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Confinement in QCD and generic Yang-Mills theories with matter representations

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ABSTRACT

We derive the low-energy limit of quantum chromodynamics (QCD) and provide evidence that in the 't Hooft limit, i.e. for a very large number of colors and increasing 't Hooft coupling, quark confinement is recovered. The low energy limit of the theory turns out to be a non-local Nambu–Jona-Lasinio (NJL) model. The effect of non-locality, arising from a gluon propagator that fits quite well to the profile of an instanton liquid, is to produce a phase transition from a chiral condensate to an instanton liquid, as the coupling increases with lower momentum. This phase transition suffices to move the poles of the quark propagator to the complex plane. As a consequence, free quarks are no longer physical states in the spectrum of the theory.

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1. Introduction

One of the most important open problems in physics is the question of quark confinement. Quarks are never seen free but only in bound states (cf. Ref. [1,2] and references therein). Several mechanisms have been proposed but none of them has ever been derived fully analytically from theory, i.e. quantum chromodynamics (QCD). An exception can be found in Kenneth G. Wilson's semianalytic approach to QCD regularized on the lattice for which an area law for confinement at strong couplings has been demonstrated [3]. Some criteria have been obtained for confinement in four dimensions. Kugo and Ojima were able to obtain a well-known criterion starting from a reformulation of BRST invariance [4,5]. Similarly, Nishijima and his collaborators pointed out some constraints to grant confinement [6–10]. A proof of confinement exists in supersymmetric models where a condensate of monopoles like in a type II superconductors is seen [11,12]. Indeed, the exact β function for the Yang-Mills theory is known for some supersymmetric and the non-supersymmetric models [13–16]. Lattice simulations have calculated the Yang-Mills theory beta function by using the RG evolution (in several schemes) in order to connect the regularization scale with the infrared behavior, cf. e.g. Ref. [17]. By tuning empirical parameters, the beta functions of the

models can be made consistent with the lattice results. Different confinement criteria and their overlapping regions are presented in Ref. [18].

Due to the discovery of Gribov copies [19] and their possible handling as proposed by Zwanziger [20], studies on confinement in Landau gauge seemed to indicate a gluon propagator running to zero in the infrared while the ghost propagator had to run to infinity faster than in the free case. This qualitative picture is essential to have an idea of the potential between quarks. Measures of the gluon and ghost propagators [21–23] and the spectrum [24,25] on the lattice have shown that in a non-Abelian gauge theory without fermions a mass gap appears, in evident contrast with the scenario devised by Gribov and Zwanziger that in the original formulation is not able to accommodate this mass gap. As shown in several theoretical works, the behavior seen on the lattice should be expected [26–31]. These works provide closed form formulas for the gluon propagator with a number of fitting parameters. Indeed, a closed analytical formula for the gluon propagator is an important element to obtain the low-energy behavior of QCD in a manageable effective theory to prove confinement. For the same aim, the behavior of the running coupling in the infrared limit is essential [32,33] (see also the review [34]). An instanton liquid picture seems to play a relevant role [35,36]. Confinement in its simplest form can be seen as the combined effect of a potential obtained from the Wilson loop of a Yang-Mills theory without fermions and the running coupling yielding a linearly increasing potential in agreement with lattice data [37]. Note that in spacetime 2+1 di-

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mensions, the theory is only marginally confining, as there is no running coupling and the potential increases only logarithmically. Still, also in this lower dimension confinement is granted [38].

An essential part in our understanding of confinement in QCD is strongly linked to a proper derivation of the low-energy limit of the theory. In this direction, a couple of seminal papers were written by Gerard 't Hooft for 1+1 spacetime dimensions [39,40]. Two important results were obtained by 't Hooft in these papers: 1) A good understanding of the low-energy behavior of the theory could be obtained in principle by considering the limit of the number of colors N_c running to infinity and keeping the product $N_c g^2$ constant, with $\alpha_s = g^2/(4\pi)$ the strong coupling. In this limit the coupling goes to zero, facilitating a perturbative approach. 2) In 1+1 spacetime dimensions the theory can be solved and provides the meson spectrum of the theory. One of the conclusions for 3+1 spacetime dimensions was that a discrete spectrum of the Hamiltonian can grant the appearance of condensates providing the right set-up for quark confinement through a string model [39,41]. Indeed, in Refs. [38,42] a discrete spectrum was obtained for a Yang-Mills theory without quarks, confirming lattice results. Without providing explicitly the spectrum, this was also proved mathematically [46].

In a series of works [31,47,48], it was recently proved that in the 't Hooft limit of a large number of colors the low energy limit of QCD is given by a non-local Nambu–Jona-Lasinio (NJL) model [49–52,57]. The local version of the NJL-model, as initially conceived in Refs. [49,50], does not confine. For bounded states obtained after bosonization [58], there is a threshold for the decay into quark and antiquark as free states that have never been observed. Non-locality can help to remove such a problem [59]. The aim of this paper is to provide evidence that the non-local NJL-model derived from QCD in Refs. [31,47,48] is indeed confining with the principles given in Refs. [59,60].

This paper is phenomenological in nature, and some relevant approximations are involved to solve the QCD equations in the low-energy limit. The most relevant of these is the 't Hooft limit N_c with the 't Hooft coupling $\lambda = N_c g^2$ kept finite but large. Accordingly, we expand in $1/\lambda$ and terminate the expansion at leading order, neglecting higher-order correlations between fermionic degrees of freedom caused by the gluonic field.

The paper is structured as follows. In Sec. 2, we derive the NJL-model from QCD and apply the bosonization and the mean field approximation, leading to the gap equation. In Sec. 3, we present the proof of quark confinement for the low-energy limit of QCD based on the gap equation we obtained previously. In Sec. 4 we give our conclusions.

2. Low-energy limit of QCD and NJL model

We consider the QCD lagrangian

$$\mathcal{L}_{QCD} = \sum_i \bar{q}_i (i\gamma^\mu D_\mu + m) q_i - \frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a - \frac{1}{2\xi} (\partial_\mu A_a^\mu) (\partial_\nu A_a^\nu) \quad (1)$$

where $D_\mu = \partial_\mu + igT_a A_\mu^a$ is the covariant derivative and $F_{\mu\nu}^a$ are the field strength tensor components. These can be obtained by $igT_a F_{\mu\nu}^a = [D_\mu, D_\nu]$. The sum over i is quite generic as it implies the sum over quark flavors and colors. Our Minkowskian metric reads $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

From the Euler–Lagrange equations, we obtain

$$0 = \frac{\partial \mathcal{L}_{QCD}}{\partial A_\nu^a} - \partial_\mu \frac{\partial \mathcal{L}_{QCD}}{\partial (\partial_\mu A_\nu^a)} \\ = \partial_\mu (\partial^\mu A_a^\nu - \partial^\nu A_a^\mu) + \frac{1}{\xi} \partial^\nu (\partial_\mu A_a^\mu) + g f_{abc} \partial_\mu (A_b^\mu A_c^\nu)$$

$$+ g f_{abc} (\partial^\mu A_b^\nu - \partial^\nu A_b^\mu) A_c^\mu + g^2 f_{abc} f_{cde} A_b^\mu A_d^\nu A_e^\mu \\ - g \sum_i \bar{q}_i \gamma^\nu T_a q_i, \\ 0 = \frac{\partial \mathcal{L}_{QCD}}{\partial \bar{q}_i} - \partial_\mu \frac{\partial \mathcal{L}_{QCD}}{\partial (\partial_\mu \bar{q}_i)} = (i\gamma^\mu D_\mu + m) q_i. \quad (2)$$

From the equations of motion we can derive, in principle, the full hierarchy of Dyson–Schwinger equations. We solve this hierarchy by a method proposed by Bender, Milton and Savage [61], recently exploited in Refs. [31,42,62–64]. Note that if a source term is added to the Lagrangian describing the vacuum expectation values, translational invariance is broken, as it is expressed by the separate arguments in the Green functions. In this case, the vacuum expectation values of products of field operators expressing those Green functions are nonvanishing even in the case of a single field operator. This gives sense to starting to solve the tower of Dyson–Schwinger equations with just this one-point Green function. In the end, setting the source term to zero restores the observable physical picture. This procedure is quite similar to the approach provided by the generating functional.

Accordingly, to the Lagrangian we add source terms like $A_\mu^a J_a^\mu$, $\bar{q}_i \eta_i$ and $\bar{\eta}_i q_i$. For the sake of simplicity, we omit details on BRST ghosts. After this addition we can evaluate the functional derivatives with respect to these sources. Such a procedure yields the Dyson–Schwinger equations [62]

$$\partial^2 A_\nu^{1a}(x) + g f_{abc} \left(\partial^\mu A_{\mu\nu}^{2bc}(x, x) + \partial^\mu A_\mu^{1b}(x) A_\nu^{1c}(x) \right. \\ \left. - \partial^\nu A_{\mu\nu}^{2bc}(x, x) - \partial_\nu A_\mu^{1b}(x) A_\nu^{1c}(x) \right) \\ + g f_{abc} \partial^\mu A_{\mu\nu}^{2bc}(x, x) + g f_{abc} \partial^\mu (A_\mu^{1b}(x) A_\nu^{1c}(x)) \\ + g^2 f_{abc} f_{cde} \left(g^{\mu\rho} A_{\mu\nu\rho}^{3bde}(x, x, x) \right. \\ \left. + A_{\mu\nu}^{2bd}(x, x) A_e^{1\mu}(x) + A_{\nu\rho}^{2eb}(x, x) A_d^{1\rho}(x) + A_{\mu\nu}^{2de}(x, x) A_b^{1\mu}(x) \right. \\ \left. + A_b^{1\mu}(x) A_\mu^{1d}(x) A_\nu^{1e}(x) \right) \\ = g \sum_i \gamma_\nu T^a q_{ii}^2(x, x) + g \sum_i \bar{q}_i^1(x) \gamma_\nu T^a q_i^1(x), \\ (i\partial - m) q_i^1(x) + g \gamma^\mu A_\mu^{1a}(x) T_a q_i^1(x) = 0, \quad (3)$$

where we have introduced the one-, two- and three-point functions as $A_\mu^{1a}(x) = \langle A_\mu^a(x) \rangle$, $A_{\mu\nu}^{2ab}(x, y) = \langle A_\mu^a(x) A_\nu^b(y) \rangle$ and $A_{\mu\nu\rho}^{3abc}(x, y, z) = \langle A_\mu^a(x) A_\nu^b(y) A_\rho^c(z) \rangle$ for the gauge fields, and $q_i^1(x) = \langle q_i(x) \rangle$ and $q_{ij}^2(x, y) = \langle q_i(x) q_j(y) \rangle$ for the quark fields. We can use the exact solutions already provided in Ref. [31] to write

$$A_\nu^{1a}(x) = \eta_\nu^a \phi(x), \quad A_{\mu\nu}^{2ab}(x, y) = \left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \right) \delta^{ab} \Delta(x - y), \quad (4)$$

where η_μ^a are the coefficients of the polarization vector with $\eta_\mu^a \eta_b^\mu = \delta_{ab}$, $\phi(x)$ is a scalar field and $\Delta(x - y)$ is the propagator of the scalar field. The ansatz we can afford so far, namely the gluon field as a constant polarization vector times a scalar function, is based on Refs. [29,30],¹ and can help us to reach up to essential statements on the confinement. One obtains

¹ The mapping between a quartic scalar field and the Yang-Mills theory has been a matter of discussion with the mathematician Terence Tao (Fields medallist) who accepted the proof given in Ref. [30] showing that such a mapping is exact in the Landau gauge but just asymptotic for other gauge choices for the coupling running to infinity.

$$\begin{aligned} & \eta_\nu^a \partial^2 \phi(x) + 2N_c g^2 \Delta(0) \eta_\nu^a \phi(x) + N_c g^2 \eta_\nu^a \phi^3(x) \\ &= g \sum_i \gamma_\nu T_a q_{ii}^2(x, x) + g \sum_i \bar{q}_i^1(x) \gamma_\nu T_a q_i^1(x), \\ & (i\gamma^\mu \partial_\mu - m_q) q_i^1(x) + g \gamma^\mu \eta_\mu^a T_a \phi(x) q_i^1(x) = 0. \end{aligned} \quad (5)$$

Using the properties of the symbols η , one has $\eta_\mu^a \eta_\mu^a = N_c^2 - 1$ and $\sum_i q_{ii}^2(x, x) = N_c N_f S(0)$. Accordingly, the first differential equation (5) takes the form

$$\begin{aligned} & \partial^2 \phi(x) + 2N_c g^2 \Delta(0) \phi(x) + N_c g^2 \phi^3(x) \\ &= \frac{g}{N_c^2 - 1} \left[N_c N_f \gamma^\nu \eta_\nu^a T_a S(0) + \sum_i \bar{q}_i^1(x) \gamma^\nu \eta_\nu^a T_a q_i^1(x) \right]. \end{aligned} \quad (6)$$

At this point we can consider the 't Hooft limit $N_c \rightarrow \infty$ with $\lambda := N_c g^2 \gg 1$ being finite but large. Next, we can rescale the space variable as $x \rightarrow \sqrt{N_c g^2} x$ and look for a perturbative series $\phi(x) = \phi_0(x) + (N_c g^2)^{-1} \phi_1(x) + O((N_c g^2)^{-2})$ in the 't Hooft coupling, yielding at leading order (after reverting the rescaling)

$$\partial^2 \phi_0(x) + 2\lambda \Delta(0) \phi_0(x) + \lambda \phi_0^3(x) = 0, \quad (7)$$

while the next-to-leading order yields

$$\begin{aligned} & \partial^2 \phi_1(x) + 2\lambda \Delta(0) \phi_1(x) + 3\lambda \phi_0^2(x) \phi_1(x) = \\ &= \frac{g}{N_c^2 - 1} \left[N_c N_f \gamma^\nu \eta_\nu^a T_a S(0) + \sum_i \bar{q}_i^1(x) \gamma^\nu \eta_\nu^a T_a q_i^1(x) \right]. \end{aligned} \quad (8)$$

Truncating the series expansion in inverse powers of $N_c g^2$ at the first order, one ends up with a NJL model. Higher orders would generate interactions with more than four fermions involved. An attempt in this direction was presented in Ref. [65] where the next-to-leading order terms turn out to depend on products of the gluon Green functions and higher powers of pairs of fermionic fields.

2.1. Zeroth order solution and Green function

As a constant, $m^2 = 2\lambda \Delta(0)$ can be considered as the mass squared. Even though the leading order equation $\partial^2 \phi_0(x) + m^2 \phi_0(x) + \lambda \phi_0^3(x) = 0$ is nonlinear, we have found a solution expressed by Jacobi's elliptic function [66],

$$\phi_0(x) = \mu \operatorname{sn}(k \cdot x + \theta | \kappa) \quad (9)$$

with

$$k^2 = \frac{1}{2} \lambda \mu^2 + m^2 \quad \text{and} \quad \kappa = \frac{m^2 - k^2}{k^2}, \quad (10)$$

where μ and θ are integration constants. $\operatorname{sn}(z|\kappa)$ is Jacobi's elliptic function of the first kind. Inserting this solution into the Green's equation

$$\left(\partial^2 + m^2 + 3\lambda \phi_0^2(x) \right) \Delta(x - y) = \delta^4(x - y) \quad (11)$$

for the two-point function obtained as the next element of the tower of Dyson-Schwinger equations, this equation can be solved in momentum space by

$$\begin{aligned} \tilde{\Delta}(p) &= - \sum_{n=0}^{\infty} \frac{B_n(\kappa)}{p^2 - m_n^2 + i\epsilon}, \\ B_n(\kappa) &= \frac{(2n+1)^2 \pi^3}{2(1-\kappa)K(\kappa)^3 \sqrt{\kappa}} \frac{(-1)^n e^{-(n+1/2)\varphi(\kappa)}}{1 - e^{-(2n+1)\varphi(\kappa)}}, \end{aligned} \quad (12)$$

where

$$\varphi(\kappa) = \frac{K^*(\kappa)}{K(\kappa)} \pi, \quad K^*(z) = K(1-z). \quad (13)$$

The corresponding mass spectrum reads

$$m_n = \frac{(2n+1)\pi}{2K(\kappa)} \sqrt{2k^2} =: (2n+1)m_G(\kappa). \quad (14)$$

This procedure ends up with a gap equation by inserting the Fourier transform of the propagator (12) back into $m^2 = 2\lambda \Delta(0)$, yielding

$$m^2 = -2\lambda \int \frac{d^4 p}{(2\pi)^4} \sum_{n=0}^{\infty} \frac{B_n(\kappa)}{p^2 - (2n+1)^2 m_G^2(\kappa) + i\epsilon}. \quad (15)$$

It can be shown that by this gap equation the spectrum of the theory without fermions is correctly given [42] in excellent agreement with lattice data. Note that the momentum scale k^2 of the one-point function and the momentum scale p^2 of the two-point function are independent. Because of this, in the following section we use an empirical value to fix k^2 .

In order to complete this section, we argue that the zeros of the gluon propagator are genuine glueball colorless states. We start by considering the correlation function for the scalar glueballs that is given by [43,44]

$$\mathcal{O}(x) = \langle F^{a\mu\nu}(x) F_{\mu\nu}^a(x) F^{b\rho\eta}(0) F_{\rho\eta}^b(0) \rangle. \quad (16)$$

Using methods explained in Ref. [31], one can see that according to Ref. [45] the four-point correlator defining the correlation function of the glueball can be reduced to convolutions over one- and two-point functions. As the one-point function has no poles but zeros, the poles of the glueball four-point correlator are given by the poles of the two-point correlator. Therefore, these poles represent true colorless glueball states.

2.2. First order solution

By convoluting the propagator Δ with the right hand side of Eq. (8) we obtain

$$\begin{aligned} \phi_1(x) &= \frac{g}{N_c^2 - 1} \int d^4 y \Delta(x - y) \\ &\times \left[N_c N_f \gamma^\nu \eta_\nu^a T_a S(0) + \sum_i \bar{q}_i^1(y) \gamma^\nu \eta_\nu^a T_a q_i^1(y) \right]. \end{aligned} \quad (17)$$

We observe that the first term is just a renormalization of the fermion mass and can be chosen to be zero via the condition $S(0) = 0$. The second term is the expected NJL interaction in the equation of motion of the quarks.

The solution $\phi(x)$ we obtained above can be inserted into Eq. (5). In the 't Hooft limit, we note that the term ϕ_0 is negligibly small compared to the NJL term ϕ_1 . One can see this by observing that $\phi_0 \sim \lambda^{1/4}$ and $\phi_1 \sim \lambda$. Therefore, in the strong coupling limit $\lambda \gg 1$ the equation for the one-point function of the quark has just the NJL term. We can write

$$\begin{aligned} & (i\gamma^\mu \partial_\mu - m_q) q_i^1(x) + \frac{g^2}{N_c^2 - 1} \sum_\eta \int d^4 y \Delta(x - y) \gamma^\mu \eta_\mu^a T_a q_i^1(x) \\ & \times \sum_j \left[\bar{q}_j^1(y) \gamma^\nu \eta_\nu^b T_b q_j^1(y) \right] = 0. \end{aligned} \quad (18)$$

Such an equation can be recognized as the Euler-Lagrange equation for the one-point function of the quark obtained from a NJL model with a non-local interaction [52,59]

$$\begin{aligned} \mathcal{L}_{\text{NJL}}'' &= \sum_i \bar{q}_i^1(x) (i\gamma^\mu \partial_\mu - m_q) q_i^1(x) \\ &+ \frac{g^2}{N_c^2 - 1} \sum_\eta \sum_i \left[\bar{q}_i^1(x) \gamma^\mu \eta_\mu^a T_a q_i^1(x) \right] \int d^4 y \Delta(x-y) \\ &\times \sum_j \left[\bar{q}_j^1(y) \gamma^\nu \eta_\nu^b T_b q_j^1(y) \right]. \end{aligned} \quad (19)$$

We see that $\sum_\eta \eta_\mu^a \eta_\nu^b = \delta_{ab} g_{\mu\nu}$, where η symbolizes the polarizations. Tracing out the color degrees of freedom with $\text{tr}(T_a T_a) = N_c C_F$, $C_F = (N_c^2 - 1)/(2N_c)$, and

$$\sum_i \bar{q}_i^1(x) \gamma^\mu q_i^1(x) = N_c \sum_i \bar{\psi}_i(x) \gamma^\mu \psi_i(x),$$

$$\sum_i \bar{q}_i^1(x) q_i^1(x) = N_c \sum_i \bar{\psi}_i(x) \psi_i(x),$$

with $\psi_i(x)$ being spinors in Dirac and flavor space only, we are led to the NJL Lagrangian

$$\begin{aligned} \mathcal{L}'_{\text{NJL}} &= \sum_i \bar{\psi}_i(x) (i\gamma^\mu \partial_\mu - m_q) \psi_i(x) \\ &+ \frac{N_c g^2}{2} \sum_i \left[\bar{\psi}_i(x) \gamma^\mu \psi_i(x) \right] \int d^4 y \Delta(x-y) \sum_j \left[\bar{\psi}_j(y) \gamma_\mu \psi_j(y) \right]. \end{aligned} \quad (20)$$

After a Fierz rearrangement of the quark fields we obtain (cf. e.g. Refs. [53–57])

$$\begin{aligned} \mathcal{L}'_{\text{NJL}} &= \sum_i \bar{\psi}_i(x) (i\gamma^\mu \partial_\mu - m_q) \psi_i(x) \\ &+ \frac{N_c g^2}{2} \int d^4 y \Delta(x-y) \sum_{i,j} \bar{\psi}_i(x) \psi_j(y) \bar{\psi}_j(y) \psi_i(x) \\ &+ \frac{N_c g^2}{2} \int d^4 y \Delta(x-y) \sum_{i,j} \bar{\psi}_i(x) i\gamma_5 \psi_j(y) \bar{\psi}_j(y) i\gamma_5 \psi_i(x) \\ &- \frac{N_c g^2}{4} \int d^4 y \Delta(x-y) \sum_{i,j} \bar{\psi}_i(x) \gamma^\mu \psi_j(y) \bar{\psi}_j(y) \gamma_\mu \psi_i(x) \\ &- \frac{N_c g^2}{4} \int d^4 y \Delta(x-y) \sum_{i,j} \bar{\psi}_i(x) \gamma^\mu \gamma_5 \psi_j(y) \bar{\psi}_j(y) \gamma_\mu \gamma_5 \psi_i(x). \end{aligned} \quad (21)$$

2.3. Bosonization

Γ_α are understood as a set of combined Dirac and flavor matrices given by 1 , $i\gamma_5$, γ_μ and $\gamma_\mu \gamma_5$ after the Fierz rearrangement and the flavor matrices $\mathbb{1}$ and $\frac{1}{2}\lambda_\alpha$ relating quarks of equal and different flavor i and j in adjoint representation. For Γ_α we have the conjugation rule $\gamma^0 \Gamma_\alpha^\dagger \gamma^0 = \Gamma_\alpha$, where α denotes the components of the adjoint flavor representation. Accordingly, the spinor $\psi(x)$ spans over all these spaces. As the coefficients of these two contributions are the same, the sum over these given $1 + 3 = 4$ degrees of freedom can be reinterpreted as a sum over the components of a four-vector. Next we apply the bosonization procedure shown in Ref. [57] by adding scalar–isoscalar and pseudoscalar–isovector mesonic fields at an intermediate space-time location $w = (x+y)/2$ as auxiliary fields $M_\alpha(w) = (\sigma(w); \vec{\pi}(w))$ coupled to the nonlocal fermionic currents. After Fierz rearrangement, this sums up to the NJL action

$$\begin{aligned} \mathcal{S}_{\text{NJL}} &= -\frac{N_c g^2}{2G^2} \int d^4 z \Delta(z) \int d^4 w M_\alpha^*(w) M^\alpha(w) \\ &+ \int d^4 x \left[\bar{\psi}(x) (i\gamma^\mu \partial_\mu - m_q) \psi(x) \right. \\ &\left. + \frac{N_c g^2}{2} \int d^4 y \Delta(x-y) \bar{\psi}(x) \Gamma_\alpha \psi(y) \bar{\psi}(y) \Gamma^\alpha \psi(x) \right] \end{aligned} \quad (22)$$

($G = 2 \int d^4 z \Delta(z) = 2\tilde{\Delta}(0)$). By performing a nonlocal functional shift

$$M_\alpha \left(\frac{x+y}{2} \right) \rightarrow M_\alpha \left(\frac{x+y}{2} \right) + G \bar{\psi}(x) \Gamma_\alpha \psi(y), \quad (23)$$

the nonlocal quartic fermionic interaction can be removed. Instead, the fermion field starts to interact nonlocally with the mesonic fields,

$$\begin{aligned} \mathcal{S}_{\text{NJL}} &= -\frac{N_c g^2}{2G^2} \int d^4 z \Delta(z) \int d^4 w M_\alpha^*(w) M^\alpha(w) \\ &+ \int d^4 x \bar{\psi}(x) (i\gamma^\mu \partial_\mu - m_q) \psi(x) \\ &- \frac{N_c g^2}{2G} \int d^4 x \int d^4 y \bar{\psi}(x) \Delta(x-y) \\ &\times \left(M_\alpha \left(\frac{x+y}{2} \right) + M_\alpha^* \left(\frac{x+y}{2} \right) \right) \Gamma^\alpha \psi(y). \end{aligned} \quad (24)$$

After Fourier transform, in momentum space one obtains

$$\begin{aligned} \mathcal{S}_{\text{NJL}} &= -\frac{N_c g^2}{4G} \int \frac{d^4 q}{(2\pi)^4} \tilde{M}_\alpha^*(q) \tilde{M}^\alpha(q) \\ &+ \int \frac{d^4 p}{(2\pi)^4} \tilde{\bar{\psi}}(p) (\not{p} - m_q) \tilde{\psi}(p) \\ &- \frac{N_c g^2}{2G} \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 p'}{(2\pi)^4} \tilde{\bar{\psi}}(p) \tilde{\Delta} \\ &\times \left(\frac{p+p'}{2} \right) \left(\tilde{M}_\alpha(p-p') + \tilde{M}_\alpha^*(p-p') \right) \Gamma^\alpha \tilde{\psi}(p'), \end{aligned} \quad (25)$$

where the symbols with tilde are used for the Fourier transformed quantities.

2.4. Mean field approximation

Out of the many different contributions obtained after Fierz rearrangement, the mean field approximation makes a choice that is phenomenologically justified. We can expand the physical mesonic fields $\tilde{\sigma}(p) = \bar{\sigma} + \delta\tilde{\sigma}(p)$ and $\tilde{\vec{\pi}}(p) = \delta\tilde{\vec{\pi}}(p)$ about the vacuum expectation value $\bar{\sigma} = \langle \sigma \rangle$ where the expansion coefficient to zeroth order is the mean field approximation. This approximation without any information about possible correlations is sufficient for our means as it leads directly to the mass gap equation. In this approximation one obtains the simplified NJL action

$$\begin{aligned} \mathcal{S}_{\text{NJL}} &= -\frac{N_c g^2 \bar{\sigma}^2}{4G} V^{(4)} + \int \frac{d^4 p}{(2\pi)^4} \tilde{\bar{\psi}}(p) (\not{p} - m_q) \tilde{\psi}(p) \\ &- \frac{N_c g^2 \bar{\sigma}}{G} \int \frac{d^4 p}{(2\pi)^4} \tilde{\bar{\psi}}(p) \tilde{\Delta}(p) \tilde{\psi}(p) \\ &= -\frac{N_c g^2 \bar{\sigma}^2}{4G} V^{(4)} + \int \frac{d^4 p}{(2\pi)^4} \tilde{\bar{\psi}}(p) (\not{p} - M_q(p)) \tilde{\psi}(p) \end{aligned} \quad (26)$$

with $G = 2\tilde{\Delta}(0)$ and the unit space-time volume $V^{(4)}$, where

$$M_q(p) = m_q + \frac{N_c g^2}{G} \tilde{\Delta}(p) \bar{\sigma} = m_q + \frac{N_c g^2 \tilde{\Delta}(p)}{2\tilde{\Delta}(0)} \bar{\sigma} \quad (27)$$

is the dynamical mass of the quark. The bosonization procedure yields

$$\frac{\mathcal{S}_{\text{bos}}}{V^{(4)}} = -\frac{N_c g^2 \bar{\sigma}^2}{4G} - i \int \frac{d^4 p}{(2\pi)^4} \ln \det(\not{p} - M_q(p)), \quad (28)$$

where \det denotes the direct product of a functional and an analytical determinant, the former in the Fock space transition between space-time points x and y , the latter in the Dirac and flavor indices. On the other hand, one has

$$\ln \det(\not{p} - M_q(p)) = \text{tr} \ln(\not{p} - M_q(p)) = \frac{1}{2} 4N_f \ln(p^2 - M_q^2(p)). \quad (29)$$

Taking the variation of the action \mathcal{S}_{bos} with respect to $\bar{\sigma}$ yields the latter quantity. Accounting for the dependence of $M_q(p)$ on $\bar{\sigma}$, we have

$$0 = -\frac{N_c g^2 \bar{\sigma}}{2G} - 2iN_f \int \frac{d^4 p}{(2\pi)^4} \frac{2M_q(p)}{p^2 - M_q^2(p)} \frac{N_c g^2}{G} \tilde{\Delta}(p) \\ \Rightarrow \bar{\sigma} = -8iN_f \int \frac{d^4 p}{(2\pi)^4} \frac{\tilde{\Delta}(p) M_q(p)}{p^2 - M_q^2(p)}. \quad (30)$$

Finally, this result can be re-inserted to Eq. (27) to obtain the dynamical mass equation

$$M_q(p) = m_q - 4iN_f N_c g^2 \frac{\tilde{\Delta}(p)}{\tilde{\Delta}(0)} \int \frac{d^4 p'}{(2\pi)^4} \frac{\tilde{\Delta}(p') M_q(p')}{p'^2 - M_q^2(p')}. \quad (31)$$

In this way, we have derived this equation directly from the QCD Lagrangian. Following Ref. [48] we do not consider the explicit dependence of $M_0 = M_q$ on the momentum. We use this choice to obtain a qualitative picture of the dynamics, giving up the possibility of wavefunction renormalization and the possibility to estimate of the size of the error. After that, performing the Wick rotation, we obtain the mass gap equation in Euclidean space,

$$M_0 = m_q + C\alpha_s \int_0^{\Lambda^2} \frac{M_0 \tilde{\Delta}(p') p'^2 dp'^2}{p'^2 + M_0^2}, \quad C = 4N_f N_c. \quad (32)$$

3. Quark confinement

The idea to understand quark confinement is strongly linked to the expected behavior of the roots of the gap equation (32), i.e. the poles of the quark propagator. The idea presented here is identical with the idea presented in Ref. [68], though without employing a general model for the non-locality. To represent physical propagating degrees of freedom, for these poles one should expect solutions on the real axis. The effect of the gluonic interaction is to move such poles in the complex plane so that no decay into such degrees of freedom is ever expected and the free quarks never propagate. This moves the solution from a chiral condensate phase to a confining phase for quarks at increasing coupling. Indeed, a full comprehension of such roots can only be achieved through the non-approximated gluon propagator $\tilde{\Delta}(p)$. Therefore, we performed an analysis of the lowest zero of $\tilde{\Delta}(p)$, finding out that for $M_0 < 0.39m_0$ there are two distinct real zeros while above these two zeros are given by two complex conjugate numbers. This can be seen in Fig. 1. The threshold at $0.39m_0$ should be seen as the point beyond which, mathematically speaking, a chiral condensate could possibly appear. We observe that the result depends critically on the mass gap value m_0 . In turn, this value depends

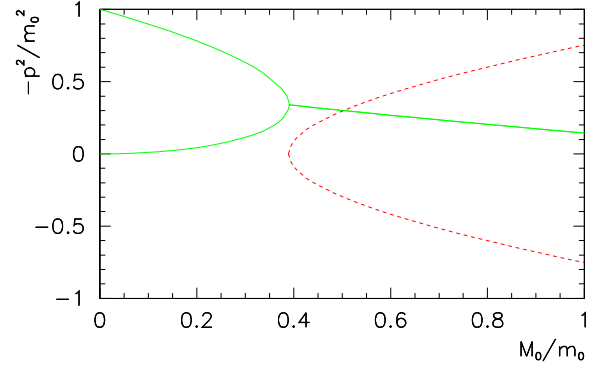


Fig. 1. The two lowest zeros of $-p^2 = M_q(p)$ in Euclidean domain in units of m_0^2 , splitted into real parts (green straight lines) and imaginary parts (red dashed lines), in dependence on the ratio M_0/m_0 . The zeros become complex for approximately $M_0/m_0 > 0.39$.

on an arbitrary integration constant and, therefore, should be fixed by the experiment. This situation is similar to that of the constant Λ entering in asymptotic freedom, and it is possible that these two constants are related. Our best choice to fix this value is via the mixed gluonic-quark state $f_0(500)$ that could in principle be identified with the σ meson in the NJL model, giving rise to the breaking of chiral symmetry.

The gap equation (32) can be simplified by choosing $m_q = 0$, and the factor M_q can be cancelled from the numerator to remove the trivial case. This technique permits to find the fixed point solution of the given iterative integral equation avoiding the awkward issue to resume all the iterates. One obtains

$$1 = C\alpha_s \int_0^{\Lambda^2} \frac{\Delta(p') p'^2 dp'^2}{p'^2 + M_0^2} \\ = C\alpha_s \sum_{n=0}^{\infty} \int_0^{\Lambda^2} \frac{B_n p'^2 dp'^2}{(p'^2 + (2n+1)^2 m_0^2)(p'^2 + M_0^2)} \\ = C\alpha_s \sum_{n=0}^{\infty} \int_0^{\Lambda^2} \frac{B_n dp'^2}{(2n+1)^2 m_0^2 - M_0^2} \\ \times \left(\frac{(2n+1)^2 m_0^2}{p'^2 + (2n+1)^2 m_0^2} - \frac{M_0^2}{p'^2 + M_0^2} \right) \\ = C\alpha_s \sum_{n=0}^{\infty} \frac{B_n}{(2n+1)^2 m_0^2 - M_0^2} \\ \times \left((2n+1)^2 m_0^2 \ln \left(1 + \frac{\Lambda^2}{(2n+1)^2 m_0^2} \right) - M_0^2 \ln \left(1 + \frac{\Lambda^2}{M_0^2} \right) \right). \quad (33)$$

For $\Lambda = 1 \text{ GeV}$ and $m_0 = 0.512(15) \text{ GeV}$ one obtains $M_0 = 0.427(29) \text{ GeV}$ as in Ref. [48] which is clearly above the threshold $0.39m_0 = 0.2 \text{ GeV}$ and, therefore, indicates quark confinement.

3.1. Understanding quark confinement

As we have seen, with increasing coupling of the theory the quark confinement arises at the point where the chiral condensates of different flavors perform a transition to a confined phase with an instanton liquid of gluon degrees of freedom. Neglecting the bare quark masses we have found the critical effective quark mass for which such a transition happens. Our choice of the ground state for the gluon field represents quite well a Fubini instanton [67].

Therefore, we expect that the chiral condensate changes into an instanton ground state that could condensate into a liquid.

The presence of the instanton liquid removes single quarks from the physical spectrum of the theory. From the mathematical point of view this means that the poles in the quark propagator become complex. The vacuum of the theory appears to undergo a series of phase transitions while the gluon sector generates a mass gap by itself in a dynamical way. The presence of the mass gap in the gluon sector is pivotal for the appearance of the phase transitions in the quark sector and, in the last instance, to the confinement of quarks.

4. Conclusions and outlook

There are different approaches to understand the confinement of quarks. One of these is given by solutions of the gap equation of the dynamical quark mass. With a reasonable UV cutoff of $\Lambda = 1 \text{ GeV}$ and the glueball spectrum starting at the mass $m_0 = 0.512(15) \text{ GeV}$ of the $f_0(500)$ resonance, the gap equation provides a value $M_0 = 0.427(29) \text{ GeV}$ for the dynamical quark mass which turns out to be too large to allow for real valued poles of the quark propagator. As a consequence, free quarks are no longer physical states of the theory and the quarks can be expected to be confined. Even though the multiple approximations applied in this approach do not allow for quantitative estimates, we have shown how this scenario is realized in the low-energy regime of QCD by taking into account the 't Hooft limit. By doing so, the low-energy limit of QCD turns out to be a well-defined non-local NJL model with all the parameters obtained from QCD. This entails a scenario where several condensates are formed that are expected to provide confinement in a regime of very low momentum and strong coupling.

Having a low-energy limit of QCD permits to do several computations to be compared with experiments. Indeed, our first application was to the $g - 2$ problem [48] with a very satisfactory agreement with data. Further research has to show whether and to what extent our description of quark confinement depends on the given parameter values and whether a stricter derivations of observables allows for a comparison with experiments.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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