



Interpolating matrix models for WLZZ series

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Abstract We suggest a two-matrix model depending on three (infinite) sets of parameters which interpolates between all the models proposed in Wang et al. (Eur Phys J C 82:902, [arXiv:2206.13038](#), 2022) and defined there through W -representations. We also discuss further generalizations of the WLZZ models, realized by W -representations associated with infinite commutative families of generators of w_∞ -algebra which are presumably related to more sophisticated multi-matrix models. Integrable properties of these generalizations are described by what we call the skew hypergeometric τ -functions.

1 Introduction

Recently in [1] a new class of τ -functions was introduced with the help of W -representations generated by an extended class of operators. These τ -functions possess superintegrability property [2], but more standard features of the (eigenvalue) matrix model partition functions [3–5] are not immediately obvious: those from the Virasoro-like constraints to topological ($1/N$) expansion [6]. Moreover, at least half of these models (“positive branch”) seem to violate the usual relation between these features: the AMM/CEO topological recursion [7–13] originating from the Virasoro/ W -algebra constraints expanded around a given spectral curve [14] seem to be related to N rather than to $1/N$ expansion [6].

The goal of this letter is to begin unraveling the intriguing mysteries of the WLZZ models. Our first claim is that these models are indeed the matrix models: we suggest explicit two-matrix model integrals for them, which reduce to single matrix integrals in particular cases (like the $n = \pm 2$ members of the WLZZ model sequences). The second claim is that the class of the models can be further extended to include other W -operators from a Borel subalgebra of the w_∞ -algebra. The third claim is that though all these models are naturally united into just two branches, positive and negative ones, and, though the first one is a particular case of the second one, the W -representations for these two branches still need to be treated differently.

This reformulation of the story looks quite illuminating and should be a nice starting point for a serious study of this unifying approach to non-perturbative partition functions.

The structure of the letter is as follows. In Sect. 2, we introduce the most general partition function of the type we are interested in: this type of partition function is associated with a τ -function of the Toda lattice hierarchy [15] and generalizes hypergeometric τ -functions introduced earlier in [16, 17]. We call such τ -functions skew hypergeometric. Surprisingly, even for this rather general class of partition functions there is a W -representation, though it is realized by the W -operators which are not always easy to find in an explicit form.

In Sect. 3, we specify further the partition functions to the class that contains the whole set of the WLZZ models. These partition functions can be realized by a two-matrix integral that depends on two sets of variables $\tilde{p}_k$ and g_k and on an external matrix Λ , and interpolates between all the WLZZ models. This matrix integral can be realized by a W -representation given by explicit differential operators in variables $p_k = \text{Tr } \Lambda^k$. Moreover, the matrix integral that

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describes a particular case when all $p_k = \Lambda = 0$ (which interpolates between the WLZZ models of the negative branch) can be also realized by another W -representation.

In Sect. 4, we discuss an extension of the WLZZ model series, and discuss its W -representation. It turns out that all W -representations discussed in the letter are associated with generators from a Borel subalgebra of the w_∞ -algebra.

Section 5 contains some concluding remarks. In the Appendices, we also added technically important issues related to action of generators of the W -representations on the basis of the Schur functions, and some simple examples of these generators.

Notation. We use the notation $S_R\{p\}$ for the Schur functions, which are graded polynomials of arbitrarily many variables p_k of grading k . The Schur function $S_R\{p\}$ is labelled by the Young diagram (partition) R : $R_1 \geq R_2 \geq \dots \geq R_{l_R} > 0$, and the grading of this Schur function is $|R| := \sum_i R_i$. Similarly, we denote $S_{R/P}\{p\}$ the skew Schur functions [18].

The Schur function $S_R\{p\}$ is a linear combination of monomials $p_\Delta := \prod_i^{\Delta} p_i$ parameterized by the Young diagrams Δ such that $|\Delta| = |R|$. One can also write the same monomial in the form $p_\Delta = \prod_{k=1}^{\infty} p_k^{m_k}$. This latter parameterization of p_Δ is related to the quantity $z_\Delta := \prod_k k^{m_k} m_k!$, which is the standard symmetric factor of the Young diagram (order of the automorphism).

We will use the scalar product $\langle \dots \rangle$, which the standard Schur scalar product [18] given by $\langle p_\Delta | p_{\Delta'} \rangle = z_\Delta \delta_{\Delta, \Delta'}$ and extended to the Schur functions by linearity. In particular,

$$\begin{aligned} \langle S_R | S_Q \rangle &= \delta_{R,Q}, \quad \langle p_k S_R | S_Q \rangle = \langle S_R | k \frac{\partial S_Q}{\partial p_k} \rangle, \\ \sum_{\Delta} \frac{g_\Delta}{z_\Delta} \langle p_\Delta \cdot S_Q | S_R \rangle &= S_{R/Q}\{g\} \end{aligned} \quad (1)$$

2 Skew hypergeometric τ -functions and their integrable properties

2.1 Definition

We start with considering the most general partition functions which we need in this paper. They are of the form

$$\begin{aligned} Z_f(\bar{p}, p, g) &= \sum_{R,Q} \frac{\prod_{i,j \in R} f(j-i)}{\prod_{i,j \in Q} f(j-i)} S_{R/Q}\{\bar{p}\} S_R\{g\} S_Q\{p\} \\ &:= \sum_{R,Q} \prod_{i,j \in R/Q} f(j-i) S_{R/Q}\{\bar{p}\} S_R\{g\} S_Q\{p\} \end{aligned} \quad (2)$$

with some (arbitrary) function $f(x)$. Such a function Z_f is not a hypergeometric τ -function [16, 17, 19], unless $p_k = 0$ and there are no skew functions in (2). However, it turns out that it is still a τ -function of the KP hierarchy w.r.t. the both

sets of times¹ p_k and g_k . Hence, we call it **skew hypergeometric τ -function**.

Let us explain that, for an arbitrary function f , the partition function (2) is a τ -function w.r.t. to both p_k and g_k variables, moreover, making substitution $f(x) \rightarrow f(x+N)$, even a stronger statement is correct: Z_f is a τ -function of the Toda lattice hierarchy with N being the Toda zeroth time. In order to prove this, let us note that, in accordance with [20], the sum

$$\tau_N(g, p|\xi) = \sum_{R,Q} \xi_{R,Q}(N) S_R\{g\} S_Q\{p\} \quad (3)$$

is a τ -function of the Toda lattice hierarchy iff

$$\xi_{R,Q}(N) = \det_{i,j \leq N} F(R_i - i + N + 1, Q_j - j + N + 1) \quad (4)$$

with some function F , and N playing the role of the zeroth time.

Now we just remind that the skew Schur function has the Jacobi-Trudi determinant representation

$$S_{R/Q}\{\bar{p}\} = \begin{cases} \det_{i,j \leq l_R} h_{R_i - i - Q_j + j} \{\bar{p}\} & \text{if } Q \in R \\ 0 & \text{if } Q \notin R \end{cases} \quad (5)$$

where h_k are the complete homogeneous symmetric polynomials, that is, $h_k = S_{[k]}$. Thus, for the partition function (2), we obtain representation (3) with

$$\xi_{R,Q}(N) = \begin{cases} \det_{i,j \leq N} \frac{G_{R_i}(N+1-i)}{G_{Q_j}(N+1-j)} h_{R_i - i - Q_j + j} \{\bar{p}\} & \text{if } Q \in R \\ 0 & \text{if } Q \notin R \end{cases} \quad (6)$$

where

$$G_k(x) := \prod_{i=0}^{k-1} f(x+i) \quad (7)$$

2.2 W -Representation

Using W -representations for generating partition functions is the standard idea, which goes back to [7, 8, 21–27], and, in application to matrix models, to [28–34]. It turns out that the partition function (2) also admits such a description.

As usual for matrix model partition functions, it can be described by the action of exponential of a W -operator on exponential of time variables:

$$Z_f(\bar{p}, p, g) = \exp \left(\sum_{k=1}^{\infty} \frac{\bar{p}_m \hat{W}_m^f[p]}{m} \right) \cdot \exp \left(\sum_{k=1}^{\infty} \frac{p_k g_k}{k} \right) \quad (8)$$

¹ Note that the traditional choice t_k of time variables of the KP hierarchy as compared with power sums p_k of variables in symmetric functions is $t_k = \frac{p_k}{k}$.

This operator acts on the variables p_k , and action of the commuting operators $\hat{W}_m^f[p]$ can be manifestly described in the basis of the Schur functions:

$$\begin{aligned} \hat{W}_m^f[p] S_R\{p\} \\ = \sum_{R'} \prod_{i,j \in R'/R} f(j-i) \left\langle m \frac{\partial S_R\{p\}}{\partial p_m} \middle| S_{R'}\{p\} \right\rangle S_{R'}\{p\} \end{aligned} \quad (9)$$

2.3 Hypergeometric τ -functions

An interesting particular case of the partition function (2) is at the point where all $p_k = 0$. Then, the partition function becomes

$$Z_f(\bar{p}, p = 0, g) = \sum_R \prod_{i,j \in R} f(j-i) S_R\{\bar{p}\} S_R\{g\} \quad (10)$$

It is a τ -function of the hypergeometric type. However, in order to construct a W -representation, one can not just make a reduction of (9), since this latter is an operator acting on variables p_k . Hence, in this case, one needs another W -representation, acting on variables g_k . Such a representation

$$Z(N; \bar{p}, p, g) = \int \int_{N \times N} dX dY \exp \left(-\text{Tr} XY + \text{Tr} Y \Lambda + \sum_k \frac{g_k}{k} \text{Tr} X^k + \sum_k \frac{\bar{p}_k}{k} \text{Tr} Y^k \right) \quad (16)$$

does exist, and is given by action of exponential of a W -operator on unity:

$$Z_f(\bar{p}, p = 0, g) = \exp \left(\sum_{k=1} \frac{\bar{p}_k \hat{W}_{-m}^f[g]}{m} \right) \cdot 1 \quad (11)$$

This operator acts on the variables g_k , and action of the commuting operators $\hat{W}_{-m}^f[g]$ can be manifestly described in the basis of the Schur functions (meaning of negative subscript will become clear in the next section):

$$\begin{aligned} \hat{W}_{-m}^f[g] S_R\{g\} \\ = \sum_{R'} \prod_{i,j \in R'/R} f(j-i) \left\langle g_m S_R\{g\} \middle| S_{R'}\{g\} \right\rangle S_{R'}\{g\} \end{aligned} \quad (12)$$

With concrete choices of the function $f(x)$, as we shall demonstrate below, one can also construct these operators as very explicit differential operators.

3 Interpolating matrix model for the WLZZ series

3.1 Interpolating partition function

Let us specify to the case of a linear function $f(x) = N + x$:

$$\begin{aligned} Z(N; \bar{p}, p, g) \\ = \sum_{R,Q} \prod_{i,j \in R/Q} (N + j - i) S_{R/Q}\{\bar{p}\} S_R\{g\} S_Q\{p\} \end{aligned} \quad (13)$$

Choosing here $\bar{p}_k = \delta_{k,m}$, we arrive at the positive branch Z_m of the WLZZ models [1]:

$$\begin{aligned} Z_m(N; p, g) \\ = \sum_{R,Q} \prod_{i,j \in R/Q} (N + j - i) S_{R/Q}\{\delta_{k,m}\} S_R\{g\} S_Q\{p\} \end{aligned} \quad (14)$$

Moreover, choosing further $p_k = 0$, we arrive at the negative branch $Z_{-,m}$ of the WLZZ models [1],

$$Z_{-,m}(N; g) = \sum_R \prod_{i,j \in R} (N + j - i) S_R\{\delta_{k,m}\} S_R\{g\} \quad (15)$$

3.2 Matrix model

We propose that the model (13), which interpolates between all the WLZZ models and includes all of them at particular values of parameters, is described by the two-matrix model that depends on two (infinite) sets of variables $\bar{p}_k, g_k$ and an external matrix Λ :

with $p_k = \text{Tr} \Lambda^k$. Here the integration is understood as power series in $g_k, \bar{p}_k$ and $\text{Tr} \Lambda^k$, and X are Hermitian matrices, while Y are anti-Hermitian ones. This means that this formula can be described in the pure combinatorics terms of Feynman diagrams, with the propagator being $\langle X_{ij} Y_{kl} \rangle = \delta_{il} \delta_{jk}$. One can make a thorough computer check that these Feynman diagrams, indeed, give the expansion (13). A complete and direct derivation of (16) will be provided elsewhere: it is more technical and can overshadow the simple pattern described in this letter.

Let us check that particular cases are also correctly reproduced from this expression.

First of all, at $p_k = 0$, i.e. at $\Lambda = 0$, one obtains the interpolating model for the negative branch of the WLZZ models. This model is $Z_{2,1}$ in the notation of [35] and has a two-matrix model representation [36], [35, Eq.(53)], [37]:

$$\begin{aligned} Z_{-,m}(N; \bar{p}, g) = \int \int_{N \times N} dX dY \exp \left(-\text{Tr} XY + \sum_k \frac{g_k}{k} \text{Tr} X^k \right. \\ \left. + \sum_k \frac{\bar{p}_k}{k} \text{Tr} Y^k \right) \end{aligned} \quad (17)$$

which, indeed, follows from (16) at $\Lambda = 0$.

The second particular case emerges at $\bar{p}_k = \delta_{k,2}$, when one arrives at the particular WLZZ model of the positive

branch at $m = 2$. Indeed, in this case, (16) reduces to formulas [38, Eq. (2.40)], [1, Eq. (30)], [6, Eqs. (93)-(94)] after performing the Gaussian integration over Y :

$$Z_2 = \int dX \exp \left(-\frac{1}{2} \text{Tr} X^2 + \sum_k \frac{g_k}{k} \cdot \text{Tr} (X + \Lambda)^k \right) \quad (18)$$

The third particular case is just the trivial one: at all $\bar{p}_k = 0$. Then, integration over Y gives the δ -function, $\delta(X - \Lambda)$, and

$$Z(N; 0, p, g) = \exp \left(\sum_k \frac{g_k p_k}{k} \right) \quad (19)$$

which, indeed, follows from (2).

3.3 W -representation

With the concrete choice of the function $f(x) = x + N$ as in (13), one can realize the W -representations operators (9) and (12) as differential operators. Operators $\hat{W}_m[p]$ from (9) are constructed using three auxiliary operators: the cut-and-join operator [39, 40]

$$\begin{aligned} \hat{W}_0 &= \hat{W}_0(N) \\ &= \frac{1}{2} \sum_{a,b} \left((a+b) p_a p_b \frac{\partial}{\partial p_{a+b}} + ab p_{a+b} \frac{\partial^2}{\partial p_a \partial p_b} \right) \\ &\quad + N \sum_k k p_k \frac{\partial}{\partial p_k} \end{aligned} \quad (20)$$

and operators

$$\hat{F}_1 = \left[\hat{W}_0, \frac{\partial}{\partial p_1} \right], \quad \hat{F}_2 = \left[\hat{W}_0, \hat{F}_1 \right] \quad (21)$$

With these operators, one constructs recurrently

$$\hat{W}_{m+1} = \frac{1}{m} \left[\hat{W}_m, \hat{F}_2 \right] \quad (22)$$

with the initial condition $\hat{W}_1 = \hat{F}_1$. Thus, finally, the partition function (13), or, equivalently, the matrix model (16) is generated by these operators

$$Z(N; \bar{p}, p, g) = \exp \left(\sum_{k=1} \frac{\bar{p}_m \hat{W}_m[p]}{m} \right) \cdot \exp \left(\sum_{k=1} \frac{p_k g_k}{k} \right) \quad (23)$$

Similarly, operators $\hat{W}_{-m}[p]$ from (12) are also constructed using three auxiliary operators: the same cut-and-join operator (20) and operators

$$\hat{E}_1 = \left[\hat{W}_0, p_1 \right], \quad \hat{E}_2 = \left[\hat{W}_0, \hat{E}_1 \right] \quad (24)$$

With these operators, one constructs recurrently

$$\hat{W}_{-m} = \frac{1}{m-1} \left[\hat{E}_2, \hat{W}_{-m+1} \right] \quad (25)$$

with the initial condition $\hat{W}_{-1} = \hat{E}_1$. Thus, finally, the partition function (13), or, equivalently, the matrix model (17) is generated by these operators

$$\begin{aligned} Z_{-}(N; \bar{p}, g) &= Z(N; \bar{p}, p=0, g) \\ &= \exp \left(\sum_{k=1} \frac{\bar{p}_m \hat{W}_{-m}[g]}{m} \right) \cdot 1 \end{aligned} \quad (26)$$

4 Generalizing the WLZZ series

Now we are ready to present a simple generalization of the interpolating model discussed in the previous section. This generalization is related with higher roots of the w_∞ -algebra. The partition function of this model is specified by choosing a polynomial $f(x) = \prod_{l=1}^n (N_l + x)$:

$$\begin{aligned} Z^{(n)}(\{N\}_n; \bar{p}, p, g) \\ = \sum_{R,Q} \left(\prod_{l=1}^n \frac{\prod_{i,j \in R} (N_l + j - i)}{\prod_{i,j \in Q} (N_l + j - i)} \right) S_{R/Q}\{\bar{p}\} S_R\{g\} S_Q\{p\} \end{aligned} \quad (27)$$

The WLZZ-models correspond to $n = 1$. This partition function celebrates the same integrability properties, however, its matrix model representation is more involved: it depends on a set $\{N\}_n$ of n integers N_l , which are sizes of matrices in the multi-matrix model, which is quite involved even in the case of $p_k = 0$, i.e. that corresponding to the negative branch of the WLZZ models: see [35, Eq.(59)].

4.1 W -representation

Though the matrix model representation becomes very involved for the class of models $Z^{(n)}$ as compared with $Z^{(1)} = Z(N; \bar{p}, p, g)$ in Sect. 3, their W -representations are still of the same complexity. Let us construct the operators $\hat{W}_m^{(n)}[p]$ from (9). They can be produced from operators already constructed in the previous section.

We will need a whole set of new auxiliary operators $\hat{F}_k(\{N\}_k)$ in addition to the cut-and-join operator (20). These operators depend on the set of integers $N_l, l = 1, \dots, k$, and are constructed iteratively:

$$\hat{F}_{k+1}(\{N\}_{k+1}) = \left[\hat{W}_0(N_{k+1}), \hat{F}_k(\{N\}_k) \right] \quad (28)$$

With these operators, one constructs recurrently the set

$$\hat{W}_{m+1}^{(n)} = \frac{1}{m} \left[\hat{W}_m^{(n)}, \hat{F}_{n+1} \right] \quad (29)$$

with the initial condition $\hat{W}_1^{(n)} = \hat{F}_n$.

Thus, finally, the partition function (13), or, equivalently, the matrix model (16) is generated by these operators

$$Z^{(n)}(\{N\}_n; \bar{p}, p, g) = \exp\left(\sum_{k=1} \frac{\bar{p}_m \hat{W}_m^{(n)}[p]}{m}\right) \cdot \exp\left(\sum_{k=1} \frac{p_k g_k}{k}\right) \quad (30)$$

Similarly, operators $\hat{W}_{-m}^{(n)}[p]$ from (12) are also constructed using a set of auxiliary operators: this time they are

$$\hat{E}_{k+1}(\{N\}_{k+1}) = [\hat{W}_0(N_{k+1}), \hat{E}_k(\{N\}_k)] \quad (31)$$

With these operators, one again constructs recurrently the set

$$\hat{W}_{-m}^{(n)} = \frac{1}{m-1} [\hat{E}_{n+1}, \hat{W}_{-m+1}^{(n)}] \quad (32)$$

with the initial condition $\hat{W}_{-1}^{(n)} = \hat{E}_n$.

Thus, finally, the partition function (13), or, equivalently, the matrix model (17) is generated by these operators

$$Z_-^{(n)}(\{N\}_n; \bar{p}, g) = Z^{(n)}(\{N\}_n; \bar{p}, p=0, g) = \exp\left(\sum_{k=1} \frac{\bar{p}_m \hat{W}_{-m}^{(n)}[g]}{m}\right) \cdot 1 \quad (33)$$

4.2 w_∞ -algebra

To bring a little more order in the zoo of considered operators, let us separately discuss their relation to each other. For the sake of simplicity, we put $N_l = 0$ in all operators. The motivation for this is that turning on N amounts to the shift

$$\hat{W}_0(N) = \hat{W}_0(0) + N \hat{L}_0. \quad (34)$$

which then translates to other formulas. This does not respect the grading.

At $N = 0$, we will drop the dependence on N in the notation. Then the operators, constructed above can be represented as the graded generators of the w_∞ -algebra.

The w_∞ algebra is formed by the operators of the form $p_{a_1} \dots p_{a_m} \frac{\partial^n}{\partial p_{b_1} \dots \partial p_{b_n}}$, and they are basically classified by two gradings: the spin, which is the net number of p 's, $n+m$, and the second grading: the sum of indices $a_1 + \dots + a_n - b_1 - \dots - b_m$, which we plot at the vertical and the horizontal axes respectively.

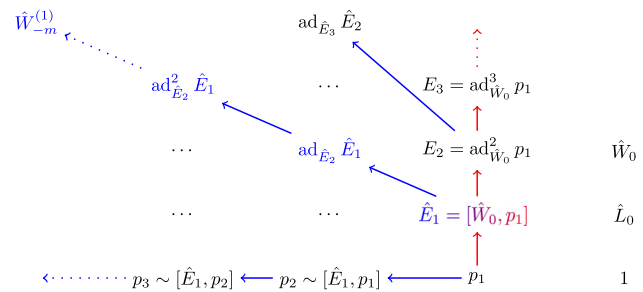
Let us denote as $V_{(m,n)}$ an operator of spin m and of the second grading n . Then:

$$\hat{E}_k = V_{(k+1,1)}, \quad p_k = V_{(1,k)}, \quad \hat{W}_0 = \frac{1}{2} V_{(3,0)}, \quad \hat{W}_{-m}^{(n)} = V_{(mn+1,m)} \quad (35)$$

These operators satisfy the w_∞ algebra relations [42]:

$$[V_{(m_1,n_1)}, V_{(m_2,n_2)}] = ((m_1-1)n_2 - (m_2-1)n_1) V_{(m_1+m_2-2, n_1+n_2)} \quad (36)$$

All the w_∞ -operators can be drawn on the following diagram of generators of the Borel subalgebra of w_∞ -algebra:



Moving to the upwards and left is achieved by taking iterated commutators with $\hat{E}_{n+1}$. Hence, each line of blue operators are building blocks of the $Z_{-m}^{(n)}$ -series at a concrete n . The more we move upwards (along the red arrows), commuting iteratively with $\hat{W}_0$, the higher n we choose.

The operators of our interest here occupy not that much part of the table, though one can generate the whole table and realize that, in fact, the whole set of operators forms another, more involved algebra, $W_{1+\infty}$ [43–48], the commutation relations (36) and the w_∞ algebra are obtained from it by a contraction.

In particular, one can construct a similar picture for the positive branch, identifying $k \frac{\partial}{\partial p_k} = V_{(1,-k)}$, $\hat{F}_m = V_{(m+1,-1)}$, etc. However, the w_∞ -algebra does not allow a central extension except for the Virasoro generators $V_{(2,k)}$. At the same time, $k \frac{\partial}{\partial p_k} = V_{(1,-k)}$ does not commute with $V_{(1,k)} = p_k$. Hence, one can not sew the two branches together within w_∞ , but in $W_{1+\infty}$ only. Note also that the corresponding W -representations act as differential operators on different sets of variables: p_k and g_k . To have a unique picture, one would better construct a W -representation of $Z^{(n)}(N_l; \bar{p}, p, g)$ in terms of differential operators acting on the variables g_k . Unfortunately, such a formulation is not known so far.

From the commutation relations (36), one can again check that the infinite families of operators $\hat{W}_{-m}^{(n)} = V_{(mn+1,m)}$ at each n are commutative, and similarly for $\hat{W}_m^{(n)}$. Let us introduce an operator $\hat{O}(N)$ that has the Schur functions as its eigenfunctions:

$$\hat{O}(N) \cdot S_R = \left(\prod_{i,j \in R} (N+j-i) \right) \cdot S_R \quad (37)$$

Such an operator has been manifestly constructed in [35, Eqs. (21),(26)]. As is clear from (12) and (1), the operators $\hat{W}_{-m}^{(1)}$ can be constructed using this operator:

$$\hat{W}_{-m}^{(1)} = \hat{O}(N) \cdot p_m \cdot \hat{O}^{-1}(N) \quad (38)$$

and, more generally,

$$\hat{W}_{-m}^{(n)} = \left(\prod_{l=1}^n \hat{O}(N_l) \right) \cdot p_m \cdot \left(\prod_{l=1}^n \hat{O}(N_l)^{-1} \right) \quad (39)$$

Hence, the powers of the operator $\hat{O}(N)$ just provide automorphisms of the w_∞ -algebra that map the commutative family $\{p_m\}$ to $\hat{W}_{-m}^{(n)}$, each automorphism producing a blue line in the picture. The vertical line: $1, \hat{L}_0, \hat{W}_0$ is also commutative, and, in terms of the generalized cut-and-join operators $\hat{W}_\Delta$ of [40,41] is just $\hat{W}_{[m]}$.

A similar picture is also correct for the $\hat{W}_m^{(n)}$ operators, which are generated from $\frac{\partial}{\partial p_m}$ with rotation by the same operator $\hat{O}(N)$: $\hat{W}_m^{(1)} = \hat{O}(N)^{-1} \cdot m \frac{\partial}{\partial p_m} \cdot \hat{O}(N)$, etc.

5 Conclusion

This letter describes an important step in the study of non-perturbative partition functions. It brings the new class of WLZZ models [1] into accordance with the traditional matrix model approach, by suggesting a two-matrix model representation for them. At the present letter, the evidence is provided just by computer calculation and comparison of averages, provided by integrals and by the W -representations, and by obvious reductions to simpler one-matrix and two-matrix integrals at distinguished points in the space of parameters, while a direct derivation will be explained elsewhere. Now the road is open for application of the standard matrix-model techniques, which, however, should make new twists because of the broken relation [6] between the Virasoro/ W -based topological recursion [7,10] and the standard $1/N$ topological expansion [49–51].

We also explained integrability [3] and superintegrability [2] of the new matrix models by relating them to a new class of τ -functions, which we named skew hypergeometric. Most important, we now have a unified description of a huge variety of matrix models, and unification is achieved in terms of the w_∞ -algebra, as was long expected. It is now clear that the crucial lacking point was the need to switch to two-matrix models (this was anticipated many years ago in [52,53]) and, for this, to find an adequate class among them, which appeared to include the mixture of two ordinary potentials and the Kontsevich background field (in the character phase [54]). This unifying model with three independent sets of time variables should now be carefully investigated and extended in various directions.

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Appendix A: Explicit formulas for W -operators

In the Appendices, we give explicit expressions for some of the constructed operators, briefly describe action of various operators on the Schur functions, and also comment on obtaining the Schur function expansion for the partition functions from W -operators following the lines of [38,55].

First of all, we list explicit expressions for first few operators. To begin, we repeat the diagonal operator $\hat{W}_0(N)$ (20):

$$\hat{W}_0(N) = \frac{1}{2} \sum_{a,b} \left((a+b) p_a p_b \frac{\partial}{\partial p_{a+b}} + a b p_{a+b} \frac{\partial^2}{\partial p_a \partial p_b} \right) + N \sum_k k p_k \frac{\partial}{\partial p_k} \quad (40)$$

where:

$$L_0 = \sum_k k p_k \frac{\partial}{\partial p_k} \quad (41)$$

The positive branch is generated by the operators:

$$\begin{aligned} \hat{E}_1 &= \sum_a a p_{a+1} \frac{\partial}{\partial p_a} + N p_1 \\ \hat{E}_2 &= \sum_{a,b} \left(a b p_{a+b+1} \frac{\partial^2}{\partial p_a \partial p_b} + (a+b-1) p_a p_b \frac{\partial}{\partial p_{a+b-1}} \right) \\ &\quad + 2N \sum_a a p_{a+1} \frac{\partial}{\partial p_a} + N^2 p_1 \end{aligned} \quad (42)$$

where the N dependent terms come from the L_0 shifts. This is in agreement with $\hat{E}_1$ being linear in N as it is given by a single commutator, while $\hat{E}_2$ is quadratic as it given by a double commutator. We can clearly see a mix of operators with different spins and the restoration of homogeneity in the spin after formally assigning spin 1 to the parameter N .

Further operators get increasingly complicated, we list just one representative for an illustration:

$$\begin{aligned}\hat{W}_{-2} &= [\hat{E}_2, \hat{W}_{-1}] \\ &= \sum_{a,b} \left(ab p_{a+b+2} \frac{\partial^2}{\partial p_a \partial p_b} + (a+b-2) p_a p_b \frac{\partial}{\partial p_{a+b-2}} \right) \\ &\quad + 2N \sum a p_{a+2} \frac{\partial}{\partial p_a} + N^2 p_2 + N p_1^2\end{aligned}\quad (43)$$

This is the W -representation operator for the Gaussian Hermitian matrix model.

The operators from the positive branch are given by:

$$\begin{aligned}\hat{F}_1 &= - \sum_b (b+1) p_b \frac{\partial}{\partial p_{b+1}} - N \frac{\partial}{\partial p_1} \\ \hat{F}_2 &= \sum_{a,b} \left(p_a p_b (a+b+1) \frac{\partial}{\partial p_{a+b+1}} + ab p_{a+b-1} \frac{\partial^2}{\partial p_a \partial p_b} \right) \\ &\quad + 2N \sum (b+1) p_b \frac{\partial}{\partial p_{b+1}} + N^2 \frac{\partial}{\partial p_1}\end{aligned}\quad (44)$$

Appendix B: Partition functions from action on the Schur functions

Action of the operators described in Appendix A on the Schur functions looks as follows. The $\hat{W}_0(N)$ -operator acts diagonally:

$$\hat{W}_0(N) S_R = \left(\sum_{(i,j) \in R} (j-i+N) \right) S_R \quad (45)$$

From this, one easily finds:

$$\begin{aligned}p_1 S_R &= \sum_{R+\square} S_{R+\square} \\ \hat{E}_1 S_R &= \sum_{R+\square} (j_\square - i_\square + N) S_{R+\square} \\ \hat{E}_2 S_R &= \sum_{R+\square} (j_\square - i_\square + N)^2 S_{R+\square}\end{aligned}\quad (46)$$

where the sum goes over all possible ways to add a single box to the partition R . The next series of operators of degree 2 and of increasing spin would go as:

$$\begin{aligned}p_2 S_R &= [\hat{E}_1, p_1] S_R \\ &= \sum_{R+\square_1+\square_2} \langle p_2 S_R | S_{R+\square_1+\square_2} \rangle S_{R+\square_1+\square_2} \\ \hat{W}_{-2} S_R &= [\hat{E}_2, \hat{E}_1] S_R = \sum_{R+\square_1+\square_2} \langle p_2 S_R | S_{R+\square_1+\square_2} \rangle \\ &\quad \times (j_{\square_1} - i_{\square_1} + N)(j_{\square_2} - i_{\square_2} + N) S_{R+\square_1+\square_2} \\ \hat{W}_{-2}^{(2)} S_R &= [\hat{E}_3, \hat{E}_2] S_R = \sum_{R+\square_1+\square_2} \langle p_2 S_R | S_{R+\square_1+\square_2} \rangle\end{aligned}$$

$$\begin{aligned}&\times (j_{\square_1} - i_{\square_1} + N)^2 (j_{\square_2} - i_{\square_2} + N)^2 \\ &\times S_{R+\square_1+\square_2}\end{aligned}\quad (47)$$

The coefficient $\langle p_2 S_R | S_{R+\square_1+\square_2} \rangle$ is in fact quite simple, and is given by:

$$\langle p_2 S_R | S_{R+\square_1+\square_2} \rangle = \begin{cases} 1 & j_{\square_2} = j_{\square_1}, i_{\square_2} = i_{\square_1} + 1 \\ -1 & j_{\square_2} = j_{\square_1} + 1, i_{\square_2} = i_{\square_1} \end{cases} \quad (48)$$

and vanishes otherwise.

Similar formulas hold for the lowering operators:

$$\begin{aligned}\frac{\partial}{\partial p_1} S_R &= \sum_{R-\square} S_{R-\square} \\ \hat{F}_1 S_R &= \sum_{R-\square} (j_\square - i_\square + N) S_{R-\square} \\ \hat{F}_2 S_R &= \sum_{R-\square} (j_\square - i_\square + N)^2 S_{R-\square}\end{aligned}\quad (49)$$

Note that the spin grading of a given operator corresponds to the power of $(j-i)$ in terms of its action on the Schur functions.

Appendix C: Obtaining (2) and (10) from the W -representations

Now let us briefly recollect how to obtain the Schur function expansion for partition functions (8) and (11) from (9) and (12) respectively. This is just a straightforward calculation. First, consider the negative branch, and expand the exponential in the W -representation (11):

$$\begin{aligned}Z_f(\bar{p}, p=0, g) &= \exp \left(\sum_{k=1} \frac{\bar{p}_m \hat{W}_{-m}^f[g]}{m} \right) \cdot 1 \\ &= \sum_{\Delta} \frac{\bar{p}_{\Delta}}{z_{\Delta}} \hat{W}_{-\Delta}^f[g] \cdot 1 \\ &= \sum_{\Delta} \frac{\bar{p}_{\Delta}}{z_{\Delta}} \sum_R \left(\prod_{(i,j) \in R} f(j-i) \right) \langle g_{\Delta} \cdot 1 | S_R \rangle S_R\{g\} = \\ &= \sum_R \prod_{(i,j) \in R} f(j-i) S_R\{g\} \sum_{\Delta} \frac{\bar{p}_{\Delta}}{z_{\Delta}} \langle g_{\Delta} \cdot 1 | S_R \rangle \\ &= \sum_R \prod_{i,j \in R} f(j-i) S_R\{\bar{p}\} S_R\{g\}\end{aligned}\quad (50)$$

where we used formulas (1) and the notation

$$\hat{W}_{-\Delta}^f[g] = \prod_{i=1}^{l(\Delta)} \hat{W}_{-\Delta_i}^f[g] \quad (51)$$

For the positive branch, one has:

$$\begin{aligned}
 Z_f(\bar{p}, p, g) &= \exp\left(\sum_{k=1} \frac{\bar{p}_m \hat{W}_m^f[p]}{m}\right) \cdot \exp\left(\sum_{k=1} \frac{p_k g_k}{k}\right) \\
 &= \sum_{\Delta} \frac{\bar{p}_{\Delta}}{z_{\Delta}} \hat{W}_{\Delta}^f[g] \cdot \sum_R S_R\{p\} S_R\{g\} \\
 &= \sum_R S_R\{g\} \sum_{R'} \left(\prod_{(i,j) \in R/Q} f(j-i) \right) S_Q(p) \\
 &\quad \times \sum_{\Delta} \frac{\bar{p}_{\Delta}}{z_{\Delta}} \left\langle \frac{\partial S_R}{\partial p_{\Delta}} \middle| S_Q \right\rangle \\
 &= \sum_{R, Q} \prod_{i, j \in R/Q} f(j-i) S_{R/Q}\{\bar{p}\} S_R\{g\} S_Q\{p\}
 \end{aligned} \tag{52}$$

where we again used formulas (1) and $\frac{\partial}{\partial p_{\Delta}} = \prod_{i=1}^{l(\Delta)} \Delta_i \frac{\partial}{\partial p_{\Delta_i}}$
 $\hat{W}_{\Delta}^f[p] = \prod_{i=1}^{l(\Delta)} \hat{W}_{\Delta_i}^f[p].$

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