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Zero modes of fermions trapped by giant vortices

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Abstract: Zero-energy solutions of the Dirac equation for the fermions bound to giant vortices of large winding number n are studied in the abelian Higgs and Chern-Simons Higgs models. The case of Jackiw-Rossi theory of the Majorana states in topological superconductors is discussed in detail. By expanding in inverse powers of n we find an analytic result for asymptotically all n solutions required by the index theorem. In the abelian Higgs model the zero modes fill the vortex core and reveal a universal structure independent of fine details of the gauge and scalar field interactions which, in particular, determines the general properties of the large-n superconducting cosmic strings. On the contrary, for the Chern-Simons Higgs vortices the zero modes are localized on the core boundary and the explicit solution is obtained for the supersymmetric couplings in a self-dual background.

Keywords: Solitons Monopoles and Instantons, Topological States of Matter

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1 Introduction

The zero-energy bound states of Dirac fermions [1] in the field of a vortex [2, 3] emerge in many physical problems ranging from superconducting cosmic strings in various extensions of the standard model [4] to the effective Majorana states in topological superconductors [5] and Chern-Simons effective field theory of the fractional quantum Hall effect [6]. These *zero modes* have been extensively studied in various gauge models supporting the vortex solutions [7–10]. Their existence is often predicted by the index theorem [11] and can be verified by a qualitative analysis of the corresponding linear equations. Such an approach, however, is too coarse to catch subtle dynamical effects which significantly affect the structure of the solutions, while the brute-force numerical simulations [12, 13] may be insufficient to identify their universal properties and characteristic features. At the same time the absence of an analytic solution of the system of nonlinear equations for the background vortex field has prevented finding an explicit analytic form of the zero modes in the past. However, the structure of the *giant* vortices carrying a large winding number n drastically simplifies in the limit $n \to \infty$ [14, 15]. In some cases the vortex equations become integrable and the analytic solution can be found by a systematic expansion in the inverse powers of the winding number about the asymptotic result which converges remarkably well down to very low n [16]. The resulting background field can be used to solve the corresponding Dirac equation explicitly. This may be of particular interest since the giant vortices are observed experimentally in a variety of quantum condensed matter systems [17–20] and zero-energy states can be experimentally detected [21]. Recently such an analysis of the renowned Jackiw-Rossi model [1] has been presented in ref. [22]. The model is widely used for the description of the zero-energy self-conjugate quasiparticle excitations, known as Majorana zero modes, bound to the vortices in topological superconductors or heterostructures [23–25]. In this paper we give a detailed account of the analysis and extend it to the other classes of gauge models. In section 2 we discuss the abelian Higgs theory with two different types of the fermion coupling to the scalar field: the one motivated by the $N = 2$ supersymmetric

QED in $2+1$ dimensions and the one by Jackiw and Rossi. In section 3 we apply our formalism to the $N=2$ supersymmetric Chern-Simons theory.

2 Abelian Higgs model

Let us start with a brief review of the giant vortex solution [15, 16]. We consider the standard Lagrangian for the abelian Higgs (Ginzburg-Landau) model of a scalar field of gauge charge e, quartic self-coupling λ, and vacuum expectation value η in $2+1$ dimensions

$$L_H = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D^\mu\phi)^\dagger D_\mu\phi - \frac{\lambda}{2}\left(|\phi|^2-\eta^2\right)^2 , \tag{2.1}$$

where $D_\mu = \partial_\mu + ieA_\mu$. It is convenient to introduce the rescaled dimensionless quantities

$$e\eta r \to r, \quad \lambda/e^2 \to \lambda, \quad \phi/\eta \to \phi, \quad A_\mu/\eta \to A_\mu\,, \tag{2.2}$$

so that in the new variables $e=\eta=1$. Vortices are static finite-energy topologically nontrivial solutions of the corresponding equations of motion. The axially symmetric solutions of winding number n in polar coordinates can be written as follows $\phi(r,\theta) = f(r)e^{in\theta}$, $A_\theta = -na(r)$, $A_r = 0$, $f(0) = a(0) = 0$ and $f(\infty) = a(\infty) = 1$. For a given winding number the solution carries n quanta of magnetic flux $\Phi = -\int F_{12}\mathrm{d}^2\boldsymbol{r} = 2\pi n$. In the limit $n\to\infty$ the vortices evolve into the *thin-wall flux tubes* of energy $T = 2\pi\sqrt{\lambda}n\eta^2$ with the nonlinear dynamics confined to a finite-depth boundary layer wrapping the vortex core of radius $r_n = \sqrt{2n}/\lambda^{1/4}$. Outside the boundary layer the dynamics of the gauge and scalar fields linearizes up to the corrections exponentially suppressed for large n. In the vortex core $r < r_n$ it is determined solely by the gauge interaction giving the universal solution

$$\begin{aligned} f(r) &= F\exp\left[\frac{n}{2}\left(\ln\left(\frac{r^2}{r_n^2}\right) - \frac{r^2}{r_n^2} + 1\right)\right], \\ a(r) &= \frac{r^2}{r_n^2}\,, \end{aligned} \tag{2.3}$$

where F is an integration constant and $a(r)$ corresponds to a homogeneous magnetic field. At $r > r_n$ the scalar and gauge fields exponentially approach their vacuum values

$$\begin{aligned} f(r) &\sim 1 + \frac{\nu}{2\pi}K_0(\sqrt{2\lambda}r) + \dots\,, \\ a(r) &\sim 1 + \frac{\mu}{2\pi n}\sqrt{2}rK_1(\sqrt{2}r) + \dots\,, \end{aligned} \tag{2.4}$$

where $K_m(z)$ is the mth modified Bessel function, and the vortex scalar charge ν and magnetic dipole moment μ read

$$\nu \sim -e^{2\sqrt{n}\,\lambda^{1/4}+\dots}\,, \qquad \mu \sim -e^{2\sqrt{n}/\lambda^{1/4}+\dots}\,. \tag{2.5}$$

Inside the boundary layer the asymptotic vortex solution does not depend on the winding number and gets the corrections in powers of $1/\sqrt{n}$ [16]. Throughout the paper we

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consistently use the universal aspects of the leading order result and neglect the model-dependent corrections. The system of the second order non-linear boundary layer equations becomes integrable in the limits of large, critical, and small scalar self-coupling. For the critical coupling $\lambda = 1$ corresponding to the $N = 2$ supersymmetric QED [26] the order of the equations reduces [27] and the *self-dual* solution can be found analytically.[1] For the functions

$$\begin{aligned} w(x) &= \ln f(r_n + x)\,, \\ \gamma(x) &= n\,(a(r_n + x) - 1)\,/r_n\,, \end{aligned} \tag{2.6}$$

it reads

$$\begin{aligned} \int_{w_0}^{w(x)} \frac{\mathrm{d}w}{(e^{2w} - 2w - 1)^{1/2}} &= x\,, \\ \gamma(x) &= -(e^{2w(x)} - 2w(x) - 1)^{\frac{1}{2}}\,, \end{aligned} \tag{2.7}$$

where

$$w_0 = -0.2997174398\ldots\,, \tag{2.8}$$

with the following asymptotic behavior at $x \to \infty$

$$\begin{aligned} w(-x) &\sim -\frac{x^2}{2} - \frac{1}{2}\,, \\ w(x) &\sim w_0 \exp\left[\int_{w_0}^{0} \left(\frac{\sqrt{2}}{(e^{2w} - 2w - 1)^{1/2}} + \frac{1}{w}\right) \mathrm{d}w\right] e^{-\sqrt{2}x}\,. \end{aligned} \tag{2.9}$$

In the limit $1 \ll \lambda \ll n^2$ it becomes

$$\begin{aligned} f(r_n + x) &= \begin{cases} \left(1 - 2\,\mathrm{sech}^2\left(\sqrt{2}(x + 1 - x_0)\right)\right)^{\frac{1}{2}}\,, & x \geq -1\,, \\ 0\,, & x < -1\,, \end{cases} \\ \gamma(x) &= -\sqrt{2\lambda}\,\mathrm{sech}\left(\sqrt{2}(x + 1 - x_0)\right)\,, \end{aligned} \tag{2.10}$$

where $x_0 = -\mathrm{arcsinh}(1)/\sqrt{2}$. The solution in the opposite limit $1/n^2 \ll \lambda \ll 1$ is discussed in detail in [16]. As we discuss in the next section in the abelian Higgs model only a finite number i.e. a vanishing fraction of the total number of zero modes are affected by the nonlinear dynamics of the boundary layer, hence the absence of the general solution is not crucial.

Yukawa interaction of the massless Dirac fermions to the scalar field is mandatory for the existence of the normalizable zero modes in the vortex background [1]. Below we consider two basic types of the Yukawa interaction: the fermion number preserving coupling inspired by the $N = 2$ supersymmetry and the fermion number violating coupling describing the quasiparticle interaction to the Cooper pair condensate in topological superconductors.

[1] For $N = 2$ supersymmetric models with nonvanishing superpotential and flat directions of the scalar potential [28] the structure of the vortex solution is significantly different [29] and is not discussed in the present paper.

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2.1 Supersymmetry motivated fermion coupling

The fermion number preserving Yukawa interaction to the charged scalar field requires a pair of charged and neutral fermions ψ and χ with the Lagrangian density [26]

$$L_F = i\bar{\psi}\not{D}\psi + i\bar{\chi}\not{\partial}\chi + i\sqrt{2}g\left(\bar{\psi}\chi\phi - \bar{\chi}\psi\phi^*\right), \tag{2.11}$$

where the three-dimensional Dirac matrices reduce to the Pauli matrices $\gamma^\mu = (\sigma_3, i\sigma_2, -i\sigma_1)$, and the gauge charges of the fermion and the scalar fields are equal to e. As before we introduce the rescaled dimensionless quantities

$$g/e \to g, \quad \psi/\sqrt{e\eta} \to \psi, \quad \chi/\sqrt{e\eta} \to \chi\,. \tag{2.12}$$

Then for $\lambda = g = 1$ the Lagrangian eqs. (2.1), (2.11) is invariant under the $N = 2$ supersymmetry transformations with ψ (χ) being a part of the chiral (vector) supermultiplet. For arbitrary g the static zero-energy modes are determined by the following equations for the spinor components

$$\begin{aligned} D_+\psi^+ - \sqrt{2}g\phi\chi^- &= 0\,,\\ \partial_-\chi^- - \sqrt{2}g\phi^*\psi^+ &= 0\,, \end{aligned} \tag{2.13}$$

$$\begin{aligned} D_-\psi^- + \sqrt{2}g\phi\chi^+ &= 0\,,\\ \partial_+\chi^+ + \sqrt{2}g\phi^*\psi^- &= 0\,, \end{aligned} \tag{2.14}$$

where the chiral derivative is

$$D_\pm = D_1 \pm iD_2 = e^{\pm i\theta}\left[\partial_r \pm \left(i\partial_\theta + na(r)\right)/r\right]. \tag{2.15}$$

By the index theorem [11] for a nonsingular scalar and gauge field configuration of a winding number n (not necessarily a solution of the field equations) the difference in the number of the normalizable solutions of eqs. (2.13) and (2.14) is equal to n. At the same time the analysis of the regular solutions to the above linear equations at $r \to 0$ and $r \to \infty$ along the lines of ref. [1] shows that eq. (2.13) and eq. (2.14) have exactly $|n|$ normalizable complex solutions for positive and negative values of n, respectively. Below we consider the case $n > 0$. Then by the decomposition

$$\begin{aligned} \psi^+ &= \sum_{l=0}^{n-1} \frac{e^{il\theta}}{\sqrt{2\pi}}\psi_l^+\,,\\ \chi^- &= \sum_{l=-n+1}^{0} \frac{e^{il\theta}}{\sqrt{2\pi}}\chi_l^-\,, \end{aligned} \tag{2.16}$$

eq. (2.13) can by transformed into the equation for the partial waves

$$\begin{aligned} \left(\frac{d}{dr} - \frac{l}{r} + \frac{na}{r}\right)\psi_l^+ - \sqrt{2}gf\chi_{-n+l+1}^- &= 0\,,\\ \left(\frac{d}{dr} - \frac{n-l-1}{r}\right)\chi_{-n+l+1}^- - \sqrt{2}gf\psi_l^+ &= 0\,. \end{aligned} \tag{2.17}$$

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Outside the core for $r > r_n$ the vortex fields approach the vacuum values $a(r),\ f(r) \approx 1$ and eq. (2.17) describes free fermions of the mass $\sqrt{2}|g|$. Its normalizable solution exponentially decays as $\psi_l^+,\ \chi_{-n+l+1}^- \propto e^{-\sqrt{2}|g|r}$ so that the zero modes are localized in the vortex core or on its boundary. Inside the core the function f is exponentially suppressed at large n. Neglecting the Yukawa term in eq. (2.17) we find $\chi_{-n+l+1}^- \propto r^{n-l-1}$ exponentially suppressed too, except the partial waves with $n - l = \mathcal{O}(1)$ which will be discussed separately. Thus, the neutral component of the zero mode asymptotically vanishes and the equation for the remaining charged component at $r < r_n$ becomes

$$\left(\frac{d}{dr} - \frac{l}{r} + \frac{\lambda^{1/2}}{2} r\right) \psi_l^+ = 0 \tag{2.18}$$

with the solution

$$\psi_l^+(r) \sim N_{l,2}\, r^l e^{-\lambda^{1/2} r^2/4}\,, \tag{2.19}$$

where

$$N_{l,c} = \left(\frac{2\lambda^{\frac{l+1}{2}}}{c^{l+1} l!}\right)^{\frac{1}{2}} \tag{2.20}$$

is the normalization factor. Eq. (2.19) describes the approximately Gaussian peaks of width $\sigma = 1/\lambda^{1/4}$ centered at r_l

$$\psi_l^+(r) \approx N'_{l,2}\, e^{-\lambda^{1/2}(r-r_l)^2/2}\,, \tag{2.21}$$

where the normalization factor is

$$N'_{l,c} = \left(\frac{\lambda}{c^2 \pi^2 l}\right)^{\frac{1}{4}}\,. \tag{2.22}$$

This solution has a clear physical interpretation — the charged fermion forms the lowest Landau level states in the homogeneous magnetic field of the giant vortex core with each state encircling an integer number of the flux quanta. However, a few states with $l \approx n$ are localized near the core boundary where the nonlinear effects become crucial and the approximation breaks down. In this case the closed form analytic solution is not available but for $n \to \infty$ the fraction of the zero modes localized at the boundary vanishes. Moreover, for $\lambda = g = 1$ the $l = n - 1$ mode can be obtained by a supersymmetry transformation. Indeed, the vortex solution breaks a half of the $N = 2$ supersymmetries. The unbroken symmetry imposes the first order Bogomolny equations on the bosonic fields (see e.g. [28])

$$\begin{aligned} D_+\phi &= 0\,, \\ F_{12} - \left(|\phi|^2 - 1\right) &= 0\,, \end{aligned} \tag{2.23}$$

and the broken symmetry transformation of the vortex solution gives the fermionic zero mode [10]

$$\begin{aligned} \psi^+ &= D_-\phi\,, \\ \chi^- &= \frac{1}{\sqrt{2}}\left[F_{12} + \left(|\phi|^2 - 1\right)\right]\,. \end{aligned} \tag{2.24}$$

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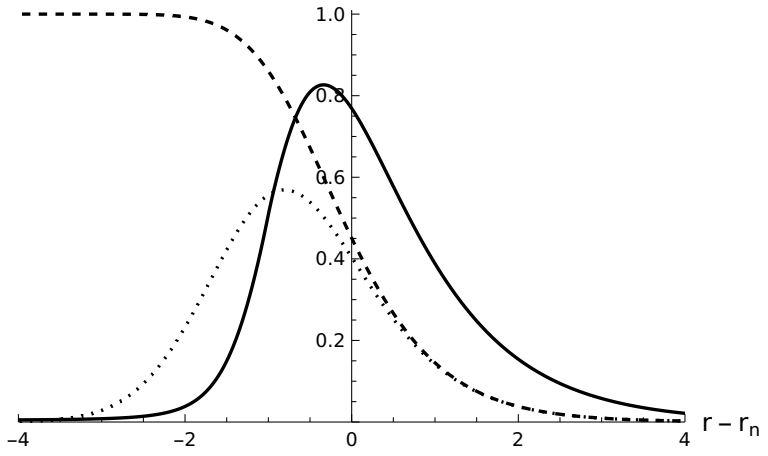


Figure 1. The supersymmetric zero modes $\sqrt{n}\psi^+_{n-1}(r)$ (dotted line) and $-\sqrt{n}\chi^-_0(r)$ (dashed line) of the abelian Higgs model, eq. (2.25). The normalized boundary layer zero mode $\sqrt{r_n}\psi^+_{n/2}(r)$ (solid line) of the Jackiw-Rossi model for $\lambda = 16$ and $g = 1$, eq. (2.39).

By substituting the critical vortex solution of the previous section in the boundary layer region $r - r_n = \mathcal{O}(1)$ we get the normalized supersymmetric zero mode

$$\begin{aligned} \psi^+_{n-1} &= \frac{e^w(e^{2w} - 1 - 2w)^{\frac{1}{2}}}{\sqrt{n/2}}\,, \\ \chi^-_0 &= \frac{e^{2w} - 1}{\sqrt{n}}\,, \end{aligned} \tag{2.25}$$

where $w(r - r_n)$ is given by eq. (2.7). In eq. (2.25) the result for the neutral component χ^-_0 matches the nonvanishing constant core solution, and the shape of the charged component is a non-Gaussian peak at $r = r_n$ which decays as $e^{-(r-r_n)^2/2}$ for $r < r_n$ and only as $e^{-\sqrt{2}(r-r_n)}$ for $r > r_n$. This gives a qualitative picture of the boundary layer zero modes for nonsupersymmetric couplings $g,\ \lambda = \mathcal{O}(1)$ as well. The shape of the supersymmetric solution is shown in figure 1.

The $\lambda = 1$ case is of particular interest since for critical coupling any vortex solution of a winding number n has $|n|$ complex bosonic zero modes [30], which define the vortex moduli space and are related to the fermionic modes by the $N = 2$ supersymmetry [9]. For the axially symmetric giant vortex background the bosonic zero modes in the core have been found explicitly in ref. [31] and agree with eq. (2.19). The bosonic zero mode localized on the vortex boundary reads [16]

$$\begin{aligned} \delta f &= w' e^w\,, \\ \delta a &= -\sqrt{\frac{2}{n}}\, w''\,, \end{aligned} \tag{2.26}$$

and up to an overall normalization agrees with eq. (2.25).

The Lagrangian eq. (2.11) can be used to construct the models of superconducting cosmic strings [4]. In a minimal model ψ and χ are associated with the four-dimensional

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chiral fermions of a charge q and $-q$ with respect to the unbroken electromagnetic gauge group. To make the model anomaly free one needs the second pair of fermions $\tilde{\psi}$ and $\tilde{\chi}$, with the broken gauge symmetry charge of $\tilde{\psi}$ equal to $-e$. Then in the presence of an external electric field the zero modes eq. (2.19) develop a supercurrent along the string and the sum of the partial wave contributions results in a constant radial current density at $n \to \infty$. In general, more fermion pairs of different gauge charges can be accommodated into the model [4], which may result in a different spatial distribution of the current density inside the vortex core (see the next section for an example).

2.2 Jackiw-Rossi fermion coupling

The Lagrangian density of the Jackiw-Rossi model reads [1]

$$L_F = i\bar{\psi}\not{D}\psi + \frac{g}{2}\left(\bar{\psi}\psi^c\phi + \bar{\psi}^c\psi\phi^*\right) , \tag{2.27}$$

where $\psi^c = -i\sigma_1\psi^*$ is the charge conjugate spinor and the gauge invariance requires the charge of the fermion to be a half of the scalar field charge. The zero-mode equation now takes the form

$$D_\pm\psi^\pm + g\phi\psi^{*\pm} = 0\,. \tag{2.28}$$

The calculation of the index for eq. (2.28) gives a half of its value for the model discussed in section 2.1, which implies the existence of $|n|$ self-conjugate zero modes [11]. For positive n the negative chirality equation does not have a normalizable solution. The n zero modes of positive chirality do not have definite angular momentum and are given by the linear combinations of the partial waves

$$\begin{aligned} \xi_l^+ &= \frac{1}{2\sqrt{\pi}}\left(e^{il\theta}\psi_l^+ + e^{i(n-1-l)\theta}\psi_{n-1-l}^+\right) , \\ \eta_l^+ &= \frac{i}{2\sqrt{\pi}}\left(e^{il\theta}\psi_l^+ - e^{i(n-1-l)\theta}\psi_{n-1-l}^+\right) , \end{aligned} \tag{2.29}$$

where $0 \le l \le n/2-1$ for even n and $0 \le l \le (n-1)/2$, $\eta_{(n-1)/2}^+ = 0$ for odd n. The equations for the partial waves read

$$\begin{aligned} \left(\frac{d}{dr} - \frac{l}{r} + \frac{na}{2r}\right)\psi_l^+ + gf\psi_{n-1-l}^+ = 0\,, \\ \left(\frac{d}{dr} - \frac{n-l-1}{r} + \frac{na}{2r}\right)\psi_{n-1-l}^+ + gf\psi_l^+ = 0\,, \end{aligned} \tag{2.30}$$

and differ from eq. (2.17) by the gauge field term. After identification of ψ_l and ψ_{n-1-l} with the components of the Nambu spinor, and of f with the pair potential the above system reproduces the Bogoliubov-de-Gennes equations for the Majorana vortex zero modes of the effective Dirac fermion at zero chemical potential in the condensed matter systems (see e.g. [23]). It is convenient to decouple the gauge field by a field redefinition $\psi_l^+(r) = u_l(r)G(r)$, where the gauge factor

$$G(r) = \exp\left(-\frac{n}{2}\int_0^r \frac{a(r')}{r'}\mathrm{d}r'\right) \tag{2.31}$$

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at large n evaluates to

$$G(r) \sim \begin{cases} e^{-\lambda^{\frac{1}{2}} r^2/8}\,, & r < r_n\,, \\ (r_n/r)^{\frac{n}{2}} e^{-\lambda^{\frac{1}{2}} r_n^2/8}\,, & r > r_n\,. \end{cases} \tag{2.32}$$

Then the system eq. (2.30) can be transformed into the second order equation

$$\left[\frac{d^2}{dr^2} - \left(\frac{n-1}{r} + \frac{f'}{f}\right)\frac{d}{dr} + \frac{l}{r}\left(\frac{n-l}{r} + \frac{f'}{f}\right) - g^2 f^2\right] u_l = 0\,, \tag{2.33}$$

where $f' = df/dr$. Outside the core $f \approx 1$, $f' \approx 0$, and its normalizable solution is $u_l(r) = r^{n/2} K_\mu(|g|r)$, $\mu = \sqrt{n^2/4 - l(n-l)}$, which gives

$$\psi_l^+(r) \propto K_\mu(|g|r)\,. \tag{2.34}$$

Though inside the vortex core the scalar field is exponentially suppressed, it is a singular perturbation since the order of the system eq. (2.30) for vanishing f is reduced. It cannot be in general neglected to get two solutions regular at $r = 0$ which are required to match eq. (2.34) for $r \to \infty$. For the model discussed in the previous section the regular solution missing for vanishing f is exponentially suppressed. However, in the case under consideration both solutions are relevant. Indeed for $r < r_n$ the logarithmic derivative term $f'/f = n/r\,(1 - r^2/r_n^2)$ in eq. (2.33) is not suppressed and the two regular solutions are

$$\begin{aligned} u_l^{(1)}(r) &= r^l\,, \\ u_l^{(2)}(r) &= r^{2n-l} \mathrm{E}_\nu\left(nr^2/2r_n^2\right)\,, \end{aligned} \tag{2.35}$$

where $\mathrm{E}_\nu(z)$ is the νth exponential integral with $\nu = 1 + l - n$. The behavior of the second solution at large n is quite peculiar. For $l < n/2$ it reduces to $u_l^{(2)}(r) \sim r^l$ i.e. the two solutions are degenerate up to the exponentially suppressed terms. For $l > n/2$, however, it transforms into $u_l^{(2)}(r) \sim r^{2n-l} e^{-nr^2/2r_n^2}$. For such l the first solution is exponentially suppressed and is the only solution which gives an unsuppressed contribution to $\psi_l^+(r)$ is $u_l^{(2)}$. For the partial waves in the large-n limit we finally get

$$\psi_l^+(r) \sim \begin{cases} N_{l,4}\, r^l e^{-\lambda^{\frac{1}{2}} r^2/8}\,, & l < n/2\,, \\ N_{2n-l,\frac{4}{3}}\, r^{2n-l} e^{-3\lambda^{\frac{1}{2}} r^2/8}\,, & l > n/2\,. \end{cases} \tag{2.36}$$

Eq. (2.36) describes two groups of approximately Gaussian peaks

$$\psi_l^+(r) \approx \begin{cases} N'_{l,4}\, e^{-\lambda^{\frac{1}{2}}(r-\bar{r}_l)^2/4}\,, & l < n/2\,, \\ N'_{2n-l,\frac{4}{3}}\, e^{-3\lambda^{\frac{1}{2}}(r-\bar{r}'_l)^2/4}\,, & l > n/2\,. \end{cases} \tag{2.37}$$

For $l < n/2$ the peaks of the width $\sigma = \sqrt{2}/\lambda^{1/4}$ are centered at $\bar{r}_l = \sqrt{2l/n}\, r_n$, while for $l > n/2$ the peaks have the width $\sigma' = \sigma/\sqrt{3}$ and are centered at $\bar{r}'_l = \sqrt{(2/3)(2 - l/n)}\, r_n$. The physical origin of the second group of states is as follows. Due to $e/2$ charge each of

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the lowest Landau level states now encircles an even number of the flux quanta. Hence, only about $n/2$ of the Landau states fit into the vortex core and are not affected by the vortex scalar field. For larger l the effect of the scalar field on the localization of the states increases, as it can be seen in eq. (2.33), and for $l \approx n$ it exceeds the effect of the magnetic field, resulting in a set of *warped* lowest Landau level states. Remarkably such a significant effect is achieved in the region where the scalar field is exponentially small, which can be attributed to the singular character of the $f \to 0$ limit of the zero-mode equations discussed above. The solutions with $l \approx n/2$ are localized inside the boundary layer where the nonlinear effects are essential and an explicit analytical solution is not available. At the same time for known functions $f(r)$ and $a(r)$ the solution in the region $r - r_n = \mathcal{O}(1)$ is given by

$$\psi^+_{n/2}(r) \propto r^{\frac{n}{2}} \exp\left[-\int^r \left(\frac{na(r')}{2r'} + |g| f(r')\right) \mathrm{d}r'\right], \tag{2.38}$$

where n is assumed to be even. Note that the asymptotic solution eq. (2.38) in the limit $n \to \infty$ does not depend on n. For example, in the integrable case $\lambda \gg 1$ we get

$$\psi^+_{n/2}(r_n + x) \propto \exp\left\{\int_0^{x+1} \left[\sqrt{\lambda/2}\,\operatorname{sech}\left(\sqrt{2}(x' - x_0)\right) - |g|\left(1 - 2\operatorname{sech}^2\left(\sqrt{2}(x' - x_0)\right)\right)^{\frac{1}{2}}\right] \mathrm{d}x'\right\}. \tag{2.39}$$

The shape of the solution eq. (2.39) for some typical values of the coupling constants is shown in figure 1. At the same time, the radial density of the zero modes $\rho(r) = \sum_{l=0}^{n-1} \left(\psi^+_l(r)\right)^2$, which is the primary object in the condensed matter applications, for large n is not sensitive to the nonuniversal contribution of the $l \approx n/2$ states and in the $n \to \infty$ limit it converges to a piecewise constant function

$$\rho(r) \sim \sqrt{\lambda} \begin{cases} 1/2\,, & r/r_n < \sqrt{2/3}\,, \\ 2\,, & \sqrt{2/3} < r/r_n < 1\,, \end{cases} \tag{2.40}$$

with $\int_0^\infty \rho(r) r \mathrm{d}r = n$. The application of the results of this section to the condensed matter systems is discussed in detail in ref. [22].

3 Supersymmetric Chern-Simons model

We consider the $N = 2$ supersymmetric model with the Lagrangian density [32]

$$L_{CS} = \frac{\kappa}{4}\epsilon^{\mu\nu\lambda} F_{\mu\nu} A_\lambda + \left(D^\mu \phi\right)^\dagger D_\mu \phi - \frac{e^4}{\kappa^2} |\phi|^2 \left(|\phi|^2 - \eta^2\right)^2 + i\bar{\psi} \not{D} \psi + \frac{e^2}{\kappa}\left(3|\phi|^2 - \eta^2\right)\bar{\psi}\psi\,, \tag{3.1}$$

where the field equations can be solved for the nondynamical scalar potential

$$A_0 = -\frac{\kappa}{2e^2\eta^2} \frac{F_{12}}{|\phi|^2}\,. \tag{3.2}$$

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This theory supports the self-dual vortex solutions [33] and the corresponding first-order equations read

$$\begin{aligned} D_+\phi &= 0\,, \\ F_{12} - \frac{2e^3}{\kappa^2}\,|\phi|^2\left(|\phi|^2-\eta^2\right) &= 0\,. \end{aligned} \tag{3.3}$$

For the axially symmetric vortex configuration with the winding number n they reduce to

$$\begin{aligned} \frac{df}{dr} - \frac{n}{r}\,(1-a)\,f &= 0\,, \\ \frac{da}{dr} + \frac{r}{n}\frac{2}{\kappa^2}f^2(f^2-1) &= 0\,, \end{aligned} \tag{3.4}$$

where the rescaled dimensionless parameter $\eta\kappa/e \to \kappa$ is introduced along with the dimensionless fields and couplings defined in the previous sections. To get the large-n asymptotic solution of eq. (3.4) we follow the approach [15]. In contrast to the Abrikosov giant vortices the boundary layer develops at $r_n = n\kappa$. Inside the core the dynamics of the scalar field linearizes and eq. (3.4) with the exponential accuracy can be approximated as follows

$$\begin{aligned} \frac{df}{dr} - \frac{n}{r}f &= 0\,, \\ \frac{da}{dr} - \frac{r}{n}\frac{2}{\kappa^2}f^2 &= 0\,, \end{aligned} \tag{3.5}$$

where the leading nonvanishing term is left in the second line. The solution of eq. (3.5) is

$$\begin{aligned} f(r) &\sim (r/r_n)^n\,, \\ a(r) &\sim (r/r_n)^{2n}\,, \end{aligned} \tag{3.6}$$

where the integration constant is found by matching to the boundary layer solution. To get the asymptotic result inside the boundary layer for $r - r_n = \mathcal{O}(1)$ one neglects the difference between r and r_n in the coefficients of eq. (3.4), which transforms into

$$\begin{aligned} w'' - 2e^{4w} + 2e^{2w} &= 0\,, \\ a(r_n + x) &= 1 - w'(x/\kappa)\,, \end{aligned} \tag{3.7}$$

where $w(x/\kappa) = \ln(f(r_n + x))$. The first line of eq. (3.7) has a first integral $I = w'^2 - e^{4w} + 2e^{2w} = 1$ and can be easily integrated with proper boundary conditions at $x = \pm\infty$ yielding

$$\begin{aligned} f(r_n + x) &= \left(1 + e^{-2x/\kappa}\right)^{-1/2}\,, \\ a(r_n + x) &= \left(1 + e^{-2x/\kappa}\right)^{-1}\,, \end{aligned} \tag{3.8}$$

which coincides with eq. (3.6) in the matching region $1 \ll |x| \ll r_n$ for negative x, where both approximations are valid. Eq. (3.8) agrees with the result of ref. [34]. The $\mathcal{O}(1/n)$ corrections to this result can be obtained within the approach [16] but are not discussed here. Outside the core the solution of the linearized equations is similar to eq. (2.4)

$$\begin{aligned} f(r) &\sim 1 + \frac{\nu}{2\pi}K_0(2r/\kappa)\,, \\ a(r) &\sim 1 + \frac{\mu}{2\pi}\frac{2r}{\kappa n}K_1(2r/\kappa)\,. \end{aligned} \tag{3.9}$$

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The values of the vortex magnetic moment and scalar charge are obtained by matching eq. (3.9) to eq. (3.8) in the second matching region $1 \ll x \ll r_n$ with the result

$$\mu = \nu = -2\sqrt{\pi} e^{2n+\ln(n)/2} \,. \tag{3.10}$$

Besides the core radius, the main difference between the giant vortices in the abelian Higgs and Chern-Simons Higgs models is in the magnetic flux distribution. In the latter the magnetic flux is confined to the boundary layer rather than inside the vortex core. This results in a qualitatively different structure and localization of the fermion zero modes as they cannot be trapped inside the core by the magnetic field. The index calculation predicts the existence of $|n|$ complex fermionic zero modes and their bosonic superpartners [9]. The zero-mode equations for the Lagrangian eq. (3.1) are

$$\begin{aligned} D_+\psi^+ + i\frac{2}{\kappa}|\phi|^2\psi^- &= 0\,, \\ D_-\psi^- + i\frac{2}{\kappa}\left(1-2|\phi|^2\right)\psi^+ &= 0\,. \end{aligned} \tag{3.11}$$

In terms of the partial waves

$$\begin{aligned} \psi^+ &= \sum_{l=0}^{n-1} \frac{e^{il\theta}}{\sqrt{2\pi}}\psi_l^+\,, \\ \psi^- &= \sum_{l=1}^{n} \frac{e^{il\theta}}{\sqrt{2\pi}}\psi_l^-\,, \end{aligned} \tag{3.12}$$

they read

$$\begin{aligned} \left(\frac{d}{dr} - \frac{l}{r} + \frac{na}{r}\right)\psi_l^+ + i\frac{2}{\kappa}|\phi|^2\psi_{l+1}^- &= 0\,, \\ \left(\frac{d}{dr} + \frac{l+1}{r} - \frac{na}{r}\right)\psi_{l+1}^- + i\frac{2}{\kappa}\left(1-2|\phi|^2\right)\psi_l^+ &= 0\,. \end{aligned} \tag{3.13}$$

To get the large-n asymptotic result for the zero modes localized inside the boundary layer we can approximate r by r_n in the coefficients of eq. (3.13), which converts into the system

$$\begin{aligned} \kappa^2\psi_l^{+\prime\prime} - 2\kappa\left(1-f^2\right)\psi_l^{+\prime} + \left(1-\alpha^2+4f^2-9f^4\right)\psi_l^+ &= 0\,, \\ \psi_{l+1}^- &= -i\psi_l^+\,, \end{aligned} \tag{3.14}$$

where $\alpha = 1 - l/n$, and we use the relations $\kappa f'/f = (1-f^2)$, $a = f^2$ following from the boundary layer equations eq. (3.7). In eq. (3.14) we consider $\alpha \neq 1$ and neglect the terms suppressed by $1/l$ and $1/n$ in the $n \to \infty$ limit, where $0 \leq \alpha < 1$ becomes a continuous parameter. For a finite number of the partial waves with $l/n \approx 0$, the zero modes extend beyond the boundary layer and the above approximation breaks down. The fraction of such solutions, however, vanishes when $n \to \infty$. After a change of the variables

$$\psi_l^+ = N(\alpha)\frac{z^{1/2}}{(1+z)^{3/2}}g(z)\,, \tag{3.15}$$

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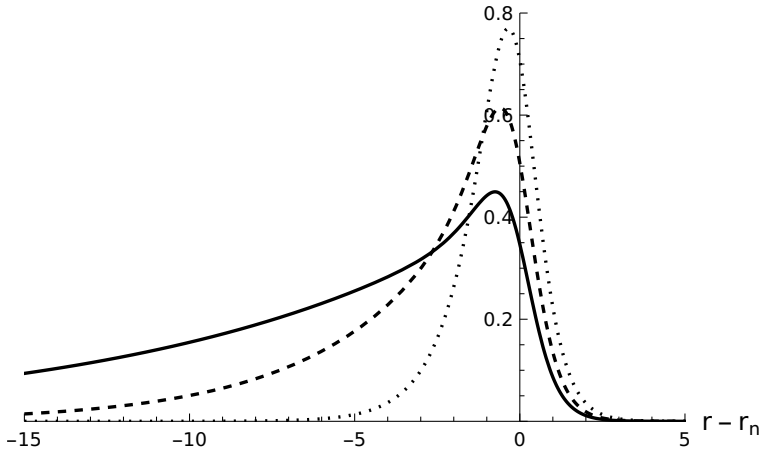


Figure 2. The normalized zero modes $\sqrt{2r_n}\psi_l^+(r)$ of the Chern-Simons model, eq. (3.15), for $\kappa = 1$ and $\alpha = 0$ (dotted line), $\alpha = 0.75$ (dashed line), $\alpha = 0.9$ (solid line).

where $z = e^{2(r-r_n)/\kappa}$ and $N(\alpha)$ is the normalization factor, we get the following equation on $g(z)$

$$\frac{d^2g}{dz^2} + \frac{1-z}{z(1+z)}\frac{dg}{dz} - \frac{\alpha^2}{4z^2}g = 0\,. \tag{3.16}$$

Its normalizable solution is given by a linear combination of the hypergeometric functions

$$g(z) = \frac{1}{2\sqrt{1-\alpha}}\Bigg[z^{\alpha/2}{}_2F_1\left(-1-(\sqrt{4+\alpha^2}-\alpha)/2, -1+(\sqrt{4+\alpha^2}+\alpha)/2; 1+\alpha; -z\right) \\ -\frac{D(\alpha)}{D(-\alpha)}z^{-\alpha/2}{}_2F_1\left(-1-(\sqrt{4+\alpha^2}+\alpha)/2, -1+(\sqrt{4+\alpha^2}-\alpha)/2; 1-\alpha; -z\right)\Bigg], \tag{3.17}$$

where

$$D(\alpha) = \Gamma(1+\alpha)\,\Gamma\left(-1+(\sqrt{4+\alpha^2}-\alpha)/2\right)\Gamma\left(2+(\sqrt{4+\alpha^2}-\alpha)/2\right)\,. \tag{3.18}$$

The normalization factor cannot be computed analytically for general α. We have found that it monotonically increases with increasing α from $N(0) = \sqrt{2}/(\kappa\sqrt{n})$ to $\lim_{\alpha\to 1} N(\alpha) = \sqrt{2}N(0)$. The shape of the solution eq. (3.15) for some typical values of κ and α is shown in figure 2. Let us discuss the general properties of the solution. For $\alpha = 0$ it reduces to $g(z) = 1$ and up to normalization describes the zero modes

$$\begin{aligned}\psi_{n-1}^+ &= D_-\phi\,,\\ \psi_n^- &= i\frac{2}{\kappa}\left(|\phi|^2-1\right)\phi\,,\end{aligned} \tag{3.19}$$

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resulting from the supersymmetry transformation of the vortex solution [9]. For $\alpha \neq 0$ the asymptotic behavior of the solution reads

$$
\begin{aligned}
g(z) &\sim C_\infty(\alpha) z^{1-\sqrt{1+(\alpha/2)^2}}\,, \qquad && z \to \infty\,,\\
g(z) &\sim -\frac{1}{2\sqrt{1-\alpha}}\frac{D(\alpha)}{D(-\alpha)} z^{-\alpha/2}\,, && z \to 0\,,
\end{aligned}
\tag{3.20}
$$

where the constant $C_\infty(\alpha)$ has a lengthy analytic expression which is not required for our analysis. As it follows from eqs. (3.15), (3.20) for $1-\alpha = \mathcal{O}(1/n)$, which corresponds to $l = \mathcal{O}(1)$, the zero mode solution extends to $x \approx -r_n$, i.e. outside the region where the boundary layer approximation can be used. For such partial waves the result eq. (3.15) is not valid as it was pointed out above.

4 Summary

Thus, we have studied the zero-energy solutions of the Dirac equation in the giant vortex background in the limit of large winding number n. For the critical scalar coupling corresponding to the models with $N = 2$ extended supersymmetry the explicit asymptotic solutions of vortex equations are available for both abelian Higgs and Chern-Simons Higgs gauge theories. In this case the analytic result has been derived for almost all of n zero modes required by the index theorem with the exception of only a finite number of modes with the angular momentum $l/n \approx 1$ for the abelian Higgs and $l/n \approx 0$ for the Chern-Simons case. We have found that the zero modes fill the core of the Abrikosov vortices but are localized on the vortex boundary in the Chern-Simons model. The characteristic thin-wall flux tube structure of the giant vortices in the abelian Higgs model allows for the analytic calculation of the zero modes even for the nonsupersymmetric couplings. The general structure of the solution is quite robust and independent of fine details of the gauge and scalar field interactions. Thus our analysis gives the answer to the question posed in ref. [4] about the structure of the superconducting cosmic strings with $n > 1$ for the case of large n: such strings become the thick thin-wall flux tubes with the radius growing as $\sqrt{n}$ and filled with the superconducting current density. For the most phenomenologically interesting case of the Jackiw-Rossi fermion coupling describing the Majorana states in topological superconductors, the zero modes are formed through a fine interplay between the effects of the magnetic and scalar fields resulting in a set of the warped lowest Landau level states. The remarkable profile of the resulting density of states eq. (2.40) can be used as a smoking gun signature for experimental observation of the (effective) Majorana fermions in two dimensions.

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