinger equation DSE,

- When $\lambda \to 0$, $\{\rho_\lambda(X_{\mathrm{DSE}}(c_g))\}_\lambda$ is a sequence of large Feynman diagrams generated by the equation DSE under increasing running coupling constants $e^{-\lambda} c_g$ which converges to $X_{\mathrm{DSE}}(c_g)$ with respect to the cut-distance topology. Thanks to Corollary 3.2, the condition $\lambda \to 0$ encodes transitions between perturbative and non-perturbative phases in Φ.
- When $\lambda \to \infty$, $\{\rho_\lambda(X_{\mathrm{DSE}}(c_g))\}_\lambda$ is a sequence of large Feynman diagrams generated by the equation DSE under decreasing running coupling constants $e^{-\lambda} c_g$ which converges to the quantum motion in a phase of the physical theory without interaction part. □

3.2. KMS states

Thanks to Corollaries 3.1, 3.2, we consider solutions of combinatorial Dyson–Schwinger equations under the rescaling of running coupling constants in terms of the strongly continuous one-parameter semigroup $\{\rho_\lambda\}_{\lambda\in\mathbb{R}_+}$ given by Lemma 3.3. However, the rescaling of running coupling constants can be performed by other strongly continuous one-parameter semigroups and Lemma 3.3 is only an example of these rescaling methods.

Theorem 3.4. *The Banach* $*$*-algebra* $\mathcal{B}(\mathcal{S}^{\Phi,g}_{\approx})$ *encodes equilibrium states of non-perturbative sector of* Φ *under the rescaling of strong running coupling constants.*

Proof. Each state $\omega:\mathcal{B}(\mathcal{S}^{\Phi,g}_{\approx})\to\mathbb{C}$ is a linear functional such that

$$||\omega||=1\ ,\ \omega(f\circ f^*)=1\ . \tag{3.22}$$

We generalize the method given in [13] to characterize equilibrium states on $\mathcal{B}(\mathcal{S}^{\Phi,g}_{\approx})$ with respect to the evolution $c_g\mapsto e^{-\lambda}c_g$ of running coupling constants $c_g\geq 1$ governed by $\{\rho_\lambda\}_{\lambda\in\mathbb{R}_+}$.

A KMS state ω_{c_g} on $\mathcal{B}(\mathcal{S}^{\Phi,g}_{\approx})$ with respect to $\{\rho_\lambda\}_{\lambda\in\mathbb{R}_+}$ is given by

$$(\lambda_1,...,\lambda_n)\mapsto \omega_{c_g}\bigg(\rho_{\lambda_1}(X_{\mathrm{DSE}_1})...\rho_{\lambda_n}(X_{\mathrm{DSE}_n})\bigg)\ ,\ X_{\mathrm{DSE}_1},...,X_{\mathrm{DSE}_n}\in\mathcal{S}^{\Phi,g}_{\approx}\ , \tag{3.23}$$

which should have an analytic continuation to the region

$$\mathrm{Dom}^{c_g}_n=\bigg\{(z_1,...,z_n)\in\mathbb{C}^n:0<\mathrm{Im}(z_i)-\mathrm{Im}(z_j)<c_g,\ 1\leq i<j\leq n\bigg\}\ . \tag{3.24}$$

In addition, these analytic continuations should be continuous bounded on the boundary such that

$$\begin{aligned}&\omega_{c_g}\bigg(\rho_{\lambda_1}(X_{\mathrm{DSE}_1})...\rho_{\lambda_{k-1}}(X_{\mathrm{DSE}_{k-1}})\rho_{\lambda_k+ic_g}(X_{\mathrm{DSE}_k})...\rho_{\lambda_n+ic_g}(X_{\mathrm{DSE}_n})\bigg)\\&=\omega_{c_g}\bigg(\rho_{\lambda_k}(X_{\mathrm{DSE}_k})...\rho_{\lambda_n}(X_{\mathrm{DSE}_n})\rho_{\lambda_1}(X_{\mathrm{DSE}_1})...\rho_{\lambda_{k-1}}(X_{\mathrm{DSE}_{k-1}})\bigg)\ .\end{aligned} \tag{3.25}$$

Therefore

$$\omega_{c_g}(\rho_\lambda(X_{\mathrm{DSE}}))=\omega_{c_g}(X_{\mathrm{DSE}})\ ,\ \forall X_{\mathrm{DSE}}\in\mathcal{S}^{\Phi,g}_{\approx}\ , \tag{3.26}$$

such that the corresponding infinitesimal generator $Z^\rho_{c_g}$ is given by the limit

$$Z^\rho_{c_g}(X)=\lim_{\lambda\to 0}\frac{1}{\lambda}\bigg(\rho_\lambda-\mathrm{Id}_{\mathcal{S}^{\Phi,g}_{\approx}}\bigg)(X) \tag{3.27}$$

for any X in a dense linear subspace of $\mathcal{S}^{\Phi,g}_{\approx}$.

For any sequence $\{c_{g,n}\}_{n\geq 1}$ of running coupling constants, a sequence $\{\omega_{c_{g,n}}\}_{n\geq 1}$ of KMS states on $\mathcal{B}(\mathcal{S}^{\Phi,g}_{\approx})$ with respect to $\{\rho_\lambda\}_{\lambda\in\mathbb{R}_+}$ is convergent to a new KMS state ω iff for any $X_{\mathrm{DSE}}\in\mathcal{S}^{\Phi,g}_{\approx}$, $\{\omega_{c_{g,n}}(X_{\mathrm{DSE}})\}_{n\geq 1}$ is convergent to $\omega(X_{\mathrm{DSE}})$ whenever n tends to infinity.

Consider a sequence $\{\omega_{e^{-\lambda_n}g}\}_{n\geq 1}$ of KMS states such that for any $X_{\mathrm{DSE}}\in\mathcal{S}^{\Phi,g}_{\approx}$, $\{\omega_{e^{-\lambda_n}g}(X_{\mathrm{DSE}})\}_{n\geq 1}$ converges to $\omega_0(X_{\mathrm{DSE}})$ whenever running coupling constants $e^{-\lambda_n}g$ tend to 0 or λ_n tends to infinity. ω_0 is called the KMS state at zero coupling with respect to $\{\rho_\lambda\}_{\lambda\in\mathbb{R}_+}$. □

The GNS representations of $\mathcal{B}(\mathcal{S}_{\approx}^{\Phi,g})$ are useful to build a new universal C^*-algebra. The functional (3.17) leads us to relate a new Hilbert space H_ω, defined by the inner product

$$< U_1, U_2 >_\omega := \omega(U_2^* \circ U_1) \ , \ U_1, U_2 \in \mathcal{B}(\mathcal{S}_{\approx}^{\Phi,g}) \ , \tag{3.28}$$

to any state ω. The C^*-enveloping algebra $E^{\Phi,g}$ associated with $\mathcal{B}(\mathcal{S}_{\approx}^{\Phi,g})$ is given by the norm closure of the universal representation

$$\Pi = \bigoplus_{\omega} \pi_\omega : \mathcal{B}(\mathcal{S}_{\approx}^{\Phi,g}) \to \mathcal{B}(\bigoplus_{\omega} H_{\pi_\omega}) \tag{3.29}$$

as the direct sum of all GNS representations of $\mathcal{B}(\mathcal{S}_{\approx}^{\Phi,g})$ such that the sum is taken over all states ω.

We apply $\{\rho_\lambda\}_{\lambda \in \mathbb{R}_+}$ to build a new strongly continuous one-parameter semigroup defined by $\tilde{\rho} : \mathbb{R}_+ \to E^{\Phi,g}$. For a given KMS state ω_{c_g} on $\mathcal{B}(\mathcal{S}_{\approx}^{\Phi,g})$ with respect to $\{\rho_\lambda\}_{\lambda \in \mathbb{R}_+}$, its corresponding KMS state on $E^{\Phi,g}$ with respect to $\{\tilde{\rho}_\lambda\}_{\lambda \in \mathbb{R}_+}$ is a new state $\tilde{\omega}_{c_g}$ such that for any $\theta_1, \theta_2 \in E^{\Phi,g}$, there exists a bounded function $\mathcal{F}_{\theta_1,\theta_2}$ which is holomorphic on the region

$$\mathrm{Dom}^{c_g} := \{z \in \mathbb{C} : 0 < \mathrm{Im}(z) < c_g\} \ , \tag{3.30}$$

and a continuous function on the boundary $\partial \ \mathrm{Dom}^{c_g}$. In addition, for any $\lambda \in \mathbb{R}_+$,

$$\mathcal{F}_{\theta_1,\theta_2}(\lambda) = \tilde{\omega}_{c_g}(\theta_1 \circ \tilde{\rho}_\lambda(\theta_2)) \ , \ \ \mathcal{F}_{\theta_1,\theta_2}(\lambda + i c_g) = \tilde{\omega}_{c_g}(\tilde{\rho}_\lambda(\theta_2) \circ \theta_1) \ . \tag{3.31}$$

Corollary 3.5. *KMS states on $E^{\Phi,g}$ with respect to all running coupling constants and any strongly continuous one-parameter semigroup $\tilde{\rho}$ on $E^{\Phi,g}$ make a convex set with respect to the weak topology.*

Proof. It is a direct result of [13] and Theorem 3.4. □

4. C^*-dynamical systems associated with the solution space of quantum motions

Thanks to Feynman graphon representations of non-perturbative solutions of quantum motions and the mathematical structure of intermediate phases in non-perturbative sector of a gauge field theory Φ, here we formulate a new class of C^*-dynamical systems.

4.1. Observables

In this part we introduce the noncommutative algebra of coordinates of the noncommutative space of commensurability classes of $\mathbb{Q}$-lattices in $\mathcal{S}_{\approx}^{\Phi,g}$ up to the scale of the running coupling constant in a strongly coupled gauge field theory Φ with the bare coupling constant g. This particular algebra leads us to build a new class of C^*-dynamical systems which encodes observables associated with the solution space of quantum motions in Φ.

Theorem 4.1. *There exists a Banach lattice structure on $\mathcal{S}_{\approx}^{\Phi,g}$.*

Proof. On the one hand, for combinatorial Dyson–Schwinger equations DSE_1, DSE_2 defined by the corresponding sets $\{\gamma_n^{(1)}\}_{n\geq 1}$, $\{\gamma_n^{(2)}\}_{n\geq 1}$ of primitive Feynman diagrams and the sequences of partial sums $\{Y_m^{(1)}\}_{m\geq 1}$, $\{Y_m^{(2)}\}_{m\geq 1}$, define a new combinatorial Dyson–Schwinger equation

$$\mathrm{DSE}_{12} := \mathrm{DSE}_1 \oplus \mathrm{DSE}_2 = < \{\gamma_n^{(1)}\}_{n\geq 1} \sqcup \{\gamma_n^{(2)}\}_{n\geq 1} > \ . \tag{4.1}$$

The Feynman graphon $W_{\mathrm{DSE}_{12}}$ is defined as the cut-distance convergent limit of the sequence $\{W_{Y_m^{(1)}+Y_m^{(2)}}\}_{m\geq 1}$ such that $Y_m^{(1)}+Y_m^{(2)}$ is the disjoint union of graphs $Y_m^{(1)}$ and $Y_m^{(2)}$. The space $\mathcal{S}_{\approx}^{\Phi,g}$ is equipped with the operation $\oplus$ such that for large Feynman diagrams X_{DSE_1} and X_{DSE_2}, the large Feynman diagram $X_{\mathrm{DSE}_{12}}$ is the smallest infinite graph which encodes both X_{DSE_1} and X_{DSE_2}. For any combinatorial Dyson–Schwinger equation DSE with the solution X_{DSE}, the inverse object $X_{\mathrm{DSE}}^{-1^{\oplus}}$ with respect to $\oplus$ is a combinatorial Dyson–Schwinger equation $\mathrm{DSE}^{-1^{\oplus}}$ such that the sequence $\{W_{Y_m^{(\mathrm{DSE})}+Y_m^{(\mathrm{DSE}^{-1^{\oplus}})}}\}_{m\geq 1} \subset \mathcal{S}_{\mathrm{graphon}}^{\Phi,[0,1]}(\mathbb{R})$ of partial sums is cut-distance convergent to the zero graphon $W_{\mathbb{I}}$. Up to the weakly isomorphic relations (3.3) and (3.4), $\mathrm{DSE}^{-1^{\oplus}}$ is the unique inverse of DSE with respect to $\oplus$. Therefore $(\mathcal{S}_{\approx}^{\Phi,g}, \oplus)$ is an abelian torsion free group such that $\mathbf{0}$, as the empty graph, is the neutral element.

On the other hand, it is shown that the solution $X_{\mathrm{DSE}} = \sum_{n\geq 0} c_g^n X_n$ of an equation DSE determines a graded connected commutative Hopf subalgebra H_{DSE} of $H_{\mathrm{FG}}(\Phi)$ generated by $\{\mathbb{I}, X_1, X_2, ...\}$ [18]. These Hopf subalgebras allow us to define a partial order relation on $\mathcal{S}_{\approx}^{\Phi,g}$ given by

$$X_{\mathrm{DSE}_1} \leq X_{\mathrm{DSE}_2} \ \Leftrightarrow \tag{4.2}$$

There exists an injective Hopf algebra homomorphism from H_{DSE_1} to H_{DSE_2}.

For $X_{\mathrm{DSE}_1} \leq X_{\mathrm{DSE}_2}$, we have

$$\begin{aligned} X_{\mathrm{DSE}_1} \vee X_{\mathrm{DSE}_2} &:= \sup\{X_{\mathrm{DSE}_1}, X_{\mathrm{DSE}_2}\} = X_{\mathrm{DSE}_2} \ , \\ X_{\mathrm{DSE}_1} \wedge X_{\mathrm{DSE}_2} &:= \inf\{X_{\mathrm{DSE}_1}, X_{\mathrm{DSE}_2}\} = X_{\mathrm{DSE}_1} \ . \end{aligned} \tag{4.3}$$

Thanks to Definition 1 and the norm (3.1), for $\Big|X_{\mathrm{DSE}}\Big| := X_{\mathrm{DSE}} \vee X_{\mathrm{DSE}}^{-1^{\oplus}}$, it is observed that

$$\Big|X_{\mathrm{DSE}_1}\Big| \leq \Big|X_{\mathrm{DSE}_2}\Big| \Rightarrow ||X_{\mathrm{DSE}_1}||_{\mathrm{cut}} \leq ||X_{\mathrm{DSE}_2}||_{\mathrm{cut}} \ . \quad \square \tag{4.4}$$

Definition 6.

- A lattice L^n of rank n in $\mathcal{S}_{\approx}^{\Phi,g}$ is a cocompact free abelian subgroup of rank n generated by non-weakly isomorphic large Feynman diagrams $X_{\mathrm{DSE}_1}, ..., X_{\mathrm{DSE}_n}$ (i.e. $X_{\mathrm{DSE}_i} \not\approx X_{\mathrm{DSE}_j}$, $i \neq j$).
- An n-dimensional $\mathbb{Q}$-lattice in $\mathcal{S}_{\approx}^{\Phi,g}$ is a pair (L^n, ϕ_n) such that L^n is a lattice of rank n while $\phi_n : \mathbb{Q}^n/\mathbb{Z}^n \to \mathbb{Q}L^n/L^n$ is a homomorphism of abelian groups. It is called an invertible $\mathbb{Q}$-lattice, if ϕ_n is an isomorphism.
- n-dimensional $\mathbb{Q}$-lattices $(L_1^n, \phi_{n,1}), (L_2^n, \phi_{n,2})$ in $\mathcal{S}_{\approx}^{\Phi,g}$ are called commensurable, iff $\mathbb{Q}L_1^n = \mathbb{Q}L_2^n$ and $\phi_{n,1} = \phi_{n,2} \mod L_1^n + L_2^n$. Define the quotient space $\mathcal{L}_n^{\Phi,g}$ as the collection of equivalence classes of n-dimensional $\mathbb{Q}$-lattices in $\mathcal{S}_{\approx}^{\Phi,g}$ up to the commensurable relation $\sim$.

For any combinatorial Dyson–Schwinger equation DSE, $X_{\mathrm{DSE}} \in \mathcal{S}_{\approx}^{\Phi,g}$, generated by the set $\{\gamma_n\}_{n\geq 1}$ of primitive (1PI) Feynman diagrams in $H_{\mathrm{FG}}(\Phi)$, define

$$\gamma_n^{(j)} := \gamma_1^{(j-1)} + ... + \gamma_n^{(j-1)} \ , \ j \geq 1 \ , \ \gamma_n^{(0)} = \gamma_n \ . \tag{4.5}$$

The set of primitive elements of the renormalization Hopf algebra $H_{\mathrm{FG}}(\Phi)$ is a graded Lie algebra which means that for each $j \geq 1$, $\gamma_n^{(j)}$ is a primitive graph. For each $j \geq 0$, we build a new equation $\mathrm{DSE}_{(j)}$ generated by the set $\{\gamma_n^{(j)}\}_{n\geq 1}$ of primitive Feynman diagrams such that $\mathrm{DSE}_{(0)} = \mathrm{DSE}$. For each $j \geq 0$, there exists a natural injective Hopf algebra homomorphism $\psi_j : H_{\mathrm{DSE}_{(j)}} \to H_{\mathrm{DSE}_{(j+1)}}$. Therefore for each $j \geq 0$, $\{X_{\mathrm{DSE}_{(0)}}, ..., X_{\mathrm{DSE}_{(j)}}\}$ determines a $j+1$-dimensional $\mathbb{Q}$-lattice in $\mathcal{S}_{\approx}^{\Phi,g}$.

Corollary 4.2. *The space of 1-dimensional $\mathbb{Q}$-lattices in $\mathcal{S}_{\approx}^{\Phi,g}$ up to scaling of running coupling constants encodes observables associated with the solution space of quantum motions in Φ.*

Proof. Fix a strongly continuous one-parameter semigroup $\rho := \{\rho_\lambda\}_\lambda$ on $\mathcal{B}(\mathcal{S}_{\approx}^{\Phi,g})$ which governs the evolution of running coupling constants. For any $X_{\mathrm{DSE}}(c_g) \in \mathcal{S}_{\approx}^{\Phi,g}$,

$$L^1_{\rho, X_{\mathrm{DSE}}(c_g)} := \Big\{ \rho_\lambda (X_{\mathrm{DSE}}(c_g))\mathbb{Z} \Big\}_{\lambda \in \mathbb{R}_+^*} \tag{4.6}$$

is a 1-dimensional $\mathbb{Q}$-lattice. Consider the groupoid

$$\mathcal{L}_{1,\rho}^{\Phi,g} :=< \Big\{ [L^1_{\rho, X_{\mathrm{DSE}}(c_g)}]_\sim \ : \ X_{\mathrm{DSE}}(c_g) \in \mathcal{S}_{\approx}^{\Phi,g} \ , \ c_g \in \mathbb{R}_+ \Big\} > \tag{4.7}$$

generated by the commensurable equivalence classes of 1-dimensional $\mathbb{Q}$-lattices of the form (4.6) in $\mathcal{S}_{\approx}^{\Phi,g}$ up to the scaling of running coupling constants. Thanks to Chapter 3: Section 4 in [12], define $\mathcal{A}_{1,\rho}^{\Phi,g}$ as the groupoid C^*-algebra generated by the quotient groupoid $\mathcal{L}_{1,\rho}^{\Phi,g}/\mathbb{R}_+^*$.

$\mathcal{A}_{1,\rho}^{\Phi,g}$ is the space of functions $\eta(L^1_{\rho, X_{\mathrm{DSE}_1}(c_{1,g})}, L^1_{\rho, X_{\mathrm{DSE}_2}(c_{2,g})})$ of pairs of commensurable $\mathbb{Q}$-lattices $L^1_{\rho, X_{\mathrm{DSE}_1}(c_{1,g})} \sim L^1_{\rho, X_{\mathrm{DSE}_2}(c_{2,g})}$ of the form (4.6) such that

$$\forall \lambda \in \mathbb{R}_+^* \ : \ \eta(\lambda L^1_{\rho, X_{\mathrm{DSE}_1}(c_{1,g})}, \lambda L^1_{\rho, X_{\mathrm{DSE}_2}(c_{2,g})}) = \eta(L^1_{\rho, X_{\mathrm{DSE}_1}(c_{1,g})}, L^1_{\rho, X_{\mathrm{DSE}_2}(c_{2,g})}) \ . \tag{4.8}$$

It is the $*$-algebra equipped with the convolution product

$$(\eta_1 \star \eta_2)(L^1_{\rho, X_{\mathrm{DSE}_1}(c_{1,g})}, L^1_{\rho, X_{\mathrm{DSE}_2}(c_{2,g})}) := \\ \sum \eta_1(L^1_{\rho, X_{\mathrm{DSE}_1}(c_{1,g})}, L^1_{\rho, X_{\mathrm{DSE}_3}(c_{3,g})}) \eta_2(L^1_{\rho, X_{\mathrm{DSE}_3}(c_{3,g})}, L^1_{\rho, X_{\mathrm{DSE}_2}(c_{2,g})}) \tag{4.9}$$

such that the sum is taken over 1-dimensional $\mathbb{Q}$-lattices $L^1_{\rho, X_{\mathrm{DSE}_3}(c_{3,g})}$ with

$$L^1_{\rho, X_{\mathrm{DSE}_1}(c_{1,g})} \sim L^1_{\rho, X_{\mathrm{DSE}_3}(c_{3,g})} \sim L^1_{\rho, X_{\mathrm{DSE}_2}(c_{2,g})} \ . \tag{4.10}$$

In addition, $\mathcal{A}_{1,\rho}^{\Phi,g}$ is equipped with the involution

$$\eta^*(L^1_{\rho, X_{\mathrm{DSE}_1}(c_{1,g})}, L^1_{\rho, X_{\mathrm{DSE}_2}(c_{2,g})}) = \overline{\eta(L^1_{\rho, X_{\mathrm{DSE}_2}(c_{2,g})}, L^1_{\rho, X_{\mathrm{DSE}_1}(c_{1,g})})} \ , \tag{4.11}$$

and topologically completed with respect to the norm

$$||\eta|| := \ \sup_{w_0} ||\pi_{w_0}(\eta)||_{\mathcal{B}(\mathcal{H}_{w_0})} \ . \tag{4.12}$$

The supremum is controlled in terms of representations π_{w_0} of $\mathcal{A}^{\Phi,g}_{1,\rho}$ on the Hilbert space $\mathcal{H}_{w_0} = \ell^2(\mathcal{L}^{\Phi,g}_{1,\rho,w_0})$ such that $\mathcal{L}^{\Phi,g}_{1,\rho,w_0}$ is the space of commensurable equivalence classes of 1-dimensional $\mathbb{Q}$-lattices in $\mathcal{S}^{\Phi,g}_{\approx}$ of the form (4.6) with the source $w_0 \in (\mathcal{L}^{\Phi,g}_{1,\rho}/\mathbb{R}^*_+ \times \mathcal{L}^{\Phi,g}_{1,\rho}/\mathbb{R}^*_+)^{(0)}$.

The covolume parameter of lattice is applied to define the time evolution

$$\sigma_t(\eta)(L^1_{\rho,X_{\mathrm{DSE}_1}(c_{1,g})}, L^1_{\rho,X_{\mathrm{DSE}_2}(c_{2,g})}) := \tag{4.13}$$
$$\left(\frac{\mathrm{covol}(L^1_{\rho,X_{\mathrm{DSE}_2}(c_{2,g})})}{\mathrm{covol}(L^1_{\rho,X_{\mathrm{DSE}_1}(c_{1,g})})}\right)^{it} \eta(L^1_{\rho,X_{\mathrm{DSE}_1}(c_{1,g})}, L^1_{\rho,X_{\mathrm{DSE}_2}(c_{2,g})})$$

on $\mathcal{A}^{\Phi,g}_{1,\rho}$. The C^*-dynamical system $(\mathcal{A}^{\Phi,g}_{1,\rho}, \{\sigma_t\}_t)$ encodes observables associated with the solution space of the combinatorial Dyson–Schwinger equation DSE(c_g) under different running coupling constants generated by $\rho := \{\rho_\lambda\}_\lambda$ during a continuum of time t. □

4.2. Nuclearity

In this part, we address an interesting application of the C^*-dynamical system $(\mathcal{A}^{\Phi,g}_{1,\rho}, \{\sigma_t\}_t)$ built on the space of 1-dimensional $\mathbb{Q}$-lattices of large Feynman diagrams. We determine a new class of nuclear C^*-dynamical systems associated with the non-perturbative sector of Φ.

Theorem 4.3. *Phase transitions between open bounded cut-distance topological regions of large Feynman diagrams in $\mathcal{S}^{\Phi,g}_{\approx}$ are encoded by a dynamical system of nested C^*-algebras governed by time evolution.*

Proof. This is a result of [3,4] and Corollary 4.2. For any open bounded cut-distance topological region $\mathcal{V}$ of large Feynman diagrams in $\mathcal{S}^{\Phi,g}_{\approx}$, consider the groupoid

$$\mathcal{L}^{\Phi,g}_{1,\rho,\mathcal{V}} :=< \left\{ [L^1_{\rho,X_{\mathrm{DSE}}(c_g)}]_\sim \ : \ X_{\mathrm{DSE}}(c_g) \in \mathcal{V} \ , \ c_g \in \mathbb{R}_+ \right\} > \tag{4.14}$$

generated by the commensurable equivalence classes of 1-dimensional $\mathbb{Q}$-lattices of the form (4.6) in $\mathcal{V}$ up to the scaling of running coupling constants. Define $\mathcal{A}^{\Phi,g}_{1,\rho,\mathcal{V}}$ as the C^*-algebra generated by the quotient groupoid $\mathcal{L}^{\Phi,g}_{1,\rho,\mathcal{V}}/\mathbb{R}^*_+$.

If $\mathcal{V}_1 \subset \mathcal{V}_2$, then $\mathcal{A}^{\Phi,g}_{1,\rho,\mathcal{V}_1} \subset \mathcal{A}^{\Phi,g}_{1,\rho,\mathcal{V}_2}$. $\mathcal{A}^{\Phi,g}_{1,\rho}$ is the smallest C^*-algebra containing $\mathcal{A}^{\Phi,g}_{1,\rho,\mathcal{V}}$ for all open bounded cut-distance topological regions $\mathcal{V}$. For any family $\{\mathcal{V}_l\}_{l=1}^{\infty}$ of nested cut-distance open regions in $\mathcal{S}^{\Phi,g}_{\approx}$, define the new system $\{\mathcal{A}^{\Phi,g}_{1,\rho,\mathcal{V}_l}\}_{\mathcal{V}_l \subset \mathcal{S}^{\Phi,g}_{\approx}}$ of nested C^*-algebras with respect to the time evolution $\{\sigma_t\}_t$ given by (4.13).

Consider an open bounded cut-distance topological region $\mathcal{V}_l$ in $\mathcal{S}^{\Phi,g}_{\approx}$, the dual space $\mathcal{A}^{\Phi,g,*}_{1,\rho,\mathcal{V}_l}$ and the Hilbert space $\mathcal{H}_\Phi$ of states. Thanks to (3.1) and (4.12),

$$X_{\mathrm{DSE}} \in \mathcal{S}^{\Phi,g}_{\approx} \ : \ ||X_{\mathrm{DSE}}||_{\mathrm{cut}} < \infty \Rightarrow \phi \in \mathcal{A}^{\Phi,g,*}_{1,\rho,\mathcal{V}_l} \ : \ ||\phi||_{\mathrm{op}} < \infty \ , \tag{4.15}$$

such that

$$||\phi||_{\mathrm{op}} = \inf \left\{ c \geq 0 \ : \ |\phi(\eta)| \leq c||\eta|| \ : \ \eta \in \mathcal{A}^{\Phi,g}_{1,\rho,\mathcal{V}_l} \right\} . \tag{4.16}$$

For any sequence $\{\phi_n\}_{n\geq 1}$ in $\mathcal{A}^{\Phi,g,*}_{1,\rho,\mathcal{V}_l}$ which converges to some linear functional $\phi \in \mathcal{A}^{\Phi,g,*}_{1,\rho}$, choose a sequence $\{x_n\}_{n\geq 1}$ of states in $\mathcal{H}_\Phi$ such that

$$||x_n|| = \frac{||\phi - \phi_n||^T_{\text{op}}}{||\phi_n||_{\text{op}}} \ , \ T \in \mathbb{R}_{>1} \ , \tag{4.17}$$

while there exists some order N_ϵ where for $n \geq N_\epsilon$, $||\phi - \phi_n|| < \epsilon < 1$. Therefore $\sum_{n=0}^\infty ||\phi_n||_{\text{op}}.||x_n|| < \infty$ which leads us to define a new functional

$$\Psi_{\mathcal{V}_l,T} : \mathcal{A}^{\Phi,g}_{1,\rho,\mathcal{V}_l} \rightarrow \mathcal{H}_\Phi \ , \tag{4.18}$$

$$\eta(L^1_{\rho,X_{\text{DSE}_1}(c_{1,g})}, L^1_{\rho,X_{\text{DSE}_2}(c_{2,g})}) \mapsto \sum_{n=1}^\infty \phi_n(\eta(L^1_{\rho,X_{\text{DSE}_1}(c_{1,g})}, L^1_{\rho,X_{\text{DSE}_2}(c_{2,g})})).x_n$$

for any pair $(L^1_{\rho,X_{\text{DSE}_1}(c_{1,g})}, L^1_{\rho,X_{\text{DSE}_2}(c_{2,g})})$ of commensurable 1-dimensional $\mathbb{Q}$-lattices with $X_{\text{DSE}_1}(c_{1,g}), X_{\text{DSE}_2}(c_{2,g}) \in \mathcal{V}_l$.

Therefore the C^*-dynamical system $\{\mathcal{A}^{\Phi,g}_{1,\rho,\mathcal{V}_l}\}_{\mathcal{V}_l \subset \mathcal{S}^{\Phi,g}_{\approx}}$ in the non-perturbative sector of Φ follows the nuclearity condition (given in [3,4]) such that the value

$$||\Psi_{\mathcal{V}_l,T}||_1 = \inf \sum_{n=1}^\infty ||\phi_n||_{\text{op}}.||x_n|| \tag{4.19}$$

can be addressed as the replacement of the partition function for this class of dynamical systems where T plays the role of the temperature parameter.

For $0 < T < 1$, we consider a sequence $\{x_n\}_{n\geq 1}$ of states in $\mathcal{H}_\Phi$ with $||x_n|| = \frac{||\phi - \phi_n||^{1/T}_{\text{op}}}{||\phi_n||_{\text{op}}}$.

For $T = 1$, we consider a sequence $\{x_n\}_{n\geq 1}$ of states in $\mathcal{H}_\Phi$ with $||x_n|| = \frac{||\phi - \phi_n||^{T+\delta}_{\text{op}}}{||\phi_n||_{\text{op}}}$ for a fixed $\delta > 0$.

If $T = 0$, then $\sum_{n=0}^\infty ||\phi_n||_{\text{op}}.||x_n|| = \sum_{n=0}^\infty 1$ is divergent. However, by applying some resummability methods [44], we can associate a finite value to this series. In other words, consider the analytic continuation from a certain complex integral representation for the Riemann zeta function $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$ to values s with $\text{real}(s) > 1$. The resulting analytically continued function $\zeta_{a.c.}$ with the corresponding Ramanujan sums $\mathcal{R}$ is applied to have

$$(\sum_{n=0}^\infty 1)(\mathcal{R}) = \zeta_{a.c.}(0) = -\frac{1}{2} \ . \tag{4.20}$$

Therefore it is possible to have a well-defined functional (4.18) for the case $T = 0$. □

5. Conclusion

On the one hand, the separable Banach manifold $\mathcal{S}^{\Phi,g}_{\approx}$ of weakly isomorphic classes of large Feynman diagrams which contribute to quantum motions in a strongly coupled gauge field theory Φ is a new mathematical tool for the study of non-perturbative aspects and intermediate phase transitions in this sector of Φ [31–36]. Homomorphism densities, as continuous functionals on the cut-distance topological space $\mathcal{S}^{\Phi,\Omega}_{\text{graphon}}(\mathbb{R})$ of Feynman graphons, enable us to characterize intermediate phases of deconfined–confined phase transition. Recently, a theory of complexity for the solution space of quantum motions is addressed [37,38] which leads us to associate a new complexity parameter to these intermediate phases. On the other hand, Quantum Field Theory and Quantum Statistical Mechanics are different platforms for the study of physical systems. Possible interrelation between these platforms are explored in terms of determining thermodynamical equilibrium states of (local) quantum field theories at positive temperatures in the context of nuclearity condition [3,4]. The stability and passivity of KMS states are applied to consider these states as equilibrium states. Thanks to these backgrounds, this research provided a new statistical mechanical model for the study of non-perturbative aspects of Φ in terms of relating two classes of C^*-dynamical systems to $\mathcal{S}^{\Phi,g}_{\approx}$ which encode equilibrium states and observables associated with the solution space of quantum motions under the evolution of running coupling constants in a continuum of time.

- The evolution of running coupling constants in the structure of quantum motions is encoded by strongly continuous one-parameter semigroups to characterize equilibrium states of non-perturbative sector of Φ in terms of the functional analysis of $\mathcal{B}(\mathcal{S}^{\Phi,g}_{\approx})$. The GNS representations of $\mathcal{B}(\mathcal{S}^{\Phi,g}_{\approx})$ are applied to associate the C^*-dynamical system $(E^{\Phi,g}, \{\tilde{\rho}_\lambda\}_\lambda)$ to Φ.
- $\mathcal{S}^{\Phi,g}_{\approx}$ is equipped with a new Banach lattice structure where the commensurable quotient space of n-dimensional $\mathbb{Q}$-lattices of large Feynman diagrams is introduced. The space of commensurable classes of 1-dimensional $\mathbb{Q}$-lattices in $\mathcal{S}^{\Phi,g}_{\approx}$ is applied to build a new C^*-algebra structure to formulate a new dynamical system $(\mathcal{A}^{\Phi,g}_{1,\rho}, \{\sigma_t\}_t)$ with respect to the time evolution. This C^*-dynamical system encodes non-perturbative observables such that KMS states on $\mathcal{A}^{\Phi,g}_{1,\rho}$ characterize equilibrium states associated with the solution space of quantum motions under the time evolution.
- The package
$$\Big\{(E^{\Phi,g}, \{\tilde{\rho}_\lambda\}_\lambda), (\mathcal{A}^{\Phi,g}_{1,\rho}, \{\sigma_t\}_t), \big(\{\mathcal{A}^{\Phi,g}_{1,\rho,\mathcal{V}_l}, \Psi_{\mathcal{V}_l,T}\}_{l=1}^{\infty}, \{\sigma_t\}_t, \mathcal{V}_l \subset_{\text{open}} \mathcal{S}^{\Phi,g}_{\approx}, T \in \mathbb{R}_{\geq 0}\big)\Big\} \tag{5.1}$$
provides a new statistical mechanical model for the study of deconfined–confined phase transitions and intermediate phase transitions of non-perturbative sector of Φ in terms of the behavior of solutions of quantum motions under the evolution of running coupling constants in a continuum of time at different temperatures.

CRediT authorship contribution statement

Ali Shojaei-Fard: Conceptualization, Original Motivation, Methodology, Research, Writing / Reviewing / Editing: Original draft preparation, Writing / Reviewing / Editing: Final preparation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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