

Phase transition in von Neumann entropy from replica symmetry breaking

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ABSTRACT: We study the entanglement transition in monitored Brownian SYK chains in the large- N limit. Without measurement the steady state n -th Rényi entropy is obtained by summing over a class of solutions, and is found to saturate to the Page value in the $n \rightarrow 1$ limit. In the presence of measurements, the analytical continuation $n \rightarrow 1$ is performed using the cyclic symmetric solution. The result shows that as the monitoring rate increases, a continuous von Neumann entanglement entropy transition from volume-law to area-law occurs at the point of replica symmetry unbreaking.

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1 Introduction

Starting from an unentangled product state, a chaotic Hamiltonian or circuit generates entanglement between different parts of the system, and eventually leads to a state with volume-law entanglement at late time. On the contrary, local measurements extract information and diminish the entanglement between the measured qubits and the rest of the system. The competition between these two effects leads to a transition in the degree of entanglement in the long-time steady state, the so-called measurement-induced phase transition [1–5].

Following this realization, there has been a surge of investigations of this transition and of related phenomena [6–24]. In principle, one can diagnose transitions in entanglement properties by introducing replicas and using them to compute the n -th Rényi entropy of the long-time steady state. In this context, it was found that the entanglement transition can be understood as an unbreaking transition of replica symmetry [6, 8, 9, 14]. Nevertheless, as the von Neumann entanglement entropy is obtained in the $n \rightarrow 1$ limit, where n is the

number of replicas, how the replica symmetry unbreaking defined for $n \geq 2$ interplays with the replica limit $n \rightarrow 1$ is still an outstanding question. Part of the difficulty lies in the hardness of analytical continuation. In the context of Haar random hybrid circuits, the transition can be effectively mapped to a statistical model in the replica space, which in the limit of infinite local Hilbert space dimension is described by bond percolation [3, 8, 9], and the replica limit $n \rightarrow 1$ is notoriously subtle even in this limit. In this paper, we fill this gap by introducing a monitored Brownian SYK model [21, 25–33], and calculating the von Neumann entanglement entropy using the replica trick.

This model, which features continuous weak monitoring, admits a path integral representation, and consequently allows saddle-point analysis in the large- N limit. We show explicitly how summing over a class of replica symmetry broken solutions can correctly produce the Page value [34] of the entanglement entropy in the absence of monitoring. A crucial step in the calculation is a mapping of part of the saddle point evaluation to the evaluation of a transition amplitude of a quantum mechanical model. Turning on measurement decreases the degree of replica symmetry breaking and eventually leads to its unbreaking. We then take the $n \rightarrow 1$ limit based on the cyclic symmetric solution and extract the behavior of the entanglement entropy per site as a function of measurement strength. The transition in the von Neumann entropy density in the $n \rightarrow 1$ limit coincides with the unbreaking transition of replica symmetry for $n \geq 2$.

The paper is organized as follows. In section 2, we define the quasi entropy of a monitored system [35]. Because of the non-unitary nature of measurements, a proper density matrix needs to be normalized after each measurement. This leads to difficulties in analytical calculations as the normalization is a nonlinear procedure that depends on the measurement outcome. The quasi entropy is the Rényi entropy of the unnormalized density matrix weighted such that it vanishes for a pure state. The quasi entropy has two crucial properties: it admits a path integral representation and, like the Rényi entropy, the $n \rightarrow 1$ limit gives the von Neumann entanglement entropy averaged over quantum trajectories.

In section 3, we introduce our model which contains both a chaotic unitary part and a measurement part. The unitary part is given by two independent Brownian SYK chains. The Brownian SYK Hamiltonian is able to efficiently generate entanglement among different parts of the chain. The measurement part amounts to weakly projecting onto the Fermi parity of the complex fermions formed by the left-right pair of Majoranas at each site. Importantly, it is a local measurement that extracts information about a single site of the left-right coupled system, and its dark state corresponds to a product state between different sites with definite Fermi parity. Therefore, the competition between the two parts will result in a measurement-induced entanglement transition.

In section 4, we first study a simpler version of our model that only consists of the unitary part. It is given by two coupled Brownian SYK clusters. After properly defining the initial state, the entanglement entropy between the two clusters can be obtained via a large- N path integral in the replica space. We find a class of saddle points that are described by permutation matrices. The permutation structure has appeared in the entanglement entropy of Haar random circuits, too [36]. The calculation of the onshell action can be done by decomposing the permutation into cycles, and then mapping each cycle to a transition

amplitude of a Kitaev chain [37]. An interesting point arising in the calculation is that the Pfaffian effectively counts the number of cycles in the solution given by a general permutation matrix. Then, by summing over this class of saddle points, we get a result that predicts maximal entanglement at leading order with subleading corrections consistent with the Page value [34] and the symmetry of the model.

In section 5, we generalize the calculation into two Brownian SYK chains with a measurement that couples the two chains. The aforementioned class of saddle points that saturate the Page value are changed in the presence of the measurement. When the measurement rate gets larger than a critical value, all saddle points reduce to a diagonal solution where no correlation between different replica exists, i.e., the replica symmetry is restored. The calculation of quasi entropy now involves two coupled Kitaev chains, where the hopping within each chain is given by the correlation between different replicas, and the coupling between two chains is proportional to the measurement rate. After the coupling increases to a critical point, the hopping within each chain vanishes, and the eigenmodes become local. We take an analytical continuation of the cyclic symmetric solution to get the von Neumann entanglement entropy. The entropy density is obtained, and it shows a continuous phase transition right at the point where the replica symmetry is restored.

We conclude the paper in section 6, where we also study the Landau-Ginzburg theory. We leave the technical details of the permutation operator that is used to calculate the quasi entropy, the explicit saddle-point solution for $n = 2$, and the product of special trigonometric functions that is used to evaluate the transition amplitude to the appendices.

2 Quasi entropy and trajectory averaged entanglement entropy

We consider time evolution generated by unitary evolution and measurements, and are interested in the entanglement entropy of the long-time steady state. The state itself is not steady in a microscopic sense, but its entanglement structure is expected to stabilize at long times. Importantly, to really assess the entanglement properties of the resulting state, one needs to keep track of each measurement record. A specific evolution with given measurement outcomes forms a quantum trajectory. Ultimately, we will calculate the von Neumann entanglement entropy averaged over quantum trajectories.

As the unitary evolution part of the evolution is well known, here we focus on the measurement part. The measurement is described by a set of Kraus operator $\{K_\nu\}$, $\sum_\nu K_\nu^\dagger K_\nu = 1$ [8, 38], where ν numerates possible measurement outcomes. Starting from an initial density matrix ρ , given a sequence of measurement outcomes $\boldsymbol{\nu} = \nu_1 \nu_2 \dots \nu_m$, the corresponding quantum trajectory is

$$\tilde{\rho}_\nu = K_{\nu_m} \dots K_{\nu_2} K_{\nu_1} \rho K_{\nu_1}^\dagger K_{\nu_2}^\dagger \dots K_{\nu_m}^\dagger. \tag{2.1}$$

Note that $\tilde{\rho}_\nu$ is an unnormalized density matrix, whose trace gives the probability of the measurement outcome $\boldsymbol{\nu}$. Summing the probabilities of all possible measurement outcomes gives unity, i.e., $\sum_\nu \text{Tr}(\tilde{\rho}_\nu) = 1$. We are interested in calculating the quasi entropy of bipartite system $A\bar{A}$ [35],

$$S_A^{(n)} = \frac{1}{1-n} \log \frac{\sum_\nu \text{Tr}[(\tilde{\rho}_\nu)^{\otimes n} M_{\text{cyc}}(A)]}{\sum_\nu (\text{Tr}[\tilde{\rho}_\nu])^n}, \tag{2.2}$$

where $M_{\text{cyc}}(A)$ is the cyclic permutation operator acting on the subsystem A of the n replicas. Note that the summation over quantum trajectories appears both in the numerator and denominator.

Although the quasi entropy is distinct from the Rényi entropy of the normalized state with measurement outcome ν , the limit $n \rightarrow 1$ of the quasi entropy converges to the averaged von Neumann entanglement entropy of the normalized state. To show this, recall that the Rényi entropy of subsystem A for a given quantum trajectory ν is

$$S_A^{(n)}(\nu) = \frac{1}{1-n} \log \frac{\text{Tr}[(\tilde{\rho}_\nu)^{\otimes n} M_{\text{cyc}}(A)]}{(\text{Tr}[\tilde{\rho}_\nu])^n}. \quad (2.3)$$

The von Neumann entanglement entropy can be obtained by analytically continuing the Rényi entropy to $n \rightarrow 1$, i.e., $S_A(\nu) = \lim_{n \rightarrow 1} S_A^{(n)}(\nu)$. We have the following relation

$$\frac{\text{Tr}[(\tilde{\rho}_\nu)^{\otimes n} M_{\text{cyc}}(A)]}{(\text{Tr}[\tilde{\rho}_\nu])^n} \approx 1 + (1-n)S_A(\nu) + \mathcal{O}((n-1)^2). \quad (2.4)$$

Using this relation, we can take $n \rightarrow 1$ limit of the quasi entropy defined in (2.2),

$$\begin{aligned} \lim_{n \rightarrow 1} S_A^{(n)} &= \lim_{n \rightarrow 1} \frac{1}{1-n} \log \frac{\sum_\nu (1 + (1-n)S_A(\nu)) (\text{Tr}[\tilde{\rho}_\nu])^n}{\sum_\nu (\text{Tr}[\tilde{\rho}_\nu])^n} \\ &= \lim_{n \rightarrow 1} \frac{1}{1-n} \log \sum_\nu (1 + (1-n)S_A(\nu)) \text{Tr}[\tilde{\rho}_\nu] \\ &= \lim_{n \rightarrow 1} \frac{1}{1-n} \log \left(1 + (1-n) \sum_\nu \text{Tr}[\tilde{\rho}_\nu] S_A(\nu) \right) \\ &= \sum_\nu \text{Tr}[\tilde{\rho}_\nu] S_A(\nu), \end{aligned} \quad (2.5)$$

where in the first step we use (2.4) and in the second step and third step we use $\sum_\nu \text{Tr}[\tilde{\rho}_\nu] = 1$. Therefore, the $n \rightarrow 1$ limit of the quasi entropy is the quantum trajectory averaged von Neumann entanglement entropy. In the following, we will calculate the quasi entropy for arbitrary n , and then take the $n \rightarrow 1$ limit to get the von Neumann entanglement entropy.

3 Model and setup

Our model consists of two coupled Brownian SYK chains, with a unitary part and non-unitary monitoring part. The unitary evolution is governed by the following Brownian SYK Hamiltonian describing left (L) and right (R) chains,

$$H(\psi) = \sum_{x;a=L,R} \left(\sum_{i < j} i J_{a,ij}^x(t) \psi_{x,a,i} \psi_{x,a,j} + \sum_{j_1, \dots, j_q} i^{q/2} U_{a,j_1 \dots j_q}^{x,x+1}(t) \psi_{x,a,j_1} \dots \psi_{x,a,j_{q/2}} \psi_{x+1,a,j_{q/2+1}} \dots \psi_{x+1,a,j_q} \right), \quad (3.1)$$

where $\psi_{x,a,i}$ $i = 1, \dots, N$ denotes i -th of N Majorana fermions at each site $x = 1, \dots, L$ of the $a = L, R$ chains, $\{\psi_{x,a,j}, \psi_{x',a',j'}\} = \delta_{xx'} \delta_{aa'} \delta_{jj'}$. L is the number of sites (it should not be confused with the L chain), and periodic boundary conditions in real space are assumed

in this paper. $J_{a,ij}^x$ is two Majorana coupling within each site, and U_{a,j_1,\dots,j_q}^x is q Majorana interaction between nearest-neighbor sites. Here we consider $q = 4k, k \in \mathbb{Z}^+$ (\mathbb{Z}^+ denotes positive integers) to conserve local Fermi parity. The couplings in the left and right chains are *independent* Gaussian variables with mean zero and variances

$$\overline{J_{a,ij}^x(t_1)J_{a',ij}^{x'}(t_2)} = \frac{4J}{N}\delta(t_{12})\delta_{aa'}\delta^{x,x'}, \quad (3.2)$$

$$\overline{U_{a,j_1\dots j_q}^{x,x+1}(t_1)U_{a',j_1\dots j_q}^{x',x'+1}(t_2)} = \frac{2^q(q/2)!^2U}{qN^{q-1}}\delta(t_{12})\delta_{aa'}\delta^{x,x'}, \quad (3.3)$$

where $t_{12} = t_1 - t_2$. The Dirac δ function $\delta(t_{12})$ indicates the Brownian nature of the couplings. In the unitary evolution, the left and right chains do not couple, but the evolution does scramble information over the entire individual chains [23].

To describe the monitoring part, we first divide the continuous time into infinitesimal steps and define proper Kraus operators, then we take the continuum limit to get an effective action description. In each infinitesimal time step δt , we consider the following measurement. The local measurement operator couples the L and R fermions at each site, as described by the Kraus operators [21]

$$\{K_1^{x,i}, K_2^{x,i}\} = \left\{ \pi_{x,i}^- + \sqrt{1 - \mathfrak{s}^2} \pi_{x,i}^+, \mathfrak{s} \pi_{x,i}^+ \right\}, \quad (3.4)$$

where $\pi_{x,i}^\pm = \frac{1}{2}(1 \mp i2\psi_{x,L,i}\psi_{x,R,i})$ is the projection onto one of the Fermi parity eigenstates and $0 \leq \mathfrak{s} \leq 1$ is the measurement strength. We will see in the following that the appropriate continuous monitoring strength should be $\mathfrak{s} \propto \sqrt{\delta t}$. The Kraus operators satisfy the normalization condition $\sum_{j=1}^2 (K_j^{x,i})^\dagger K_j^{x,i} = 1$.

During a single time step in δt , the evolution of the unnormalized density matrix for a quantum trajectory with measurement outcome $\nu_{x,i}$ for each flavor at each site is

$$\tilde{\rho}_\nu = (\otimes_{x,i} K_{\nu_{x,i}}) \rho (\otimes_{x,i} K_{\nu_{x,i}}^\dagger), \quad (3.5)$$

where on the right-hand side we suppress the superscript for the flavor and site in Kraus operators $K_j^{x,i}$ as they can be inferred from the measurement outcome. The quasi entropy requires n replicas, i.e.,

$$\tilde{\rho}_\nu^{\otimes n} = \otimes_{\alpha=1}^n (\otimes_{x,i} K_{\nu_{x,i}}^\alpha) \rho^\alpha (\otimes_{x,i} K_{\nu_{x,i}}^{\alpha\dagger}), \quad (3.6)$$

where we have introduced a new superscript Greek letter $\alpha = 1, \dots, n$ to denote different replicas (it should not be confused with the original superscript of the Kraus operator that is used to denote site and flavor). We will use the superscript Greek letter as the replica index throughout the paper. According to the definition of quasi entropy, we can change the order of the trajectory average and the trace over the density matrix, and consider the trajectory average first, namely, $\sum_\nu \text{Tr}[(\tilde{\rho}_\nu)^{\otimes n} \dots] = \text{Tr}[\sum_\nu (\tilde{\rho}_\nu)^{\otimes n} \dots]$ for any \dots independent of ν . The effect of the monitoring operator on the replicas is given by the following operator,

$$\begin{aligned} & \sum_\nu \otimes_{\alpha=1}^n (\otimes_{x,i} K_{\nu_{x,i}}^\alpha) \otimes (\otimes_{x,i} K_{\nu_{x,i}}^{\alpha\dagger}) \\ & = \otimes_{x,i} \sum_{\nu_{x,i}=1,2} \otimes_{\alpha=1}^n \otimes_{s=\pm} K_{\nu_{x,i},s}^\alpha, \end{aligned} \quad (3.7)$$

where we have introduced a Keldysh-like contour index $s = \pm$ to denote $K_{\nu_{x,i},+}^\alpha = K_{\nu_{x,i}}^\alpha$, and $K_{\nu_{x,i},-}^\alpha = K_{\nu_{x,i}}^{\alpha\dagger}$. Finally, the average over measurement outcomes is [21]

$$\begin{aligned} & \otimes_{x,i} \sum_{\nu_{x,i}=1,2} \otimes_{\alpha,s} K_{\nu_{x,i},s}^\alpha \approx \otimes_{x,i} \left(1 - \frac{\mathfrak{s}^2}{2} \sum_{\alpha=1}^n \sum_{s=\pm} \pi_{x,i,s}^{+,\alpha} \right) \\ & \approx \otimes_{x,i} \exp \left(-\frac{\mathfrak{s}^2}{2} \sum_{\alpha,s} \pi_{x,i,s}^{+,\alpha} \right) \\ & = \exp \left(\frac{\delta t \mu}{2} \sum_{x,i,\alpha,s} i \psi_{x,L,i,s}^\alpha \psi_{x,R,i,s}^\alpha \right), \end{aligned} \tag{3.8}$$

in which we have used the relation $\pi_{x,a,j}^+ + \pi_{x,a,j}^- = 1$ and assumed $\mathfrak{s} \ll 1$ and kept orders up to $\mathcal{O}(\mathfrak{s}^2)$.¹ In the last line, we introduce $\mu = \frac{\mathfrak{s}^2}{\delta t}$ as the effective monitoring strength. When the continuum limit $\delta t \rightarrow 0$ is taken, μ is kept fixed. This means that the appropriate measurement strength is $\mathfrak{s} \propto \sqrt{\delta t}$. Constants are neglected because they will not affect the dynamics. After taking the continuum limit, we arrive at the following action description of the monitoring,

$$-I_\mu = \int dt \frac{i\mu}{2} \sum_{x,i,\alpha,s} \psi_{x,L,i,s}^\alpha(t) \psi_{x,R,i,s}^\alpha(t), \tag{3.9}$$

where the summation over time steps results in an integral in the continuum limit.

Since our model contains not only trajectory average but also disorder average due to the Brownian coupling, we generalize the quasi entropy to also include disorder average,

$$S_A^{(n)} = \frac{1}{1-n} \log \frac{\mathbb{E} \text{Tr}[(\tilde{\rho}_\nu)^{\otimes n} M_{\text{cyc}}(A)]}{\mathbb{E}(\text{Tr}[\tilde{\rho}_\nu])^n}, \tag{3.10}$$

where \mathbb{E} denotes both the trajectory average and the disorder average in (3.2).

4 Entanglement entropy in two coupled Brownian SYK clusters

Before we consider the full calculation of entanglement entropy with monitoring, let us first consider a simpler case without measurement: two coupled Brownian SYK clusters, considered as a single chain with only two sites called $x = 1, 2$. Because the R chain plays no role in this calculation, we will suppress the chain index in this section, so the relevant fermions are $\psi_{x,j} = \psi_{x,L,j}$. We will divide the system into two parts, with all fermions at site $x = 1$ in A and all fermions at site $x = 2$ in the complement \bar{A} , and refer to $x = 1$ as $x \in A$, or simply A , and $x = 2$ as $x \in \bar{A}$ or simply \bar{A} .

4.1 Rényi entropy of a time-evolved EPR state

We consider the following setup: starting from the tensor product of thermofield double (TFD) states in each of the clusters ($x = 1, 2$), we compute the quasi entropy between

¹In this limit, the results from the second measurement operator are neglected. So it is effectively a postselection on the first measurement outcome. Actually for replica number $n \leq 2$, the second measurement operator dominates. Thus, the calculation of the von Neumann entropy is for the scenario of postselection, and the transition is the forced measurement-induced phase transition.

the two clusters at late time [32, 39–41]. Because Brownian random interactions do not conserve energy, we simply consider an infinite temperature TFD state or, equivalently, a fermionic EPR state for each cluster. To describe such a state, we double the Hilbert space by introducing two copies of the fermions, the original fermions $\psi_{x,j}$ and copies $\chi_{x,j}$, for both clusters $x = 1, 2$. The initial density matrix is given by

$$\rho = |\text{EPR}\rangle\langle\text{EPR}|, \quad (\psi_{x,j} + i\chi_{x,j})|\text{EPR}\rangle = 0, \quad (4.1)$$

$\forall x = 1, 2, \forall j = 1, \dots, N$. The time evolution is generated by the sum of ψ and χ Hamiltonians, $H(\psi)$ and $H(-i\chi)$. This choice implies that $H(\psi)|\text{EPR}\rangle = H(-i\chi)|\text{EPR}\rangle$. The time evolved density matrix becomes

$$\rho(T) = U(T)\rho U^\dagger(T), \quad (4.2)$$

where $U(T) = \mathcal{T}e^{-i\int_0^{T/2} dt(H(\psi)+H(-i\chi))}$, \mathcal{T} denotes time ordering. This joint ψ - χ evolution of the EPR state for time $T/2$ is equivalent to just evolving with $U_\psi(T) = \mathcal{T}e^{-i\int_0^T dtH(\psi)}$ or $U_\chi(T) = \mathcal{T}e^{-i\int_0^T dtH(-i\chi)}$ for time T by virtue of the identity $H(\psi)|\text{EPR}\rangle = H(-i\chi)|\text{EPR}\rangle$.

Without measurement, the quasi entropy is same as annealed average of the Rényi entropy,

$$S_A^{(n)} = \frac{1}{1-n} \log \mathbb{E} \text{Tr}[\rho^{\otimes n} M_{\text{cyc}}(A)]. \quad (4.3)$$

For the EPR state, this quantity is related to the two-point function of cyclic permutation operators. Notice that the cyclic permutation operator factorizes, $M_{\text{cyc}}(A) = M_{\text{cyc}}^\psi(A) \otimes M_{\text{cyc}}^\chi(A)$ for the doubled Hilbert space, where each of them is given by (see appendix A)

$$M_{\text{cyc}}^\psi(A) = \prod_{x \in A} \prod_{j=1}^N e^{\frac{\pi}{2} \psi_{x,j}^1 \psi_{x,j}^2} e^{\frac{\pi}{2} \psi_{x,j}^3 \psi_{x,j}^4} \dots e^{\frac{\pi}{2} \psi_{x,j}^{n-3} \psi_{x,j}^{n-2}} e^{\frac{\pi}{2} \psi_{x,j}^{n-1} \psi_{x,j}^n}, \quad (4.4)$$

where the superscript denotes replica index, and the same for the χ fermions. Then the trace can be cast into

$$\begin{aligned} & \text{Tr}[\rho(T)^{\otimes n} M_{\text{cyc}}(A)] \\ &= \langle \text{EPR} |^{\otimes n} U_\chi^\dagger(T)^{\otimes n} [M_{\text{cyc}}^\psi(A) \otimes M_{\text{cyc}}^\chi(A)] U_\chi(T)^{\otimes n} | \text{EPR} \rangle^{\otimes n} \\ &= \langle \text{EPR} |^{\otimes n} M_{\text{cyc}}^{\chi\dagger}(A) U_\chi^\dagger(T)^{\otimes n} M_{\text{cyc}}^\chi(A) U_\chi(T)^{\otimes n} | \text{EPR} \rangle^{\otimes n} \\ &= 2^{-nN/2} \text{Tr}[M_{\text{cyc}}^{\chi\dagger}(A) U_\chi^\dagger(T)^{\otimes n} M_{\text{cyc}}^\chi(A) U_\chi(T)^{\otimes n}], \end{aligned} \quad (4.5)$$

where in the first line we use $\rho(T) = U_\chi(T)|\text{EPR}\rangle\langle\text{EPR}|U_\chi^\dagger(T)$, in the second line we note $M_{\text{cyc}}^\chi(A)|\text{EPR}\rangle^{\otimes n} = M_{\text{cyc}}^{\psi\dagger}(A)|\text{EPR}\rangle^{\otimes n}$, and in the last line because the operators contain only ψ fields, we can trace over the χ Hilbert space. As is seen from the last expression, the quasi entropy of the EPR state at time T is given by the averaged two-point correlation function of the cyclic permutation operators. If $T = 0$, it is easy to see the trace is one, consistent with the initial state being a pure state.

To get a path integral representation of the trace, it is useful to write it as

$$\text{Tr}[\rho(T)^{\otimes n} M_{\text{cyc}}(A)] = \langle \text{EPR} |^{\otimes n} M_{\text{cyc}}^{\chi\dagger}(A) [U_\chi^\dagger(T) \otimes U_\psi(T)]^{\otimes n} M_{\text{cyc}}^\chi(A) | \text{EPR} \rangle^{\otimes n}. \quad (4.6)$$

If there were not permutation operators, then the trace would be one since these two time evolutions cancel for the EPR state. One then recognizes $U_\psi(T)$ and $U_\chi^\dagger(T)$ as the forward and backward evolution in Keldysh contour language. Indeed, the boundary conditions for the fields in conventional Keldysh field theory are [42]

$$\psi_{x,j,+}^\alpha(0) = -\psi_{x,j,-}^\alpha(0), \quad \psi_{x,j,+}^\alpha(T) = \psi_{x,j,-}^\alpha(T), \quad (4.7)$$

$\forall j = 1, \dots, N, \forall \alpha = 1, \dots, n$, where the minus sign at time $t = 0$ is from the fermionic coherent state path integral. Recall that for an EPR state,

$$(\psi_{x,j}^\alpha + i\chi_{x,j}^\alpha)|\text{EPR}\rangle^{\otimes n} = 0, \quad \langle \text{EPR} |^{\otimes n} (\psi_{x,j}^\alpha - i\chi_{x,j}^\alpha) = 0, \quad (4.8)$$

$\forall j = 1, \dots, N, \forall \alpha = 1, \dots, n$, which suggests that by taking $\psi_{x,j}^\alpha \rightarrow \psi_{x,j,+}$ and $i\chi_{x,j}^\alpha \rightarrow \psi_{x,j,-}$, we can write the trace as a n -replicated Keldysh field theory with a total of $2n$ contours. In the following, we will use notation like ψ_+, ψ_- for Keldysh fields, while notation like ψ, χ for operators defined in EPR states.

The presence of the permutation operator then changes the boundary conditions for the fermions in subsystem A , for $x \in A$,

$$\begin{aligned} \psi_{x,j}^\alpha M_{\text{cyc}}^\chi(A) |\text{EPR}\rangle^{\otimes n} &= M_{\text{cyc}}^\chi(A) (-i\chi_{x,j}^\alpha) |\text{EPR}\rangle^{\otimes n} \\ &= M_{\text{cyc}}^\chi(A) (-i\chi_{x,j}^\alpha) M_{\text{cyc}}^{\chi^\dagger}(A) M_{\text{cyc}}^\chi(A) |\text{EPR}\rangle^{\otimes n} \\ &= \sum_{\beta} \text{sgn}(\alpha - \beta) \delta^{\alpha+1,\beta} (-i\chi_{x,j}^\beta) M_{\text{cyc}}^\chi(A) |\text{EPR}\rangle^{\otimes n}. \end{aligned} \quad (4.9)$$

In the last line, when $\alpha = n$, the symbol means $\delta^{n+1,\beta} = \delta^{1,\beta}$. The n -replicated Keldysh field theory is coupled through boundary conditions due to the permutation operator. We can redefine the fields in the backward contours to have the conventional boundary conditions, and so bring the effect of the twisted boundary conditions to the bulk action, i.e.,

$$\sum_{\beta} \text{sgn}(\alpha - \beta) \delta^{\alpha+1,\beta} \psi_{x,j,-}^\beta \rightarrow \psi_{x,j,-}^\alpha, \quad (4.10)$$

$\forall x \in A, \forall \alpha = 1, \dots, n, \forall j = 1, \dots, N$.

With this field redefinition, we can derive the path integral, $\mathbb{E}\text{Tr}[\rho^{\otimes n} M_{\text{cyc}}(A)] = \int DGD\Sigma e^{-I}$, where the G - Σ action reads [21, 32, 39]

$$\begin{aligned} -\frac{I}{N} &= \sum_{x=1,2} \left[\log \text{Pf}[\hat{S}\partial_t - \hat{\Sigma}_x] + \int dt_1 dt_2 \left(-\frac{1}{2} \Sigma_{x,ss'}^{\alpha\beta} G_{x,ss'}^{\alpha\beta} + \frac{J}{8} \delta(t_{12}) c_{ss'} (2G_{x,ss'}^{\alpha\beta}(t_1, t_2))^2 \right) \right] \\ &\quad + \frac{U}{2q} \int dt_1 dt_2 \delta(t_{12}) c_{ss'} (2G_{1,ss'}^{\alpha\beta}(t_1, t_2))^{q/2} M_s^{\alpha\gamma} M_{s'}^{\beta\delta} (2G_{2,ss'}^{\gamma\delta}(t_1, t_2))^{q/2}, \end{aligned} \quad (4.11)$$

where $t_{12} \equiv t_1 - t_2$, and $s = \pm$ is introduced to denote the forward and backward evolution. $S_{++} = 1, S_{--} = -1, S_{+-} = S_{-+} = 0$, and $c_{++} = c_{--} = -1, c_{+-} = c_{-+} = 1$ capture the structure of the forward and backward evolutions. The summation over the replica indices and the contour indices is implicit. The M matrix is defined as $M_+^{\alpha\beta} = \delta^{\alpha\beta}$, $M_-^{\alpha\beta} = \epsilon^{\alpha\beta}$, where $\epsilon^{\alpha\beta} \equiv \delta^{\alpha+1,\beta}$ is the cyclic permutation matrix originating from the

redefinition in (4.10). Because our model conserves the local Fermi parity, the sign from the redefinition (4.10) disappears. The boundary conditions are the same as (4.7). In the large- N action, there is an emergent $\text{SO}(N)$ symmetry among the flavors of Majorana $\psi_{x,j,s}^\alpha$, $j = 1, \dots, N$. For simplicity, we will often talk about the Majorana fermions simply as $\psi_{x,s}^\alpha$ without referring to each individual flavor.

4.2 Saddle-point solutions given by permutations

The saddle-point equations following from (4.11) are

$$\hat{G}_x^{-1} = \hat{S}\partial_t - \hat{\Sigma}_x, \quad (4.12)$$

$$\begin{aligned} \Sigma_{x,ss'}^{\alpha\beta} &= c_{ss'}\delta(t_{12}) \left[J(2G_{x,ss'}^{\alpha\beta}) \right. \\ &\quad \left. + U(2G_{x,ss'}^{\alpha\beta})^{q/2-1} [M_{(x)}]_s^{\alpha\gamma} [M_{(x)}]_{s'}^{\beta\delta} (2G_{x,ss'}^{\gamma\delta})^{q/2} \right], \end{aligned} \quad (4.13)$$

where we have defined $[M_{(1)}]_s^{\alpha\beta} = M_s^{\alpha\beta}$, $[M_{(2)}]_s^{\alpha\beta} = M_s^{\beta\alpha}$, and $\bar{1} = 2$, $\bar{2} = 1$. Because we are interested in the steady state in the long-time limit, we assume the solution depends only on the time difference. Then the first equation can be solved by Fourier transform,

$$\hat{G}_x^{-1}(\omega) = -i\omega\hat{S} - \hat{\Sigma}_x(\omega), \quad (4.14)$$

where $\hat{G}_x(t_1, t_2) = \int \frac{d\omega}{2\pi} \hat{G}_x(\omega) e^{i\omega t_{12}}$, $\hat{\Sigma}_x(t_1, t_2) = \int \frac{d\omega}{2\pi} \hat{\Sigma}_x(\omega) e^{i\omega t_{12}}$, with $t_{12} = t_1 - t_2$. The Fourier transform is over the continuous frequency as we take $T \rightarrow \infty$. Moreover, because the self energy is proportional to a Dirac δ function, $\delta(t_{12})$, in frequency space it is just a constant, $\hat{\Sigma}_x(\omega) = \hat{\Sigma}_x$. In the following, we will always use $\hat{\Sigma}_x$ without argument to denote the constant self energy in frequency space.

An inspection of the self energy equation of motion,

$$\begin{aligned} \Sigma_{1,-+}^{\alpha\beta} &= J(2G_{1,-+}^{\alpha\beta}) + \\ &\quad U(2G_{1,-+}^{\alpha\beta})^{q/2-1} \sum_{\gamma} \epsilon^{\alpha\gamma} (2G_{2,-+}^{\gamma\beta})^{q/2}, \end{aligned} \quad (4.15)$$

$$\begin{aligned} \Sigma_{2,-+}^{\alpha\beta} &= J(2G_{2,-+}^{\alpha\beta}) + \\ &\quad U(2G_{2,-+}^{\alpha\beta})^{q/2-1} \sum_{\gamma} (\epsilon^T)^{\alpha\gamma} (2G_{2,-+}^{\gamma\beta})^{q/2}, \end{aligned} \quad (4.16)$$

suggests that the solution is given by permutation matrices. Let τ denote a general n -by- n permutation matrix. It is easy to check for any pairs of permutation matrices, $(\tau_A, \tau_{\bar{A}})$, satisfying $\tau_A = \epsilon\tau_{\bar{A}}$, there is a solution given by the Green's function, $\hat{G}_A(t_1, t_2) = \hat{G}(t_1, t_2, \tau_A)$, $\hat{\Sigma}_A = \hat{\Sigma}(\tau_A)$, and $\hat{G}_{\bar{A}}(t_1, t_2) = \hat{G}(t_1, t_2, \tau_{\bar{A}})$, $\hat{\Sigma}_{\bar{A}} = \hat{\Sigma}(\tau_{\bar{A}})$, where the functions are defined by

$$\hat{G}(t_1, t_2, \tau) = \frac{e^{-\Lambda|t_{12}|}}{2} \begin{pmatrix} \text{sgn}(t_{12}) & -\tau^T \\ \tau & -\text{sgn}(t_{12}) \end{pmatrix}, \quad (4.17)$$

$$\hat{\Sigma}(\tau) = \Lambda \begin{pmatrix} 0 & -\tau^T \\ \tau & 0 \end{pmatrix}, \quad (4.18)$$

where $t_{12} = t_1 - t_2$, $\Lambda = J + U$, the solution is written in the Keldysh space and τ is a n -by- n permutation matrix in the replica space.

To show these are the center-of-mass time invariant solutions, we take the transpose of the first equation and use the antisymmetric property of the bilocal field $\hat{G}^T = -\hat{G}$ to get [33]

$$\partial_{t_1} \hat{S} \hat{G}_x(t_1, t_2) + \partial_{t_2} \hat{G}_x(t_1, t_2) \hat{S} = [\hat{\Sigma}_x(t_1, t_2), \hat{G}_x(t_1, t_2)]. \quad (4.19)$$

For the Brownian model, we have further $\hat{\Sigma}_x(t_1, t_2) = \hat{\Sigma}_x(t_1) \delta(t_1 - t_2)$. Setting $t_1 = t_2$, the differential equation for center-of-mass time is

$$\partial_t \hat{S} \hat{G}_x(t, t) \hat{S} = [\hat{\Sigma}_x(t), \hat{G}_x(t, t)]. \quad (4.20)$$

The right-hand side vanishes for the saddle-point solutions (4.17).

The solution given by (4.17) is onshell in the bulk $0 < t < T$, but does not satisfy the boundary condition. The full dynamics set by the equation of motion (4.20) with the boundary condition (4.7) is a complicated problem for general n , involving multiple-variable non-linear differential equations for which a general solution does not exist. The solution given by a permutation matrix (4.17) is actually a fixed point of the equation of motion in the long-time limit. We show in appendix B that the real onshell solution in the long-time limit is given by (4.17) in the bulk, with deviations from it near the boundary to account for the correct boundary conditions. The deviation is exponentially suppressed as time moves away from the boundary, with a time scale set by $\frac{1}{\sqrt{U(2J+U)}}$. In the following, we will see that the solution (4.17) gives an essential contribution to the entanglement of the steady state. In appendix B, we also argue that the solution (4.17) correctly captures the time-independent part of the onshell action at the late time. There are two further remarks on the solution given by (4.17) being the relevant one. First, summing over these solutions correctly leads to the Page value in the long-time limit. If there were a contribution due to the deviation near the boundary, the Rényi entropy would have involved microscopic parameters like U/J , which is not reasonable for a scrambling system at late time. Second, it is reasonable to expect that when the $n \rightarrow 1$ limit is taken, the effect of the boundary conditions on the solutions vanishes as there is nothing to permute at $n = 1$. This is similar to the vanishing of backreaction in the gravity setup [39, 43].

Actually, for any choice of $s_x^{\alpha\beta} = \pm 1$, $\alpha \neq \beta$ (we define $s_x^{\alpha\alpha} = 1$ for completeness), there is a center-of-mass time invariant solution given by the modified permutation matrix $\tau_x^{\alpha\beta} \equiv s_x^{\alpha\beta} \tau^{\alpha\beta}$, namely, $G_{x \in A} = G(t_1, t_2, \tau_{A,x})$, $G_{x \in \bar{A}} = G(t_1, t_2, \tau_{\bar{A},x})$, where we have referred to $x = 1$ as $x \in A$, and $x = 2$ as $x \in \bar{A}$. The emergence of these solutions is due to the Fermi parity symmetry. The theory (3.1) has Fermi parity conservation at each site, whose transformation law is $\psi_{x,j} \rightarrow -\psi_{x,j}$, $j = 1, \dots, N$. When we extend the theory to n replicas, the symmetry is also extended to independent transformation in each replica, namely, $\psi_{x,\pm}^\alpha \rightarrow -\psi_{x,\pm}^\alpha$ in terms of the Keldysh field.² The solutions given by $\tau_x^{\alpha\beta}$ can be divided into classes that within each class they are related to each other via these transformations. There are inequivalent classes of solutions distinguished by,

$$P_x = \prod_{\alpha=1}^n (2i \chi_x^\alpha \psi_x^\alpha). \quad (4.21)$$

²Note that the large- N action has an emergent $SO(N)$ symmetry, and we can talk about $\psi_{x,j,s}^\psi$ without referring to an individual j .

Because of the $\text{SO}(N)$ symmetry, the Fermi parity is conserved for each individual flavor. Thus, for each site there are two distinct classes of solutions for each permutation matrix τ_x distinguished by $P_x = \pm 1$.

Put it in another way, the saddle-point solution spontaneously breaks the local Fermi parity symmetry, and the parity transformation can bring one solution to the other. Those solutions that can be connected by the transformation is distinguished by $P_x = \pm 1$ in (4.21), i.e., there are two inequivalent classes for each site. Since the initial state is a product EPR state between each site where for each site, the Fermi parity is $P_x = 1, \forall x$, there is a unique class of solutions that is allowed by the initial state.

4.3 Summing over all saddle points

Let us now compute the onshell action (4.11) by plugging in the saddle-point solution. First notice that the forward and backward evolution cancels because of $c_{ss'}$,

$$\int dt \sum_x \frac{J}{8} c_{ss'} (2G_{x,ss'}^{\alpha\beta}(t,t))^2 + \frac{U}{2q} c_{ss'} (2G_{1,ss'}^{\alpha\beta}(t,t))^{q/2} M_s^{\alpha\gamma} M_{s'}^{\beta\delta} (2G_{2,ss'}^{\gamma\delta}(t,t))^{q/2} = 0. \quad (4.22)$$

There is also an exponential decaying part,

$$\exp\left(-\frac{1}{2} \int dt \sum_x \Sigma_{x,ss'}^{\alpha\beta} G_{x,ss'}^{\alpha\beta}(t,t)\right) = \exp(-n\Lambda T). \quad (4.23)$$

We discuss the Pfaffian in detail. The evaluation of the Pfaffian can be mapped to a Kitaev chain problem. Because the large- N action (4.11) has an $\text{SO}(N)$ symmetry, the Pfaffian at each site (we first suppress the site index, and later restore it when we are familiar with the evaluation for a single site) is equivalent to a transition amplitude of an EPR state of an effective single flavor Majorana fermion for two contours of each replica, ψ_{\pm}^{α} . The EPR state is $(\psi^{\alpha} + i\chi^{\alpha})|\text{EPR}\rangle = 0, \forall \alpha = 1, \dots, n$, and the Pfaffian is

$$\text{Pf}[\hat{S}\partial_t - \Sigma] = \langle \text{EPR} | e^{TH} | \text{EPR} \rangle, \quad (4.24)$$

$$H = \frac{1}{2} (\psi, \chi) \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \Sigma \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix}. \quad (4.25)$$

Here the self-energy Σ is given by some n -by- n permutation matrix.

Now, all permutations of n elements can be decomposed into cyclic permutations (or cycles) with length $n_{\check{\tau}(i)}$ such that $\sum_i n_{\check{\tau}(i)} = n$. We use a check over the letter, $\check{\tau}$, to denote cycles. Here since the problem is quadratic, we can indeed simplify the Hamiltonian by taking the permutation matrix τ into block diagonal individual cycles $\tau = \oplus_i \check{\tau}(i)$, and for each of these cycles, we can bring them to the following canonical form,

$$\check{\tau}_{(i)}^{\alpha\beta} = \begin{cases} \delta^{\alpha+1,\beta}, & n_{\check{\tau}(i)} = \text{odd} \\ \text{sgn}(\beta - \alpha) \delta^{\alpha+1,\beta}, & n_{\check{\tau}(i)} = \text{even} \end{cases}, \quad (4.26)$$

where we defined $\delta^{n+1,\beta} = \delta^{1,\beta}$. Note that for the even length cycles, it has an additional minus sign to be consistent with the even Fermi parity $P_x = 1$.

Now the Hamiltonian decomposes into individual cycles. Let us consider a cycle $\check{\tau}$ in the canonical form with length $n_{\check{\tau}}$. We define

$$H(\check{\tau}) = \frac{\Lambda}{2}(\psi, \chi) \begin{pmatrix} 0 & -i\check{\tau}^T \\ i\check{\tau} & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \quad (4.27)$$

which is nothing but a Kitaev chain [37] with length $n_{\check{\tau}}$. Using the complex fermion,

$$c_\alpha = \frac{\psi^\alpha + i\chi^\alpha}{\sqrt{2}}, \quad c_\alpha^\dagger = \frac{\psi^\alpha - i\chi^\alpha}{\sqrt{2}}, \quad (4.28)$$

satisfying $\{c_\alpha, c_\beta^\dagger\} = \delta_{\alpha\beta}$, we can bring the Hamiltonian into

$$\begin{aligned} H(\check{\tau}) &= \frac{\Lambda}{2} \sum_{\alpha=1}^{n_{\check{\tau}}} [(c_\alpha^\dagger c_{\alpha+1} + c_\alpha c_{\alpha+1} + h.c.) + ((-1)^{n_{\check{\tau}}+1} c_n c_1^\dagger + h.c.)] \\ &= \Lambda \sum_k (c_k^\dagger, c_{-k}) \begin{pmatrix} -\cos k & -i \sin k \\ i \sin k & \cos k \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} + \text{mod}(n_{\check{\tau}}, 2) \Lambda \left(c_0^\dagger c_0 + \frac{1}{2} \right), \end{aligned} \quad (4.29)$$

where we have the BdG Hamiltonian [42] in the second line. The choice of momentum needs some explanation because the odd and even lengths have different boundary conditions. When $n_{\check{\tau}}$ is an odd integer, the fermion satisfies periodic boundary condition, $k = \frac{2j\pi}{n_{\check{\tau}}}$, and the summation is over $j = 1, \dots, \frac{n_{\check{\tau}}-1}{2}$. When $n_{\check{\tau}}$ is an even integer, the fermion satisfies anti-periodic boundary condition, $k = \frac{(2j-1)\pi}{n_{\check{\tau}}}$, and the summation is over $j = 1, \dots, \frac{n_{\check{\tau}}}{2}$. The last term with zero momentum in the second line appears only for odd integer $n_{\check{\tau}}$ because it is allowed under periodic boundary conditions.

It is easy to diagonalize the Hamiltonian with the Bogoliubov quasiparticle operators,

$$d_{-,k} = -i \cos \frac{k}{2} c_k + \sin \frac{k}{2} c_{-k}^\dagger \quad (4.30)$$

$$d_{+,k} = i \sin \frac{k}{2} c_k + \cos \frac{k}{2} c_{-k}^\dagger, \quad (4.31)$$

such that

$$\begin{aligned} H(\check{\tau}) &= \Lambda \sum_k (d_{+,k}^\dagger d_{+,k} - d_{-,k}^\dagger d_{-,k}) \\ &\quad + \text{mod}(n_{\check{\tau}}, 2) \Lambda \left(c_0^\dagger c_0 + \frac{1}{2} \right). \end{aligned} \quad (4.32)$$

Each of the eigenmodes is independent for the choice of momentum discuss in above, and the exponential of the Hamiltonian is given by

$$e^{TH(\check{\tau})} = e^{\text{mod}(n_{\check{\tau}}, 2) \frac{\Lambda}{2}} \left(1 + (e^{\Lambda T} - 1) c_0^\dagger c_0 \right) \quad (4.33)$$

$$\begin{aligned} &\times \prod_k \left(1 + (e^{\Lambda T} - 1) d_{+,k}^\dagger d_{+,k} + 1 + (e^{-\Lambda T} - 1) d_{-,k}^\dagger d_{-,k} \right) \\ &\ni e^{\text{mod}(n_{\check{\tau}}, 2) \frac{\Lambda}{2}} \prod_k e^{\Lambda T} \cos^2 \frac{k}{2} c_k c_k^\dagger, \end{aligned} \quad (4.34)$$

where because the EPR state is the vacuum of a complex fermion (4.28), we express the Bogoliubov quasiparticle using the complex fermion and show the largest nonvanishing

component for each k in the second line. Then it is straightforward to evaluate the transition amplitude of the EPR state,

$$\langle \text{EPR} | e^{TH(\check{\tau})} | \text{EPR} \rangle = e^{\frac{n\check{\tau}}{2}\Lambda T} \prod_k \cos^2 \frac{k}{2} = e^{\frac{n\check{\tau}}{2}\Lambda T} 2^{-n\check{\tau}+1}, \quad (4.35)$$

where the range of product of k is the same as the summation we discussed before.

We now know the contribution for a cycle $\check{\tau}$. It is time to discuss the full solution including different sites. For the full solution, recall that the solution in $x \in A$ and $x \in \bar{A}$ are related by the cyclic permutation ϵ , namely, $\tau_A = \epsilon\tau_{\bar{A}}$. Suppose that they can be decomposed into m_{cyc} number cycles, i.e., those two permutation matrices can be built using $\check{\tau}_{(i)}$, $i = 1, 2, \dots, m_{\text{cyc}}$. The sum of the length of each cycles should satisfy $\sum_{j=1}^{m_{\text{cyc}}} n\check{\tau}_{(j)} = 2n$. Then the Pfaffian becomes

$$\prod_{x=1,2} \text{Pf}[\hat{S}\partial_t - \Sigma_x] = \langle \text{EPR} | e^{T \sum_{i=1}^{m_{\text{cyc}}} H(\check{\tau}_{(i)})} | \text{EPR} \rangle \quad (4.36)$$

$$= \prod_{i=1}^{m_{\text{cyc}}} e^{\frac{n\check{\tau}}{2}\Lambda T} 2^{-n\check{\tau}+1} \quad (4.37)$$

$$= e^{n\Lambda T} 2^{-2n+m_{\text{cyc}}}. \quad (4.38)$$

Because n is the number of replicas, it is natural to expect the action to have a factor of n . What is interesting is that the Pfaffian depends on the number of cycles in the solution, m_{cyc} , and does not depend on other details of the permutation matrix.

For a pair of permutations, (σ, τ) , that are related by $\sigma = \epsilon\tau$, it is known that the maximal possible number of cycles that can be decomposed from them is $m_{\text{cyc}} = n+1$ [44, 45]. Then we have

$$\prod_{x=1,2} \text{Pf}[\hat{S}\partial_t - \Sigma_x] = \langle \text{EPR} | e^{T \sum_{i=1}^{n+1} H(\check{\tau}_{(i)})} | \text{EPR} \rangle \quad (4.39)$$

$$= e^{n\Lambda T} 2^{-n+1}. \quad (4.40)$$

The exponential growing in time part cancels the decaying part in (4.23), and the onshell action becomes $e^{-I_{\text{onshell}}} = e^{(1-n)N \log 2}$.

For those pairs with less cycles, the contribution will be exponentially suppressed, i.e., $2^{Nm_{\text{cyc}}}$, so we ignore them. There are C_n pairs of permutation matrices, (σ, τ) , that are related by $\sigma = \epsilon\tau$, having the maximal number of cycles, where C_n is the Catalan number [44, 45]. In addition to that, we have also a degeneracy coming from the local Fermi parity symmetry $\psi_j^\alpha \rightarrow -\psi_j^\alpha$, $j = 1, \dots, N$ independently for each replica $\alpha = 1, \dots, n$, which leads to a number $\prod_j (\frac{1}{2} \times 2^{n\check{\tau}_j}) = 2^{n-1}$.³ Taking into account these degenerate saddle points, the steady state Rényi entropy is given by

$$S_A^{(n)} = (N-2) \log 2 + \frac{1}{1-n} \log C_n. \quad (4.41)$$

³The 1/2 inside the parentheses is due to the fact that the Fermi parity of the initial state is even at each site, so that only half of solutions are allowed.

To get the von Neumann entanglement entropy, it is useful to represent the Catalan number as $C_n = \frac{1}{2\pi} \int_0^4 dx x^{n-1} \sqrt{(4-x)x}$. We can analytically continue the number to $n \rightarrow 1$, and this leads to

$$\begin{aligned} & \lim_{n \rightarrow 1} \frac{1}{1-n} \log C_n \\ &= \lim_{n \rightarrow 1} \frac{1}{1-n} \log \left(\frac{1}{2\pi} \int_0^4 dx \sqrt{(4-x)x} \right. \\ & \quad \left. - (1-n) \frac{1}{2\pi} \int_0^4 dx \log x \sqrt{(4-x)x} \right) \end{aligned} \quad (4.42)$$

$$= \lim_{n \rightarrow 1} \frac{1}{1-n} \log \left(1 - \frac{1}{2}(1-n) \right) = -\frac{1}{2}, \quad (4.43)$$

so the von Neumann entanglement entropy is $S_A = N \log 2 - 2 \log 2 - \frac{1}{2}$. The first $1/N$ is due to the Fermi parity symmetry, i.e., starting from an EPR state the steady state can explore half of the total Hilbert space at each site [32]. The second $1/N$ correction is consistent with the Page value in the limit of large Hilbert space dimension [34].

We can do better than just calculating the von Neumann entanglement entropy. This explicit form (4.41) allows us to calculate the entanglement spectrum density of states. This is done by considering the resolvent of reduced density matrix of subsystem A [39],

$$R(\lambda) = \frac{1}{\lambda - \rho_A} = \frac{1}{\lambda} + \sum_{k=1}^{\infty} \rho_A^k, \quad \rho_A = \text{Tr}_{\bar{A}} \rho. \quad (4.44)$$

The entanglement spectrum density of states is given by the resolvent through⁴

$$D(\lambda) = \frac{1}{2\pi i} \lim_{\delta \rightarrow 0} (\text{Tr}[R(\lambda - i\delta)] - \text{Tr}[R(\lambda + i\delta)]). \quad (4.45)$$

The trace of resolvent can be obtained by the saddle-point action,

$$\text{Tr}[R(\lambda)] = \frac{1}{\lambda} 2^{N-1} + \sum_{k=1}^{\infty} \frac{\text{Tr}[\rho_A^k(\lambda)]}{\lambda^{k+1}} \quad (4.46)$$

$$= \frac{1}{\lambda} 2^{N-1} + \sum_{k=1}^{\infty} \frac{\text{Tr}[\rho^{\otimes k} M_{\text{cyc}}(A)]}{\lambda^{k+1}} \quad (4.47)$$

$$= \frac{1}{\lambda} 2^{N-1} + \sum_{k=1}^{\infty} \frac{C_k e^{(1-k)(N-1) \log 2}}{\lambda^{k+1}} \quad (4.48)$$

$$= 2^{2(N-1)-1} \left(1 - \sqrt{1 - \frac{2^{2-(N-1)}}{\lambda}} \right), \quad (4.49)$$

where in the third equality we used the saddle-point results and in the last equality we performed the summation. The density of state function becomes

$$D(\lambda) = \frac{2^{2(N-1)-1}}{\pi} \left(\sqrt{\frac{2^{2-(N-1)}}{\lambda}} - 1 \right), \quad (4.50)$$

⁴As we know the steady state can only explore the Hilbert space with even Fermi parity at each site, we restrict the trace to that subspace, e.g., $\text{Tr}[1] = 2^{N-1}$.

which is equivalent to the result of a random pure state [34] in the subsector accessible from the initial EPR state. It is expected since unitary evolution generated from the Brownian SYK Hamiltonian approaches the Haar random unitary at late time [23, 33]. This result strongly supports that the saddle-point solutions given by the permutation matrix (4.17) play an essential role in late-time entanglement entropy.

If there is no cyclic permutation operator in the path integral, we are just calculating the n -th power of the trace of a density matrix. In this case, the solutions described by the permutation matrix also exists. However, the relation between solution in A and \bar{A} changes to $\tau_A = \tau_{\bar{A}}$, so the maximal decomposition to cycles is given by the identity matrix, where each diagonal element is a trivial cycle with length one. Taking τ_A and $\tau_{\bar{A}}$ to be the identity matrix where the number of cycles is $m_{\text{cyc}} = 2n$ in total, we have from (4.36)

$$\prod_{x=1,2} \text{Pf}[\hat{S}\partial_t - \Sigma_x] = e^{n\Lambda T}. \tag{4.51}$$

The exponential growing part gets cancelled exactly by (4.23), so we get one, which is n -th power of the trace of a density matrix.

Finally, although here we only consider two coupled clusters, it is straightforward to generalize the calculation to the half-chain quasi entropy of a chain with L sites. The calculation is the same: the solution in A and \bar{A} are given by pairs of permutation matrices, $(\tau_A, \tau_{\bar{A}})$ with $\tau_A = \epsilon\tau_{\bar{A}}$, where now A is half of the chain and \bar{A} is the rest. Taking into account the length of the chain and of the subregion A , the half-chain entanglement is given by $S_A = \frac{NL}{2} \log 2 - \frac{L}{2} \log 2 - \frac{1}{2}$, where the first $1/N$ is due to the Fermi parity symmetry at each site, i.e., starting from the EPR state, the steady state can only explore the subspace of even Fermi parity at each site.

5 Entanglement phase transition in the monitored Brownian SYK chains

In this section, we will consider the monitored Brownian SYK chains introduced in section 3. We focus on the effect of the measurement on the saddle-point solutions and thus the quasi entropy. Then we take analytical continuation $n \rightarrow 1$ to get the von Neumann entanglement entropy for the cyclic symmetric solution.

5.1 Saddle-point analysis

To generalize the story in the coupled Brownian SYK clusters into monitored Brownian SYK chains, we double the degrees of freedom into left and right chains, and add the effective action description (3.9) for continuous monitoring. Similar to the unitary case, we will consider EPR initial states. It is described by two copies of the fermions, denoted by $\psi_{x,a,j}$ and $\chi_{x,a,j}$ at every site in both chains. The initial density matrix is given by

$$\rho = |\text{EPR}\rangle\langle\text{EPR}|, \quad (\psi_{x,a,j} + i\chi_{x,a,j})|\text{EPR}\rangle = 0, \tag{5.1}$$

$\forall x = 1, \dots, L, a = L, R, j = 1, \dots, N$. We divide the chain into A and \bar{A} , and compute the quasi entropy of the steady state (3.10). The only modification of the quasi entropy is the presence of monitoring, and we have shown that the effect of monitoring part is described

by (3.9). Note that the redefinition (4.10) will not change the form of the effective action (3.9). It is then straightforward to get the G - Σ action for $\mathbb{E}\text{Tr}[\hat{\rho}^{\otimes n} M_{\text{cyc}}(A)] = \int DG D\Sigma e^{-I(A)}$,

$$\begin{aligned}
 & -\frac{I(A)}{N} = \\
 & \log \text{Pf}[\hat{S}\partial_t - \hat{\Sigma}_x] + \int dt_1 dt_2 \left(-\frac{1}{2} \Sigma_{x,ab,ss'}^{\alpha\beta} G_{x,ab,ss'}^{\alpha\beta} + \frac{J}{8} \delta(t_{12}) \delta_{ab} c_{ss'} (2G_{x,ab,ss'}^{\alpha\beta}(t_1, t_2))^2 \right) \\
 & + \frac{U}{4q} \int dt_1 dt_2 \delta(t_{12}) \delta_{ab} c_{ss'} (2G_{x,ab,ss'}^{\alpha\beta}(t_1, t_2))^{q/2} M_s^{\alpha\gamma}(x, x+1) M_{s'}^{\beta\delta}(x, x+1) (2G_{x+1,ab,ss'}^{\gamma\delta}(t_1, t_2))^{q/2} \\
 & + \frac{i\mu}{2} \int dt_1 dt_2 \delta(t_{12}) \delta_{ss'} \delta^{\alpha\beta} (\delta_{aL} \delta_{bR} - \delta_{aR} \delta_{bL}) G_{x,ab,ss'}^{\alpha\beta}(t_1, t_2), \tag{5.2}
 \end{aligned}$$

where the summation over the site indices, the chain indices, the replica indices, and the contour indices, $x, a, b, \alpha, \beta, \gamma, \delta, s, s'$, is implicit. The third line is the monitoring part from (3.9). In the second line, $M_s(x, y)$ is the cyclic permutation operator at the boundary of intervals A and \bar{A} , $M_+^{\alpha\beta}(x, y) = \delta^{\alpha\beta}$, and

$$M_-^{\alpha\beta}(x, y) = \begin{cases} \delta^{\alpha\beta} & x, y \in A \text{ or } x, y \in \bar{A} \\ \epsilon^{\alpha\beta} & x \in A \text{ and } y \in \bar{A} \\ \epsilon^{\beta\alpha} & x \in \bar{A} \text{ and } y \in A \end{cases}, \tag{5.3}$$

which is only nontrivial when x and y are located at different subsystems, and is introduced to calculate the quasi entropy between A and \bar{A} . We assume the length of A and \bar{A} are equal and given by $L/2$. The boundary conditions of the fields are similar to (4.7) with a trivial extension to left and right chains.

The equations of motion from the action read and

$$\begin{aligned}
 \hat{G}_x^{-1} &= \hat{S}\partial_t - \hat{\Sigma}_x, \tag{5.4} \\
 \Sigma_{x,ab,ss'}^{\alpha\beta} &= \delta(t_{12}) \left[J \delta_{ab} c_{ss'} (2G_{x,ab,ss'}^{\alpha\beta}) + i\mu \delta_{ss'} \delta^{\alpha\beta} (\delta_{aL} \delta_{bR} - \delta_{aR} \delta_{bL}) \right. \\
 & \quad + \frac{U}{2} \delta_{ab} c_{ss'} \left((2G_{x,ab,ss'}^{\alpha\beta})^{q/2-1} M_s^{\alpha\gamma}(x, x+1) M_{s'}^{\beta\delta}(x, x+1) (2G_{x+1,ab,ss'}^{\gamma\delta})^{q/2} \right. \\
 & \quad \left. \left. + (2G_{x-1,ab,ss'}^{\gamma\delta})^{q/2} M_s^{\gamma\alpha}(x-1, x) M_{s'}^{\delta\beta}(x-1, x) (2G_{x,ab,ss'}^{\alpha\beta})^{q/2-1} \right) \right]. \tag{5.5}
 \end{aligned}$$

We assume that the solutions are uniform in A and \bar{A} up to the parity transformation. The invariant quantity that distinguishes different classes of solutions now becomes the total Fermi parity of left and right Majorana fermions at each site, $P_{x,L} P_{x,R}$. In the presence of measurement, only the total parity of left and right Majorana fermions at each site is conserved, i.e., the theory (5.2) is invariant under transformation that acts on both left and right Majorana fermions, $\psi_{x,L/R,\pm}^\alpha \rightarrow -\psi_{x,L/R,\pm}^\alpha$.

The solution is now modified by the presence of the monitoring. Similar to the discussion in the previous section, it is easy to check that the ansatz $G_{x \in A} = G(t_1, t_2, \tau_{A,x})$, $\Sigma_{x \in A}(t_1, t_2) = \Sigma(\tau_{A,x})$, and $G_{x \in \bar{A}}(t_1, t_2) = G(t_1, t_2, \tau_{\bar{A},x})$, $\Sigma_{x \in \bar{A}} = \Sigma(\tau_{\bar{A},x})$ solves the

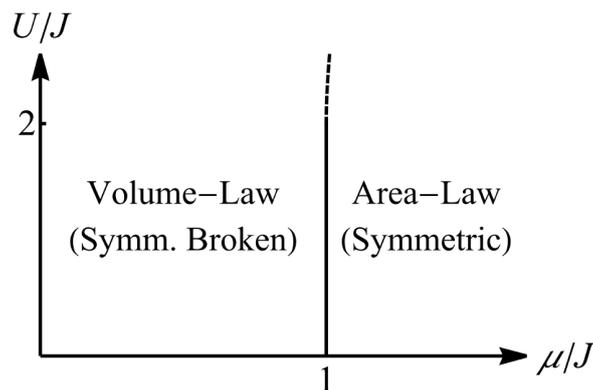


Figure 1. The phase diagram as a function of measurement strength for $q = 4$. The phase boundary is determined by (5.8). The solid line indicates a continuous transition, while the dashed line indicates a first-order transition⁵.

equation of motion, with $\tau_A = \epsilon\tau_{\bar{A}}$, where $G(t_1, t_2, \tau)$ and $\Sigma(\tau)$ are now given by

$$G(t_1, t_2, \tau) = \frac{e^{-\Lambda|t_{12}|}}{2} \times \begin{pmatrix} \text{sgn}(t_{12}) & -i \sin \theta & -\cos \theta \tau^T & 0 \\ i \sin \theta & \text{sgn}(t_{12}) & 0 & -\cos \theta \tau^T \\ \cos \theta \tau & 0 & -\text{sgn}(t_{12}) & -i \sin \theta \\ 0 & \cos \theta \tau & i \sin \theta & -\text{sgn}(t_{12}) \end{pmatrix}, \quad (5.6)$$

and

$$\Sigma(\tau) = \begin{pmatrix} 0 & i\mu & -\Lambda\tau^T & 0 \\ -i\mu & 0 & 0 & -\Lambda\tau^T \\ \Lambda\tau & 0 & 0 & i\mu \\ 0 & \Lambda\tau & -i\mu & 0 \end{pmatrix}. \quad (5.7)$$

The solution is written in the $(\psi_{L,+}^\alpha, \psi_{R,+}^\alpha, \psi_{L,-}^\alpha, \psi_{R,-}^\alpha)$ basis, and for the Green's function, we have introduced $\tan \theta = \frac{\mu}{\Lambda}$, such that when $\theta = 0$, it reduces to the unmeasured case. Recall that $\tau_x^{\alpha\beta} = s_x^{\alpha\beta} \tau^{\alpha\beta}$ is the permutation matrix that characterizes the correlation between different replicas. The EPR initial state allows the solutions with $P_{x,L} P_{x,R} = 1$, but only the permutation solution for $P_{x,L} = P_{x,R} = 1$ can smoothly connect to the zero measurement limit. Note that in the non-measurement case, the Fermi parity at two chains is conserved separately. So, we consider this sector of solutions, which includes $(2^n/2)^L = 2^{L(n-1)}$,⁶ degenerate solutions from the parity transformation.

In the presence of measurement, the parameter Λ satisfies

$$\Lambda = U \left(\frac{\Lambda}{\sqrt{\Lambda^2 + \mu^2}} \right)^{q-1} + J \frac{\Lambda}{\sqrt{\Lambda^2 + \mu^2}}. \quad (5.8)$$

⁵Near the transition point, $\Lambda = 0$, we can expand (5.8) to get $\Lambda = (U - 2J) \left(\frac{\Lambda}{\mu} \right)^3 + J \frac{\Lambda}{\mu}$. When $U > 2J$, there is no real solutions near $\Lambda = 0$. Hence, it is a first order transition.

⁶Because the parity transformation acts simultaneously on left and right Majorana fermions, we have a similar degeneracy as before.

For small interaction strength $U \ll J$, Λ is given by

$$\Lambda = J \left(1 - \frac{\mu^2}{J^2}\right)^{\frac{1}{2}} + U \left(1 - \frac{\mu^2}{J^2}\right)^{\frac{q-3}{2}} + \mathcal{O}(U^2). \quad (5.9)$$

Λ vanishes at $\mu = J$. At this point $\theta = \frac{\pi}{2}$, and the correlation between left and right chains is maximal while the correlation between different replicas vanishes. The solution restores the replica symmetry. We will see in the following section that this corresponds to an entanglement transition, namely, when the replica symmetry is broken, the von Neumann entanglement entropy satisfies a volume law, while when the replica symmetry is unbroken, the von Neumann entanglement entropy satisfies an area law. A plot of the phase diagram for $q = 4$ as a function of the monitoring rate is shown in figure 1.

5.2 Quasi entropy by summing over all saddle points

Now we evaluate the onshell action on the above solution. Due to the presence of monitoring, the forward and backward evolution no longer cancel,

$$\begin{aligned} & \int dt \left[\frac{U}{4q} \delta_{ab} c_{ss'} (2G_{x,ab,ss'}^{\alpha\beta}(t,t))^{q/2} M_s^{\alpha\gamma}(x,x+1) M_{s'}^{\beta\delta}(x,x+1) (2G_{x+1,ab,ss'}^{\gamma\delta}(t,t))^{q/2} \right. \\ & \quad \left. + \frac{J}{8} \delta_{ab} c_{ss'} (2G_{x,ab,ss'}^{\alpha\beta}(t,t))^q \right] \\ & = \left[\frac{J}{2} \left[\left(\frac{\Lambda}{\sqrt{\Lambda^2 + \mu^2}} \right)^2 - 1 \right] + \frac{U}{q} \left[\left(\frac{\Lambda}{\sqrt{\Lambda^2 + \mu^2}} \right)^q - 1 \right] \right] nLT. \end{aligned} \quad (5.10)$$

From this expression, we can check that when the monitoring strength is zero, $\mu = 0$, the contributions from the forward and backward evolution do cancel. The monitoring part itself also leads to a contribution,

$$\begin{aligned} & \int dt \frac{i\mu}{2} \delta_{ss'} \delta^{\alpha\beta} (\delta_{aL} \delta_{bR} - \delta_{aR} \delta_{bL}) G_{x,ab,ss'}^{\alpha\beta}(t,t) \\ & = -\frac{\mu^2}{\sqrt{\Lambda^2 + \mu^2}} nLT. \end{aligned} \quad (5.11)$$

Similar to the unitary case, we have also the following contribution,

$$-\frac{1}{2} \int dt \Sigma_{x,ab,ss'}^{\alpha\beta} G_{x,ab,ss'}^{\alpha\beta}(t,t) = -\sqrt{\Lambda^2 + \mu^2} nLT. \quad (5.12)$$

In the unitary case, this exponential decaying part is cancelled by the Pfaffian. We will see that it is also true for monitoring case. Nevertheless, the other parts (5.10) and (5.11) cannot be canceled. This is consistent with $\tilde{\rho}_\nu$ being an unnormalized density matrix.

The evaluation of the Pfaffian can again be mapped to a transition amplitude of Kitaev chains, but now we have two coupled Kitaev chains. Similarly, because the large- N action (5.2) has an $\text{SO}(N)$ symmetry, the Pfaffian is equivalent to the transition amplitude of an EPR state of an effective single flavor Majorana fermion for two contours of each replica, $\psi_{x,a,s}^\alpha$. The EPR state is $(\psi_{x,a}^\alpha + i\chi_{x,a}^\alpha)|\text{EPR}\rangle = 0, \forall x = 1, \dots, L, \forall \alpha = 1, \dots, n$, and $\forall a = L, R$. The Pfaffian is given by the following transition amplitude,

$$\prod_{x=1}^L \text{Pf}[\hat{S}\partial_t - \Sigma_x] = \langle \text{EPR} | e^T \sum_{x=1}^L H_x | \text{EPR} \rangle, \quad (5.13)$$

with the Hamiltonian given by

$$H_x = \sum_{a=L,R} \frac{1}{2} (\psi_{x,a}, \chi_{x,a}) \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \Sigma_x \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \psi_{x,a} \\ \chi_{x,a} \end{pmatrix}. \quad (5.14)$$

Similar to the unitary case, the self-energy Σ_x is given by some permutation matrix which can be decomposed into cycles with length $n_{\tilde{\tau}(i)}$ such that $\sum_i n_{\tilde{\tau}(i)} = n$, and for each of cycles, we can bring them to the canonical form (4.26).

Consider a cyclic permutation $\tilde{\tau}$ at one site (so we suppress the site index), by extending the previous calculation to left and right chains, the Pfaffian can be mapped to the amplitude of EPR state with the following Hamiltonian

$$H(\tilde{\tau}) = \frac{1}{2} (\psi_L, \chi_L, \psi_R, \chi_R) \times \begin{pmatrix} 0 & -i\Lambda\tilde{\tau}^T & i\mu & 0 \\ i\Lambda\tilde{\tau} & 0 & 0 & -i\mu \\ -i\mu & 0 & 0 & -i\Lambda\tilde{\tau}^T \\ 0 & i\mu & i\Lambda\tilde{\tau} & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \chi_L \\ \psi_R \\ \chi_R \end{pmatrix}. \quad (5.15)$$

Using the complex fermion,

$$c_{a,\alpha} = \frac{\psi_a^\alpha + i\chi_a^\alpha}{\sqrt{2}}, \quad c_{a,\alpha}^\dagger = \frac{\psi_a^\alpha - i\chi_a^\alpha}{\sqrt{2}}, \quad (5.16)$$

satisfying $\{c_{a,\alpha}, c_{b,\beta}\} = \delta_{ab}\delta_{\alpha\beta}$, we can bring the Hamiltonian to

$$H(\tilde{\tau}) = \frac{\Lambda}{2} \sum_{a,\alpha} (c_{a,\alpha}^\dagger c_{a,\alpha+1} + c_{a,\alpha} c_{a,\alpha+1} + h.c.) + \frac{\mu}{2} \sum_{\alpha} (i c_{L,\alpha} c_{R,\alpha} + i c_{L,\alpha}^\dagger c_{R,\alpha}^\dagger + h.c.) \quad (5.17)$$

$$= \sum_k (c_{L,k}^\dagger, c_{L,-k}, c_{R,k}^\dagger, c_{R,-k}) \begin{pmatrix} -\Lambda \cos k & -i\Lambda \sin k & 0 & i\mu \\ i\Lambda \sin k & \Lambda \cos k & i\mu & 0 \\ 0 & -i\mu & -\Lambda \cos k & -i\Lambda \sin k \\ -i\mu & 0 & i\Lambda \sin k & \Lambda \cos k \end{pmatrix} \begin{pmatrix} c_{L,k} \\ c_{L,-k}^\dagger \\ c_{R,k} \\ c_{R,-k}^\dagger \end{pmatrix}. \quad (5.18)$$

Because the odd and even length have different boundary conditions, the choice of momentum is: when $n_{\tilde{\tau}}$ is an odd integer, the fermion satisfies periodic boundary condition, $k = \frac{2j\pi}{n_{\tilde{\tau}}}$, and the summation is over $j = 0, 1, \dots, \frac{n_{\tilde{\tau}}-1}{2}$. When $n_{\tilde{\tau}}$ is an even integer, the fermion satisfies anti-periodic boundary condition, $k = \frac{(2j-1)\pi}{n_{\tilde{\tau}}}$, and the summation is over $j = 1, \dots, \frac{n_{\tilde{\tau}}}{2}$.

The Bogoliubov quasiparticle annihilation operators [42] are given by

$$d_{L,-k} = \cos \frac{\theta}{2} \left(i \cos \frac{k}{2} c_{L,k} - \sin \frac{k}{2} c_{L,-k}^\dagger \right) + \sin \frac{\theta}{2} \left(i \sin \frac{k}{2} c_{R,k} + \cos \frac{k}{2} c_{R,-k}^\dagger \right), \quad (5.19)$$

$$d_{R,-k} = \sin \frac{\theta}{2} \left(i \sin \frac{k}{2} c_{L,k} + \cos \frac{k}{2} c_{L,-k}^\dagger \right) + \cos \frac{\theta}{2} \left(-i \cos \frac{k}{2} c_{R,k} + \sin \frac{k}{2} c_{R,-k}^\dagger \right), \quad (5.20)$$

$$d_{L,+k} = \cos \frac{\theta}{2} \left(-i \sin \frac{k}{2} c_{L,k} - \cos \frac{k}{2} c_{L,-k}^\dagger \right) + \sin \frac{\theta}{2} \left(-i \cos \frac{k}{2} c_{R,k} + \sin \frac{k}{2} c_{R,-k}^\dagger \right), \quad (5.21)$$

$$d_{R,+k} = \sin \frac{\theta}{2} \left(-i \cos \frac{k}{2} c_{L,k} + \sin \frac{k}{2} c_{L,-k}^\dagger \right) + \cos \frac{\theta}{2} \left(i \sin \frac{k}{2} c_{R,k} + \cos \frac{k}{2} c_{R,-k}^\dagger \right), \quad (5.22)$$

where $\tan \theta = \frac{\mu}{\Lambda}$, and in terms of these operators, the Hamiltonian becomes diagonal,

$$\begin{aligned}
 H(\check{\tau}) &= \sqrt{\Lambda^2 + \mu^2} \sum_{a=L,R;k>0} (d_{a,+k}^\dagger d_{a,+k} - d_{a,-k}^\dagger d_{a,-k}) \\
 &\quad + \text{mod}(n_{\check{\tau}}, 2) \sqrt{\Lambda^2 + \mu^2} (d_{L,+0}^\dagger d_{L,+0} - d_{R,-0}^\dagger d_{R,-0}). \quad (5.23)
 \end{aligned}$$

Note that only the odd $n_{\check{\tau}}$ can have zero momentum mode because even $n_{\check{\tau}}$ should satisfies the anti-periodic boundary condition. With the help of the Bogoliubov quasiparticle operator, the exponential of the Hamiltonian can be simplified by

$$\begin{aligned}
 e^{TH(\check{\tau})} &= \left(1 + \left(e^{\sqrt{\Lambda^2 + \mu^2}T} - 1\right) d_{L,+0}^\dagger d_{L,+0} + 1 + \left(e^{-\sqrt{\Lambda^2 + \mu^2}T} - 1\right) d_{R,-0}^\dagger d_{R,-0}\right)^{\text{mod}(n_{\check{\tau}}, 2)} \\
 &\quad \times \prod_{a=L,R;k>0} \left(1 + \left(e^{\sqrt{\Lambda^2 + \mu^2}T} - 1\right) d_{a,+k}^\dagger d_{a,+k} + 1 + \left(e^{-\sqrt{\Lambda^2 + \mu^2}T} - 1\right) d_{a,-k}^\dagger d_{a,-k}\right) \quad (5.24)
 \end{aligned}$$

$$\begin{aligned}
 \ni \left(e^{\sqrt{\Lambda^2 + \mu^2}T} \cos^2 \frac{\theta}{2} c_{L,0} c_{L,0}^\dagger\right)^{\text{mod}(n_{\check{\tau}}, 2)} \prod_{a=L,R;k>0} e^{\sqrt{\Lambda^2 + \mu^2}T} \left(\cos^2 \frac{\theta}{2} \cos^2 \frac{k}{2} + \sin^2 \frac{\theta}{2} \sin^2 \frac{k}{2}\right) c_{a,k} c_{a,k}^\dagger, \quad (5.25)
 \end{aligned}$$

where in the first line we explicit show the contribution of $k = 0$ sector that is present for odd $n_{\check{\tau}}$ only. Remember the transition amplitude is between the vacuum state of the original complex fermion (5.16), so we only show the dominant contribution for each k in the second equality. Therefore, the transition amplitude for a cycle $\check{\tau}$ becomes

$$\langle \text{EPR} | e^{TH(\check{\tau})} | \text{EPR} \rangle = e^{n_{\check{\tau}} \sqrt{\Lambda^2 + \mu^2}T} \prod_k \left(\cos^2 \frac{\theta}{2} \cos^2 \frac{k}{2} + \sin^2 \frac{\theta}{2} \sin^2 \frac{k}{2}\right). \quad (5.26)$$

Here the choice of momentum in the product is that for an odd $n_{\check{\tau}}$, $k = \frac{2j\pi}{n_{\check{\tau}}}$, with $j = -\frac{n_{\check{\tau}}-1}{2}, \dots, -1, 0, 1, \dots, \frac{n_{\check{\tau}}-1}{2}$, and for an even $n_{\check{\tau}}$, $k = \frac{(2j-1)\pi}{n_{\check{\tau}}}$, with $j = -\frac{n_{\check{\tau}}}{2} + 1, \dots, -1, 0, 1, \dots, \frac{n_{\check{\tau}}}{2}$, by noticing each term in the product is an even function of k . The product is evaluated to be (see appendix C)

$$\begin{aligned}
 \prod_k \left(\cos^2 \frac{\theta}{2} \cos^2 \frac{k}{2} + \sin^2 \frac{\theta}{2} \sin^2 \frac{k}{2}\right) &= \prod_k \frac{1}{2} (1 + \cos \theta \cos k) \quad (5.27) \\
 &= 2^{1-2n_{\check{\tau}}} \cos^{n_{\check{\tau}}} \theta \left[{}_2F_1 \left(n_{\check{\tau}}, -n_{\check{\tau}}, \frac{1}{2}; \frac{1}{2}(1 - \sec \theta) \right) + 1 \right],
 \end{aligned}$$

where ${}_2F_1(a, b, c; z)$ is the Gaussian hypergeometric function.

We now know the contribution from a cycle $\check{\tau}$, which can be used to obtain the result of the full solution. Again, the solution in A and \bar{A} are related by the cyclic permutation ϵ , $\tau_A = \epsilon \tau_{\bar{A}}$. As we discussed before, the maximal number of cycles is $n + 1$, i.e., $\check{\tau}_{(i)}$, $i = 1, \dots, n + 1$, where the sum of the length of each cyclic permutation satisfies $\sum_{j=1}^{n+1} n_{\check{\tau}_{(j)}} = 2n$. Thus we have

$$\begin{aligned}
 \prod_{x=1}^L \text{Pf}[\hat{S}\partial_t - \Sigma_x] &= \left(\langle \text{EPR} | e^{T \sum_{i=1}^{n+1} H(\check{\tau}_{(i)})} | \text{EPR} \rangle \right)^{L/2} \\
 &= \left[e^{2n \sqrt{\Lambda^2 + \mu^2}T} 2^{1-3n} \cos^{2n} \theta \right. \\
 &\quad \left. \times \prod_{i=1}^{n+1} \left[{}_2F_1 \left(n_{\check{\tau}_{(i)}}, -n_{\check{\tau}_{(i)}}, \frac{1}{2}; \frac{1}{2}(1 - \sec \theta) \right) + 1 \right] \right]^{L/2}, \quad (5.28)
 \end{aligned}$$

where in the first line we take a $L/2$ power as there are $L/2$ contributions for the half-chain quasi entropy, i.e., both A and \bar{A} have $L/2$ number of sites.

Due to the presence of the measurement, we have to consider the denominator in quasi entropy (3.10), $\mathbb{E}\text{Tr}[\tilde{\rho}^{\otimes n}]$. The calculation is paralleled to $\mathbb{E}\text{Tr}[\tilde{\rho}^{\otimes n} M_{\text{cyc}}(A)]$. The difference is that the relation between τ_A and $\tau_{\bar{A}}$ should be modified from $\tau_A = \epsilon\tau_{\bar{A}}$ to $\tau_A = \tau_{\bar{A}}$, and as we discussed around (4.51) in the previous section, to maximize the number of cycles, it is given by the identity matrix. So taking τ_A and $\tau_{\bar{A}}$ to be the identity which have $2n$ cycles in total, we have gotten the Pfaffian as follows,

$$\begin{aligned} \prod_{x=1}^L \text{Pf}[\hat{S}\partial_t - \Sigma_x] &= \left(\langle \text{EPR} | e^{T \sum_{i=1}^{2n} H(1)} | \text{EPR} \rangle \right)^{L/2} \\ &= \left[e^{\sqrt{\Lambda^2 + \mu^2} T} 2^{-1} \cos \theta \left[{}_2F_1\left(1, -1, \frac{1}{2}; \frac{1}{2}(1 - \sec \theta)\right) + 1 \right] \right]^{nL} \\ &= e^{n\sqrt{\Lambda^2 + \mu^2} TL} \left(\cos \frac{\theta}{2} \right)^{2nL}, \end{aligned} \quad (5.29)$$

where $H(1)$ in the first line denotes (5.15) with trivial cycle $\check{\tau} = 1$. On the other hand, the calculation of (5.10), (5.11) and (5.12) is the same as before. Thus, all the time dependent contributions cancel when we take the ratio between these two quantities, namely, $\frac{\mathbb{E}\text{Tr}[\tilde{\rho}^{\otimes n} S_{\text{cyc}}(A)]}{\mathbb{E}\text{Tr}[\tilde{\rho}^{\otimes n}]}$. And finally, we get the quasi entropy,

$$e^{(1-n)S_A^{(n)}} = 2^{L(n-1)} \sum_{\mu=1}^{C_n} \left[\frac{2^{1-3n} \cos^{2n} \theta \prod_{i=1}^{n+1} \left[{}_2F_1\left(n_{\check{\tau}_{(i)}^\mu}, -n_{\check{\tau}_{(i)}^\mu}, \frac{1}{2}; \frac{1}{2}(1 - \sec \theta)\right) + 1 \right]}{\cos^{4n} \frac{\theta}{2}} \right]^{NL/2}, \quad (5.30)$$

where $\{\check{\tau}_{(i)}^\mu\}$, $i = 1, \dots, n+1$, is the cycle decomposition of the solution τ_A^μ and $\tau_{\bar{A}}^\mu$, and there are C_n number of such solutions, $\mu = 1, \dots, C_n$. C_n is the Catalan number. In the prefactor we also include the degeneracy coming from the local Fermi parity symmetry.

Without measurements, i.e., $\theta = 0$, (5.30) reduces to two decoupled unitary chains discussed at the end of section 4. With the measurement, we are not able to evaluate the product explicitly. It is however clear that as $\theta \rightarrow \frac{\pi}{2}$, the different saddle-point solutions tend to the same form with no correlation between replicas, because $\Lambda = 0$ in (5.6). And according to

$$\lim_{\theta \rightarrow \frac{\pi}{2}} \cos^n \theta \left[{}_2F_1\left(n_{\check{\tau}_{(i)}^\mu}, -n_{\check{\tau}_{(i)}^\mu}, \frac{1}{2}; \frac{1}{2}(1 - \sec \theta)\right) + 1 \right] = 2^{n-1} e^{i2\pi n}, \quad (5.31)$$

we have $e^{(1-n)S_A^{(n)}} = \left(\frac{2^{1-3n} 2^{n-1} e^{i2\pi n}}{(1/\sqrt{2})^{4n}} \right)^{NL/2} = 1$ from (5.30) where all degeneracy disappears as the replica symmetric solution is unique, for any integer $n \geq 2$. This implies that the quasi entropy of any order vanishes at $\theta = \frac{\pi}{2}$ in the large- N limit. In the next section, we will consider a special solution, the cyclic symmetric one, and analytically continue the solution to $n \rightarrow 1$ to show the von Neumann entanglement entropy transition.

5.3 von Neumann entanglement entropy from replica trick

Among the C_n different permutation that satisfy $\tau_A = \epsilon\tau_{\bar{A}}$ and have the maximal number of cycles, there is a cyclic symmetric one, where $\tau_{\bar{A}}$ is identity, and τ_A is itself a cyclic

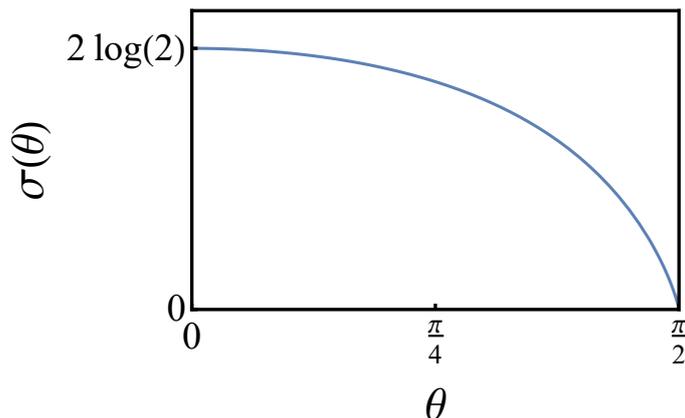


Figure 2. The plot of von Neumann entanglement entropy per site per flavor $\sigma(\theta)$ as a function of the monitoring rate $\theta = \arctan \frac{\mu}{\Lambda}$.

permutation of length n .⁷ So the number of cycles is maximal, $n + 1$. This is the only nontrivial solution when $n = 2$, i.e., $C_2 = 1$, and we expect it to dominate for $n < 2$ in the replica limit. Focusing on the contribution of this special solution, we make the following analytic continuation,

$$e^{(1-x)S_A^{(x)}} = \left[\frac{\left(\cos^2 \frac{\theta}{2}\right)^x \left(\frac{\cos^x \theta}{2^{2x-1}} \left[{}_2F_1\left(x, -x, \frac{1}{2}; \frac{1}{2}(1 - \sec \theta)\right) + 1 \right] \right)}{\cos^{4x} \frac{\theta}{2}} \right]^{NL/2}, \quad (5.32)$$

where x is a real number.⁸ In the numerator inside the parentheses, the first factor is the contribution from identity $\tau_A^{\alpha\beta} = \delta^{\alpha\beta}$, and the second factor is the contribution from the cyclic permutation $\tau_A^{\alpha\beta} = \epsilon^{\alpha\beta}$. We expand the result near $x = 1$ to get the von Neumann entanglement entropy

$$S_A = \lim_{x \rightarrow 1} \frac{1}{1-x} \log [1 + (1-x)\sigma(\theta)]^{NL/2} = \sigma(\theta) \frac{NL}{2}, \quad (5.33)$$

where the von Neumann entanglement entropy per site per flavor is

$$\sigma(\theta) = \left(\log 2(1 + \sec \theta) + \tan \frac{\theta}{2} \log(\sec \theta - \tan \theta) \right), \quad (5.34)$$

$\tan \theta = \frac{\mu}{\Lambda}$. A plot of this function is shown in figure 2. At $\theta = 0$, we recover the maximal von Neumann entanglement entropy $S_A = NL \log 2$. The von Neumann entanglement entropy density vanishes smoothly at $\theta = \frac{\pi}{2}$, where the von Neumann entanglement entropy becomes area-law. It shows clearly that the measurement-induced phase transition occurs at the unbreaking point of replica symmetry. Near the transition, we can expand the von Neumann entanglement entropy to get

$$S_A = \frac{NL}{2} \left[\log \left(\frac{\pi}{2} - \theta \right) - \log 2e \right] \left(\frac{\pi}{2} - \theta \right). \quad (5.35)$$

⁷In the gravity setup [39], a similar type of solution is considered in taking the $n \rightarrow 1$ limit.

⁸We neglect the degeneracy from the Fermi parity transformation since it is $\mathcal{O}(1)$ contribution, and we are interested in $\mathcal{O}(N)$ contribution.

Thus, by noticing the relation between θ and Λ in (5.9), the entanglement density vanishes as $(1 - \frac{\mu}{J})^{1/2} \log(1 - \frac{\mu}{J})$ near the critical point $\mu = J$. This result differs from the known universality classes.

It is instructive to consider a more general case for $L_A \neq L_{\bar{A}}$. When $L_A < L_{\bar{A}}$, the solution is given by the identity in $L_{\bar{A}}$ and the replica symmetric permutation matrix in L_A , because the identity is onshell in the bulk and satisfies the boundary condition (see appendix B). Then the von Neumann entanglement entropy for this solution is

$$e^{(1-n)S_A^{(n)}} = \frac{\left(\cos^{2n} \frac{\theta}{2}\right)^{NL_{\bar{A}}} \left[\frac{\cos^n \theta}{2^{2n-1}} \left[{}_2F_1\left(n, -n, \frac{1}{2}; \frac{1}{2}(1 - \sec \theta)\right) + 1\right]\right]^{NL_A}}{\left(\cos^{2n} \frac{\theta}{2}\right)^{N(L_A+L_{\bar{A}})}}. \quad (5.36)$$

It is clear that the numerator has two factors, the first one from $L_{\bar{A}}$ and the second one from L_A . Taking a similar analytic continuation, we get the same result, $S_A = \sigma(\theta)NL_A$.

6 Conclusion and discussion

To conclude, we introduced a large- N model in which the von Neumann entropy can be calculated analytically. We obtained the Rényi entropy by summing over a class of solutions related to the permutation group for the non-measured case, and we analytically established a von Neumann entanglement entropy transition using the replica trick in the measured case. The result in section 4 strongly suggests that the late-time von Neumann entanglement entropy of a chaotic unitary system is dominated by saddle points that connect different replicas. Interestingly, the degeneracy of these replica non-diagonal solutions play an important role in determining the entanglement entropy, and a single replica non-diagonal solution can lead to an inaccurate density spectrum density of state. For instance, if we neglected the degeneracy given by the Catalan number, we would get $D(\lambda) = 2^{N-2}\delta(\lambda - 2^{-N+2})$. This would imply that all Rényi entropies are given by the same value, similar to stabilizer states generated by Clifford circuits. It would be interesting to understand if there is any larger significance to this observation, for example, related to a path integral representation of Clifford dynamics.

The continuous transition between replica symmetry broken and unbroken solutions studied in section 5 should be contrasted with the discontinuous transition between two similar large- N solutions in the unitary time evolution of Rényi entropy [32]. As the replica non-diagonal solution is closely related to the replica wormhole observed in the context of black hole information paradox [39, 43], it is worth speculating about the physics of a monitored black hole. In the current setup, the monitoring is implemented on the full system and causes the restoration of replica symmetry. We expect that by increasing the monitoring of the black hole and its environment at late time, the replica wormhole continuously disappears. Because the so-called entanglement island is obtained by continuing the cyclic symmetric wormhole [39], it should also disappear with sufficiently frequent measurements. An interesting question is to monitor the environment, and study the effect of such measurements on the black hole. Though the measurement destroys the entanglement between black hole and radiation, the state remaining in black hole is expected to be maximally scrambled. We leave this exploration as a future work.

More broadly, the approach of mapping the path integral of multi-replicas to the transition amplitude of quantum states provides a general tool in evaluations of entropy related quantities. We expect generalization of such an approach to other models that admit saddle-point analysis to be straightforward. For example, it is of value to generalize and utilize this tool in bosonic models [20]. In this case, the emergent transition amplitude will be governed by a spin model living in the replica space. The transition amplitude for a general n might be complicated, but we expect that a similar cyclic symmetric replica non-diagonal solution dominates near $n \approx 1$, which could be analysed in detail.

Finally, the model defined by the action (5.2) is distinct from the model we considered previously in ref. [21]. The major distinction lies in that the action (3.1) conserves local Fermi parity, whereas the model studied in ref. [21] breaks the local Fermi parity by two Majorana hopping between sites. If we consider $n = 2$, the Landau-Ginzburg action for $q = 4$ would be

$$\frac{I_{\text{eff}}}{N} = \int dt dx \left(\frac{1}{2} [(\partial_t \vec{\phi})^2 + \phi_1^2 (\partial_x \phi_1)^2 + \phi_2^2 (\partial_x \phi_2)^2] + r \vec{\phi}^2 + \lambda |\vec{\phi}|^4 + \lambda' (\phi_1^4 + \phi_2^4) \right), \quad (6.1)$$

where $\vec{\phi} = (\phi_1, \phi_2)$ is a two-dimensional order parameter related to rotations between two replicas. The space derivative appears in the fourth order term, and it leads to a distinct universality class. It will be interesting to derive the Landau-Ginzburg action in the limit $n \rightarrow 1$.

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A Permutation operator for fermionic state

Consider a Hilbert space consisting of N Majorana fermions, ψ_1, \dots, ψ_N , $\{\psi_i, \psi_j\} = \delta_{ij}$, and N is an even integer. It is convenient to double the Hilbert space and work in a state language. To do that, we introduce another set of Majorana fermion called χ_1, \dots, χ_N , $\{\chi_i, \chi_j\} = \delta_{ij}$, and the maximally entangled state between these two sets of Majorana fermions, $|EPR\rangle$, defined by

$$(\psi_j + i\chi_j)|EPR\rangle = 0, \quad \forall j = 1, \dots, N. \quad (A.1)$$

Then any operator acting on the ψ Hilbert space can be uniquely mapped to a state in the doubled Hilbert space by acting it on the EPR state $O_\psi \rightarrow O_\psi |EPR\rangle$.

Let us first consider a simpler case with $N = 2$. By pairing the two Majorana into a single complex fermion, $c_\psi = \frac{\psi_1 + i\psi_2}{\sqrt{2}}$, $c_\psi^\dagger = \frac{\psi_1 - i\psi_2}{\sqrt{2}}$ and $c_\chi = i\frac{\chi_1 - i\chi_2}{\sqrt{2}}$, $c_\chi^\dagger = -i\frac{\chi_1 + i\chi_2}{\sqrt{2}}$ the Hilbert space is spanned by

$$\{|00\rangle, |10\rangle = c_\psi^\dagger |00\rangle, |01\rangle = c_\chi^\dagger |00\rangle, |11\rangle = c_\psi^\dagger c_\chi^\dagger |00\rangle\}, \quad (A.2)$$

with the EPR state expressed as $|\text{EPR}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$. It is convenient to label it as $|\text{EPR}\rangle = \frac{|0\bar{0}\rangle + |1\bar{1}\rangle}{\sqrt{2}} = \sum_a \frac{|a\bar{a}\rangle}{\sqrt{2}}$. For arbitrary even integer N , we have the similar expression $|\text{EPR}\rangle = \sum_a \frac{|a\bar{a}\rangle}{2^{N/2}}$.

We take n copies of the doubled Hilbert space, and denote the Majorana operators as $\psi_i^\alpha, \chi_i^\alpha, i = 1, 2, \dots, N, \alpha = 1, 2, \dots, n$. The cyclic permutation operator is defined to take

$$M_{\text{cyc}}|a_1\bar{a}_1a_2\bar{a}_2a_3\bar{a}_3 \dots a_n\bar{a}_n\rangle \rightarrow |a_1\bar{a}_na_2\bar{a}_1a_3\bar{a}_2 \dots a_n\bar{a}_{n-1}\rangle, \quad (\text{A.3})$$

and it is trivial to show that

$$\text{Tr}[\rho^n] = \langle \text{EPR} |^{\otimes n} \rho^{\otimes n} M_{\text{cyc}} | \text{EPR} \rangle^{\otimes n}, \quad (\text{A.4})$$

$$|\text{EPR}\rangle^{\otimes n} = 2^{-nN/2} \sum_{a_1, \dots, a_n} |a_1\bar{a}_1a_2\bar{a}_2a_3\bar{a}_3 \dots a_n\bar{a}_n\rangle. \quad (\text{A.5})$$

It is also easy to show that the following operator gives (A.3),

$$M_{\text{cyc}} = \prod_{i=1}^N e^{\frac{\pi}{2}\chi_i^n \chi_i^{n-1}} e^{\frac{\pi}{2}\chi_i^{n-1} \chi_i^{n-2}} \dots e^{\frac{\pi}{2}\chi_i^3 \chi_i^2} e^{\frac{\pi}{2}\chi_i^2 \chi_i^1}. \quad (\text{A.6})$$

In terms of the operator in the ψ Hilbert space, the cyclic permutation becomes

$$M_{\text{cyc}} = \prod_{i=1}^N e^{\frac{\pi}{2}\psi_i^1 \psi_i^2} e^{\frac{\pi}{2}\psi_i^2 \psi_i^3} \dots e^{\frac{\pi}{2}\psi_i^{n-2} \psi_i^{n-1}} e^{\frac{\pi}{2}\psi_i^{n-1} \psi_i^n}, \quad (\text{A.7})$$

with the action on the ψ_i^α field

$$M_{\text{cyc}} \psi_i^\alpha M_{\text{cyc}}^\dagger = \sum_{\beta} \text{sgn}(\alpha - \beta) \delta^{\alpha+1, \beta} \psi_i^\beta, \quad \forall i = 1, \dots, N, \quad (\text{A.8})$$

where we define $\delta^{\alpha+1, \beta} = \delta^{1, \beta}$ when $\alpha = n$. Thus, (A.8) is the cyclic permutation operator for fermionic states, and is used in section 4.

B Saddle-point solutions

The equations of motion and boundary conditions for the saddle-point analysis given in the following for general n is a complicated dynamical problem,

$$\partial_t \hat{S} \hat{G}_x(t, t) \hat{S} = [\hat{\Sigma}_x(t), \hat{G}_x(t, t)], \quad (\text{B.1})$$

$$\Sigma_{x, ss'}^{\alpha\beta}(t) = c_{ss'} \left[J(2G_{x, ss'}^{\alpha\beta}) + \text{U}(2G_{x, ss'}^{\alpha\beta})^{q/2-1} [M(x)]_s^{\alpha\gamma} [M(x)]_{s'}^{\beta\delta} (2G_{\bar{x}, ss'}^{\gamma\delta})^{q/2} \right], \quad (\text{B.2})$$

$$\psi_{x,+}^\alpha(0) = -\psi_{x,-}^\alpha(0), \quad \psi_{x,+}^\alpha(T) = \psi_{x,-}^\alpha(T), \quad \forall \alpha = 1, \dots, n. \quad (\text{B.3})$$

One can show that there is a conserved quantity for each site, given by

$$\text{Tr}(\hat{S} \hat{G}_x \hat{S} \hat{G}_x) = \text{const}, \quad (\text{B.4})$$

where the constant is set by the boundary condition. This follows from

$$\frac{d}{dt} \text{Tr}(\hat{S}\hat{G}_x\hat{S}\hat{G}_x) = 2\text{Tr}\left(\frac{d(\hat{S}\hat{G}_x\hat{S})}{dt}\hat{G}_x\right) \quad (\text{B.5})$$

$$= 2\text{Tr}\left([\Sigma_x, G_x]\hat{G}_x\right) \quad (\text{B.6})$$

$$= 0. \quad (\text{B.7})$$

Except for this conserved quantity, in general, the problem becomes a coupled nonlinear differential equation that has no general solution. For $n = 2$, we explicitly write down the equations of motion

$$\frac{dx_1}{dt} = 4Jx_2z_1 + 2Ux_2z_1(y_1^2 + w^2), \quad (\text{B.8})$$

$$\frac{dx_2}{dt} = -4Jx_1z_1 - 2Ux_1z_1(y_2^2 + w^2), \quad (\text{B.9})$$

$$\frac{dz_1}{dt} = 2Ux_1x_2(y_1^2 - y_2^2), \quad (\text{B.10})$$

$$\frac{dy_1}{dt} = 4Jy_2w_1 + 2Uy_2w_1(x_1^2 + z^2), \quad (\text{B.11})$$

$$\frac{dy_2}{dt} = -4Jy_1w_1 - 2Uy_1w_1(x_2^2 + z^2), \quad (\text{B.12})$$

$$\frac{dw_1}{dt} = 2Uy_1y_2(x_1^2 - x_2^2), \quad (\text{B.13})$$

where the coefficients are defined via

$$G_A(t, t) = \frac{1}{2} \left(\sum_{k=1}^n x_k(t) X_k + \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} z_m(t) Z_m \right), \quad (\text{B.14})$$

$$G_{\bar{A}}(t, t) = \frac{1}{2} \left(\sum_{k=1}^n y_k(t) X_k + \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} w_m(t) Z_m \right), \quad (\text{B.15})$$

with

$$X_k = \begin{pmatrix} 0 & -\sigma^{-k} \\ \sigma^k & 0 \end{pmatrix}, \quad k = 1, \dots, n, \quad (\text{B.16})$$

$$Z_m = \frac{1}{2} \begin{pmatrix} \sigma^m - \sigma^{-m} & 0 \\ 0 & \sigma^{-m} - \sigma^m \end{pmatrix}, \quad m = 1, \dots, \lfloor \frac{n}{2} \rfloor. \quad (\text{B.17})$$

Here $\lfloor \frac{n}{2} \rfloor$ is the largest integer that is less than $\frac{n}{2}$, and $\sigma^{\alpha\beta} = \text{sgn}(\alpha - \beta)\delta^{\alpha+1, \beta}$ is the permutation matrix with a proper sign to be consistent with the even Fermi parity. When $\alpha = n$, the symbol means $\delta^{n+1, \beta} = \delta^{1, \beta}$.

Although we have only explicitly write down the equation for $n = 2$, it can be seen from the definition of the matrices that these type of coefficients form a closed set of dynamical variables. It reduces the number of variable from $\sim n^2$ to $\sim n$. For $n = 2$, this

representation actually captures all independent variables. In terms of this set of variables, the conserved quantity is given by

$$\sum_{k=1}^n x_k^2 - \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} z_m^2 = \text{const}, \quad (\text{B.18})$$

$$\sum_{k=1}^n y_k^2 - \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} w_m^2 = \text{const}. \quad (\text{B.19})$$

To further simplify the problem, we assume that in \bar{A} the conserved quantity is given by $y_1^2 + y_2^2 - w_1^2 = 1$ (as we discuss in the following, this is required to have a time-independent action), and since the boundary condition is $y_2 = -1$, it uniquely determines the steady solution $y_1 = w_1 = 0$ and $y_2 = -1$. The equations of motion reduce to

$$\frac{dx_1}{dt} = 4Jx_2z_1, \quad (\text{B.20})$$

$$\frac{dx_2}{dt} = -(4J + 2U)x_1z_1, \quad (\text{B.21})$$

$$\frac{dz_1}{dt} = -2Ux_1x_2, \quad (\text{B.22})$$

and $x_1^2 + x_2^2 - z_1^2$ is conserved. There is a discrete symmetry given by multiplying any two of the three variables x_1, x_2, z_1 by a minus one, leading to four-fold degenerate nontrivial solutions. Among these solutions, half of them are determined by the initial value of x_2 , i.e., by the boundary condition, and given the boundary condition, there are still two solutions related by the Fermi parity transformation discussed in the main text. In the following, we will discuss one explicit solution.

These differential equations can be solved via Jacobi elliptic functions,

$$x_1 = \sqrt{c_1} \text{sn} \left(2\sqrt{U(2J+U)}c_2(t-t_0), \frac{c_1}{c_2} \right), \quad (\text{B.23})$$

$$x_2 = -\sqrt{\frac{(2J+U)c_1}{2J}} \left(1 - \text{sn} \left(2\sqrt{U(2J+U)}c_2(t-t_0), \frac{c_1}{c_2} \right) \right)^{1/2}, \quad (\text{B.24})$$

$$z_1 = -\sqrt{\frac{Uc_2}{2J}} \text{dn} \left(2\sqrt{U(2J+U)}c_2(t-t_0), \frac{c_1}{c_2} \right), \quad (\text{B.25})$$

where $\text{sn}(u, c)$ and $\text{dn}(u, c)$ are the Jacobi elliptic functions. c_1, c_2, t_0 are integral constants to be determined below. t_0 is the shift of time argument, and c_1, c_2 are related to the conserved quantity by $x_1^2 + x_2^2 - z_1^2 = c_1 + \frac{U}{2J}(c_1 - c_2)$.

The conserved quantity is related to the time dependence of the onshell action. Because the action is dimensionless, it can only depend on JT and UT . Taking the derivative with respect to T , we have

$$\begin{aligned} \frac{dI(JT, UT)}{dT} &= \frac{J}{T} \frac{\partial I}{\partial J} + \frac{U}{T} \frac{\partial I}{\partial U} \\ &= -c_{ss'} \int dt \left[\frac{J}{8T} \sum_x (2G_{x,ss'}^{\alpha\beta}(t,t))^2 + \frac{U}{8T} (2G_{1,ss'}^{\alpha\beta}(t,t))^2 M_s^{\alpha\gamma} M_{s'}^{\beta\delta} (2G_{2,ss'}^{\gamma\delta}(t,t))^2 \right] \\ &= - \int dt \left[\frac{J}{2T} (x_1(t)^2 + x_2(t)^2 - z_1(t)^2 - 1) + \frac{U}{2T} (x_1(t)^2 - 1) \right], \end{aligned} \quad (\text{B.26})$$

where we have plugged in $y_1 = w_1 = 0$ and $y_2 = -1$. In the third line, the first term in parentheses is given by the conserved quantity. To have a time-independent result, the conserved quantity should be $x_1(t)^2 + x_2(t)^2 - z_1(t)^2 = 1$. The second term vanishes at long times when $x_1(t) \rightarrow 1$. With these two facts, c_1, c_2 can be determined to be $c_1 = 1$ and $c_2 = 1$ as $T \rightarrow \infty$. The boundary condition at $t = 0$ fixes $t_0 = \frac{1}{4\sqrt{U(2J+U)}} \cosh^{-1} \frac{J+U}{J}$ such that $x_2(0) = -1$. An example is shown in figure 3, where we allow c_2 to be slightly less than 1 to capture the boundary condition at $t = T$. As c_2 approaches 1, the time T approaches ∞ . In the long-time limit, the solution becomes,

$$x_1 = \tanh \left(2\sqrt{U(2J+U)}(t-t_0) \right), \tag{B.27}$$

$$x_2 = -\sqrt{\frac{2J+U}{2J}} \frac{1}{\cosh \left(2\sqrt{U(2J+U)}(t-t_0) \right)}, \tag{B.28}$$

$$z_1 = -\sqrt{\frac{U}{2J}} \frac{1}{\cosh \left(2\sqrt{U(2J+U)}(t-t_0) \right)}, \tag{B.29}$$

where indeed the solution in the bulk takes the form of (4.17), and the effect of the boundary conditions decreases exponentially fast into the bulk. Now in the long-time limit, the onshell action tends to a constant determined by the ratio U/J because it is the only dimensionless quantity when $T \rightarrow \infty$. The long-time onshell action cannot depend on $U(2J+U)$, so we can take this advantage to send $U(2J+U) \rightarrow \infty$ without changing the onshell action. In this case, the solution is given by (4.17). Therefore, although (4.17) does not satisfy the boundary condition in any finite $U(2J+U)$, the action evaluated via (4.17) gives the correct answer in the long-time limit. The argument implicitly states

$$\begin{aligned} \lim_{E \rightarrow \infty} I[G_{\text{exact}}, \Sigma_{\text{exact}}] &= I \left[\lim_{E \rightarrow \infty} G_{\text{exact}}, \lim_{E \rightarrow \infty} \Sigma_{\text{exact}} \right] \\ &= I[G, \Sigma], \end{aligned} \tag{B.30}$$

where E is a microscopic energy scale in the model, i.e., $U = uE, J = jE$.⁹ In the above limit, $U/J = u/j$ is kept fixed. G_{exact} and Σ_{exact} denote the exact saddle-point solution satisfying the boundary condition, and G and Σ in the last equality is the solution given in (4.17). This must be true because the long-time limit of the Rényi entropy and von Neumann entanglement entropy of the Brownian SYK clusters saturate to a constant set by the dimension of Hilbert space and the symmetry of the model independent of any microscopic energy scale [23, 33, 34]. And the evaluation of the von Neumann entanglement entropy from the permutation solution (4.17) correctly gives the Page value (see section 4).

For general n , we expect the same situation happens. The conserved quantity (B.18) is given by 1, and the bulk is dominated by one of the permutation matrix X_k given by (B.16) to have a time-independent onshell action. When the measurement is turned on, the addition of (3.9) is an integration over the Green's function, for which we expect (B.30) still holds. Thus, in section 5, we consider a generalization of solutions of the form (4.17) to the measurement case, cf. (5.6), and study how the measurement affects the solution and causes the entanglement transition.

⁹This is same as taking $T \rightarrow \infty$, because the action is dimensionless and depends on TE only.

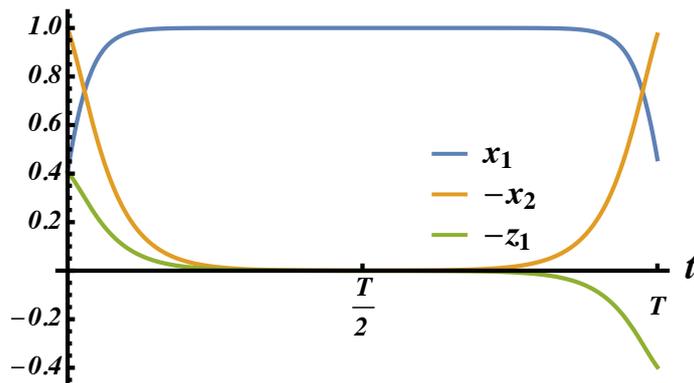


Figure 3. A plot of the solution of x_1, x_2, z_1 . We choose $J = 1, U = 0.4, c_1 = 1, c_2 = 1 - 10^{-7}$.

C Finite products of trigonometric functions

In this section we first show that the following identity holds and then use it to derive (5.27),

$$2(T_n(a) + 1) = \begin{cases} \prod_{k=-\frac{n}{2}+1}^{\frac{n}{2}} \left(2a + 2 \cos \frac{(2k+1)\pi}{n}\right), & n = \text{even}, \\ \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \left(2a + 2 \cos \frac{2k\pi}{n}\right), & n = \text{odd}. \end{cases} \quad (\text{C.1})$$

for k being integers, where $T_n(a)$ is the Chebyshev polynomial of the first kind, namely, $T_n(\cos \phi) = \cos n\phi$. For an even integer n , note that (C.1) is equivalent to

$$x^n + x^{-n} + 2 \cos n\varphi = \prod_{k=-\frac{n}{2}+1}^{\frac{n}{2}} \left[x + x^{-1} - 2 \cos \left(\varphi + \frac{(2k-1)\pi}{n} \right) \right], \quad (\text{C.2})$$

by setting $\varphi = \pi$ and $x = e^{i\phi}$. The left-hand side of (C.2) has the roots,

$$x = e^{i\left(\varphi + \frac{(2k-1)\pi}{n}\right)}, \quad k \in \left\{ -\frac{n}{2} + 1, -\frac{n}{2} + 2, \dots, \frac{n}{2} \right\}. \quad (\text{C.3})$$

By noticing

$$x + x^{-1} = 2 \cos \left(\varphi + \frac{(2k-1)\pi}{n} \right), \quad (\text{C.4})$$

we arrive the right-hand side of (C.2). The proof for an odd integer n is similar. (C.1) is equivalent to

$$x^n + x^{-n} - 2 \cos n\varphi = \prod_{k=-\frac{n-1}{2}}^{-\frac{n-1}{2}} \left[x + x^{-1} - 2 \cos \left(\varphi + \frac{2k\pi}{n} \right) \right], \quad (\text{C.5})$$

by setting $\varphi = \pi$ and $x = e^{i\phi}$. The left-hand side of (C.5) has the roots,

$$x = e^{i\left(\varphi + \frac{2k\pi}{n}\right)}, \quad k \in \left\{ -\frac{n-1}{2}, -\frac{n-1}{2} + 1, \dots, \frac{n-1}{2} \right\}. \quad (\text{C.6})$$

Then by noticing

$$x + x^{-1} = 2 \cos \left(\varphi + \frac{2k\pi}{n} \right), \quad (\text{C.7})$$

we get the right-hand side of (C.2).

Using the identity (C.1), we can derive (5.27). The derivation is simply given here. For even integers n ,

$$\begin{aligned} & \prod_{k=-\frac{n}{2}+1}^{\frac{n}{2}} \frac{1}{2} (1 + \cos \theta \cos k) \\ &= 2^{-2n} \cos^n \theta \prod_{k=-\frac{n}{2}+1}^{\frac{n}{2}} \left(\frac{2}{\cos \theta} + 2 \cos k \right) \\ &= 2^{1-2n} \cos^n \theta (T_n(\sec \theta) + 1) \\ &= 2^{1-2n} \cos^n \theta \left[{}_2F_1 \left(n, -n, \frac{1}{2}; \frac{1}{2}(1 - \sec \theta) \right) + 1 \right], \end{aligned} \quad (\text{C.8})$$

and for odd integers n ,

$$\begin{aligned} & \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{1}{2} (1 + \cos \theta \cos k) \\ &= 2^{-2n} \cos^n \theta \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \left(\frac{2}{\cos \theta} + 2 \cos k \right) \\ &= 2^{1-2n} \cos^n \theta (T_n(\sec \theta) + 1) \\ &= 2^{1-2n} \cos^n \theta \left[{}_2F_1 \left(n, -n, \frac{1}{2}; \frac{1}{2}(1 - \sec \theta) \right) + 1 \right], \end{aligned} \quad (\text{C.9})$$

where in the last step of the derivation, we use the fact that the Chebyshev polynomial is equal to the Gaussian hypergeometric function,

$$T_x(\cos \phi) = {}_2F_1 \left(x, -x, \frac{1}{2}; \frac{1}{2}(1 - \cos \phi) \right), \quad (\text{C.10})$$

for integer x . Then we use the fact that the Gaussian hypergeometric function is defined for real number x to make analytical continuations.

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