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Virasoro symmetries of the constrained dispersionless mKP hierarchy

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ABSTRACT

This paper aims at the auto-Bäcklund transformations for the dispersionless mKP hierarchy, dispersionless Miura map between the constrained dispersionless mKP hierarchy and the constrained dispersionless Harry Dym hierarchy and Virasoro symmetries of the constrained dispersionless mKP hierarchy. The additional symmetries of the constrained dispersionless mKP hierarchy are constructed in a manner similar to that of the dispersionless mKP hierarchy. By comparison, a new Laurent series $\mathcal{Y}_l$ is introduced. The additional flows form a subalgebra of the Virasoro algebra.

1. Introduction

The Kadomtsev-Petviashvili (KP) hierarchy is one of the most important integrable hierarchies and it appears in many different fields of mathematics and physics, such as the enumerative algebraic geometry, topological field and string theory [1]. The KP hierarchy has two important symmetries, they are the Sato's Bäcklund symmetry generated by vertex operator and the additional symmetries by Orlov-Schulman operator [2]. The KP hierarchy has many generalizations and extensions including universal character hierarchy [3–8]. The mKP hierarchy is a modified KP hierarchy. As one of the most important research objects in mathematical physics and integrable systems, the mKP hierarchy is of great significance [9]. Dispersionless integrable hierarchies (KP, Toda etc.) are important systems in integrable field theories, especially applied in mathematics, physics, topological field theory and matrix model theory [10]. We introduce the Planck constant ϵ into the ordinary KP hierarchy, and the Planck constant ϵ is restricted to approach 0. The order of microdifferential operators with ϵ are defined by

$$ord^{\epsilon}(\Sigma a_{l,j}(t)\epsilon^{l}\partial^{j}) = max\{j-l|a_{l,j}(t)\neq 0\}.$$

In particular, $ord^{\epsilon}(\epsilon) = -1$, $ord^{\epsilon}(\partial) = 1$. The principal symbol of a microdifferential operator $A = \Sigma a_{l,j}(t)\epsilon^{n}\partial^{m}$ is

$$\sigma^{\epsilon}(A) = \epsilon^{-ord^{\epsilon}(A)} \sum_{j-l=ord^{\epsilon}(A)} a_{l,j}k^{j},$$

and Lie bracket has become Poisson bracket.

Gauge transformation is one kind of powerful methods to construct the solutions of the integrable systems, it has significant significance and is widely used [11–17]. For soliton theory, the Miura map has played a significant role in its development [18–20]. The Miura map is generally not easy to solve and is a transformation between two nonlinear evolution equations [21,22]. However, the solution of one nonlinear system can be obtained from the solution of the other nonlinear system through the Miura map. Bäcklund transformations can be used to study the soliton hierarchies, which is a very effective tool. On the basis of appropriate gauge transformation, using commonly used dressing method can obtain the $(2+1)$-dimensional transformations [23]. In [19], the Miura map between the dispersionless KP hierarchy and dispersionless modified KP hierarchy is given. This article has constructed

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$$\mathcal{L}_{dmKP} = e^{-ad\phi}\mathcal{L}_{dKP},$$

which can also be used for the Miura map between the constrained dispersionless KP hierarchy and constrained dispersionless modified KP hierarchy and the dKP hierarchy with self-consistent sources to the dmKP hierarchy with self-consistent sources [23]. In [20], the Miura map between the dDym hierarchy and the dmKP hierarchy is given. Its structure can also be used for the Miura map between the constrained dispersionless Harry Dym hierarchy and constrained dispersionless modified KP hierarchy [22]. The constrained dispersionless dmKP hierarchy can be thought of as the quasi-classical limit of the constrained mKP hierarchy [24].

Additional symmetries are important symmetries that explicitly depend on time and space [25–27]. Additional symmetries do not commute with each other and the additional symmetry flows of the KP hierarchy can form an infinite-dimensional algebra $W_{1+\infty}$. Additional symmetry also has wide applications [28,29]. In [30], the additional symmetries of the dispersionless cKP hierarchy are given by introducing a new Laurent series $\mathcal{Y}_l$. Additional symmetry can be used to derive string equations and Virasoro constraints in string theory [31,32].

This paper is organized in the following way. In Section 2, some basic knowledge will be introduced. We introduce the mKP hierarchy, the constrained mKP hierarchy, the dmKP hierarchy, the constraints of dispersionless mKP hierarchy, the Harry Dym hierarchy, the dispersionless Harry Dym hierarchy and the constrained dispersionless Harry Dym hierarchy. In Section 3, we give the auto-Bäcklund transformations for dmKP hierarchy. In Section 4, we give dispersionless Miura map between the constrained dispersionless mKP hierarchy and the constrained dispersionless Harry Dym hierarchy. In Section 5, the additional symmetries of the constrained dispersionless mKP hierarchy are given. In Section 6, the conclusions and discussions are given.

2. Background materials

In section 2, we introduce some basic knowledge, the mKP hierarchy, the constrained mKP hierarchy, the dmKP hierarchy, the constrained dispersionless mKP hierarchy, the Harry Dym hierarchy, the dispersionless Harry Dym hierarchy and the constrained dispersionless Harry Dym hierarchy, which we will use them later on.

The Lax operator of the mKP hierarchy is as

$$L_{mKP} = \partial + v_0 + v_1\partial^{-1} + v_2\partial^{-2} + \ldots,$$

where $\partial = \partial_x$, $v_i = v_i(x = t_1, t_2, \cdots)$.

And it satisfies the Lax equations

$$\partial_{t_k} L_{mKP} = [Q_k, L_{mKP}], \tag{2.1}$$

where $Q_k = (L^k_{mKP})_{\geq 1}$ means the part of order ≥ 1 of L^k_{mKP}.

The Lax equation (2.1) is equivalent to the existence of wave function $\Psi(t, \lambda)$ such that $(t = (t_1 = x, t_2, \cdots))$

$$L_{mKP}\Psi = \lambda\Psi,$$
$$\partial_{t_k}\Psi = Q_k\Psi,$$

where λ is the eigenvalue of L_{mKP}.

Let $p = \sum_{i=-\infty}^{m} p_i\partial^i$ be an arbitrary pseudo-differential operators, $p^* = \sum_{i=-\infty}^{m}(-1)^i\partial^i p_i$ is its formal adjoint operator, and $(AB)^* = B^*A^*$ for pseudo-differential operators A and B. The adjoint wave function satisfies the equations

$$L^*_{mKP}\Psi^* = \lambda\Psi^*,$$
$$\partial_{t_k}\Psi^* = -Q^*_k\Psi^*.$$

The Orlov-Schulman operator of the mKP hierarchy is defined by

$$M = \sum_{n=1}^{\infty} nt_n L^{n-1}_{mKP} + \sum_{i=1}^{\infty} v_i L^{-i-1}_{mKP}. \tag{2.2}$$

The constrained mKP hierarchy is

$$\partial_{t_k} L^n = [Q_k, L^n],$$

the Lax operator L^n has the following relationship,

$$L^n = (L^n)_{\geq 1} + \sum_{i=1}^{N} v_i(t)\partial^{-1} r_i(t)\partial, \tag{2.3}$$

where $v_i(t)$, $r_i(t)$ and Q_k satisfy

$$v_{i,t_k} = Q_k(v_i),$$
$$r_{i,t_k} = -(\partial^{-1}Q^*_k\partial)(r_i), i = 1, 2, 3, \ldots, N,$$
$$Q_k = [((L^n)_{\geq 1} + \sum_{i=1}^{N} v_i(t)\partial^{-1} r_i(t)\partial)^{k/n}]_{\geq 1}.$$

The Lax equation of the dispersionless mKP hierarchy is defined by ($\mathcal{L}_{dmKP} = \lambda$)

$$\mathcal{L}_{dmKP} = P + V_0 + V_1 P^{-1} + V_2 P^{-2} + \cdots, \tag{2.4}$$

$\mathcal{L}_{dmKP}$ satisfies the Lax equations

$$\partial_{T_n} \mathcal{L}_{dmKP} = \{\mathcal{Q}_n, \mathcal{L}_{dmKP}\}, \tag{2.5}$$

where $\mathcal{Q}_n(P)$ is defined by

$$\mathcal{Q}_n = (\mathcal{L}^n_{dmKP})_{\geq 1}.$$

The Poisson bracket $\{,\}$ is defined by

$$\{f(X,P), g(X,P)\} = \frac{\partial f}{\partial P}\frac{\partial g}{\partial X} - \frac{\partial f}{\partial X}\frac{\partial g}{\partial P}. \tag{2.6}$$

The dmKP hierarchy can be also written in the zero-curvature form

$$\frac{\partial \mathcal{Q}_n(P)}{\partial T_m} - \frac{\partial \mathcal{Q}_m(P)}{\partial T_n} + \{\mathcal{Q}_n(P), \mathcal{Q}_m(P)\} = 0.$$

We consider the quasiclassical limit of the constrained mKP hierarchy to obtain the constrained dispersionless mKP hierarchy, by taking $t_n \longrightarrow \epsilon t_n = T_n$ [24],

$$L^n_\epsilon = Q_{\epsilon,n} + \sum_{i=1}^{N} v_i(\frac{T}{\epsilon})(\epsilon\partial)^{-1} r_i(\frac{T}{\epsilon})\epsilon\partial,$$
$$Q_{\epsilon,n} = (L^n_\epsilon)_{\geq 1},$$

where $v_i(\frac{T}{\epsilon}), r_i(\frac{T}{\epsilon})$ and $Q_{\epsilon,k}$ satisfy

$$\epsilon(v_i(\frac{T}{\epsilon}))_{T_k} = Q_{\epsilon,k}(v_i(\frac{T}{\epsilon})),$$
$$\epsilon(r_i(\frac{T}{\epsilon}))_{T_k} = -((\epsilon\partial)^{-1} Q^*_{\epsilon,k}(\epsilon\partial))(r_i(\frac{T}{\epsilon})), i = 1, 2, 3, \ldots, N,$$
$$Q_{\epsilon,k} = ((L^n_\epsilon)^{k/n})_{\geq 1}.$$

It can be proved that $v_i(\frac{T}{\epsilon})$ and $r_i(\frac{T}{\epsilon})$ have the following WKB asymptotic expansion as $\epsilon \longrightarrow 0$,

$$v_i(\frac{T}{\epsilon}) \sim \exp[\frac{S(T,\lambda_i)}{\epsilon} + a_{i1} + O(\epsilon)],$$
$$r_i(\frac{T}{\epsilon}) \sim \exp[-\frac{S(T,\lambda_i)}{\epsilon} + a_{i2} + O(\epsilon)].$$

By calculation,

$$v_i(\frac{T}{\epsilon})(\epsilon\partial)^{-1} r_i(\frac{T}{\epsilon})(\epsilon\partial) = -\frac{e^{a_{i1}+a_{i2}}}{S(T,\lambda_i)_X} \times (1 + (S(T,\lambda_i)_X + O(\epsilon))(\epsilon\partial)^{-1} + \ldots$$
$$+ ((S(T,\lambda_i)_X)^n + O(\epsilon))(\epsilon\partial)^{-n} + \ldots).$$

We can obtain the constrained dispersionless mKP hierarchy for $\mathcal{L}$ as

$$\mathcal{L}^n = \mathcal{Q}_n - \sum_{i=1}^{N} (\frac{a_i}{P_i} + \frac{a_i}{P - P_i}), \tag{2.7}$$

$$\mathcal{L} = \{\mathcal{Q}_n, \mathcal{L}\}, \tag{2.8}$$

where $\mathcal{Q}_n = (\mathcal{L}^n)_{\geq 1}$, a_i, P_i satisfy

$$a_i = e^{a_{i1}+a_{i2}}, \tag{2.9}$$

$$P_i = S(T,\lambda_i)_X, \tag{2.10}$$

$$\frac{\partial a_i}{\partial T_n} = [a_i(\frac{\partial \mathcal{Q}_n}{\partial P})|_{P=P_i}]_X, \tag{2.11}$$

$$\frac{\partial P_i}{\partial T_n} = [\mathcal{Q}_n|_{P=P_i}]_X. \tag{2.12}$$

When $n = 1, N = 1$,

$$\mathcal{L} = P - (\frac{a_1}{P_1} + \frac{a_1}{P - P_1}), \tag{2.13}$$

we can get its Taylor expansion

$$\mathcal{L} = P - a_1 \sum_{i=0}^{+\infty} P_1^{i-1} P^{-i}. \tag{2.14}$$

As we all know, the symmetry constraint for Harry Dym hierarchy is given by

$$L^n_{cDym} = B_n + \sum_{i=1}^{N} q_i \partial^{-1} u_i \partial^2,$$

where q_i, u_i and B_n satisfy

$$\frac{\partial q_i}{\partial t_n} = B_n(q_i),$$
$$\frac{\partial u_i}{\partial t_n} = -\partial^{-2} B_n^* \partial^2(u_i), i = 1, 2, 3, \ldots, N,$$
$$B_n = (L^n_{cDym})_{\geq 2},$$

where B_n^* is the adjoint operator of B_n.

The dispersionless Harry Dym hierarchy is given by

$$\mathcal{L}_{dDym} = U_1 P + U_0 + \frac{U_{-1}}{P} + \frac{U_{-2}}{P^2} + \ldots,$$

it satisfies the Lax equations

$$\partial_{T_k} \mathcal{L}_{dDym} = \{\mathcal{B}_k(P), \mathcal{L}_{dDym}\},$$

where

$$\mathcal{B}_k(P) = (\mathcal{L}^k_{dDym})_{\geq 2}.$$

Using the same method, consider the quasiclassical limit of the constrained Harry Dym hierarchy, we can obtain the constrained dispersionless Harry Dym hierarchy [20],

$$\mathfrak{L}^n = \mathcal{B}_n + \sum_{i=1}^{N} \left(\frac{\alpha_i P}{\beta_i^2} + \frac{\alpha_i}{\beta_i} + \frac{\alpha_i}{P - \beta_i}\right), \tag{2.15}$$

$$\mathfrak{L}_{T_n} = \{\mathcal{B}_n, \mathfrak{L}\}, \tag{2.16}$$

where $\mathcal{B}_n = (\mathfrak{L}^n)_{\geq 2}$, α_i, β_i satisfy

$$\alpha_i = e^{\alpha_{i1} + \alpha_{i2}}, \tag{2.17}$$

$$\beta_i = S(T, \lambda_i)_X, \tag{2.18}$$

$$\frac{\partial \alpha_i}{\partial T_n} = [\alpha_i (\frac{\partial \mathcal{B}_n}{\partial P})|_{P=P_i}]_X, \tag{2.19}$$

$$\frac{\partial \beta_i}{\partial T_n} = [\mathcal{B}_n|_{P=P_i}]_X. \tag{2.20}$$

3. Auto-Bäcklund transformations for dmKPH

In section 3, we introduce the auto-Bäcklund transformations for dmKP hierarchy by using the method in [19].

We have

$$\lambda = P^m + a_{m-1} P^{m-1} + a_{m-2} P^{m-2} + \ldots + a_1 P + a_0 + a_{-1} P^{-1} + a_{-2} P^{-2} + \ldots,$$

where $a_{m-1}, a_{m-2}, \ldots, a_1, a_0, a_{-1}, a_{-2}, \ldots$ are functions of $T = (T_1 = x, T_2, T_3, \ldots)$. $\phi(T)$ is arbitrary function of T, and it is independent of P.

We define

$$\tilde{\lambda} = e^{-ad\phi} \lambda = \lambda - \{\phi, \lambda\} + \frac{1}{2}\{\phi, \{\phi, \lambda\}\} - \frac{1}{3}\{\phi, \{\phi, \{\phi, \lambda\}\}\} + \ldots,$$

where the Poisson bracket is defined by (2.6). By calculation, we can obtain

$$\tilde{\lambda} = \sum_{n=0}^{\infty} \frac{1}{n!} (\phi_x)^n \partial_P^n \lambda.$$

Lemma 3.1. *The following identities hold true [19]*

$$e^{-ad\phi}(\mu_1\mu_2) = e^{-ad\phi}\mu_1 \cdot e^{-ad\phi}\mu_2,$$
$$\begin{aligned}\tilde{\mu}_{\geq 1} &= e^{-ad\phi}(\mu_{\geq 0}) - \mu_{\geq 0}|_{P=\phi_x}\\ &= \mu_{\geq 0} + \phi_x\partial_p(\mu_{\geq 0}) + \tfrac{1}{2!}\phi_x^2\partial_p^2(\mu_{\geq 0}) + \ldots\\ &\quad - (\phi_x^m + a_{m-1}\phi_x^{m-1} + \ldots + a_1\phi_x + a_0)\\ &= e^{-ad\phi}(\mu_{\geq 1}) - \mu_{\geq 1}|_{P=\phi_x},\end{aligned}$$
$$\begin{aligned}\tilde{\mu}_{T_q} - \{(\tilde{\mu}^n)_{\geq 1}, \tilde{\mu}\} &= e^{-ad\phi}(\mu_{T_q} - \{(\mu^n)_{\geq 0}, \mu\}) - \{\phi_{T_q} - (\mu^n)_{\geq 0}|_{P=\phi_x}, \tilde{\mu}\}\\ &= e^{-ad\phi}(\mu_{T_q} - \{(\mu^n)_{\geq 1}, \mu\}) - \{\phi_{T_q} - (\mu^n)_{\geq 1}|_{P=\phi_x}, \tilde{\mu}\}.\end{aligned}$$

Let

$$\mu = P + V_0 + V_1P^{-1} + V_2P^{-2} + \ldots,$$

we suppose that $V_i(T)$ satisfy the dmKP hierarchy in (2.5), and $\phi(T)$ satisfies the following equation,

$$\phi_{T_q} = (\mu^n)_{\geq 1}|_{P=\phi_x}.$$

Then, we have $\tilde{\mu} = e^{-ad\phi}\mu$ also satisfy the dmKP hierarchy.

$$\tilde{\mu} = P + \tilde{V}_0 + \tilde{V}_1P^{-1} + \tilde{V}_2P^{-2} + \ldots,$$

we can get $\tilde{V}_i = V_i$, it is the auto-Bäcklund transformations for the dispersionless mKP hierarchy.

4. Dispersionless Miura map between the constrained dispersionless mKP hierarchy and the constrained dispersionless Harry Dym hierarchy

In section 4, we introduce the dispersionless Miura map between the constrained dispersionless mKP hierarchy and the constrained dispersionless Harry Dym hierarchy by using the method in [20].

Lemma 4.1. *Let [20]*

$$x' = \psi, T'_n = T_n, \mathfrak{L}(x', T') = \mathcal{L}(x, T), P' = \psi_x^{-1}P.$$

Noting that,

$$\frac{\partial}{\partial T_n} = \frac{\partial T'_n}{\partial T_n}\frac{\partial}{\partial T'_n} + \frac{\partial x'}{\partial T_n}\frac{\partial}{\partial x'} = \frac{\partial}{\partial T'_n} + \frac{\partial \psi}{\partial T_n}\frac{\partial}{\partial x'},$$
$$A'_{\geq 2} = A'_{\geq 1} - A'_{[1]} = A'_{\geq 1} - \{A'_{\geq 1}, x'\}_{[0]}P' = A_{\geq 1} - \{\mathcal{A}_{\geq 1}, \psi\}_{[0]}P',$$

we have

$$\mathfrak{L}_{T'_n} = \frac{\partial \mathcal{L}}{\partial T_n} - \frac{\partial \psi}{\partial T_n}\frac{\partial \mathcal{L}}{\partial x'} = \mathcal{L}_{T_n} - \{\frac{\partial \psi}{\partial T_n}P', \mathfrak{L}\},$$
$$\mathcal{B}_n = \mathcal{Q}_n - \{\mathcal{Q}_n, \psi\}_{[0]}P'.$$

We can obtain

$$\begin{aligned}\mathfrak{L}_{T_n} - \{\mathcal{B}_n, \mathfrak{L}\} &= \mathcal{L}_{T_n} - \{\frac{\partial \psi}{\partial T_n}P', \mathfrak{L}\} - \{\mathcal{Q}_n - \{\mathcal{Q}_n, \psi\}_{[0]}P', \mathfrak{L}\}\\ &= \mathcal{L}_{T_n} - \{\mathcal{Q}_n, \mathcal{L}\} - \{\{(\frac{\partial \psi}{\partial T_n} - \{\mathcal{Q}_n, \psi\}_{[0]})P', \mathfrak{L}\}.\end{aligned}$$

Theorem 4.1. *If $\mathcal{L}, a_i, P_i$ satisfy the constrained dispersionless mKP hierarchy (2.14) and ψ is a function of (x, T) satisfying*

$$\frac{\partial \psi}{\partial T_n} = \{\mathcal{Q}_n, \psi\}_{[0]},$$

then $\mathfrak{L}$ satisfy the constrained dispersionless Harry Dym hierarchy (2.15).

Proof. The Lemma (4.1) indicates that $\mathcal{L}(x, T) = \mathfrak{L}(x', T')$. Then, using the same method as in [20], we can indicate that α_i, β_i satisfy the constrained dispersionless Harry Dym hierarchy through these relationships

$$\alpha_i = -\psi_x^{-1}a_i,$$
$$\beta_i = \psi_x^{-1}P_i,$$

which have been confirmed in [22]. □

5. The additional symmetries of the constrained dispersionless mKP hierarchy

In section 5, we introduce the additional symmetries of the constrained dispersionless mKP hierarchy by introducing a new Laurent series $\mathcal{Y}_l$.

The additional symmetries of the dispersionless mKP hierarchy were given by

$$\frac{\partial \mathcal{L}_{dmKP}}{\partial t^*_{l,m}} = -\{(\mathcal{M}^m \mathcal{L}^l_{dmKP})_{<1}, \mathcal{L}_{dmKP}\},$$

or they can be written as

$$\frac{\partial \mathcal{L}_{dmKP}}{\partial t^*_{l,m}} = \{(\mathcal{M}^m \mathcal{L}^l_{dmKP})_{\geq 1}, \mathcal{L}_{dmKP}\} + m\mathcal{M}^{m-1}\mathcal{L}^l_{dmKP}. \tag{5.1}$$

The function $\mathcal{M}$ satisfies the equations

$$\frac{\partial \mathcal{M}}{\partial t^*_{l,m}} = -\{(\mathcal{M}^m \mathcal{L}^l_{dmKP})_{<1}, \mathcal{M}\},$$

where the Orlov-Schulman function of the dispersionless mKP hierarchy is defined by [31]

$$\mathcal{M} = \sum_{n=1}^{+\infty} nT_n \mathcal{L}^{n-1}_{dmKP} + \sum_{i=1}^{+\infty} V_i \mathcal{L}^{-i-1}_{dmKP}, \tag{5.2}$$

we have $\{\mathcal{M}, \mathcal{L}_{dmKP}\} = -1$.

Furthermore, we have

$$\frac{\partial \mathcal{M}^n \mathcal{L}^k_{dmKP}}{\partial t^*_{l,m}} = -\{(\mathcal{M}^m \mathcal{L}^l_{dmKP})_{<1}, \mathcal{M}^n \mathcal{L}^k_{dmKP}\}.$$

Now, we apply the additional symmetric flows defined by (5.1) to $\mathcal{L}$ which is defined by (2.14). When $m = 1$, we have

$$(\frac{\partial \mathcal{L}}{\partial t^*_{l,1}})_{<1} = \{(\mathcal{M}\mathcal{L}^l)_{\geq 1}, \mathcal{L}\}_{<1} + (\mathcal{L}^l)_{<1}. \tag{5.3}$$

Next let us analyze the two terms $\{(\mathcal{M}\mathcal{L}^l)_{\geq 1}, \mathcal{L}\}_{<1}, (\mathcal{L}^l)_{<1}$.

Theorem 5.1. *The Laurent series $\mathcal{L}$ and the Orlov function $\mathcal{M}$ of the constrained dispersionless mKP hierarchy have the following relationship.*

$$\begin{aligned}(\{(\mathcal{M}\mathcal{L}^l)_{\geq 1}, \mathcal{L}\})_{<1} = &-\frac{\partial a_1}{\partial x}\sum_{i=0}^{+\infty}\sum_{n=1}^{+\infty} n(\mathcal{M}\mathcal{L}^l)_{[n]} P_1^{i+n-2} P^{-i} \\ &- a_1 \frac{\partial P_1}{\partial x}\sum_{i=0}^{+\infty}\sum_{n=1}^{+\infty} n(n+i-2)(\mathcal{M}\mathcal{L}^l)_{[n]} P_1^{n+i-3} P^{-i} \\ &- a_1 \sum_{i=0}^{+\infty}\sum_{n=1}^{+\infty}(n+i-1)\frac{\partial (\mathcal{M}\mathcal{L}^l)_{[n]}}{\partial x} P_1^{n+i-2} P^{-i},\end{aligned} \tag{5.4}$$

we define that $(A)_{[i]}$ mean A_i for arbitrary formal Laurent series $A = \sum_i A_i P^{-i}$.

Proof. By using the equation (2.6) and (2.14)

$$\begin{aligned}\{(\mathcal{M}\mathcal{L}^l)_{\geq 1}, \mathcal{L}\}_{<1} &= \frac{\partial (\mathcal{M}\mathcal{L}^l)_{\geq 1}}{\partial P}\frac{\partial \mathcal{L}_{<1}}{\partial X} - \frac{\partial (\mathcal{M}\mathcal{L}^l)_{\geq 1}}{\partial X}\frac{\partial \mathcal{L}_{<1}}{\partial P} \\ &= (\frac{\partial (\mathcal{M}\mathcal{L}^l)_{\geq 1}}{\partial P}(-\frac{\partial a_1}{\partial X}\sum_{i=0}^{+\infty} P_1^{i-1}P^{-i} - a_1\frac{\partial P_1}{\partial X}\sum_{i=0}^{+\infty}(i-1)P_1^{i-2}P^{-i}))_{<1} \\ &\quad - (\frac{\partial (\mathcal{M}\mathcal{L}^l)_{\geq 1}}{\partial X} a_1 \sum_{i=0}^{+\infty} iP_1^{i-1}P^{-i-1})_{<1} \\ &= -\sum_{n=1}^{+\infty}\sum_{i=n-1}^{+\infty} n(\mathcal{M}\mathcal{L}^l)_{[n]}\frac{\partial a_1}{\partial x} P_1^{i-1}P^{-i+n-1} - \sum_{n=1}^{+\infty}\sum_{i=n-1}^{+\infty}(i-1)n(\mathcal{M}\mathcal{L}^l)_{[n]} a_1 \frac{\partial P_1}{\partial x} P_1^{i-2}P^{-i+n-1} \\ &\quad - a_1 \sum_{n=1}^{+\infty}\sum_{i=n-1}^{+\infty}\frac{\partial (\mathcal{M}\mathcal{L}^l)_{[n]}}{\partial x} P_1^{i-1} i P^{-i+n-1}\end{aligned}$$

$$= -\frac{\partial a_1}{\partial x}\sum_{i=0}^{+\infty}\sum_{n=1}^{+\infty} n(\mathcal{M}\mathcal{L}^l)_{[n]} P_1^{i+n-2} P^{-i} - a_1 \frac{\partial P_1}{\partial x}\sum_{i=0}^{+\infty}\sum_{n=1}^{+\infty} n(n+i-2)(\mathcal{M}\mathcal{L}^l)_{[n]} P_1^{n+i-3} P^{-i}$$
$$- a_1 \sum_{i=0}^{+\infty}\sum_{n=1}^{+\infty}(n+i-1)\frac{\partial(\mathcal{M}\mathcal{L}^l)_{[n]}}{\partial x} P_1^{n+i-2} P^{-i}. \quad \square$$

Theorem 5.2. *The Laurent series $\mathcal{L}$ of the constrained dispersionless mKP hierarchy has the following relationship.*

$$(\mathcal{L}^l)_{<1} = \sum_{k=1}^{l} C_l^k \mathcal{L}_{l-k}\mathcal{L}_0^{k-1}, l = 1, 2, 3, \ldots, \tag{5.5}$$
$$(\mathcal{L}^0)_{<1} = 1. \tag{5.6}$$

We define that

$$\mathcal{L}_j = -a_1 \sum_{i=j}^{+\infty} P_1^{i-1} P^{-i+j}.$$

For $l = 0, 1, 2$, the flows can be rewritten as

$$(\frac{\partial \mathcal{L}}{\partial t^*_{l,1}})_{<1} = -\frac{\partial a_1}{\partial t^*_{l,1}}\sum_{i=0}^{+\infty} P_1^{i-1} P^{-i} - a_1 \frac{\partial P_1}{\partial t^*_{l,1}}\sum_{i=0}^{+\infty}(i-1) P_1^{i-2} P^{-i}.$$

When $l = 0$,

$$(\frac{\partial \mathcal{L}}{\partial t^*_{0,1}})_{<1} = -\frac{\partial a_1}{\partial t^*_{0,1}}\sum_{i=0}^{+\infty} P_1^{i-1} P^{-i} - a_1 \frac{\partial P_1}{\partial t^*_{0,1}}\sum_{i=0}^{+\infty}(i-1) P_1^{i-2} P^{-i},$$

using the equation (5.3),

$$(\frac{\partial \mathcal{L}}{\partial t^*_{0,1}})_{<1} = -\frac{\partial a_1}{\partial x}\sum_{i=0}^{+\infty}\sum_{n=1}^{+\infty} n(\mathcal{M}\mathcal{L}^0)_{[n]} P_1^{i+n-2} P^{-i}$$
$$- a_1 \frac{\partial P_1}{\partial x}\sum_{i=0}^{+\infty}\sum_{n=1}^{+\infty} n(n+i-2)(\mathcal{M}\mathcal{L}^0)_{[n]} P_1^{n+i-3} P^{-i}$$
$$- a_1 \sum_{i=0}^{+\infty}\sum_{n=1}^{+\infty}(n+i-1)\frac{\partial(\mathcal{M}\mathcal{L}^0)_{[n]}}{\partial x} P_1^{n+i-2} P^{-i} + 1.$$

When $l = 1$,

$$(\frac{\partial \mathcal{L}}{\partial t^*_{1,1}})_{<1} = -\frac{\partial a_1}{\partial t^*_{1,1}}\sum_{i=0}^{\infty} P_1^{i-1} P^{-i} - a_1 \frac{\partial P_1}{\partial t^*_{1,1}}\sum_{i=0}^{\infty}(i-1) P_1^{i-2} P^{-i},$$

using the equation (5.3),

$$(\frac{\partial \mathcal{L}}{\partial t^*_{1,1}})_{<1} = -\frac{\partial a_1}{\partial x}\sum_{i=0}^{+\infty}\sum_{n=1}^{+\infty} n(\mathcal{M}\mathcal{L}^1)_{[n]} P_1^{i+n-2} P^{-i}$$
$$- a_1 \frac{\partial P_1}{\partial x}\sum_{i=0}^{+\infty}\sum_{n=1}^{+\infty} n(n+i-2)(\mathcal{M}\mathcal{L}^1)_{[n]} P_1^{n+i-3} P^{-i}$$
$$- a_1 \sum_{i=0}^{+\infty}\sum_{n=1}^{+\infty}(n+i-1)\frac{\partial(\mathcal{M}\mathcal{L}^1)_{[n]}}{\partial x} P_1^{n+i-2} P^{-i} + \mathcal{L}_0.$$

When $l = 2$,

$$(\frac{\partial \mathcal{L}}{\partial t^*_{2,1}})_{<1} = -\frac{\partial a_1}{\partial t^*_{2,1}}\sum_{i=0}^{\infty} P_1^{i-1} P^{-i} - a_1 \frac{\partial P_1}{\partial t^*_{2,1}}\sum_{i=0}^{\infty}(i-1) P_1^{i-2} P^{-i},$$

using the equation (5.3),

$$(\frac{\partial \mathcal{L}}{\partial t^*_{2,1}})_{<1} = -\frac{\partial a_1}{\partial x}\sum_{i=0}^{+\infty}\sum_{n=1}^{+\infty} n(\mathcal{M}\mathcal{L}^2)_{[n]}P_1^{i+n-2}P^{-i}$$
$$- a_1\frac{\partial P_1}{\partial x}\sum_{i=0}^{+\infty}\sum_{n=1}^{+\infty} n(n+i-2)(\mathcal{M}\mathcal{L}^2)_{[n]}P_1^{n+i-3}P^{-i}$$
$$- a_1\sum_{i=0}^{+\infty}\sum_{n=1}^{+\infty}(n+i-1)\frac{\partial(\mathcal{M}\mathcal{L}^2)_{[n]}}{\partial x}P_1^{n+i-2}P^{-i} + \mathcal{L}_0^2 + 2\mathcal{L}_1.$$

We can find that the flows do not hold for $l = 0, 1, 2, \ldots$.

Therefore, we construct new flows,

$$(\frac{\partial \mathcal{L}}{\partial t^*_{l,1}})_{<1} = \{-(\mathcal{M}\mathcal{L}^l)_{<1} + \mathcal{Y}_l, \mathcal{L}\}_{<1}, \tag{5.7}$$

where

$$\mathcal{Y}_l = -a_1\sum_{k=1}^{l} C_l^k \sum_{i=l-k}^{+\infty}\frac{P_1^{i-1}P^{-i+l-k+1}}{-i+l-k+1}, l = 1, 2, 3, \ldots, \tag{5.8}$$

$$\mathcal{Y}_0 = \mathcal{M}, \tag{5.9}$$

in which a_1, P_1 satisfy the following equations

$$-\frac{\partial a_1}{\partial x}\sum_{i=0}^{+\infty}P_1^{i-1}P^{-i} - a_1\frac{\partial P_1}{\partial x}\sum_{i=0}^{+\infty}(i-1)P_1^{i-2}P^{-i} + \sum_{k_1=1}^{l}\mathcal{L}_0^{k_1-1} = 0, k_1 = k.$$

We define that

$$\sum_{k=1}^{l}\sum_{i=0}^{+\infty} i^{-1}(\frac{\partial a_1}{\partial t^*_j}P_1^i + a_1\frac{\partial P_1}{\partial t^*_j}(l-k+i)P_1^{i-1}) = -\sum_{k=1}^{l}\sum_{n=1}^{+\infty}\sum_{i=0}^{+\infty}(\frac{\partial a_1}{\partial x}\frac{n(\mathcal{M}\mathcal{L}^j)_{[n]}P_1^{n+i-1}}{-n-i+1}$$
$$+ a_1\frac{\partial P_1}{\partial x}\frac{(l-k+n+i-1)n(\mathcal{M}\mathcal{L}^j)_{[n]}}{-n-i+1}P_1^{n+i-2}$$
$$+ a_1 P_1^{n+i-1}\frac{\partial(\mathcal{M}\mathcal{L}^j)_{[n]}}{\partial x}) + (l-j)a_1\sum_{i=0}^{+\infty} i^{-1}P_1^{j+i-1}.$$

Lemma 5.1. *The Laurent series $\mathcal{L}$ and $\mathcal{Y}_l$ satisfy the following relation,*

$$\{\mathcal{Y}_l, \mathcal{L}\}_{<1} = -(\mathcal{L}^l)_{<1} + a_1\sum_{k=1}^{l}\sum_{m=0}^{+\infty}\sum_{i=0}^{+\infty}\frac{iC_l^k}{-i+1}(\frac{\partial a_1}{\partial x}P_1^{l-k+i+m-2}$$
$$+ a_1\frac{\partial P_1}{\partial x}(l-k+m-1)P_1^{l-k+i+m-3})P^{-i-m}, l = 1, 2, 3, \ldots,$$
$$\{\mathcal{Y}_0, \mathcal{L}\}_{<1} = -1.$$

The Laurent series $\mathcal{Y}_l$ also have the following relationship.

Lemma 5.2. *The formal Laurent series $\mathcal{Y}_l$ satisfy the following relation,*

$$\frac{\partial \mathcal{Y}_l}{\partial t^*_j} = \{(\mathcal{M}\mathcal{L}^j)_{\geq 1}, \mathcal{Y}_l\}_{<1} + (l-j)\mathcal{Y}_{l+j-1}, \tag{5.10}$$

where $j, l = 0, 1, 2, \ldots$.

Proof. The left-hand side of equation (5.10) is

$$\frac{\partial \mathcal{Y}_l}{\partial t^*_j} = -\sum_{k=1}^{l}\sum_{i=l-k}^{+\infty}\frac{C_l^k}{-i+l-k+1}(\frac{\partial a_1}{\partial t^*_j}P_1^{i-1} + a_1\frac{\partial P_1}{\partial t^*_j}(i-1)P_1^{i-2})P^{-i+l-k+1},$$

the first term on the right-hand side of equation (5.10) is

$$\{(\mathcal{M}\mathcal{L}^j)_{\geq 1}, \mathcal{Y}_l\}_{<1} = -\sum_{n=1}^{+\infty}\sum_{k=1}^{l}\sum_{i=l-k+n}^{+\infty} C_l^k(\frac{\partial a_1}{\partial x}\frac{n(\mathcal{M}\mathcal{L}^j)_{[n]}P_1^{i-1}}{-i+l-k+1} + a_1\frac{\partial P_1}{\partial x}\frac{(i-1)n(\mathcal{M}\mathcal{L}^j)_{[n]}}{-i+l-k+1}P_1^{i-2}$$
$$+ a_1 P_1^{i-1}\frac{\partial(\mathcal{M}\mathcal{L}^j)_{[n]}}{\partial x})P^{-i+l-k+n},$$

the second term on the right-hand side of equation (5.10) is

$$(l-j)\mathcal{Y}_{l+j-1} = -(l-j)a_1\sum_{k=1}^{l}\sum_{i=l+j-k-1}^{+\infty}\frac{C_l^k}{-i+l+j-k}P_1^{i-1}P^{-i+l-j-k}.$$

By comparison, we know that the left and right sides of the equation (5.10) are equal. □

Lemma 5.3. *The additional flows $\frac{\partial}{\partial t_l^*}$ commute with the flows $\frac{\partial}{\partial t_n}$ of the constrained dispersionless mKP hierarchy.*

Proof.

$$[\frac{\partial}{\partial t_l^*}, \frac{\partial}{\partial t_n}]\mathcal{L} = \frac{\partial}{\partial t_l^*}\{\mathcal{B}_n, \mathcal{L}\} - \frac{\partial}{\partial t_n}\{-(\mathcal{M}\mathcal{L}^l)_{<1} + \mathcal{Y}_l, \mathcal{L}\}$$
$$= \{\frac{\partial \mathcal{B}_n}{\partial t_l^*}, \mathcal{L}\} + \{\mathcal{B}_n, \frac{\partial \mathcal{L}}{\partial t_l^*}\} - \{\frac{\partial(-\mathcal{M}\mathcal{L}^l + \mathcal{Y}_l)_{<1}}{\partial t_n}, \mathcal{L}\} - \{(-\mathcal{M}\mathcal{L}^l + \mathcal{Y}_l)_{<1}, \frac{\partial \mathcal{L}}{\partial t_n}\}$$
$$= \{\{-\mathcal{M}\mathcal{L}^l + \mathcal{Y}_l)_{<1}, (\mathcal{L}^n)_{\geq 1}\}_{\geq 1}, \mathcal{L}\} + \{(\mathcal{L}^n)_{\geq 1}, \{(-\mathcal{M}\mathcal{L}^l + \mathcal{Y}_l)_{<1}, \mathcal{L}\}\}$$
$$- \{\{(\mathcal{L}^n)_{\geq 1}, (-\mathcal{M}\mathcal{L}^l + \mathcal{Y}_l)_{<1}\}_{<1}, \mathcal{L}\} - \{(-\mathcal{M}\mathcal{L}^l + \mathcal{Y}_l)_{<1}, \{(\mathcal{L}^n)_{\geq 1}, \mathcal{L}\}\}$$
$$= \{\{-\mathcal{M}\mathcal{L}^l + \mathcal{Y}_l)_{<1}, (\mathcal{L}^n)_{\geq 1}\}, \mathcal{L}\} + \{(\mathcal{L}^n)_{\geq 1}, \{(-\mathcal{M}\mathcal{L}^l + \mathcal{Y}_l)_{<1}, \mathcal{L}\}\}$$
$$- \{(-\mathcal{M}\mathcal{L}^l + \mathcal{Y}_l)_{<1}, \{(\mathcal{L}^n)_{\geq 1}, \mathcal{L}\}\}.$$

By using the Jacobi identity of the Poisson bracket $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$, we can get the last equation is 0. □

Lemma 5.4. *The additional flows act on the space of the formal Laurent series $\mathcal{L}$ of the constrained dispersionless mKP hierarchy, forming a subalgebra of the Virasoro algebra.*

Proof.

$$[\frac{\partial}{\partial t_l^*}, \frac{\partial}{\partial t_j^*}]\mathcal{L} = \frac{\partial}{\partial t_l^*}\{(-\mathcal{M}\mathcal{L}^j)_{<1}, \mathcal{L}\} - \frac{\partial}{\partial t_j^*}\{(-\mathcal{M}\mathcal{L}^l + \mathcal{Y}_l)_{<1}, \mathcal{L}\}$$
$$= \{\frac{\partial(-\mathcal{M}\mathcal{L}^j)_{<1}}{\partial t_l^*}, \mathcal{L}\} + \{(-\mathcal{M}\mathcal{L}^j)_{<1}, \frac{\partial \mathcal{L}}{\partial t_l^*}\}$$
$$- \{\frac{\partial(-\mathcal{M}\mathcal{L}^l + \mathcal{Y}_l)_{<1}}{\partial t_j^*}, \mathcal{L}\} - \{(-\mathcal{M}\mathcal{L}^l + \mathcal{Y}_l)_{<1}, \frac{\partial \mathcal{L}}{\partial t_j^*}\}$$
$$= \{\{(-\mathcal{M}\mathcal{L}^l + \mathcal{Y}_l)_{<1}, (-\mathcal{M}\mathcal{L}^j)\}_{<1}, \mathcal{L}\}$$
$$- \{\{(-\mathcal{M}\mathcal{L}^j)_{<1}, (-\mathcal{M}\mathcal{L}^l + \mathcal{Y}_l)\}_{<1}, \mathcal{L}\}$$
$$- \{\{(-\mathcal{M}\mathcal{L}^l + \mathcal{Y}_l)_{<1}, (-\mathcal{M}\mathcal{L}^j)_{<1}\}, \mathcal{L}\}$$
$$= \{\{-\mathcal{M}\mathcal{L}^l + \mathcal{Y}_l, -\mathcal{M}\mathcal{L}^j\}_{<1}, \mathcal{L}\}.$$

By the equation (5.10), we can get

$$\{-\mathcal{M}\mathcal{L}^l + \mathcal{Y}_l, -\mathcal{M}\mathcal{L}^j\}_{<1} = (j-l)(-(\mathcal{M}\mathcal{L}^{j+l-1})_{<1} + \mathcal{Y}_{j+l-1}),$$

it also means that

$$[\frac{\partial}{\partial t_l^*}, \frac{\partial}{\partial t_j^*}] = (j-l)\frac{\partial}{\partial t_{j+l-1}^*},$$

where $j, l \in \mathbb{Z}_{\geq 0}$. □

Theorem 5.3. *The additional flows acting on a_1 and P_1 of the constrained dispersionless mKP hierarchy are given by*

$$\frac{\partial a_1}{\partial t_{l,1}^*} = \frac{\partial a_1}{\partial x}\sum_{n=1}^{+\infty} n(\mathcal{M}\mathcal{L})_{[n]}P_1^{n-1},$$
$$\frac{\partial P_1}{\partial t_{l,1}^*} = -(\frac{\partial P_1}{\partial x}\sum_{n=1}^{+\infty} n(n-2)(\mathcal{M}\mathcal{L})_{[n]}P_1^{n-1} + \sum_{n=1}^{+\infty}(n-1)\frac{\partial(\mathcal{M}\mathcal{L})_{[n]}}{\partial x}P_1^{n}).$$

When $l = 0$,

$$\frac{\partial a_1}{\partial t^*_{0,1}} = \frac{\partial a_1}{\partial x}\sum_{n=1}^{+\infty} n\mathcal{M}_{[n]}P_1^{n-1},$$
$$\frac{\partial P_1}{\partial t^*_{0,1}} = -(\frac{\partial P_1}{\partial x}\sum_{n=1}^{+\infty} n(n-2)\mathcal{M}_{[n]}P_1^{n-1} + \sum_{n=1}^{+\infty}(n-1)\frac{\partial \mathcal{M}_{[n]}}{\partial x}P_1^n).$$

When $l = 1$,

$$\frac{\partial a_1}{\partial t^*_{1,1}} = \frac{\partial a_1}{\partial x}\sum_{n=1}^{+\infty} n(\mathcal{M}\mathcal{L})_{[n]}P_1^{n-1},$$
$$\frac{\partial P_1}{\partial t^*_{1,1}} = -(\frac{\partial P_1}{\partial x}\sum_{n=1}^{+\infty} n(n-2)(\mathcal{M}\mathcal{L})_{[n]}P_1^{n-1} + \sum_{n=1}^{+\infty}(n-1)\frac{\partial (\mathcal{M}\mathcal{L})_{[n]}}{\partial x}P_1^n).$$

These can be proven by the above lemmas and the equation (5.3).

In (2.7), when $n = 1$,

$$\mathcal{L} = P - \sum_{j=1}^{N}(\frac{a_j}{P_j} + \frac{a_j}{P - P_j}),$$

we can get its Taylor expansion

$$\mathcal{L} = P - \sum_{j=1}^{N} a_j \sum_{i=0}^{+\infty} P_j^{i-1}P^{-i}.$$

Now, we can construct the additional symmetries of the constrained dispersionless mKP hierarchy,

$$(\frac{\partial \mathcal{L}}{\partial t^*_{l,1}})_{<1} = \{-(\mathcal{M}\mathcal{L}^l)_{<1} + \mathcal{Y}_l^N, \mathcal{L}\}_{<1},$$

where

$$\mathcal{Y}_0^N = \mathcal{M},$$
$$\mathcal{Y}_l^N = \sum_{j=1}^{N}(-a_j\sum_{k=1}^{l} C_l^k \sum_{i=l-k}^{+\infty}\frac{P_j^{i-1}P^{-i+l-k+1}}{-i+l-k+1}), l = 1, 2, 3, \ldots,$$

in which a_j, P_j satisfy the following equations

$$-\sum_{j=1}^{N}\frac{\partial a_j}{\partial x}\sum_{i=0}^{+\infty}P_j^{i-1}P^{-i} - \sum_{j=1}^{N}a_j\frac{\partial P_j}{\partial x}\sum_{i=0}^{+\infty}(i-1)P_j^{i-2}P^{-i} + \sum_{k_1=1}^{l}\mathcal{L}_{j,0}^{k_1-1} = 0, k_1 = k.$$

We define that

$$\mathcal{L}_{j,k} = \sum_{j=1}^{N}(-a_j\sum_{i=k}^{+\infty}P_j^{i-1}P^{-i+k}).$$

The Laurent series $\mathcal{L}$ of the constrained dispersionless mKP hierarchy has the following relationship,

$$(\mathcal{L}^l)_{<1} = \sum_{k=1}^{l} C_l^k \mathcal{L}_{j,l-k}\mathcal{L}_{j,0}^{k-1}, l = 1, 2, 3, \ldots,$$
$$(\mathcal{L}^0)_{<1} = 1.$$

The additional symmetries of the constrained dispersionless mKP hierarchy are constructed.

6. Conclusions and discussions

In this article, we have discussed the auto-Bäcklund transformations for dmKP hierarchy, its construction method draws inspiration from the article [19]. In section 4, we construct the dispersionless Miura map between the constrained dispersionless mKP hierarchy and the constrained dispersionless Harry Dym hierarchy. In section 5, we construct the additional symmetries of the constrained dispersionless mKP hierarchy by introducing a new Laurent series $\mathcal{Y}_l$ in equation (5.8) and (5.9). We have been indicated that the additional flows commute with the flows of the constrained dispersionless mKP hierarchy. We also have been indicated that the additional flows act on the space of the formal Laurent series $\mathcal{L}$ of the constrained dispersionless mKP hierarchy, forming a subalgebra of the Virasoro algebra. The additional flows acting on a_1 and P_1 of the constrained dispersionless mKP hierarchy are given.

Declaration of competing interest

We declare that we have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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