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Equivalent transformation and integrability of the nonlinear Schrödinger equations with time-dependent coefficients ☆

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Abstract

The nonlinear Schrödinger (NLS) types of equations play a key role in quantum mechanics, Quantum communication and physical applications. However, how to deal with explicit solutions and other properties of the NLS equations, especially for the variable-coefficient NLS (vc-NLS) types of equations is a difficult problem. In this paper, we construct the form-preserving equivalent transformations (ETs) to transform the vc-NLS systems into constant-coefficient NLS (cc-NLS) systems, and the form-preserving ETs are given explicitly. Then, based on the equivalent transformation method, we deal with the integrability of the NLS equations, and the Lax pairs (LPs) are provided as verification of the integrability.

1. Introduction

P. Clarkson and M. Kruskal [1] proposed a direct method for reduction nonlinear partial differential equations (NLPDEs) rather than Lie group analysis method, it is the CK direct method.

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This method is effective and valuable for dealing with similarity reductions and exact solutions to NLPDEs [1–8], and the greatest advantage of this method is that it avoids the abstract Lie group theory. In the current paper, we develop the CK direct method to consider the generalized variable-coefficient nonlinear Schrödinger (vc-NLS) equation as follows:

$$iu_t + a(t)u_{xx} + b(t)u^2u^* + c(t)u = 0, \tag{1.1}$$

where $u = u(x, t)$ is the unknown complex function of the real independent variables x and t, u^* is the conjugate of u, the coefficients $a = a(t)$, $b = b(t)$ and $c = c(t)$ are real analytic functions, $ab \neq 0$ is assumed throughout this paper (otherwise, it is not a NLS equation).

In particular, if $c = c(t) = 0$, then this equation is reduced to the following classical variable-coefficient nonlinear Schrödinger equation

$$iu_t + a(t)u_{xx} + b(t)u^2u^* = 0. \tag{1.2}$$

We note first that these complex functions are of great complicated than the real functions for studying its exact solutions and other properties. For dealing with these characters, we must to transform such complex functions into real functions firstly.

Letting $u = v + iw$ and substituting it into Eq. (1.1), we have a vc-NLPDE system

$$\begin{cases} v_t + a(t)w_{xx} + b(t)(v^2 + w^2)w + c(t)w = 0, \\ w_t - a(t)v_{xx} - b(t)(v^2 + w^2)v - c(t)v = 0, \end{cases} \tag{1.3}$$

where $v = v(x, t)$ and $w = w(x, t)$ are unknown real functions of x and t. Correspondingly, Eq. (1.2) is transformed into the following vc-NLPDE system

$$\begin{cases} v_t + a(t)w_{xx} + b(t)(v^2 + w^2)w = 0, \\ w_t - a(t)v_{xx} - b(t)(v^2 + w^2)v = 0. \end{cases} \tag{1.4}$$

Thus, we transform the complex vc-NLS equations (1.1) and (1.2) into real vc-NLPDE systems (1.3) and (1.4) respectively, they are called vc-NLS systems sometimes. If the exact solutions $v = v(x, t)$ and $w = w(x, t)$ are obtained, then the exact solutions to the complex vc-NLS equations can be given by $u = v + iw$ accordingly. Thus, we only consider such real vc-NLS systems in what follows.

As is well known, the Schrödinger types of equations play a significant role in mathematical physics and physical applications such as quantum mechanics, quantum communication and solid state physics, etc. So far there are lot of studies on the NLS types of equations, but there are few clear results about the explicit solutions, especially for the vc-NLS equations as far as we know [9–13]. More generally, the vc-NLPDEs differ greatly from its constant-coefficient counterparts, and they are more involved for investigating exact solutions and other properties of the latter. For dealing with exact solutions and properties of vc-NLPDEs, a lot of methods were developed such as the Painlevé test [14–19], Lie symmetry analysis [19–23], various trial function methods based on the homogeneous balance principle (HBP) [24–26] and CK direct method, and so on. Lou, et al. considered the similarity reductions and exact solutions to some NLPDEs by using the improved CK method [2–8]. To sum up, although these aforementioned methods have their own advantages, but they have not achieved desired results for dealing with exact solutions to the vc-NLS equations actually. Recently, we studied the exact solutions to some vc-NLPDEs by using the CK reduction method [6–8], but such PDEs are single equations rather than a system of equations. In view of the significance of such nonlinear Schrödinger types of equations, it is necessary to explore some new ways to deal with integrability and other properties of these equations. In the current paper, we develop the CK reduction method for dealing with this problem. *The main idea is to transform the vc-NLS systems into cc-NLS systems by constructing the form-preserving equivalent transformations (ETs), this is the key to the study. Then the integrability of the vc-NLS equations is considered accordingly.* Summarizing, the contribution and novelty of the present paper are as follows:

- We develop the equivalent transformation method for reducing vc-NLS systems to cc-NLS systems, and the form-preserving ETs are given explicitly.
- We propose a criterion for testing integrability of the NLS equations based on the ET method, as verification of the integrability, the Lax pairs (LPs) are provided.

The rest of this paper is structured as follows. In Section 2, by constructing the equivalent transformations, we reduce the vc-NLS systems to constant-coefficient NLS (cc-NLS) systems in the same form of the former under some conditions. In Section 3, the integrability of the NLS equations is considered based on the equivalent transformation method for the first time, and the Lax pairs (LPs) of the vc-NLS equations are given as verification of the integrability. Finally, the conclusion and further discussion are given in Section 4.

2. Equivalent transformations (ETs)

In this section, by the improved CK reduction method, we transform the vc-NLS systems (1.3) and (1.4) into constant-coefficient NLS systems in the same form of the former as follows:

$$\begin{cases} v_t + \alpha w_{xx} + \beta(v^2 + w^2)w + \gamma w = 0, \\ w_t - \alpha v_{xx} - \beta(v^2 + w^2)v - \gamma v = 0, \end{cases} \tag{2.1}$$

where α, β and γ are arbitrary constants, and $\alpha\beta \neq 0$. In particular, if $\gamma = 0$, then this cc-PDE system becomes the following system:

$$\begin{cases} v_t + \alpha w_{xx} + \beta(v^2 + w^2)w = 0, \\ w_t - \alpha v_{xx} - \beta(v^2 + w^2)v = 0. \end{cases} \tag{2.2}$$

On the other hand, the above two systems can be derived from the complex nonlinear Schrödinger equations

$$iu_t + \alpha u_{xx} + \beta u^2 u^* + \gamma u = 0 \tag{2.3}$$

and

$$iu_t + \alpha u_{xx} + \beta u^2 u^* = 0 \tag{2.4}$$

through the transformation $u = v + iw$, respectively. In fact, these constant-coefficient NLS equations above are of great importance in physics and applications.

Now, we assume the CK type of transformation as follows

$$v \equiv v(x,t) = f(x,t) + g(x,t)V(p,q), \quad w \equiv w(x,t) = h(x,t) + k(x,t)W(p,q), \tag{2.5}$$

where $f = f(x,t)$, $g = g(x,t)$, $h = h(x,t)$, $k = k(x,t)$, $p = p(x,t)$ and $q = q(x,t)$ are functions of x and t to be determined.

Substituting (2.5) into system (1.3), and requiring that $V = V(p,q)$ and $W = W(p,q)$ satisfy the same types of equations as $v = v(x,t)$ and $w = w(x,t)$ with the transformations $\{v, x, t\} \to$

$\{V, p, q\}$ and $\{w, x, t\} \to \{W, p, q\}$, respectively. In other words, requiring that $\{V, p, q\}$ and $\{W, p, q\}$ satisfy the following system

$$\begin{cases} V_p + \alpha W_{qq} + \beta(V^2 + W^2)W + \gamma W = 0, \\ W_p - \alpha V_{qq} - \beta(V^2 + W^2)V - \gamma V = 0, \end{cases} \tag{2.6}$$

where the constants α, β and γ are given in system (2.1). Particularly, if $\gamma = 0$, then this system becomes the following system:

$$\begin{cases} V_p + \alpha W_{qq} + \beta(V^2 + W^2)W = 0, \\ W_p - \alpha V_{qq} - \beta(V^2 + W^2)V = 0. \end{cases} \tag{2.7}$$

In general, if a nontrivial transformation (2.5) is obtained, then it is called an equivalent transformation (ET) [5–8]. Due to the fact that the transformed equation has the same form as the original equation, so we call such equivalent transformation form-preserving. In this case, we said that systems (1.3) and (2.6) are similar systems, or equivalent systems.

Then, substituting (2.5) into system (1.3), by the direct reduction method, we have the following result:

Theorem 2.1. *If $V = V(p, q)$ and $W = W(p, q)$ are a solution to system (2.6), then*

$$\begin{cases} v = c_4 V(c_1 x + c_2, \dfrac{c_1^2}{\alpha} A(t) + c_3), \\ w = c_4 W(c_1 x + c_2, \dfrac{c_1^2}{\alpha} A(t) + c_3), \end{cases} \tag{2.8}$$

is a solution to vc-system (1.3), under the following condition

$$\alpha c_4^2 b(t) - \beta c_1^2 a(t) = 0, \quad \alpha c(t) - \gamma c_1^2 a(t) = 0, \tag{2.9}$$

where c_i $(i = 1, ..., 4)$ are arbitrary constants and $c_1 c_4 \neq 0$.

Proof. Substituting (2.5) into (1.3), by the similarity reduction method, we can get the equivalent transformation (2.8), and through this equivalent transformation, the vc-NLS system (1.3) can be transformed into cc-NLS system (2.6) under condition (2.9). The detail is omitted here. □

In other words, under the condition (2.9), the vc-NLS system (1.3) can be transformed into cc-NLS system (2.6) by the equivalent transformation (2.8).

In particular, if $c(t) = 0$, then vc-system (1.3) becomes (1.4). In this case, we have

Corollary 2.2. *If $V = V(p, q)$ and $W = W(p, q)$ are a solution to system (2.7), then*

$$\begin{cases} v = c_4 V(c_1 x + c_2, \dfrac{c_1^2}{\alpha} A(t) + c_3), \\ w = c_4 W(c_1 x + c_2, \dfrac{c_1^2}{\alpha} A(t) + c_3), \end{cases} \tag{2.10}$$

is a solution to vc-system (1.4), under the following condition

$$\alpha c_4^2 b(t) - \beta c_1^2 a(t) = 0, \tag{2.11}$$

where c_i $(i = 1, ..., 4)$ are arbitrary constants and $c_1 c_4 \neq 0$.

In other words, under the condition (2.11), the vc-NLS system (1.4) can be transformed into cc-NLS system (2.7) by the equivalent transformation (2.10).

Therefore, based on the above results, if the exact solutions to the cc-NLS systems (2.6) and (2.7), i.e. (2.1) and (2.2) are obtained, then the exact solutions to the corresponding vc-NLS systems (1.3) and (1.4) are presented through the equivalent transformations (2.8) and (2.10), respectively. So, the exact solutions to the vc-NLS equations (1.1) and (1.2) can be given by the transformation $u = v + iw$ immediately. In what follows, we only consider the symmetry reductions and exact solutions to the NLS systems (2.1) and (2.2).

Remark 2.1. More generally, we can get the ETs of the nonlinear NLS systems as follows:

$$\begin{cases} v = c_4 V(c_1 x + c_2, \pm\dfrac{c_1^2}{\alpha}A(t) + c_3), \\ w = \pm c_4 W(c_1 x + c_2, \pm\dfrac{c_1^2}{\alpha}A(t) + c_3), \end{cases} \tag{2.12}$$

under the above conditions. For the sake of simplicity, we omit the "±" in our discussion here and in what follows.

Summarizing the above discussion, we have the result:

Theorem 2.3. *vc-NLS systems (1.3) and (1.4) be transformed into cc-NLS systems (2.6) and (2.7) if and only if the conditions (2.9) and (2.11) are satisfied.*

Otherwise, these vc-systems cannot be transformed into such cc-systems by the ETs (2.8) and (2.10), so the conditions are necessary for the ET method.

3. Integrability and exact solutions

The classic nonlinear Schrödinger equation is as follows

$$iq_t + q_{xx} + 2q^2 q^* = 0, \tag{3.1}$$

where $q = q(x,t)$ is the unknown complex function. Letting $q = r + is$, and substituting it into Eq. (3.1), we get

$$\begin{cases} r_t + s_{xx} + 2(r^2 + s^2)s = 0, \\ s_t - r_{xx} - 2(r^2 + s^2)r = 0, \end{cases} \tag{3.2}$$

where $r = r(x,t)$ and $s = s(x,t)$ are unknown real functions of x and t. Clearly, if we let $\alpha = 1$ and $\beta = 2$ in Eq. (2.4) and system (2.2), then they become Eq. (3.1) and system (3.2), respectively.

Now we show that system (2.2) can be transformed into system (3.2) by the scaling transformation. In fact, in view of α and β are arbitrary constants, we can suppose $\alpha = 1$ and $\beta > 0$, thus let $v = \sqrt{\frac{2}{\beta}}r$, $w = \sqrt{\frac{2}{\beta}}s$, and substitute it into system (2.2), so this system be transformed into system (3.2), and vice versa. So, in the sense of scaling transformation, we can say that these equations are equivalent, and they have the same integrability.

Moreover, it is known that Eq. (3.1) is integrable for it can be derived through the AKNS procedure [27,28], so is the system (3.2). Thus, we know that the system (2.2) and Eq. (2.4) are integrable. In this paper, a vc-PDE or a nonlinear PDE is called integrable if it can be transformed into a given integrable PDE or a constant-coefficient linear PDE by the equivalent transformation (including scaling transformation).

Furthermore, in view of vc-NLS system (1.4) can be transformed into cc-system (2.2) by the equivalent transformation (2.10), thus we get that the vc-NLS system (1.4) is integrable under the condition (2.11), so the vc-NLS equation (1.2) is also integrable. Summarizing, we give the following definition

Definition 3.1. A vc-PDE or a nonlinear PDE is called integrable, if it can be transformed into a given integrable equation or a constant-coefficient linear equation by the equivalent transformation under some condition, the condition is called the integrable condition.

Here we assume that a constant-coefficient linear equation is integrable, and scaling transformation is a special case of equivalent transformation.

Thus, based on the definition and Theorem 2.3, we have

Theorem 3.1. *vc-NLS Eqs. (1.1) and (1.2) are integrable under the condition (2.9) and (2.11), respectively. In particular, NLS Eq. (2.4) is integrable.*

On the other hand, as a verification, we can give the Lax pair (LP) of vc-NLS Eq. (1.2) under condition (2.11), see Remark 3.1 as follows.

As for the exact solutions to vc-NLS equations, since we have transformed these vc-NLS equations into its constant-coefficient counterparts, the exact solutions to the vc-NLS equations can be obtained based on the cc-NLS equations. Particularly, the exact solutions to the integrable NLS equations are presented immediately, including the soliton types of solutions, etc.

Remark 3.1. Now we give the Lax pair of vc-NLS Eq. (1.2) under condition (2.11) by AKNS procedure [27,28]. First, in view of (2.11), we have

$$\alpha c_4^2 b(t) - \beta c_1^2 a(t) = 0, \tag{3.3}$$

or

$$\frac{b(t)}{a(t)} = \frac{\beta c_1^2}{\alpha c_4^2}, \tag{3.4}$$

for α, β, c_1 and c_4 are arbitrary nonzero constants. Clearly, we can choose these constants such that

$$\frac{\beta c_1^2}{\alpha c_4^2} = -2, \tag{3.5}$$

that is

$$\beta c_1^2 = -2\alpha c_4^2. \tag{3.6}$$

Thus, we have $b(t) = -2a(t)$. So under condition (2.11), Eq. (1.2) becomes the following form

$$iu_t + a(t)u_{xx} - 2a(t)u^2u^* = 0. \tag{3.7}$$

Then, the linear eigenvalue problems can be expressed as

$$\Phi_x = M\Phi, \quad \Phi_t = N\Phi, \tag{3.8}$$

where $\Phi = (\phi_1, \phi_2)^T$ is the complex vector function and each component of Φ is a scalar function with respect to x and t, the superscript T illustrates the transpose for a matrix, while M and N are presented in the following forms

$$M = \begin{pmatrix} -i\lambda & q \\ r & i\lambda \end{pmatrix}, \qquad N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{3.9}$$

where

$$A = -2ia\lambda^2 - iaqq^*, \quad B = 2aq\lambda + iaq_x, \quad C = 2aq^*\lambda - iaq_x^*,$$

here the spectral parameter λ is a complex constant, $r = q^*$, $q = q(x,t)$ satisfies Eq. (3.7), and q^* is the conjugate of q.

It is easy to see that Eq. (3.7) can be re-produced through the compatibility condition $M_t - N_x + [M, N] = 0$, where $[M, N] = MN - NM$. We note that the Lax pair can assure the complete integrability (Lax integrability) of Eq. (3.7), (2.11) is the integrable condition.

Similarly, the Lax pair of Eq. (3.1) can be given, the detail is omitted.

4. Further discussion and conclusions

In the current paper, the variable-coefficient nonlinear Schrodinger types of equations are investigated by the equivalent transformation method. Under some conditions, the generalized vc-NLS systems are transformed into constant-coefficient NLS systems. Furthermore, we consider the integrability of the vc-NLS equations based on the equivalent transformation method for the first time. *In summary, the basic technique there is to transform a given vc-PDE system into a simpler one (a cc-PDE system, in the present paper) and then we can write the solutions of the original equation in terms of the solutions obtained for the simpler one.* Moreover, what is the relationship between these conditions and other integrable conditions, and whether they are equivalent or not? Are there any other types of equivalent transformations for these equations? These are interesting and promising problems and we hope to investigate it in the future.

Remark 4.1. We note that the conditions for transforming vc-NLS systems into cc-NLS systems play a key role in the ET method. Conversely, if the conditions are not satisfied, then the vc-NLS systems cannot be transformed into its constant-coefficient counterparts through such transformations. By the improved equivalent transformation method [8], we transform a vc-NLPDE into different types of cc-PDEs, one of them is simpler for solving, such as a linear equation. However, we cannot transform the vc-NLS (1.1) and (1.2) into linear cc-PDEs by the this equivalent transformation method. How to transform the vc-NLS system into other more simpler types of cc-PDEs, it is an open problem and we hope to study it further.

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The authors of the paper declare that they have no conflict of interest.

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No data was used for the research described in the article.

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