



On the stability of Yang-Mills vacuum

George Savvidy

Institute of Nuclear and Particle Physics, NCSR Demokritos, GR-15310 Athens, Greece

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ABSTRACT

The (anti)self-dual covariantly constant vacuum fields have positive/stable and infinitely many zero modes, so called Leutwyler chromons. A regularisation method is suggested that allows to sum the contribution of the zero modes and to obtain the effective Lagrangian that has contribution of all positive/stable modes and zero modes. In the second approach we integrated over the collective coordinates of the nonlinearly interacting zero modes and obtained the same result. The last method is a generalisation of the t'Hooft integration over the instanton zero mode collective coordinates. It is important that both methods lead to the same result indicating the robustness of the logarithmic structure of the effective Lagrangian.

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1. Introduction

In the earlier investigations of the covariantly constant vacuum field configurations [1–4] it was realised that consideration of the vacuum polarisation in the quadratic approximation displays an apparent instability of the vacuum fields¹ due to the negative/unstable modes [5–17,19–27]. Here we are considering the (anti)self-dual covariantly constant vacuum fields (25) that have only positive/stable and infinitely many zero modes, so called Leutwyler chromons [11–13]. A regularisation method is suggested that allows to sum the contribution of the zero modes and to obtain the effective Lagrangian that contains the contribution of all positive/stable modes and zero modes (36). In the second approach we integrated over the collective coordinates of the nonlinearly interacting zero modes and obtained the same result (48). The last method is a generalisation of the t'Hooft integration over the instanton zero mode collective coordinates [18]. It is important that the zero modes contribution that was calculated in terms of infrared regularisation of the spectrum (32), (33), (35) and then by integration over collective coordinates of interacting zero modes (45), both lead to the same result indicating the robustness of the logarithmic structure of the effective Lagrangian [1].

2. Euclidean path integral

The amplitudes and the effective Lagrangians [29–31] in Yang-Mills theory [3,4,32] can be analysed in the Euclidean path integral representation [18,28]. The quantum-mechanical physical states are described by the gauge-invariant wave functionals $\psi_n[\vec{A}] = \langle \vec{A} | E_n \rangle$, and the matrix elements of the operator $\exp(-H_{YM}T)$ are represented in the following form [11,12,28]:

$$\begin{aligned} \langle \vec{A}' | e^{-H_{YM}T} | \vec{A} \rangle &= \sum_n e^{-E_n T} \psi_n[\vec{A}'] \psi_n^*[\vec{A}] \\ &= N \int_{\vec{A}}^{\vec{A}'} \mathcal{D}\vec{A}(\tau) e^{-S_E[A(\tau)]} \\ &\equiv N_f e^{-\int d^4x \mathcal{L}_{eff}(\vec{A}', \vec{A})}, \end{aligned} \quad (1)$$

where $H_{YM} \psi_n[\vec{A}] = E_n \psi_n[\vec{A}]$. The left-hand side involves quantities defined in a physical space-time with the Yang-Mills Hamiltonian

$$H_{YM} = \frac{1}{2} \int d^3x \left[(\vec{E}_i^a)^2 + (\vec{H}_i^a)^2 \right]. \quad (2)$$

While the right-hand side of the equation (1) involves an integral over the Euclidean field configurations $A_i^a(\vec{x}, \tau)$ that have the proper boundary values

$$A_i^a(\vec{x}, 0) = A_i^a(\vec{x}), \quad A_i^a(\vec{x}, T) = A_i'^a(\vec{x}). \quad (3)$$

The Euclidean action is given by

$$S_E = \frac{1}{4} \int d^4x G_{\mu\nu}^a G_{\mu\nu}^a, \quad (4)$$

^{E-mail address:} savvidy@inp.demokritos.gr.

¹ Here, and afterwards, the phrase “vacuum fields” refers to the gauge fields that are the solutions of the sourceless Yang-Mills equation and, in particular, the covariantly constant gauge fields [3].

where $x_0 \rightarrow -ix_0 = -i\tau$, $A_0 \rightarrow iA_0$ and $G_{\mu\nu}^a(A) = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$. The real electric field corresponds to an imaginary electric field in Euclidean space $\vec{E} \rightarrow i\vec{E}$. In order to analyse the behaviour of vacuum fields in the vicinity of a given interpolating-vacuum field $B_\mu^a(x)$ that starts at $A_i^a(\vec{x})$ and ends at $A_i^a(\vec{x})$ we will represent the fluctuating field in the form

$$A_\mu^a(x) = B_\mu^a(x) + a_\mu^a(x) \quad (5)$$

and expand the field strength $G_{\mu\nu}^a(A)$ in powers of quantum field $a_\mu^a(x)$ that has the zero boundary values

$$G_{\mu\nu}^a(A) = G_{\mu\nu}^a(B) + \nabla_\mu^{ab} a_\nu^b - \nabla_\nu^{ab} a_\mu^b + gf^{abc} a_\mu^b a_\nu^c, \quad (6)$$

where $\nabla_\mu^{ab} a_\nu^b = \partial_\mu a_\nu^b + gf^{abc} B_\mu^c a_\nu^b$ is a covariant derivative with respect to the vacuum field $B_\mu^a(x)$ and $[\nabla_\mu, \nabla_\nu]^{ab} = gf^{abc} G_{\mu\nu}^c(B)$. For the Euclidean action (4) we will have

$$S_E = \frac{1}{4} \int d^4x G_{\mu\nu}^a(B) G_{\mu\nu}^a(B) + \int d^4x G_{\mu\nu}^a(B) \nabla_\mu^{ab} a_\nu^a + \frac{1}{4} \int d^4x \left((\nabla_\mu^{ab} a_\nu^b - \nabla_\nu^{ab} a_\mu^b + gf^{abc} a_\mu^b a_\nu^c)^2 + 2g G_{\mu\nu}^a(B) f^{abc} a_\mu^b a_\nu^c \right). \quad (7)$$

The vacuum field B_μ^a is supposed to satisfy the Euclidean YM equation:

$$\nabla_\mu G_{\mu\nu}(B) = 0, \quad (8)$$

so that the second term in (7) that is linear in $a_\mu^a(x)$ vanishes and the effective Lagrangian is a gauge invariant functional [3]. The quadratic part of the action has the following form:

$$K_E = \frac{1}{4} \int d^4x \left[(\nabla_\mu^{ab} a_\nu^b - \nabla_\nu^{ab} a_\mu^b)^2 + 2g G_{\mu\nu}^a(B) f^{abc} a_\mu^b a_\nu^c \right]. \quad (9)$$

The positive eigenvalues of the quadratic form provide exponential decay of the amplitude e^{-K_E} and will be referred as *stable*, while the negative eigenvalues produce a divergency of the amplitude in the directions of the corresponding eigenfunctions and will be referred as *unstable*. The nonlinear terms in the Euclidean Lagrangian (7) are cubic and quartic in the quantum field $a_\mu^a(x)$:

$$V_E = \int d^4x \left(-gf^{abc} a_\nu^b a_\mu^c \nabla_\mu^{ad} a_\nu^d + \frac{g^2}{4} (f^{abc} a_\mu^b a_\nu^c)^2 \right). \quad (10)$$

The vacuum field $B_\mu^a(x)$ is stable if the action associated with it is smaller than the action of all neighbouring fields. In that respect the one-loop approximation makes sense if the amplitude e^{-K_E} (9) decay in all directions of the quantum field $a_\mu^a(x)$, that is, all eigenvalues of the quadratic form K_E are positive. If some of the eigenvalues are negative, the vacuum field $B_\mu^a(x)$ displays an apparent instability, and stability should be reconsidered by including the interaction V_E (10) of the quantum field $a_\mu^a(x)$.

We will specify the boundary fields to be equal $A_i^a(\vec{x}) = A_i^a(\vec{x})$, to be the *covariantly constant* fields [33,34] that obey the equation (8). The particular vacuum field $B_\mu^a(x)$ has the following form²:

$$B_\mu^a(x) = -\frac{1}{2} F_{\mu\nu} x_\nu \delta_3^a, \quad (11)$$

where the vectors E_i and H_i can be aligned with the z-axis by two independent rotations [11]:

² The covariantly constant chromomagnetic fields fulfil the equation $\nabla_\rho G_{\mu\nu}(B) = 0$ [3], which defines a much larger class of Lorentz invariant vacuum field configurations than the one of the form (11).

$$F_{12} = H, \quad F_{30} = E. \quad (12)$$

The Euclidean gauge invariants are: $\mathcal{F}_E = \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a = \frac{H^2 + E^2}{2}$ and the $\mathcal{G}_E = \frac{1}{4} G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a = HE$. In the quadratic approximation (9) the stability analyses reduces to the following eigenvalue problem:

$$-\nabla_\mu^{ab} (\nabla_\mu^{bc} a_\nu^c - \nabla_\nu^{bc} a_\mu^c) + gf^{abc} G_{\mu\nu}^b a_\mu^c = \lambda a_\nu^a. \quad (13)$$

It is convenient to decompose the field a_μ into the neutral and charged components³:

$$a_\mu^a = (a_\mu^3, a_\mu, a_\mu^-), \quad a_\mu = a_\mu^1 + ia_\mu^2, \quad a_\mu^- = a_\mu^1 - ia_\mu^2. \quad (14)$$

For the neutral and charged components we will get:

$$\begin{aligned} -\partial_\mu (\partial_\mu a_\nu^3 - \partial_\nu a_\mu^3) &= \lambda a_\nu^3, \\ -\nabla_\mu (\nabla_\mu a_\nu - \nabla_\nu a_\mu) + ig F_{\mu\nu} a_\mu &= \lambda a_\nu, \end{aligned} \quad (15)$$

where $\nabla_\mu = \partial_\mu - \frac{1}{2} ig F_{\mu\nu} x_\nu$. Taking into account that $[\nabla_\mu, \nabla_\nu] = ig F_{\mu\nu}$ and imposing the background gauge fixing condition on the quantum field $\nabla_\mu^{ab} a_\mu^b = 0$ [3,4,18] we will get for the charged components the following equation:

$$-\nabla_\mu \nabla_\mu a_\nu + 2ig F_{\mu\nu} a_\mu = \lambda a_\nu. \quad (16)$$

For the vacuum field (11), (12) we will get

$$\begin{aligned} H_0 &= -\nabla_\mu \nabla_\mu \\ &= -\partial_1^2 - \partial_2^2 + ig H(x_2 \partial_1 - x_1 \partial_2) + \frac{g^2}{4} H^2 (x_2^2 + x_1^2) \\ &\quad - \partial_3^2 - \partial_0^2 + ig E(x_0 \partial_3 - x_3 \partial_0) + \frac{g^2}{4} E^2 (x_3^2 + x_0^2), \end{aligned} \quad (17)$$

and H_0 is a sum of isomorphic oscillators in the (1, 2) and (3, 0) planes. The operators [11,12]

$$\begin{aligned} c_i &= \partial_i + \frac{g}{2} H x_i, \quad c_i^+ = -\partial_i + \frac{g}{2} H x_i, \quad i = 1, 2 \\ d_j &= \partial_j + \frac{g}{2} E x_j, \quad d_j^+ = -\partial_j + \frac{g}{2} E x_j, \quad j = 3, 0 \end{aligned} \quad (18)$$

allow to represent H_0 in the form $H_0 = (c_1^+ + ic_2^+)(c_1 - ic_2) + (d_3^+ + id_0^+)(d_3 - id_0) + gH + gE$. The eigenstates of the operator H_0 therefore are:

$$\begin{aligned} \psi_{nm} &= (c_1^+ + ic_2^+)^n (d_3^+ + id_0^+)^m \psi_{00} \\ &= (gH)^n (x_1 + ix_2)^n (gE)^m (x_3 + ix_0)^m \psi_{00}, \end{aligned} \quad (19)$$

where $\psi_{00} = e^{-\frac{gH}{4}(x_1^2 + x_2^2)} e^{-\frac{gE}{4}(x_3^2 + x_0^2)}$ and the corresponding eigenvalues $H_0 \psi_{nm} = \lambda_0 \psi_{nm}$ are:

$$\lambda_0 = (2n + 1)gH + (2m + 1)gE. \quad (20)$$

All eigenstates have infinite degeneracy because the states

$$\psi_{nm}(n_0, m_0) = (c_1^+ - ic_2^+)^{n_0} (d_3^+ - id_0^+)^{m_0} \psi_{nm} \quad (21)$$

have identical eigenvalues (20) and are indexed by two integers $n_0, m_0 = 0, 1, 2, \dots$. The eigenstates of the charged field (14) are defined by the operator

$$H_{\mu\nu} = -g_{\mu\nu} \nabla_\lambda \nabla_\lambda - 2ig F_{\mu\nu} \quad (22)$$

and have the following form:

$$b = a_1 + ia_2, \quad b^- = a_1 - ia_2, \quad h = a_3 + ia_0, \quad h^- = a_3 - ia_0. \quad (23)$$

³ We will consider the $SU(2)$ gauge group in the remaining part of the article.

Thus the eigenvalues of the operator $H_{\mu\nu}$ will take the following form:

$$\begin{aligned} b^- : \quad \lambda_1 &= (2n+1)gH + (2m+1)gE + 2gH \\ b : \quad \lambda_2 &= (2n+1)gH + (2m+1)gE - 2gH \\ h^- : \quad \lambda_3 &= (2n+1)gH + (2m+1)gE + 2gE \\ h : \quad \lambda_4 &= (2n+1)gH + (2m+1)gE - 2gE, \end{aligned} \quad (24)$$

where $n, m = 0, 1, 2, \dots$. The conjugate field a_μ^- has the identical eigenvalues λ_i , $i = 5, \dots, 8$.

The negative eigenvalues appear in λ_2, λ_4 when $E \neq H$, and only in the case of (anti)self-dual vacuum field [11,12]

$$E = H \quad (25)$$

there are no negative/unstable eigenvalues in the spectrum. It follows that when $n = m = 0$, there is an infinite number of zero eigenvalues $\lambda_2 = \lambda_4 = 0$ that appear due to the high symmetry of the self-dual field $E = H$ and the degeneracy (21). These are Leutwyler zero mode chromons, and they are linear combinations of (21) ($b^- = a_1 - ia_2 = 0$, $h^- = a_3 - ia_0 = 0$):

$$\begin{aligned} a_1(\xi) &= ia_2(\xi) = \sum_{n_0, m_0} \xi_{n_0 m_0} (c_1^+ - ic_2^+)^{n_0} (d_3^+ - id_0^+)^{m_0} \psi_{00} \\ a_3(\eta) &= ia_0(\eta) = \sum_{n_0, m_0} \eta_{n_0 m_0} (c_1^+ - ic_2^+)^{n_0} (d_3^+ - id_0^+)^{m_0} \psi_{00}, \end{aligned} \quad (26)$$

where we introduced the zero-mode complex amplitudes (collective coordinates) $\xi_{n_0 m_0}$ and $\eta_{n_0 m_0}$, generalising the t'Hooft instanton zero mode collective coordinates [18]. On the zero mode eigenfunctions the quadratic form (9) vanishes $K_E = 0$ and the stability of the vacuum field $B_\mu^a(x)$ is determined by the nonlinear interaction term V_E (10).

3. The effective Lagrangian and the contribution of positive/stable modes

First let us calculate the contribution of positive/stable eigenvalues of the quadratic form K_E which is defined through the determinants of the operators $H_{\mu\nu}$ and H_0 [3,4]:

$$\begin{aligned} \mathcal{L}_{\text{positive modes}}^{(1)} V_3 T &= \frac{1}{2} \ln \text{Det} H - \ln \text{Det} H_0 \\ &= \frac{1}{2} \text{Deg} \sum_{n, m, i} \ln \lambda_i(n, m) - \text{Deg} \sum_{n, m} \ln \lambda_0(n, m) \\ &= -\frac{1}{2} \text{Deg} \int \frac{ds}{s} e^{-\sum_{n, m, i} \lambda_i(n, m)s} \\ &\quad + \text{Deg} \int \frac{ds}{s} e^{-\sum_{n, m} \lambda_0(n, m)s}, \end{aligned} \quad (27)$$

where $\text{Deg} = \left(\frac{gH}{2\pi}\right)^2 V_3 T$ is the degeneracy of the eigenstates (21). Substituting the eigenvalues (24) we will get

$$\mathcal{L}_{\text{positive modes}}^{(1)} = -\frac{1}{8\pi^2} \int \frac{ds}{s} \frac{g^2 H^2}{\sinh^2 gHs}. \quad (28)$$

By using the renormalisation condition [1,2]

$$\frac{\partial \mathcal{L}}{\partial \mathcal{F}_E} \Big|_{g^2 \mathcal{F}_E = \mu^4} = 1, \quad (29)$$

where $\mathcal{F}_E = \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a = H^2$ for the contribution of the positive/stable modes, we will get:

$$\begin{aligned} \mathcal{L}_{\text{positive modes}}^{(1)} &= -\frac{1}{8\pi^2} \int \frac{ds}{s} \left[\frac{g^2 H^2}{\sinh^2 gHs} \right. \\ &\quad \left. - \frac{g^2 H^2}{\sinh^2 \mu^2 s} + \frac{g^2 H^2 \mu^2 s \cosh \mu^2 s}{\sinh^3 \mu^2 s} \right] \\ &= -\frac{g^2 H^2}{48\pi^2} \left[\ln \frac{2g^2 H^2}{\mu^4} - 1 \right]. \end{aligned} \quad (30)$$

The contribution of the zero mode eigenstates (26) should be considered separately. The zero-mode amplitudes are not suppressed by the exponent e^{-K_E} , because $K_E = 0$, and the amplitudes can grow unbound. It merely means that the integration in these directions is not Gaussian [18] and the zero modes contribution should be calculated by integration over collective coordinates (26):

$$\prod_{a, \mu} \mathcal{D}a_\mu^a = J \prod_{n_0, m_0=0}^{\text{Deg}} d\xi'_{n_0 m_0} d\xi''_{n_0 m_0} d\eta'_{n_0 m_0} d\eta''_{n_0 m_0}, \quad (31)$$

where $\xi_{n_0 m_0} = \xi'_{n_0 m_0} + i\xi''_{n_0 m_0}$, $\eta_{n_0 m_0} = \eta'_{n_0 m_0} + i\eta''_{n_0 m_0}$ and J^4 is the Jacobian factor [18,34]. In order to calculate the contribution of zero modes we will suggest two alternative methods, the method of infrared regularisation and the integration over collective coordinates of interacting zero modes. Both methods, the infrared regularisation and integration over the collective coordinates, lead to the same result (36), (48).

4. Infrared regularisation of zero modes

Let us infinitesimally elevate the zero mode spectrum (24) by adding the terms αgH and αgE to the zero eigenvalues spectrum (24):

$$\begin{aligned} \lambda_2 &= (2n+1)gH + (2m+1)gE - 2gH + \alpha gH \\ \lambda_4 &= (2n+1)gH + (2m+1)gE - 2gE + \alpha gE, \end{aligned} \quad (32)$$

where α is a dimensionless infinitesimal parameter. As we will see, the result does not depend on α . The contribution of these modes $\lambda_{2,4} = (2n+2m+\alpha)gH$ at $n = m = 0$ will be

$$\mathcal{L}_{\text{zero modes}}^{(1)} = -2 \frac{g^2 H^2}{4\pi^2} \int \frac{ds}{s} e^{-\alpha gHs}. \quad (33)$$

Thus the contribution of positive (28) and zero modes (33) will take the following form:

$$\begin{aligned} \mathcal{L}_E^{(1)} &= \mathcal{L}_{\text{positive modes}}^{(1)} + \mathcal{L}_{\text{zero modes}}^{(1)} \\ &= -\frac{1}{2\pi^2} \int \frac{ds}{s} \left[\frac{g^2 H^2}{4 \sinh^2 gHs} + g^2 H^2 e^{-\alpha gHs} \right]. \end{aligned} \quad (34)$$

By using the renormalisation condition (29) for the effective Lagrangian we will get

$$\begin{aligned} \mathcal{L}_E^{(1)} &= -\frac{1}{2\pi^2} \int \frac{ds}{s} \left[\frac{g^2 H^2}{4 \sinh^2 gHs} - \frac{g^2 H^2}{4 \sinh^2 \mu^2 s} \right. \\ &\quad \left. + \frac{g^2 H^2 \mu^2 s \cosh \mu^2 s}{4 \sinh^3 \mu^2 s} + \right. \\ &\quad \left. + g^2 H^2 \left(e^{-\alpha gHs} - e^{-\alpha \mu^2 s} + \frac{\alpha \mu^2 s}{2} e^{-\alpha \mu^2 s} \right) \right] \\ &= -\frac{g^2 H^2}{48\pi^2} \left[\ln \frac{2g^2 H^2}{\mu^4} - 1 \right] + \frac{g^2 H^2}{4\pi^2} \left[\ln \frac{2g^2 H^2}{\mu^4} - 1 \right]. \end{aligned} \quad (35)$$

⁴ The Jacobian J here is a gauge field strength independent function.

The Lagrangian does not depend on the infrared regularisation parameter α . The effective Lagrangian will take the following form:

$$\mathcal{L}_E = H^2 + \frac{11g^2 H^2}{48\pi^2} \left[\ln \frac{2g^2 H^2}{\mu^4} - 1 \right]. \quad (36)$$

There is a perfect consistency between this result and the one obtained in a physical space-time [1,17]:

$$\mathcal{L}_{YM} = -\mathcal{F} - \frac{11g^2 \mathcal{F}}{48\pi^2} \left[\ln \frac{2g^2 \mathcal{F}}{\mu^4} - 1 \right], \quad \mathcal{F} = \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a > 0. \quad (37)$$

As it was already mentioned, the electric field in a physical space-time $\vec{E}$ corresponds to an imaginary electric field in Euclidian space $\vec{E} \rightarrow i\vec{E}$, therefore one can map the gauge invariant operators as

$$\mathcal{F} = \frac{1}{2} (\vec{H}^2 - \vec{E}^2) \Rightarrow \frac{1}{2} (\vec{H}^2 + \vec{E}^2)_E = \mathcal{F}_E, \quad (38)$$

and the effective Lagrangian (37) transforms into the Euclidean effective Lagrangian (36).

5. Nonlinear interaction of zero modes

Here we will turn to the second approach taking into account a nonlinear interaction of zero modes performing the integration (31) over the collective coordinates in the zero-mode directions. The cubic self-interaction term (10) vanishes in the zero mode directions (26):

$$V_E^{(3)} = -g \int d^4x \epsilon^{abc} a_\nu^b a_\mu^c \nabla_\mu^{ad} a_\nu^d = 0. \quad (39)$$

The quartic interaction term is not vanishing, and the path integral has the following form:

$$Z_{\text{zero modes}} = N \int \exp \left[-\frac{g^2}{4} \int d^4x (\epsilon^{abc} a_\mu^b a_\nu^c)^2 \right] \prod_{a,\mu}^{zero \text{ modes}} \mathcal{D}a_\mu^a, \quad (40)$$

where the integration is over all interacting zero modes. The zero mode partition function (40), if expanded in the coupling constant g^2 , will generate an infinite number of multi-loop diagrams, and it seems impossible to calculate them. Nevertheless the exact calculation of the zero mode partition function is possible because the field strength dependence within the path integral (40) can be factorised. In terms of collective coordinates the zero mode interaction term has the following form:

$$\begin{aligned} V_E &= \frac{g^2}{4} \int d^4x (\epsilon^{abc} a_\mu^b a_\nu^c)^2 \\ &= \frac{g^2}{2} \int d^4x \left(\left| \sum_{n_0, m_0} \xi_{n_0 m_0} \psi_{00}(n_0, m_0; x) \right|^2 \right. \\ &\quad \left. + \left| \sum_{n_0, m_0} \eta_{n_0 m_0} \psi_{00}(n_0, m_0; x) \right|^2 \right), \end{aligned} \quad (41)$$

and the corresponding partition function will take the following form⁵:

$$\begin{aligned} Z_{\text{zero modes}} &= \int \exp \left\{ -\frac{g^2 (gH)^2}{2} \int \left(\left| \sum_{n_0, m_0} \xi_{n_0 m_0} \psi_{00}(n_0, m_0; y) \right|^2 \right. \right. \\ &\quad \left. \left. + \left| \sum_{n_0, m_0} \eta_{n_0 m_0} \psi_{00}(n_0, m_0; y) \right|^2 \right)^2 d^4 y \right\} \\ &\quad J \prod_{n_0, m_0}^{Deg} \mu^4 d\xi'_{n_0 m_0} d\xi''_{n_0 m_0} d\eta'_{n_0 m_0} d\eta''_{n_0 m_0}, \end{aligned} \quad (42)$$

where we introduced the dimensionless coordinates $y_i = x_i (gH)^{1/2}$ in the integral V_E allowing to factorise the field strength dependence in the exponent of the path integral (42). The wave functions $\psi_{00}(n_0, m_0; x)$ were normalised, and in terms of the dimensionless coordinates they are:

$$\begin{aligned} \psi_{00}(n_0, m_0; y) &= \frac{(y_1 - iy_2)^{n_0} (y_3 - iy_0)^{m_0}}{(\pi^2 2^{n_0+m_0+2} n_0! m_0!)^{1/2}} \exp \left\{ -\frac{y_1^2 + y_2^2 + y_3^2 + y_0^2}{4} \right\}. \end{aligned} \quad (43)$$

By introducing the dimensionless collective coordinates $(\xi, \eta) \rightarrow (\xi, \eta)/(gH)^{1/2}$ we will get

$$\begin{aligned} Z_{\text{zero modes}} &= \int \exp \left\{ -\frac{g^2}{2} \int \left(\left| \sum_{n_0, m_0} \xi_{n_0 m_0} \psi_{n_0 m_0}(y) \right|^2 \right. \right. \\ &\quad \left. \left. + \left| \sum_{n_0, m_0} \eta_{n_0 m_0} \psi_{n_0 m_0}(y) \right|^2 \right)^2 d^4 y \right\} \\ &\quad J \prod_{n_0, m_0}^{Deg} \left(\frac{\mu^2}{gH} \right)^2 d\xi'_{n_0 m_0} d\xi''_{n_0 m_0} d\eta'_{n_0 m_0} d\eta''_{n_0 m_0}. \end{aligned} \quad (44)$$

The field strength dependence completely factorises and we have

$$\begin{aligned} Z_{\text{zero modes}} &= N \prod_{n_0, m_0}^{Deg} \left(\frac{\mu^4}{g^2 H^2} \right) \\ &= N \left(\frac{\mu^4}{g^2 H^2} \right)^{Deg} = N e^{-\frac{g^2 H^2}{4\pi^2} \ln \frac{g^2 H^2}{\mu^4} V_3 T}, \end{aligned} \quad (45)$$

where

$$\begin{aligned} N &= \int \exp \left\{ -\frac{g^2}{2} \int \left(\left| \sum_{n_0, m_0} \xi_{n_0 m_0} \psi_{n_0 m_0}(y) \right|^2 \right. \right. \\ &\quad \left. \left. + \left| \sum_{n_0, m_0} \eta_{n_0 m_0} \psi_{n_0 m_0}(y) \right|^2 \right)^2 d^4 y \right\} \\ &\quad J \prod_{n_0, m_0}^{Deg} d\xi'_{n_0 m_0} d\xi''_{n_0 m_0} d\eta'_{n_0 m_0} d\eta''_{n_0 m_0}. \end{aligned} \quad (46)$$

In the limit $T \rightarrow \infty$ we have $Z_{\text{zero modes}} \rightarrow N e^{-\mathcal{L}_{eff} V_3 T}$ and from (45) for the zero modes contribution we will get

$$\mathcal{L}_{\text{interacting zero modes}} = \frac{g^2 H^2}{4\pi^2} \ln \frac{g^2 H^2}{\mu^4}. \quad (47)$$

The effective Lagrangian is a sum of positive/stable (30) and interacting zero modes (47):

$$\begin{aligned} \mathcal{L}_E^{eff} &= H^2 + \mathcal{L}_{\text{positive modes}}^{(1)} + \mathcal{L}_{\text{interacting zero modes}} \\ &= H^2 + \frac{11g^2 H^2}{48\pi^2} \left[\ln \frac{2g^2 H^2}{\mu^4} - 1 \right], \end{aligned} \quad (48)$$

⁵ I would like to thank Konstantin Savvidy for pointing out to me the possibility of such generalisation.

and coincides with our previous results (36). It is remarkable that the zero modes contribution that was calculated in terms of infrared regularisation of the spectrum (32), (33), (35) and then by integration over collective coordinates of interacting zero modes (45), both lead to the same result indicating the robustness of the logarithmic structure of the effective Lagrangian [1].

6. Conclusion

It is appealing to notice that the stability of a larger class of vacuum fields that were considered in the literature [36–46] allows to conjecture that the Yang-Mills vacuum is a highly degenerate state. Some of these vacuum fields expose a natural chaotic behaviour [36–42]. In this respect one should ask whether there are other physical systems that have high degeneracy of the vacuum state as it is the case in the Yang-Mills theory [35]. Turning to the statistical spin systems, one can observe that the classical 3D Ising system has a double degeneracy of the vacuum state and of all its excited states. It is this symmetry that allows to construct a dual gauge invariant representation of the 3D Ising model [47]. The extensions of the 3D Ising model that have a direct ferromagnetic and one quarter of the next to nearest neighbour antiferromagnetic interaction constructed in [48], as well as a model with a zero intersection coupling constant ($k = 0$) [49,50,52,53], have *exponential degeneracy of the vacuum state*. In recent publications this symmetry was referred to as the subsystem symmetry [56]. This higher symmetry allows to construct the dual representations of the same systems and in various dimensions [49–51]. As a consequence of the high degeneracy of the vacuum state, these systems have rich physical properties, including a glass behaviour [54,55], exotic fracton excitations [56] and was suggested as high density memory device in [57].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

- [1] G.K. Savvidy, Infrared instability of the vacuum state of gauge theories and asymptotic freedom, *Phys. Lett. B* 71 (1977) 133, [https://doi.org/10.1016/0370-2693\(77\)90759-6](https://doi.org/10.1016/0370-2693(77)90759-6).
- [2] S.G. Matinyan, G.K. Savvidy, Vacuum polarization induced by the intense gauge field, *Nucl. Phys. B* 134 (1978) 539, [https://doi.org/10.1016/0550-3213\(78\)90463-7](https://doi.org/10.1016/0550-3213(78)90463-7).
- [3] I.A. Batalin, S.G. Matinyan, G.K. Savvidy, Vacuum polarization by a source-free gauge field, *Sov. J. Nucl. Phys.* 26 (1977) 214, *Yad. Fiz.* 26 (1977) 407.
- [4] G.K. Savvidy, Vacuum polarisation surrounding soft gluon fields, PhD, <http://www.inp.demokritos.gr/~savvidy/phd.pdf>, 1977.
- [5] N.K. Nielsen, P. Olesen, An unstable Yang-Mills field mode, *Nucl. Phys. B* 144 (1978) 376, [https://doi.org/10.1016/0550-3213\(78\)90377-2](https://doi.org/10.1016/0550-3213(78)90377-2).
- [6] V.V. Skalozub, On restoration of spontaneously broken symmetry in magnetic field, *Yad. Fiz.* 28 (1978) 228.
- [7] J. Ambjorn, N.K. Nielsen, P. Olesen, A hidden Higgs Lagrangian in QCD, *Nucl. Phys. B* 152 (1979) 75, [https://doi.org/10.1016/0550-3213\(79\)90080-4](https://doi.org/10.1016/0550-3213(79)90080-4).
- [8] H.B. Nielsen, Approximate QCD lower bound for the bag constant B , *Phys. Lett. B* 80 (1978) 133, [https://doi.org/10.1016/0370-2693\(78\)90326-X](https://doi.org/10.1016/0370-2693(78)90326-X).
- [9] N.K. Nielsen, P. Olesen, Electric vortex lines from the Yang-Mills theory, *Phys. Lett. B* 79 (1978) 304, [https://doi.org/10.1016/0370-2693\(78\)90249-6](https://doi.org/10.1016/0370-2693(78)90249-6).
- [10] J. Ambjorn, P. Olesen, A color magnetic vortex condensate in QCD, *Nucl. Phys. B* 170 (1980) 265, [https://doi.org/10.1016/0550-3213\(80\)90150-9](https://doi.org/10.1016/0550-3213(80)90150-9).
- [11] H. Leutwyler, Vacuum fluctuations surrounding soft gluon fields, *Phys. Lett. B* 96 (1980) 154, [https://doi.org/10.1016/0370-2693\(80\)90234-8](https://doi.org/10.1016/0370-2693(80)90234-8).
- [12] H. Leutwyler, Constant gauge fields and their quantum fluctuations, *Nucl. Phys. B* 179 (1981) 129, [https://doi.org/10.1016/0550-3213\(81\)90252-2](https://doi.org/10.1016/0550-3213(81)90252-2).
- [13] P. Minkowski, On the ground state expectation value of the field strength bilinear in gauge theories and constant classical fields, *Nucl. Phys. B* 177 (1981) 203–217, [https://doi.org/10.1016/0550-3213\(81\)90388-6](https://doi.org/10.1016/0550-3213(81)90388-6).
- [14] C.A. Flory, Covariant constant chromomagnetic fields and elimination of the one loop instabilities, Preprint, SLAC-PUB-3244, <http://www-public.slac.stanford.edu/sciDoc/docMeta.aspx?slacPubNumber=SLAC-PUB-3244>, <https://lib-extopc.kek.jp/preprints/PDF/1983/8312/8312331.pdf>.
- [15] L.D. Faddeev, A.J. Niemi, Aspects of electric magnetic duality in SU(2) Yang-Mills theory, *Phys. Lett. B* 525 (2002) 195–200, [https://doi.org/10.1016/S0370-2693\(01\)01432-0](https://doi.org/10.1016/S0370-2693(01)01432-0), arXiv:hep-th/0101078 [hep-th].
- [16] G.L. Pimentel, A.M. Polyakov, G.M. Tarnopolsky, Vacua on the brink of decay, *Rev. Math. Phys.* 30 (07) (2018) 1840013, https://doi.org/10.1142/9789813233867_0020, arXiv:1803.09168 [hep-th].
- [17] G. Savvidy, From Heisenberg–Euler Lagrangian to the discovery of chromomagnetic gluon condensation, *Eur. Phys. J. C* 80 (2) (2020) 165, <https://doi.org/10.1140/epjc/s10052-020-7711-6>, arXiv:1910.00654 [hep-th].
- [18] G. 't Hooft, Computation of the quantum effects due to a four-dimensional pseudoparticle, *Phys. Rev. D* 14 (1976) 3432–3450, <https://doi.org/10.1103/PhysRevD.14.3432>.
- [19] D. Kay, R. Parthasarathy, K.S. Viswanathan, Constant self-dual Abelian gauge fields and fermions in SU(2) gauge theory, *Phys. Rev. D* 28 (1983) 3116–3120.
- [20] D. Kay, Unstable modes, zero modes, and phase transitions in QCD, Ph.D Thesis, Simon Fraser University, August 1985.
- [21] W. Dittrich, M. Reuter, Effective QCD Lagrangian with zeta function regularization, *Phys. Lett. B* 128 (1983) 321, [https://doi.org/10.1016/0370-2693\(83\)90268-X](https://doi.org/10.1016/0370-2693(83)90268-X).
- [22] D. Zwanziger, Nonperturbative modification of the Faddeev–Popov formula and banishment of the naive vacuum, *Nucl. Phys. B* 209 (1982) 336, [https://doi.org/10.1016/0550-3213\(82\)90260-7](https://doi.org/10.1016/0550-3213(82)90260-7).
- [23] D. Kay, A. Kumar, R. Parthasarathy, Savvidy vacuum in SU(2) Yang-Mills theory, *Mod. Phys. Lett. A* 20 (2005) 1655–1662.
- [24] K.I. Kondo, Stability of magnetic condensation and mass generation for confinement in SU(2) Yang-Mills theory, *PoS QCD-TNT-III* (2013) 020, <https://doi.org/10.22323/1.193.0020>, arXiv:1312.0053 [hep-th].
- [25] Y.M. Cho, M.L. Walker, Stability of monopole condensation in SU(2) QCD, *Mod. Phys. Lett. A* 19 (2004) 2707, <https://doi.org/10.1142/S0217732304015750>.
- [26] D.G. Pak, B.H. Lee, Y. Kim, T. Tsukioka, P.M. Zhang, On microscopic structure of the QCD vacuum, *Phys. Lett. B* 780 (2018) 479–484, <https://doi.org/10.1016/j.physletb.2018.03.040>, arXiv:1703.09635 [hep-th].
- [27] D.G. Pak, R.G. Cai, T. Tsukioka, P. Zhang, Y.F. Zhou, Inherent color symmetry in quantum Yang-Mills theory, *Phys. Lett. B* 839 (2023) 137804, <https://doi.org/10.1016/j.physletb.2023.137804>, arXiv:2009.13938 [hep-th].
- [28] S. Coleman, The uses of Instantons, *Erice Lectures*, 1977.
- [29] W. Heisenberg, H. Euler, Consequences of Dirac's theory of positrons, *Z. Phys.* 98 (1936) 714.
- [30] J.S. Schwinger, On gauge invariance and vacuum polarization, *Phys. Rev.* 82 (1951) 664, <https://doi.org/10.1103/PhysRev.82.664>.
- [31] S.R. Coleman, E.J. Weinberg, Radiative corrections as the origin of spontaneous symmetry breaking, *Phys. Rev. D* 7 (1973) 1888, <https://doi.org/10.1103/PhysRevD.7.1888>.
- [32] G. Savvidy, Gauge field theory vacuum and cosmological inflation without scalar field, *Ann. Phys.* 436 (2022) 168681, <https://doi.org/10.1016/j.aop.2021.168681>, arXiv:2109.02162 [hep-th].
- [33] M.J. Duff, M. Ramon-Medrano, On the effective Lagrangian for the Yang-Mills field, *Phys. Rev. D* 12 (1975) 3357, <https://doi.org/10.1103/PhysRevD.12.3357>.
- [34] K. Fujikawa, Path integral measure for gauge invariant fermion theories, *Phys. Rev. Lett.* 42 (1979) 1195–1198, <https://doi.org/10.1103/PhysRevLett.42.1195>.
- [35] R. Jackiw, C. Rebbi, Vacuum periodicity in a Yang-Mills quantum theory, *Phys. Rev. Lett.* 37 (1976) 172–175, <https://doi.org/10.1103/PhysRevLett.37.172>.
- [36] G. Basesyan, S. Matinyan, G. Savvidy, Nonlinear plane waves in the massless Yang-Mills theory, *Pisma Zh. Eksp. Teor. Fiz.* 29 (1979) 641–644.
- [37] S. Matinyan, G. Savvidy, N. Ter-Arutunyan-Savvidi, Classical Yang-Mills mechanics. Nonlinear colour oscillations, *Zh. Eksp. Teor. Fiz.* 80 (1980) 830–838.
- [38] G. Savvidy, The Yang-Mills classical mechanics as a Kolmogorov system, *Phys. Lett. B* 130 (1983) 303–307.
- [39] G. Savvidy, The Yang-Mills quantum mechanics, *Phys. Lett. B* 159 (1985) 325–329.
- [40] G. Savvidy, Classical and quantum mechanics of non-Abelian gauge fields, *Nucl. Phys.* 246 (1984) 302–334.
- [41] T. Banks, W. Fischler, S.H. Shenker, L. Susskind, M theory as a matrix model: a conjecture, *Phys. Rev. D* 55 (1997) 5112.
- [42] T. Anous, C. Coburn, Mini-BFSS matrix model in silico, *Phys. Rev. D* 100 (2019) 066023.
- [43] Y.M. Cho, A restricted gauge theory, *Phys. Rev. D* 21 (1980) 1080, <https://doi.org/10.1103/PhysRevD.21.1080236-3>.
- [44] Y. Kim, B.H. Lee, D.G. Pak, C. Park, T. Tsukioka, Quantum stability of nonlinear wave type solutions with intrinsic mass parameter in QCD, *Phys. Rev. D* 96 (5) (2017) 054025, <https://doi.org/10.1103/PhysRevD.96.054025>, arXiv:1607.02083 [hep-th].
- [45] S.M. Apenko, D.A. Kirzhnits, Y.E. Lozovik, Dynamic chaos, Anderson localization, and confinement, *JETP Lett.* 36 (1982) 213–215.

- [46] G. Savvidy, Maximally chaotic dynamical systems, *Ann. Phys.* 421 (2020) 168274, <https://doi.org/10.1016/j.aop.2020.168274>.
- [47] F.J. Wegner, Duality in generalized Ising models and phase transitions without local order parameters, *J. Math. Phys.* 12 (1971) 2259–2272, <https://doi.org/10.1063/1.1665530>.
- [48] G.K. Savvidy, F.J. Wegner, Geometrical string and spin systems, *Nucl. Phys. B* 413 (1994) 605–613, [https://doi.org/10.1016/0550-3213\(94\)90003-5](https://doi.org/10.1016/0550-3213(94)90003-5), arXiv:hep-th/9308094 [hep-th].
- [49] G.K. Savvidy, K.G. Savvidy, Selfavoiding gonihedric string and spin systems, *Phys. Lett. B* 324 (1994) 72–77, [https://doi.org/10.1016/0370-2693\(94\)00114-6](https://doi.org/10.1016/0370-2693(94)00114-6), arXiv:hep-lat/9311026 [hep-lat].
- [50] G.K. Savvidy, K.G. Savvidy, Interaction hierarchy, *Phys. Lett. B* 337 (1994) 333–339, [https://doi.org/10.1016/0370-2693\(94\)90984-9](https://doi.org/10.1016/0370-2693(94)90984-9), arXiv:hep-th/9409030 [hep-th].
- [51] G.K. Savvidy, K.G. Savvidy, F.J. Wegner, Geometrical string and dual spin systems, *Nucl. Phys. B* 443 (1995) 565–580, [https://doi.org/10.1016/0550-3213\(95\)00151-H](https://doi.org/10.1016/0550-3213(95)00151-H), arXiv:hep-th/9503213 [hep-th].
- [52] R. Pietig, F.J. Wegner, Phase transition in lattice surface systems with gonihedric action, *Nucl. Phys. B* 466 (1996) 513–526, [https://doi.org/10.1016/0550-3213\(96\)00072-7](https://doi.org/10.1016/0550-3213(96)00072-7), arXiv:hep-lat/9604013 [hep-lat].
- [53] R. Pietig, F.J. Wegner, Low temperature expansion of the gonihedric Ising model, *Nucl. Phys. B* 525 (1998) 549–570, [https://doi.org/10.1016/S0550-3213\(98\)00342-3](https://doi.org/10.1016/S0550-3213(98)00342-3), arXiv:hep-lat/9712002 [hep-lat].
- [54] A. Lipowski, D. Johnston, D. Espriu, Slow dynamics of Ising models with energy barriers, *Phys. Rev. E* 62 (2000) 3404, <https://doi.org/10.1103/PhysRevE.62.3404>.
- [55] C. Castellano, C. Chamon, D. Sherrington, Quantum mechanical and information theoretic view on classical glass transitions, *Phys. Rev. B* 81 (2010) 184303, <https://doi.org/10.1103/PhysRevB.81.184303>.
- [56] S. Vijay, J. Haah, L. Fu, Fracton topological order, generalized lattice gauge theory and duality, *Phys. Rev. B* 94 (23) (2016) 235157, <https://doi.org/10.1103/PhysRevB.94.235157>, arXiv:1603.04442 [cond-mat.str-el].
- [57] G.K. Savvidy, The system with exponentially degenerate vacuum state, arXiv:cond-mat/0003220 [cond-mat].