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High Energy Physics – Theory

A note on the geometry of hypermultiplets in the field theory of intersecting D3-branes

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Dedicated to the loving memory of my father, Mohammad Hossein Abbaspour-Tamijani, who taught me the passion for research.

ABSTRACT

We study the Kähler geometry of the hypermultiplet moduli space in the effective field theory of two intersecting D3-branes from the viewpoint of the classical RG flow equation governing the running of the Kähler potential defining the theory. We analyze this equation as a nonlinear PDE of first order and construct a general class of its solutions compatible with the symmetries in the problem. These solutions involve an arbitrary function of two independent variables and generically break the $\mathcal{N}=2$ supersymmetry of the system to $\mathcal{N}=1$ in 4 dimensions. Restricting this general class by the constraints of $\mathcal{N}=2$ supersymmetry and periodicity for magnetic charge quantization, we recover the Gibbons-Hawking geometry previously proposed for this system in ref. [5]. This provides a systematic derivation and a uniqueness proof for the geometry of the moduli space in this problem. We also give an independent proof of the preservation of the $\mathcal{N}=2$ supersymmetry by the RG flow and a reinterpretation of the periodicity in terms of a symmetry of the RG flow equation in appendices.

1. Introduction

Renormalization of field theories with extended sources/defects has attracted some recent attentions in view of its applications to phenomenological scenarios such as the brane-world models with two extra dimensions [1]. In this context, simple prototype models with codimension two defects have been analyzed and it has been shown that the brane couplings are renormalized due to the divergencies arising from the brane-to-bulk interactions [2] (see also [3]). This leads to renormalization group (RG) flows for the brane couplings describing such interactions. Divergencies in such models can arise from both classical and quantum origins. In the classical case they originate from short distance singularities caused by the zero thickness of the brane, whereas in the quantum case they are traced back to the broken translation invariance along the brane's transverse directions [2].

It has been argued that this type of renormalization is also fruitful in reinterpreting brane actions in the framework of effective field theories, which otherwise are ill-defined due to singular bulk fields sourced by the brane [4]. This point of view has recently been adapted [5] to a system of intersecting D3-branes in which two D3-branes orthogonally intersect over one common dimension [6]. This system is a 1/4 BPS state (preserving 8 supercharges) corresponding to $\mathcal{N}=2$ supersymmetry in the 4-dimensional language. In certain conditions it admits a definite decoupling limit and can be regarded as a kind of defect CFT with a certain holographic dual [7]. In this example, the locus of intersection of the two branes plays the role of a codimension 2 defect in the 4-dimensional worldvolume theory of each of the branes. Thus the bulk theory is substituted for two copies of the usual $\mathcal{N}=4$ SYM theory along the 3+1 dimensional worldvolumes of the two branes, whereas the defect's role is played by their 1+1 dimensional intersection.

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The defect (intersection) theory is defined by dimensional reduction of the theory of an $\mathcal{N}=2$ hypermultiplet from 4 to 2 dimensions.[1] Replacing the hypermultiplet with a pair of chiral superfields (B,C) in $\mathcal{N}=1$ language, the superspace action for this theory is constructed using a Kähler potential $K(B,C,\bar{B},\bar{C})$ which encodes all the information regarding the hypermultiplet interactions or couplings. The component action for the two complex scalars in (B,C) then takes the form of a nonlinear sigma model with a Kähler metric derived from K on its target space. This is the metric on the moduli space of deformations (resolutions) of a 1/4 BPS system of two flat intersecting D3-branes to a single smoothly curved D3-brane that asymptotes to the former and preserves the same supercharges. A curved brane with this property is described by a calibrated surface in the relative transverse space of the original system of branes [8]. The nonlinear sigma model action of the moduli fields then naturally emerges from the DBI action of a single D3-brane with such a calibrated geometry in flat spacetime in the limit of small curvature of the brane and it turns out to be given by a Gibbons-Hawking (GH) geometry on the moduli space (parametrized by (B,C)) [5].

Singularities of the brane fields along the intersection give rise to renormalization of this theory using the framework of classical effective field theory which leads to a classical RG flow for the Kähler potential [5], [9]. Viewed as a nonlinear partial differential equation (PDE) of first order, the RG flow equation can be regarded as the equation governing the running behaviors of the D3-D3′ system such as its UV and IR limits, the RG fixed points, existence of a Landau pole [10] and preservation of the $\mathcal{N}=2$ supersymmetry under this flow. Finding exact solutions of this equation with suitable boundary conditions, which gives the exact form of the scale dependence of the geometry, hence is essential in understanding such behaviors.

To achieve this goal we first reduce the number of independent variables in the RG flow equation using the restrictions from the gauge invariance and R-symmetry in the problem. Using simple tricks this equation then is reduced to a quasilinear PDE of first order in 3 independent variables, whose general solution is easily constructed by the Cauchy's method of characteristics in the theory of PDE's. This general solution involves an arbitrary function of 2 independent variables and generically violates the requirement of $\mathcal{N}=2$ supersymmetry expressed as the unimodularity condition for the corresponding Kähler metric. It thus corresponds to a geometry preserving only an $\mathcal{N}=1$ 4D supersymmetry of the original theory. Imposing the unimodularity condition alongside with a periodicity property required by magnetic charge quantization then reduces the general solution to a periodic GH geometry first proposed in ref. [5]. This provides a systematic derivation and a proof of uniqueness of the geometry in the problem at hand.

Towards these goals, the paper is organized as follows: after briefly reviewing the model and its RG flow equation in section 2, we present in section 3 the simplifying symmetry assumptions that constrain the form of the Kähler potential. This then reduces the RG equation to a simplified PDE with a fewer number of independent variables in section 4. We then apply the method of characteristics to derive general solution of this equation. We reformulate periodicity as a boundary condition and construct general form of the periodic solutions. We next introduce a change of variables that simplifies analysis of the simultaneous solutions of the RG equation and the unimodularity condition. We establish equivalence of the resulting geometry to a GH geometry in section 5. We discuss the UV/IR limits of the general solution and particular solutions including the RG fixed points in section 6. We conclude the paper by a summary of its results in section 7. We give a general proof of the preservation of the $\mathcal{N}=2$ condition by the RG flow in Appendix A. We reinterpret existence of periodic solutions in terms of a symmetry of the RG flow equation under the periodicity map on the space of hypermultiplets in Appendix B.

2. The model and its RG flow

The microscopic degrees of freedom of the D3⊥D3′(1) system [11] consist of the 3-3, 3′-3′ and 3-3′ open strings whose end points lie on D3 or D3′. The 3-3 and 3′-3′ strings propagate along the 3+1 dimensional worldvolumes of D3 and D3′, whereas the 3-3′ strings propagate only along their 1+1 dimensional intersection. As a consequence, the dynamics at low energies ($\alpha' \to 0$ limit) is governed by a coupled system of three field theories consisting of two copies of a 3+1 dimensional $\mathcal{N}=4$ SYM theory along D3 and D3′ worldvolumes and a 1+1 dimensional theory of $\mathcal{N}=(4,4)$ hypermultiplets along their intersection [7]. This coupled system of field theories was first introduced and studied as a kind of defect CFT with holographic properties in ref. [7]. The authors of this paper considered only the possibility of a canonical action for the hypermultiplet fields on the intersection. This formulation, however, turns out to be incomplete in some respects. For example, as noted in ref. [5], it does not admit proper solitonic solutions [12] that describe D1 strings stretched between the two branes. So the authors of ref. [5] considered a generalized non-canonical action instead of that in ref. [7] in a way to support such solitonic solutions (for explicit soliton equations see ref. [13]). Here we briefly review some relevant facts from refs. [5], [7] to set up our problem in the present paper.

We consider a pair of overlapping/intersecting perpendicular D3-branes with common temporal and spatial directions along the (x^0,x^1) coordinates. To adapt notations used in ref. [5], we choose the branes worldvolume directions to be oriented along the coordinate system as follows:

$$\begin{aligned} \mathrm{D3}:&\quad 0145 \\ \mathrm{D3}':&\quad 0167 \end{aligned} \tag{1}$$

Each D3-brane in this system is a 1/2 BPS state that preserves 16 supercharges in a separate $D=4, \mathcal{N}=4$ subalgebra of the 10-dimensional type IIB superalgebra. The two algebras share a common set of 8 supercharges belonging to a $D=4, \mathcal{N}=2$ subalgebra, making the overall system a 1/4 BPS state preserving these 8 supercharges [11]. The low-energy dynamics of this system is governed by the massless modes of the 3-3, 3′-3′ strings and the low-lying modes of the 3-3′ strings. The latter modes are also massless if the

[1] This is also referred to as an $\mathcal{N}=(4,4)$ hypermultiplet in 1+1 dimensions.

transverse (minimal) distance between the two branes is vanishing. The corresponding field contents of D3 and D3′ (the 3-3 and 3′-3′ strings) are the usual $U(1)$, $\mathcal{N}=4$ vector multiplets in 3+1 dimensions. One can represent them in terms of the $\mathcal{N}=1$ vector and chiral superfields: $(V, Q_{1,2,3})$ for D3 and $(V', S_{1,2,3})$ for D3′. The one on their intersection (the 3-3′ strings) is a 4-dimensional $\mathcal{N}=2$ hypermultiplet dimensionally reduced to 1+1 dimensions and can be represented in terms of a pair of chiral superfields (B, C) in $\mathcal{N}=1$ language.

From the perturbative string perspective, the field content of the low-energy theory at the intersection is identified with the ground states of the 3-3′ strings as follows: Its bosonic content is given by the ground states in the NS sector which consist of 4 scalars in a spinor representation of the $SO(2)_{45} \times SO(2)_{67}$ rotations in the relative transverse directions of the two branes. Its fermionic content is given by the ground states in the R sector which form 4 $SO(1,1)_{01}$ spinors transforming in a spinor representation of the $SO(4)_{2389}$ rotations in the overall transverse directions. This contains 4 fermionic degrees of freedom on-shell which matches the 4 bosonic degrees of freedom in the NS sector. The 4 bosons can be assembled into two complex scalars, whereas the 4 fermions fit into two Weyl spinors (on-shell) in 4-dimensional language. This is exactly the content of an $\mathcal{N}=2$ hypermultiplet in 4 dimensions or its equivalent $\mathcal{N}=(4,4)$ in 2 dimensions.

The chiral fields B, C carry the respective charges $(1,-1)$ and $(-1,1)$ (corresponding to two different orientations of the 3-3′ strings) under the corresponding gauge group $U(1)_V \times U(1)_{V'}$ of this brane system. They carry the charges $(1/2, 1/2)$ under the $S(2)_{45} \times SO(2)_{67}$ subgroup of the overall R-symmetry group $S(2)_{45} \times SO(2)_{67} \times SO(4)_{2389}$ for $\mathcal{N}=2$ supersymmetry.

From the $\mathcal{N}=2$ point of view, each of the pairs (V, Q_3) and (V', S_3) forms a vector multiplet, whereas (Q_1, Q_2), (S_1, S_2) and (B, C) each constitutes a hypermultiplet. All this field content can be obtained by dimensionally reducing 4D superfields from $x^{0,1,2,3}$ to $x^{0,1}$ directions [14]. So they do not depend on $x^{2,3}$ coordinates. On the other hand, the $x^{4,5}$ and $x^{6,7}$ coordinates appearing in the D3 and D3′ superfields are treated as parameters in this formulation. We will occasionally use the complex coordinates $z_1 := \frac{1}{2}(x^4 + ix^5)$ and $z_2 := \frac{1}{2}(x^6 + ix^7)$ to label these directions. The most general action for this system compatible with the above symmetries can be written in the following form:

$$\mathbf{S} = \mathbf{S}_{3-3}\left[V, Q_{1,2,3}\right] + \mathbf{S}_{3'-3'}\left[V', S_{1,2,3}\right] + \mathbf{S}_{3-3'}\left[B, C, V - V', Q_3 - S_3\right], \tag{2}$$

where $\mathbf{S}_{3-3}$ and $\mathbf{S}_{3'-3'}$ are the usual superspace actions for $\mathcal{N}=4$ SYM theories along the worldvolumes of D3 and D3′ written in terms of the $\mathcal{N}=1$ superfields, while $\mathbf{S}_{3-3'}$ is defined by:

$$\begin{aligned}\mathbf{S}_{3-3'} = \int d^2x_{\parallel}\Big[&\int d^4\theta K(Be^{V-V'}, Ce^{V'-V}, \bar{B}, \bar{C}) \\ &+ \frac{i}{\sqrt{2}}\int d^2\theta f(BC)(Q_3 - S_3) + \text{c.c.}\Big].\end{aligned} \tag{3}$$

Here K is a Kähler potential and $x_{\parallel}$ denotes the intersection directions $x_{\parallel} = (x^0, x^1)$ and the fields V, V', Q_3, S_3 are evaluated on the locus of the intersection at $z_1 = z_2 = 0$. The superpotential here is written as $W = \frac{i}{\sqrt{2}} f(BC)(Q_3 - S_3)$. This is the most general form consistent with $\mathcal{N}=2$ supersymmetry and gauge invariance. By writing this action in component form it is easy to find that it is compatible with the $SO(4)_{2389}$ R-symmetry (and hence with $\mathcal{N}=2$) if the Kähler metric $g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K$ is related to f as follows:

$$|f'|^2 = g := \det(g_{\alpha\bar{\beta}}), \tag{4}$$

where we have defined the complex coordinates by $Z^\alpha := (B, C)$ and $\bar{Z}^{\bar{\alpha}} := (\bar{B}, \bar{C})$. Further, for the superpotential action to be invariant under $SO(2)_{45} \times SO(2)_{67}$ R-symmetries, the function $f(BC)$ must have the R-charges $(1,1)$, which means that it can only be chosen (up to a normalization factor) to be $f(BC) = BC$. Eq. (4) then is expressed as the unimodularity $g = 1$ condition. This implies that the Kähler metric is Ricci flat.

An interesting consequence of the theory defined by eqs. (2), (3) is the appearance of a classical renormalization group flow [2] for the Kähler potential associated to logarithmic divergencies arising at the level of classical field equations. This is because the field equations for the D3 and D3′ fields (e.g. V, V', etc.) derived from this action are sourced by the fields (B, C) on the intersection, which is a codimension 2 defect on both of the branes. So they involve 2D delta function sources along $x^{4,5}$ and $x^{6,7}$ directions, respectively [5], [9]. This means that the brane fields involve singularities on the locus of the intersection/defect at $x^{4,5} = x^{6,7} = 0$. Since it is the values of the brane fields at $x^{4,5} = x^{6,7} = 0$ that appear in the definition of the intersection theory, inserting these fields in $\mathbf{S}_{3-3'}$ leads to logarithmic divergencies and hence an ambiguity at the level of the classical field theory. This can be avoided by renormalizing the theory using the concept of *classical* effective field theory. The idea is to integrate out all the brane fields from the theory by solving their field equations perturbatively in terms of the fields (B, C) on the intersection. Regulating the divergent momentum integrals [10] associated to singularities of the fields at $x^{4,5} = x^{6,7} = 0$ by introducing a momentum cut-off Λ then brings in an additional scale μ ($\mu < \Lambda$) in the regularized theory through powers of $\ln(\Lambda/\mu)$. The scale μ is interpreted as the scale at which one probes the intersection of the two branes. The perturbative corrections to the action computed in this manner are found to be written as a series of D-terms in powers of $\ln(\Lambda/\mu)$. The procedure thus renormalizes the Kähler potential K in the intersection theory but not the superpotential W which is an F-term. Identifying the initial Kähler potential as a bare quantity evaluated at the UV cut-off Λ, and the corrected Kähler potential as that at the scale μ then leads to a consistent renormalization procedure. The exact classical RG flow to all orders is fixed only by the first order term in $\ln(\Lambda/\mu)$ and is given by the following equation [5], [9]:

$$\mu\partial_\mu K = \frac{1}{4\pi}\left(\mathcal{P}^2 - 2|f|^2\right), \tag{5}$$

where

$$\mathcal{P} := \left(B\partial_B - C\partial_C\right) K \tag{6}$$

is the moment map for K. The flow is consistent in the sense that all higher order corrections to K are determined in terms of the first order one just by differentiation [9]. Eq. (5) can be viewed as a partial differential equation for the unknown function $K(B, C, \bar{B}, \bar{C}; \mu)$. Giving an "initial condition" for K at a fixed scale $\mu = \mu_0$ will uniquely determine the solution for K at other scales.

An important restriction on the above model is a periodicity property of the form $\mathcal{P} \sim \mathcal{P} + 8\pi$ assigned to $\mathcal{P}$ by the requirement of magnetic charge quantization. This must be equivalent to a particular identification of the points on the space of the fields (B, C). Given the specific form of the identification map, this restriction then demands existence of a particular class of solutions for the PDE (5). Choosing a suitable scale dependent normalization, the identification map turns out that preserves the above form of the RG flow equation. One may ask which Kähler geometries are simultaneously consistent with the RG flow equation, the periodicity and the $\mathcal{N} = 2$ condition. We will find that the unique solution satisfying these conditions is a Gibbons-Hawking geometry of the form proposed in ref. [5].

3. Symmetry constraints on the Kähler potential

There are two symmetry conditions that restrict the functional form of the Kähler potential $K(B, C, \bar{B}, \bar{C})$ in eq. (3). The first one comes from the invariance under the $U(1)_V \times U(1)_{V'}$ gauge transformations of the intersection theory coupled to two vector multiplets on D3-branes. These are defined by [5]

$$\begin{aligned} &V \to V + \Lambda + \bar{\Lambda} \qquad && V' \to V' + \Lambda' + \bar{\Lambda}' \\ &B \to Be^{\Lambda' - \Lambda} \qquad && C \to Ce^{\Lambda - \Lambda'}, \end{aligned} \tag{7}$$

for B and C having the gauge charges $(+1, -1)$ and $(-1, +1)$ with respect to (V, V'), respectively. The corresponding invariance property for K means

$$K(Be^{V-V'}, Ce^{V'-V}, \bar{B}, \bar{C}) \equiv K(e^{\bar{\Lambda}-\bar{\Lambda}'}Be^{V-V'}, e^{\bar{\Lambda}'-\bar{\Lambda}}Ce^{V'-V}, e^{\bar{\Lambda}'-\bar{\Lambda}}\bar{B}, e^{\bar{\Lambda}-\bar{\Lambda}'}\bar{C}). \tag{8}$$

Setting $V - V' = 0$ and expanding to first order in $(\bar{\Lambda} - \bar{\Lambda}')$, this condition implies the constraint on K:

$$\left(B\partial_B - C\partial_C - \bar{B}\partial_{\bar{B}} + \bar{C}\partial_{\bar{C}}\right) K = 0. \tag{9}$$

In terms of the holomorphic and antiholomorphic components of the $U(1)$ Killing vector

$$k^\alpha := (-iB, iC), \quad \bar{k}^{\bar{\alpha}} := (i\bar{B}, -i\bar{C}), \tag{10}$$

eq. (9) is equivalent to

$$ik^\alpha\partial_\alpha K + i\bar{k}^{\bar{\alpha}}\partial_{\bar{\alpha}} K = 0. \tag{11}$$

The second constraint comes from the invariance of K under the $SO(2)_{45}$ or $SO(2)_{67}$ R-symmetry transformations. For $SO(2)_{45}$ these are defined by

$$z_1 \to e^{i\alpha}z_1, \quad (B, C) \to (e^{i\alpha/2}B, e^{i\alpha/2}C), \tag{12}$$

for (B, C) having the R-charges $(1/2, 1/2)$ [5]. Similar relations, with z_1 replaced by z_2, define $SO(2)_{67}$ transformations. The R-symmetry invariance then means

$$K(B, C, \bar{B}, \bar{C}) \equiv K(e^{i\alpha/2}B, e^{i\alpha/2}C, e^{-i\alpha/2}\bar{B}, e^{-i\alpha/2}\bar{C}). \tag{13}$$

Upon expanding to first order in α, this gives the constraint

$$\left(B\partial_B + C\partial_C - \bar{B}\partial_{\bar{B}} - \bar{C}\partial_{\bar{C}}\right) K = 0. \tag{14}$$

The PDE's (9), (14) are indeed the necessary and sufficient conditions for the validity of the symmetry conditions (8), (13), respectively.[2] Combining them readily gives:

$$\left(B\partial_B - \bar{B}\partial_{\bar{B}}\right) K = \left(C\partial_C - \bar{C}\partial_{\bar{C}}\right) K = 0. \tag{15}$$

These imply that the generic dependence of K on $(B, C, \bar{B}, \bar{C})$ must be written in the following more specific way

[2] These are special cases of the Euler's theorem on homogeneous functions of several variables [15].

$$K = K(B\bar{B}, C\bar{C}). \tag{16}$$

Working with the complex variables (U, W) instead of (B, C), as in ref. [5],

$$U := \frac{1}{2}BC, \quad e^W := \frac{B}{C}, \tag{17}$$

and defining the real variables (u, v) by

$$\begin{aligned} e^u &:= |BC| = 2\sqrt{U\bar{U}}\,, \\ e^v &:= |\frac{B}{C}| = e^{(W+\bar{W})/2}, \end{aligned} \tag{18}$$

eq. (16) simply states that K is written as the real function $K = K(u, v)$. In particular, using these variables we find a simple expression for the moment map, eq. (6),

$$\mathcal{P} = \partial_v K. \tag{19}$$

In the following, we will use the facility provided by (u, v) variables to simplify analysis of the RG flow equation for K.

4. General solution of the RG flow equation

The RG flow equation (5) for running of the function $K(u, v, \lambda)$ in terms of the logarithmic scale variable $\lambda := -(1/2\pi)\ln(\mu/\mu_0)$ can be written as:

$$\partial_\lambda K = -\frac{1}{2}(\partial_v K)^2 + e^{2u}. \tag{20}$$

Rather than trying to solve this equation for K, it is more convenient first to transform it to an equation for $\mathcal{P} = \partial_v K$ by differentiation with respect to v and using eq. (19). This gives a more simple yet first order equation for $\mathcal{P}$:

$$\partial_\lambda \mathcal{P} + \mathcal{P}\partial_v \mathcal{P} = 0. \tag{21}$$

This has the advantage of being linear in derivatives of the unknown function $\mathcal{P}(u, v, \lambda)$. This allows one to solve it using the method of characteristics in the theory of PDE's [16]. The characteristic curves for this equation are given by the system of ODE's:

$$\frac{du}{d\lambda} = 0, \quad \frac{dv}{d\lambda} = \mathcal{P}, \quad \frac{d\mathcal{P}}{d\lambda} = 0, \tag{22}$$

which are solved by $u = a$, $v = b + c\lambda$, $\mathcal{P} = c$, with a, b, c =constants. The general solution for $\mathcal{P}(u, v, \lambda)$ is found by imposing a relation between the constants like $c = F(a, b)$, with F being an arbitrary function of two independent variables. This means that the solution for $\mathcal{P}$ is implicitly given by the algebraic equation:

$$\mathcal{P} = F(u, v - \lambda\mathcal{P}). \tag{23}$$

It is easy to verify that any solution of this equation for $\mathcal{P}$ as a function of the independent variables (u, v, λ) indeed solves the PDE (21). To see this, it is enough to differentiate eq. (23) with respect to v, λ, taking into account its explicit and implicit dependences on these variables, to get the relations

$$\begin{aligned} \partial_\lambda \mathcal{P} &= -(\mathcal{P} + \lambda\partial_\lambda \mathcal{P})\partial_v F(u, v - \lambda\mathcal{P}), \\ \partial_v \mathcal{P} &= (1 - \lambda\partial_v \mathcal{P})\partial_v F(u, v - \lambda\mathcal{P}). \end{aligned} \tag{24}$$

Eliminating F between these two equations, one recovers the PDE (21), hence verifying the validity of eq. (23) as its solution. After solving for $\mathcal{P}(u, v, \lambda)$ from eq. (23), the solution for $K(u, v, \lambda)$ simply follows a further integration with respect to v which brings in another yet arbitrary function of (u, λ). This function is fixed by inserting the solution for K in the PDE (20) and applying the initial condition.

The freedom of choosing F in the above solution can be fixed by giving an initial condition for $\mathcal{P}$. For example, stating that at $\lambda = 0$ (i.e. at the scale $\mu = \mu_0$) the Kähler metric is flat, in the variables (u, v) of eq. (18), means that

$$K|_{\lambda=0} = |B|^2 + |C|^2 = 2e^u \cosh v, \tag{25}$$

which upon using eq. (19) and (23) for $\lambda = 0$ requires the particular form of the function F to be:

$$\mathcal{P}|_{\lambda=0} = F(u, v) = 2e^u \sinh v. \tag{26}$$

The corresponding solution for $\mathcal{P}$ then is implicitly determined by solving the following algebraic equation for $\mathcal{P}$:

$$\mathcal{P} = 2e^u \sinh(v - \lambda\mathcal{P}). \tag{27}$$

Graphically, it is easy to see that this equation (for given values of u, v, λ), has a unique solution for $\mathcal{P}$, which lies in the range $|\mathcal{P}| < |v/\lambda|$ and has the same sign as v/λ. Although, it is not possible to find the exact form of this solution explicitly from eq. (27), it is possible to find it perturbatively as a power series in λ, assuming that λ is small. For this, it is enough first to expand eq. (27) in powers of λ and then solve it by iteration for $\mathcal{P}$. The result to $\mathcal{O}(\lambda^2)$ will be written as:

$$\mathcal{P} = 2e^u \sinh v - 2\lambda e^{2u} \sinh 2v + \lambda^2 e^{3u}(3\sinh 3v - \sinh v) + \cdots. \tag{28}$$

Integrating with respect to v then gives the solution for K up to an unknown additive function $G(u,\lambda)$. By inserting the result in eq. (20) and using the initial condition (25), one then finds $G(u,\lambda) = 2\lambda e^{2u}$, hence giving the solution for K as:

$$K = 2e^u \cosh v + \lambda e^{2u}(2 - \cosh 2v) + \lambda^2 e^{3u}(\cosh 3v - \cosh v) + \cdots. \tag{29}$$

Translated back to the language of (B, C, μ) variables, this gives in turn:

$$\begin{aligned} K(B, C, \bar{B}, \bar{C}; \mu) = {} & |B|^2 + |C|^2 \\ & + \left(\frac{1}{2\pi} \ln \frac{\mu}{\mu_0}\right) \frac{1}{2} \left(|B|^4 + |C|^4 - 4|B|^2|C|^2\right) \\ & + \left(\frac{1}{2\pi} \ln \frac{\mu}{\mu_0}\right)^2 \frac{1}{2} (|B|^2 - |C|^2)^2 (|B|^2 + |C|^2) \\ & + \cdots, \end{aligned} \tag{30}$$

which is the same result obtained in ref. [9]. It turns out that this solution for the flat space initial condition does not respect the periodicity condition on $\mathcal{P}$ demanded in ref. [5]. In the following subsections, using the freedom implied in the general solution (23), we construct a solution that satisfies such a periodic "boundary condition".

4.1. The periodic boundary condition

The periodicity property $\mathcal{P} \sim \mathcal{P} + 8\pi$ means to identify the points on the hypermultiplet space in a specific manner. The identification map is specified based on the two properties: the holomorphicity in (B, C) variables and the preservation of the gauge quantum numbers of these variables. Besides, it must preserve the forms of the Kähler potential (16) and the superpotential in the action (3). The map then is given by the following relations [5]

$$B \to B' = \beta f(BC) B, \quad C \to C' = \frac{C}{\beta f(BC)} \tag{31}$$

with β some real constant which may generally depend on λ. Written in terms of the (u, v) variables using eq. (18), the map (31) then means that

$$\begin{aligned} & u \to u' = u, \\ & v \to v' = v + 2u + \ln \beta^2. \end{aligned} \tag{32}$$

So, the periodicity condition on $\mathcal{P}$ as a function of (u, v, λ) means that

$$\mathcal{P}(u, v + 2u + \ln \beta^2, \lambda) = \mathcal{P}(u, v, \lambda) + 8\pi. \tag{33}$$

The parameter β here must be considered as a yet unknown function $\beta = \beta(\lambda)$. This function is determined by demanding that the function $\mathcal{P}(u, v + 2u + \ln \beta^2, \lambda) - 8\pi$ satisfies the same eq. (21) as for $\mathcal{P}(u, v, \lambda)$. Taking into account the dependence of β on λ, this requires that

$$(\partial_\lambda \mathcal{P}' + \mathcal{P}' \partial_{v'} \mathcal{P}') + (\partial_\lambda \ln \beta^2 - 8\pi) \partial_{v'} \mathcal{P}' = 0, \tag{34}$$

where $\mathcal{P}' = \mathcal{P}(u, v', \lambda)$. The term in the first parenthesis in this equation vanishes by eq. (21) whereas the remaining term gives

$$\partial_\lambda \ln \beta^2 - 8\pi = 0 \quad \to \quad \beta(\lambda) = \beta_0 e^{4\pi\lambda}. \tag{35}$$

We may choose the initial scale μ_0 in a way to set $\beta_0 = 1$. The periodicity condition (33) then will be written as

$$\mathcal{P}(u, v + 2u + 8\pi\lambda, \lambda) = \mathcal{P}(u, v, \lambda) + 8\pi. \tag{36}$$

We will use this as a boundary condition for eq. (21) to restrict the function $F(u, v)$ in its general solution by eq. (23).

4.2. Finding all solutions with periodic boundary condition

The periodic boundary condition (36) can be translated to a condition on $F(u, v)$. As a restriction on the function $\mathcal{P}(u, v, \lambda)$ implicitly defined by eq. (23), this condition means that changing $v \to v + 2u + 8\pi\lambda$ in that equation must result in changing its solution as $\mathcal{P} \to \mathcal{P} + 8\pi$. This means that $\mathcal{P}$ must satisfy the equation

$$\mathcal{P} + 8\pi = F(u, v + 2u - \lambda\mathcal{P}). \tag{37}$$

Subtracting (23) and (37) and replacing $v \to v + \lambda\mathcal{P}$ in the result, one finds a condition on F independent of $\mathcal{P}$ as follows:

$$F(u, v + 2u) = F(u, v) + 8\pi. \tag{38}$$

The most general function F with this property is written as:

$$F(u, v) = 4\pi\frac{v}{u} + \sum_{n=-\infty}^{\infty} c_n(u)e^{in\pi v/u}, \tag{39}$$

where the summation represents the Fourier expansion of a periodic function of v with the periodicity $v \sim v + 2u$, with u as a fixed parameter.

It proves useful, rather than to give the general solution for $\mathcal{P}(u, v, \lambda)$ implicitly via eq. (23), to invert this equation for $v - \lambda\mathcal{P}$ in terms of $(u, \mathcal{P})$ and write v explicitly as a function of $(u, \mathcal{P}, \lambda)$ as

$$v = \lambda\mathcal{P} + G(u, \mathcal{P}). \tag{40}$$

The arbitrary function $G(u, \mathcal{P})$ here is a substitute for $F(u, v)$ (equal to its inverse with respect to its second argument) in the previous form of the solution. By the same logic as for $F(u, v)$, one then finds a periodicity condition on $G(u, \mathcal{P})$ of the form:

$$G(u, \mathcal{P} + 8\pi) = G(u, \mathcal{P}) + 2u. \tag{41}$$

This is solved by sum of a linear function of $\mathcal{P}$ and a periodic function with the periodicity $\mathcal{P} \sim \mathcal{P} + 8\pi$,

$$G(u, \mathcal{P}) = \frac{1}{4\pi}\left(u\mathcal{P} + \sum_{n=-\infty}^{\infty} d_n(u)e^{in\mathcal{P}/4}\right) \tag{42}$$

The function $v(u, \mathcal{P}, \lambda)$ given by eq. (40) must be the inverse of the function $\mathcal{P} = \partial_v K(u, v, \lambda)$ in variable v. This allows one to express all functions of (u, v, λ) alternatively as functions of $(u, \mathcal{P}, \lambda)$. From the solution (40) together with (42) one obtains

$$\begin{aligned}\partial_{\mathcal{P}} v &= \lambda + \partial_{\mathcal{P}} G(u, \mathcal{P}) =: \frac{1}{4}V(u, \mathcal{P}, \lambda)\\ &= \lambda + \frac{u}{4\pi} + \frac{i}{4}\sum_{n=-\infty}^{\infty} n d_n(u)e^{in\mathcal{P}/4}.\end{aligned} \tag{43}$$

The function $V(u, \mathcal{P}, \lambda)$ defined by this equation also has the periodicity $V(u, \mathcal{P}+8\pi, \lambda) = V(u, \mathcal{P}, \lambda)$. After imposing the unimodularity condition, we will find that V is the same as the harmonic function (the GH potential) that appears in a GH geometry. Note that the only dependence of V on λ comes through the linear additive term 4λ in eq. (43). Given the function $V(u, \mathcal{P}, \lambda)$, one can derive $v(u, \mathcal{P}, \lambda)$ simply by integrating it with respect to $\mathcal{P}$:

$$v = \frac{1}{4}\int d\mathcal{P}\, V(u, \mathcal{P}, \lambda). \tag{44}$$

The explicit solution for $K(u, v, \lambda)$ then is obtained by inverting the above equation for $\mathcal{P}$ and carrying out another integration with respect to v:

$$K(u, v, \lambda) = \int dv\, \mathcal{P}(u, v, \lambda). \tag{45}$$

4.3. Change of variables

As explained earlier, writing the general solution for $\mathcal{P}$ in the form of eq. (40) is suggestive of a change of independent variables from (u, v, λ) to $(u, \mathcal{P}, \lambda)$. The link between the two sets is provided by eq. (19), which upon inverting gives v in terms of the second set of variables: $v = v(u, \mathcal{P}, \lambda)$. As such, any function of (u, v, λ) can be written alternatively as another function of $(u, \mathcal{P}, \lambda)$. Then partial derivatives of $v(u, \mathcal{P}, \lambda)$ can be determined in terms of those of $\mathcal{P}(u, v, \lambda)$ using the standard procedure. In particular, one finds

$$\partial_\lambda v = -\frac{\partial_\lambda \mathcal{P}}{\partial_v \mathcal{P}}. \tag{46}$$

Combining this with eq. (21) then gives

$$\partial_\lambda v = \mathcal{P}. \tag{47}$$

This is just the equivalent form of the RG eq. (20) written in terms of v as a function of the independent variables $(u, \mathcal{P}, \lambda)$. Its general solution is found simply by integration with respect to λ which reproduces the same result given earlier by eq. (40).

Let us now denote by L the same function K written in terms of the $(u, \mathcal{P}, \lambda)$ variables:

$$L(u, \mathcal{P}, \lambda) := K\left(u, v = \lambda\mathcal{P} + G(u, \mathcal{P}), \lambda\right). \tag{48}$$

We can easily compute partial derivatives of L from this definition:

$$\partial_u L = \partial_u K + \mathcal{P}\partial_u G,$$
$$\partial_{\mathcal{P}} L = \mathcal{P}\left(\lambda + \partial_{\mathcal{P}} G\right),$$
$$\partial_\lambda L = \mathcal{P}^2 + \partial_\lambda K, \tag{49}$$

where we have made multiple use of eq. (19). Using the last equation in (49), the RG eq. (20) for K then translates to an equivalent equation for L as

$$\partial_\lambda L = \frac{1}{2}\mathcal{P}^2 + e^{2u}. \tag{50}$$

This is easily integrated to

$$K = L(u,\mathcal{P},\lambda) = \lambda\left(\frac{1}{2}\mathcal{P}^2 + e^{2u}\right) + L_0(u,\mathcal{P}), \tag{51}$$

where $L_0(u,\mathcal{P}) := L(u,\mathcal{P},\lambda=0)$ is a yet arbitrary function of two variables which is related to the function $G(u,\mathcal{P})$ in eq. (40). One way to find this relation is to differentiate eq. (51) with respect to $\mathcal{P}$ and equate it to the second eq. (49). This will give

$$\partial_{\mathcal{P}} L_0(u,\mathcal{P}) = \mathcal{P}\partial_{\mathcal{P}} G(u,\mathcal{P}), \tag{52}$$

which is easily integrated for L_0 to

$$L_0(u,\mathcal{P}) = \mathcal{P}\partial_{\mathcal{P}} H(u,\mathcal{P}) - H(u,\mathcal{P}), \tag{53}$$

where we have defined $H(u,\mathcal{P})$ via

$$G(u,\mathcal{P}) =: \partial_{\mathcal{P}} H(u,\mathcal{P}). \tag{54}$$

Eliminating $\mathcal{P}$ from eq. (51) using eq. (40) then gives the desired solution for $K(u,v,\lambda)$.

4.4. The unimodularity/Ricci flatness condition

As explained in section 2, to ensure $\mathcal{N}=2$ supersymmetry invariance of the action (3), we need to impose the unimodularity ($g=1$) condition on the Kähler metric, which in turn amounts to Ricci flatness of the Kähler geometry. Here we have a 2-dimensional Kähler geometry with complex coordinates $Z^\alpha = (B,C)$ with a Kähler potential having the particular form of eq. (16). Working in terms of the (u,v) variables defined by eq. (18), one finds the metric components:

$$g_{B\bar{B}} = \frac{1}{4B\bar{B}}(\partial_u + \partial_v)^2 K, \qquad g_{B\bar{C}} = \frac{1}{4B\bar{C}}(\partial_u^2 - \partial_v^2)K,$$
$$g_{C\bar{B}} = \frac{1}{4C\bar{B}}(\partial_u^2 - \partial_v^2)K, \qquad g_{C\bar{C}} = \frac{1}{4C\bar{C}}(\partial_u - \partial_v)^2 K. \tag{55}$$

The unimodularity condition then is expressed as the equation

$$g = \frac{1}{4}e^{-2u}\left[(\partial_u^2 K)(\partial_v^2 K) - (\partial_u\partial_v K)^2\right] = 1, \tag{56}$$

which is a PDE of second order for the unknown function $K(u,v,\lambda)$, where λ enters as a parameter. We want now to determine a special class of solutions of the RG flow equation that simultaneously satisfy also this unimodularity condition. This amounts to determining the restriction on the unknown function $H(u,\mathcal{P})$ in our general solution (51), (53) for K imposed by the unimodularity condition. For this, we need the expressions for the second derivatives of K with respect to (u,v) in terms of the derivatives of H with respect to $(u,\mathcal{P})$. These are found in four steps. i) First, we use the first eq. (49) together with eqs. (51), (53) and (19) to find expressions for $\partial_u K, \partial_v K$ in terms of $(u,\mathcal{P},\lambda)$. ii) Next, by differentiating these expressions with respect to (u,v) we find new expressions involving $\partial_u\mathcal{P}$, $\partial_v\mathcal{P}$. iii) Then, by differentiating eq. (40) with respect to (u,v) we find equations that determine $\partial_u\mathcal{P}$, $\partial_v\mathcal{P}$ as functions of $(u,\mathcal{P},\lambda)$. iv) Substituting these in the expressions in the step (ii) then leads to the desired expressions for $\partial_u^2 K, \partial_u\partial_v K, \partial_v^2 K$. The resulting expressions only involve second derivatives of $H(u,\mathcal{P})$ and are written as follows:

$$\partial_u^2 K = 4\lambda e^{2u} - \partial_u^2 H + \frac{(\partial_u\partial_{\mathcal{P}} H)^2}{\lambda + \partial_{\mathcal{P}}^2 H},$$
$$\partial_u\partial_v K = \partial_u\mathcal{P} = -\frac{\partial_u\partial_{\mathcal{P}} H}{\lambda + \partial_{\mathcal{P}}^2 H},$$
$$\partial_v^2 K = \partial_v\mathcal{P} = \frac{1}{\lambda + \partial_{\mathcal{P}}^2 H} \tag{57}$$

Using these expressions in the unimodularity condition, eq. (56), then reduces it to

$$g = \frac{\lambda - \frac{1}{4}e^{-2u}\partial_u^2 H(u,\mathcal{P})}{\lambda + \partial_{\mathcal{P}}^2 H(u,\mathcal{P})} = 1. \tag{58}$$

This is satisfied identically, independent of λ, if H satisfies

$$e^{-2u}\partial_u^2 H + 4\partial_{\mathcal{P}}^2 H = 0. \tag{59}$$

This is a 3D Laplace equation in cylindrical coordinates $(\rho, \varphi, z) = (e^u/2, \varphi, \mathcal{P}/4)$ restricted to axisymmetric solutions. We will see in the next section that the harmonic function H is closely related to the one appearing in the definition of a GH geometry. Meanwhile, elimination of λ and appearance of a two-variable PDE of the form (59) is a direct indication that the RG flow preserves the unimodularity condition and hence $\mathcal{N} = 2$ supersymmetry.

5. Gibbons-Hawking geometry

Gibbons-Hawking (GH) geometry is a particular Ricci flat (hyper-) Kähler geometry with a $U(1)$ isometry in two complex dimensions [5]. It is useful in constructing a class of $\mathcal{N} = 2$ models using a pair of $\mathcal{N} = 1$ chiral superfields which together form an $\mathcal{N} = 2$ hypermultiplet. In this section, after reviewing the GH geometry along the lines of ref. [5] (and completing it in some respects), we establish its equivalence to the solution of the RG flow equation constructed in the previous section.

Denoting the $U(1)$ isometry direction by the periodic coordinate θ and the other three directions by $\mathbf{x} := (x, y, z)$ (the cartesian coordinates on $\mathbb{E}^3$), the GH metric can be expressed in the following form:

$$ds^2 = V(\mathbf{x})d\mathbf{x}\cdot d\mathbf{x} + \frac{1}{V(\mathbf{x})}\left(d\theta + \mathbf{A}(\mathbf{x})\cdot d\mathbf{x}\right)^2, \tag{60}$$

where the 3-vector $\mathbf{A}(\mathbf{x})$ is related to $V(\mathbf{x})$ by $\boldsymbol{\nabla}\times\mathbf{A} = \boldsymbol{\nabla}V$. Obviously, $V(\mathbf{x})$ must then satisfy a 3D Laplace equation $\nabla^2 V = 0$, in the absence of singular points (sources). Adding properly chosen delta function sources then gives:

$$\nabla^2 V = -4\pi \sum_i \delta^3(\mathbf{x} - \mathbf{x}_i). \tag{61}$$

Defining the 1-forms

$$A := \mathbf{A}(\mathbf{x})\cdot d\mathbf{x}, \quad \omega := A + d\theta, \quad \phi := V dz + i\omega \tag{62}$$

and the complex coordinates $U, \bar{U} := x \pm iy$, the metric (60) simply is written as

$$ds^2 = V dU d\bar{U} + \frac{1}{V}\phi\bar{\phi}, \tag{63}$$

where V is now regarded as the function $V = V(U, \bar{U}, z)$. The condition $\boldsymbol{\nabla}\times\mathbf{A} = \boldsymbol{\nabla}V$ in the language of differential forms is equivalent to $dA = * dV$, where the $*$ denotes the Hodge dual in the flat $\mathbb{E}^3$ space parametrized by the cartesian coordinates (x, y, z). One can compute $d\phi$ using this condition to obtain

$$d\phi = \left(-2\partial_U V dz + \frac{1}{2}\partial_z V d\bar{U}\right)\wedge dU. \tag{64}$$

Adding a term proportional to dU to the 1-form in the parenthesis does not change the result for $d\phi$, but can be chosen to make it an exact 1-form[3]

$$d\sigma := -2\partial_U V dz + \frac{1}{2}\partial_z V d\bar{U} + T dU, \tag{65}$$

where $T = T(U, \bar{U}, z)$ and $\sigma = \sigma(U, \bar{U}, z)$ are suitably chosen functions of three variables. Eq. (64) can then be locally (i.e. far from the points $\mathbf{x}_i$) integrated to give

$$\phi = dW + \sigma dU, \tag{66}$$

for some complex valued function $W(U, \bar{U}, z, \theta)$. Separating the θ dependence of this function as

$$W = W_1(U, \bar{U}, z) + i\theta, \tag{67}$$

and applying the definition of ϕ by eq. (62) to eq. (66) then reduces the latter to

$$V dz + iA = dW_1 + \sigma dU. \tag{68}$$

Assuming A, W_1 to be real and decomposing this equation into its $U, \bar{U}, z$ components, we get the results[4] for W_1, σ, A:

[3] This, of course, requires certain integrability conditions that can be seen that hold identically, if V satisfies the Laplace equation.

[4] Here we choose a gauge in which $A_z = 0$ and so A has only $A_U, A_{\bar{U}}$ components.

$$\partial_z W_1 = V, \tag{69}$$

$$\sigma = -2\partial_U W_1, \tag{70}$$

$$A = \frac{1}{2i}\left(\sigma dU - \bar{\sigma} d\bar{U}\right). \tag{71}$$

Given the function $V(x,y,z)$, eq. (69) fixes $W_1(x,y,z)$ up to an arbitrary additive function of (x,y) and hence σ and A via eqs. (70), (71). Alternatively, by exchanging the roles of W_1, z as the dependent and independent variables, we can express eqs. (70), (71) in the following equivalent forms:

$$\sigma = 2V\partial_U z, \tag{72}$$

$$\partial_{W_1} z = \frac{1}{V}. \tag{73}$$

Taking $(U, \bar{U}, W, \bar{W})$ as complex coordinates, instead of the real ones (x, y, z, θ), the metric now is written as

$$ds^2 = V dU d\bar{U} + \frac{1}{V}(dW + \sigma dU)\left(d\bar{W} + \bar{\sigma} d\bar{U}\right). \tag{74}$$

The Kähler $(1,1)$-form for this metric

$$\Omega := V dU \wedge d\bar{U} + \frac{1}{V}(dW + \sigma dU) \wedge \left(d\bar{W} + \bar{\sigma} d\bar{U}\right) \tag{75}$$

is closed, because

$$\begin{aligned} d\Omega &= d\left[V(dx + idy) \wedge (dx - idy) + \frac{1}{V}(Vdz + i\omega) \wedge (Vdz - i\omega)\right] \\ &= -2i(dV \wedge dx \wedge dy - d\omega \wedge dz) \equiv 0, \end{aligned} \tag{76}$$

where in going to the last equality we have used $d\omega = dA = * dV$. This amounts to saying that $\partial\Omega = \bar{\partial}\Omega = 0$, which implies that Ω can (locally) be written as $\Omega = \partial\bar{\partial}K$ for some Kähler potential $K(Z^\alpha, \bar{Z}^{\bar{\alpha}})$, where $(Z^\alpha, \bar{Z}^{\bar{\alpha}}) := (U, W, \bar{U}, \bar{W})$. This means that the metric tensor has the form of a Kähler geometry:

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K \tag{77}$$

The Kähler potential is determined by matching the metric components from eq. (74) with those of the corresponding Kähler geometry, i.e.

$$\begin{aligned} &g_{U\bar{U}} = \partial_U \partial_{\bar{U}} K = V + \frac{\sigma\bar{\sigma}}{V} \qquad g_{U\bar{W}} = \partial_U \partial_{\bar{W}} K = \frac{\sigma}{V} \\ &g_{W\bar{U}} = \partial_W \partial_{\bar{U}} K = \frac{\bar{\sigma}}{V} \qquad g_{W\bar{W}} = \partial_W \partial_{\bar{W}} K = \frac{1}{V}. \end{aligned} \tag{78}$$

The metric determinant in these coordinates equals

$$g = g_{U\bar{U}} g_{W\bar{W}} - g_{U\bar{W}} g_{W\bar{U}} = 1. \tag{79}$$

For a general Kähler metric $g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K$ the Ricci tensor has the non-vanishing components [17]:

$$R_{\alpha\bar{\beta}} = -\partial_\alpha \partial_{\bar{\beta}} \ln g, \tag{80}$$

where $g := \det(g_{\alpha\bar{\beta}})$. Thus the Ricci flatness condition means that g must be decomposable as a product of holomorphic and anti-holomorphic functions of $(Z^\alpha, \bar{Z}^{\bar{\alpha}})$; namely

$$R_{\alpha\bar{\beta}} = 0 \quad \leftrightarrow \quad g = h(Z^\alpha)\bar{h}(\bar{Z}^{\bar{\alpha}}) = |h(Z^\alpha)|^2. \tag{81}$$

The holomorphic and antiholomorphic parts are conjugate of each other to ensure reality and positive-definiteness of the metric (and hence of g). It is always possible to find a holomorphic change of coordinates $Z'^\alpha = Z'^\alpha(Z^\beta)$ in a way to transform g to a constant value, given that g generally transforms as $g' = g|\det(\partial Z'^\alpha/\partial Z^\beta)|^{-2}$. This is consistent with our need in the present context, which is stated as $g = |f'|^2$ (and so $g = 1$ for $f(BC) = BC$). Indeed, from eqs. (17), one easily finds that the Jacobian of the map $(B, C) \to (U, W)$ equals

$$\det\left(\frac{\partial(U, W)}{\partial(B, C)}\right) = -1, \tag{82}$$

which ensures validity of the unimodularity condition in (B, C) coordinates as well. This is what we need for having $\mathcal{N} = 2$ supersymmetry in the intersection theory.

5.1. The Kähler geometry with $U(1)_G \times U(1)_R$ isometry as a GH geometry

The Kähler geometry in the field theory of intersecting D3-branes possesses two different $U(1)$ isometries: a $U(1)_G$ corresponding to the $U(1)_V \times U(1)_{V'}$ gauge symmetry and a $U(1)_R$ corresponding to the $SO(2)_{45} \times SO(2)_{67}$ R-symmetry. In the complex (U, W) variables these isometries are represented by:

$$\begin{aligned} U(1)_G &: \; W \to W + i\alpha \\ U(1)_R &: \; U \to e^{i\beta} U. \end{aligned} \tag{83}$$

Identifying the complex (U, W) variables in field theory (eq. (17)) with those in GH geometry (eq. (74)), and replacing them with the real ones (x, y, z, θ) (eq. (60)), the above isometries are equivalently expressed as:

$$\begin{aligned} U(1)_G &: \; \theta \to \theta + \alpha \\ U(1)_R &: \; x + iy \to e^{i\beta}(x + iy). \end{aligned} \tag{84}$$

The corresponding invariance properties are then expressed as the fact that the Kähler potential $K(U, W, \bar{U}, \bar{W})$ must be written as a function only of the $U(1)$ invariant combinations $(U\bar{U}, W + \bar{W})$, or equivalently of the real variables (u, v) defined by eq. (18):

$$K = K(u, v) = K\left(\ln(2\sqrt{U\bar{U}}), \frac{1}{2}(W + \bar{W})\right). \tag{85}$$

Given the general solution of the RG equation for $K(u, v, \lambda)$ implicitly via the two equations (see eqs. (40), (51), (53), (54))

$$\begin{aligned} v &= \lambda \mathcal{P} + \partial_{\mathcal{P}} H(u, \mathcal{P}), \\ K &= \lambda\left(\frac{1}{2}\mathcal{P}^2 + e^{2u}\right) + \mathcal{P}\partial_{\mathcal{P}} H(u, \mathcal{P}) - H(u, \mathcal{P}), \end{aligned} \tag{86}$$

the metric components in complex (U, W) coordinates using eqs. (57) then become

$$\begin{aligned} g_{U\bar{U}} &= e^{-2u}\partial_u^2 K = 4\lambda + e^{-2u}\left(-\partial_u^2 H + \frac{(\partial_u\partial_{\mathcal{P}} H)^2}{\lambda + \partial_{\mathcal{P}}^2 H}\right), \\ g_{U\bar{W}} &= \frac{1}{4U}\partial_u\partial_v K = -\frac{1}{4U}\left(\frac{\partial_u\partial_{\mathcal{P}} H}{\lambda + \partial_{\mathcal{P}}^2 H}\right), \\ g_{W\bar{W}} &= \frac{1}{4}\partial_v^2 K = \frac{1}{4}\left(\frac{1}{\lambda + \partial_{\mathcal{P}}^2 H}\right). \end{aligned} \tag{87}$$

Subjected to the Laplace equation (unimodularity condition eq. (59)) the first equation in (87) reduces to:

$$g_{U\bar{U}} = 4(\lambda + \partial_{\mathcal{P}}^2 H) + \frac{1}{4U\bar{U}}\frac{(\partial_u\partial_{\mathcal{P}} H)^2}{\lambda + \partial_{\mathcal{P}}^2 H} \tag{88}$$

The line element corresponding to the above metric components then is written as:

$$ds^2 = 4(\lambda + \partial_{\mathcal{P}}^2 H)dU d\bar{U} + \frac{1}{4(\lambda + \partial_{\mathcal{P}}^2 H)}\left(dW - \partial_u\partial_{\mathcal{P}} H\frac{dU}{U}\right)\left(d\bar{W} - \partial_u\partial_{\mathcal{P}} H\frac{d\bar{U}}{\bar{U}}\right). \tag{89}$$

This has the form of a GH metric eq. (74) if we identify V, σ in that equation with

$$V = 4[\lambda + \partial_{\mathcal{P}}^2 H(u, \mathcal{P})], \tag{90}$$

$$\sigma = -\frac{1}{U}\partial_u\partial_{\mathcal{P}} H(u, \mathcal{P}). \tag{91}$$

Note that by eq. (59), V satisfies an equation similar to H,

$$e^{-2u}\partial_u^2 V + 4\partial_{\mathcal{P}}^2 V = 0. \tag{92}$$

Eq. (89) is equivalent to the GH metric (60) provided we can find a function $z = z(U, W, \bar{U}, \bar{W})$ satisfying eqs. (72), (73). In the axisymmetric case we have the dependences

$$V = V(u, \mathcal{P}, \lambda), \quad \mathcal{P} = \mathcal{P}(u, v, \lambda), \quad z = z(u, v, \lambda) \tag{93}$$

So eqs. (72), (73) are reduced to

$$\begin{aligned} \sigma &= \frac{V}{U}\partial_u z \\ \partial_v z &= \frac{1}{V} \end{aligned} \tag{94}$$

where $v = W_1$ is the real part of W. Combining eqs. (94) with (90), (91) then gives:

$$\partial_u z|_{v,\lambda} = -\frac{\partial_u \partial_{\mathcal{P}} H(u,\mathcal{P})}{4[\lambda + \partial^2_{\mathcal{P}} H(u,\mathcal{P})]} = -\frac{\partial_u \partial_{\mathcal{P}} H}{V},$$
$$\partial_v z|_{u,\lambda} = \frac{1}{4[\lambda + \partial^2_{\mathcal{P}} H(u,\mathcal{P})]} = \frac{1}{V}. \tag{95}$$

These equations are integrable for $z(u,v,\lambda)$ as is most easily seen by changing the variables $(u,v,\lambda) \to (u,\mathcal{P},\lambda)$ using $v = \lambda\mathcal{P} + \partial_{\mathcal{P}} H(u,\mathcal{P})$, as in section 4. This will give

$$\partial_{\mathcal{P}} z|_{u,\lambda} = \partial_v z|_{v,\lambda}(\lambda + \partial^2_{\mathcal{P}} H) = \frac{1}{V}\cdot\frac{1}{4}V = \frac{1}{4},$$
$$\partial_u z|_{\mathcal{P},\lambda} = \partial_u z|_{v,\lambda} + \partial_v z|_{u,\lambda}\partial_u\partial_{\mathcal{P}} H = -\frac{\partial_u\partial_{\mathcal{P}} H}{V} + \frac{1}{V}\cdot\partial_u\partial_{\mathcal{P}} H = 0. \tag{96}$$

So, up to an irrelevant constant $c(\lambda)$ which can be set to zero, we can integrate these two equations for z to:

$$z = \frac{1}{4}\mathcal{P}. \tag{97}$$

With this identification, eq. (92) (and eq. (59) alike) is a Laplace equation on $\mathbb{E}^3$ for axisymmetric solutions written in a variant of the cylindrical coordinates $(u,\varphi,\mathcal{P})$ instead of the Cartesian ones (x,y,z). The two sets are related by

$$u = \ln(2\sqrt{x^2+y^2}), \quad \varphi = \tan^{-1}\frac{y}{x}, \quad \mathcal{P} = 4z. \tag{98}$$

Satisfaction of the Laplace equation by V verifies that we have a consistent GH geometry in this problem. Note that the λ dependence of V in eq. (90) (and hence of the GH geometry) comes only via the linear term 4λ in this equation. Imposing the periodicity condition $\mathcal{P} \sim \mathcal{P}+8\pi$ (i.e. $z \sim z+2\pi$), beside the above λ dependence, then uniquely specifies a solution of the Laplace equation of the following form:

$$V = 4\lambda + \left(\sum_{n=-\infty}^{\infty}\frac{1}{\sqrt{x^2+y^2+(z-2\pi n)^2}}\right)_{reg}$$
$$= 4\lambda + 4\left(\sum_{n=-\infty}^{\infty}\frac{1}{\sqrt{4e^{2u}+(\mathcal{P}-8\pi n)^2}}\right)_{reg}, \tag{99}$$

where the subscript *reg* means to regularize the summation by subtracting its divergent part [5]. The solution for $H(u,\mathcal{P})$ by eq. (90) then is obtained by twice integrating the λ independent part of V with respect to $\mathcal{P}$. This completes our proof of equivalence of the GH geometry with a solution of the RG flow equation that satisfies the $\mathcal{N}=2$ supersymmetry and periodicity conditions.

6. The limiting cases and particular solutions

It is instructive first to consider the IR and UV limits of the general solution described by eq. (86) without imposing the condition of $\mathcal{N}=2$ supersymmetry. These solutions generically preserve an $\mathcal{N}=1$ supersymmetry. Interestingly, even for H not satisfying the Laplace equation, the general expression (58) for g implies that in both $\lambda \to \pm\infty$ (i.e. IR/UV) limits one has $g \to 1$, which means that the RG flow evolves even $\mathcal{N}=2$ breaking initial conditions for K to $\mathcal{N}=2$ preserving ones in these limits for all solutions in this class. This point can be seen more explicitly by considering the general solution in these limits, namely

$$v = \lambda\mathcal{P},$$
$$K = \lambda\left(\frac{1}{2}\mathcal{P}^2 + e^{2u}\right). \tag{100}$$

Upon eliminating $\mathcal{P}$ between these two equations we find the limiting solution:

$$K(u,v,\lambda) = \lambda e^{2u} + \frac{v^2}{2\lambda}. \tag{101}$$

We note that consistently in this case we have $\mathcal{P} = \partial_v K = v/\lambda$. Computing the metric components in complex (U,W) coordinates from eq. (101) then gives:

$$ds^2 = 4\lambda dU d\bar{U} + \frac{1}{4\lambda}dW d\bar{W}. \tag{102}$$

This defines a flat metric on the space of hypermultiplets in the IR limit ($\lambda \to \infty$) corresponding to a non-interacting field theory if formulated in terms of (U,W) instead of (B,C) variables. This justifies restoring $\mathcal{N}=2$ supersymmetry in this limit. The UV limit ($\lambda \to -\infty$), on the other hand, is not well-defined because it leads to a negative-definite non-physical metric in the theory. As pointed out in [5], this is a sign of a Landau pole and hence the UV incompleteness of the model with which we have started.

6.1. Particular solutions and the RG fixed points

There is a particular class of solutions of the PDE (20) which are not included in the general class of solutions we constructed in section 4. In the language of eq. (23) these are solutions for which the function $F(u,v)$ does not depend on v, so that one cannot invert that equation for $v-\lambda\mathcal{P}$ in terms of $(u,\mathcal{P})$. These solutions do not meet the periodicity condition (38) and hence are not of interest for the model considered in ref. [5], but in view of their relevance to the fixed points of the RG flow we consider them below. For this class, we have instead of eq. (40), the relation

$$\mathcal{P}=\partial_v K=F(u), \tag{103}$$

where F is now an arbitrary function of a single variable. Integrating this equation for $K(u,v,\lambda)$ then gives:

$$K(u,v,\lambda)=vF(u)+C(u,\lambda), \tag{104}$$

where $C(u,\lambda)$ is a yet unknown function of two variables. Plugging this into the RG eq. (20) then determines this function up to an arbitrary function of u and gives a new class of solutions for K of the following form:

$$K(u,v,\lambda)=E(u)+vF(u)+\lambda\left(e^{2u}-\frac{1}{2}F^2(u)\right). \tag{105}$$

These solutions are parametrized by two arbitrary functions E,F of a single variable. Fixed points of the RG flow are defined as those solutions of the RG equation that do not depend on λ. This means that they must satisfy $\partial_\lambda K=0$ and hence the PDE:

$$-\frac{1}{2}(\partial_v K)^2+e^{2u}=0, \tag{106}$$

which is easily integrated to

$$K(u,v)=E(u)\pm\sqrt{2}ve^{u}. \tag{107}$$

This is evidently a special subclass of the solutions (105) for which $F(u)=\pm\sqrt{2}e^{u}$. The metric determinant for this class in complex (B,C) coordinates is found from eq. (56) to be a negative constant value:

$$g=-\frac{1}{2}. \tag{108}$$

These solutions thus correspond to Ricci flat Kähler geometries. But they do not describe relevant $\mathcal{N}=2$ models because, having $g<0$, they explicitly break the required $SO(4)$ R-symmetry. Besides, having $g<0$ means that the corresponding Kähler metric is not positive-definite, which prevents having a ghost-free physical theory. Indeed, by computing the metric components in complex (U,W) coordinates from eq. (107), one easily finds that the line element for these solutions is:

$$ds^2_{F.P.}=e^{-2u}E''(u)dUd\bar{U}\pm\frac{\sqrt{2}}{4}(W+\bar{W})\left(\sqrt{\frac{\bar{U}}{U}}dUd\bar{W}+\sqrt{\frac{U}{\bar{U}}}dWd\bar{U}\right). \tag{109}$$

Writing it as

$$ds^2_{F.P.}=e^{-2u}E''(u)\left(|dU+AdW|^2-|AdW|^2\right), \tag{110}$$

with $A=\pm\frac{\sqrt{1}2}{4e^{-2u}E''(u)}(W+\bar{W})\sqrt{\frac{U}{\bar{U}}}$, then explicitly displays non-positive-definiteness of this metric. This is indeed a generic feature of all solutions in the class (105), because they all have a negative metric determinant independent of λ:

$$g=-\frac{1}{4}e^{-2u}F'^2(u). \tag{111}$$

So all geometries based on eq. (105) explicitly break $\mathcal{N}=2$ supersymmetry at all values of λ.

7. Conclusion

In this paper we analyzed the classical RG flow equation for the running of the Kähler potential in the classical effective field theory of two D3-branes in the case that they orthogonally intersect or overlap along one common dimension. This is a BPS static configuration with 8 preserved supercharges. Using the gauge invariance and R-symmetry properties of the model at hand, we arrived at a simplified effective flow equation for K in the form of a first order non-linear PDE in three independent variables, with one of them playing the role of a parameter. We used the method of characteristics to obtain general solution of this equation. This solution involves an arbitrary function H of two independent variables. The freedom of choosing this function was exploited to explicitly construct two specific examples of solutions of the RG flow equation: the one with flat space initial condition and the other with periodic boundary condition. In the latter case, H is not completely fixed by the boundary condition but is only restricted to a specific class of periodic+linear functions in the variable $\mathcal{P}$. This freedom was fixed by imposing the Ricci flatness condition on the Kähler geometry derived from K. This is equivalent to the unimodularity condition on the Kähler metric, which characterizes the $\mathcal{N}=2$ supersymmetry. After performing a suitable change of variables, this led us to a 3D Laplace equation satisfied by H. The formulation used here allows only solutions for H which are independent of the azimuthal angle φ due to a $U(1)$ R-symmetry in the problem. By a proper identification of the field theory and geometric quantities, we were able to show that the solution obtained in this manner exactly matches the periodic Gibbons-Hawking geometry used in ref. [5] to describe the dynamics of the hypermultiplets at the intersection. The scale dependence of the geometry was found that appears as an additive term of the form $4\lambda = -(2/\pi)\ln(\mu/\mu_0)$ in the harmonic function V defining the geometry. This leads to a negative-definite metric at large values of μ. As discussed in [5], this means the existence of a Landau pole and UV incompleteness of the model which can presumably be cured by modifying it using the full string theory (see [18] for another possible resolution). We noted that, beside the general class of solutions, there exist also a particular class of solutions not included in the former. The geometries based on these solutions, however, do not describe physical $\mathcal{N} = 2$ models because they break the required $SO(4)$ R-symmetry and violate positive-definiteness of the geometry leading to the presence of ghosts in the theory. We computed fixed points of the RG flow and showed that they form a subclass of these particular solutions and so they too violate the requirements of a physical theory. In the appendix we give a general proof of the preservation of $\mathcal{N} = 2$ supersymmetry by the RG flow without referring to any solution of the RG equation, based only on general properties of the model. This, however, does not conflict with the existence of $\mathcal{N} = 2$ breaking solutions such as the particular solutions we explained above, as solutions of this equation can fall in different classes according to the choice of the initial condition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A. General proof of preservation of $\mathcal{N} = 2$ supersymmetry by the RG flow

As we have mentioned earlier, a necessary (and sufficient) condition for having $\mathcal{N} = 2$ supersymmetry in the D3-D3′ system is given by the relation $g = |f'|^2$ (eq. (4)). Since the Kähler metric generally changes under the RG flow, it is natural to ask whether or not this condition is generally preserved by the flow. To find a general answer to this question, here we derive an RG equation governing directly the flow of g. For this we start with the RG eq. (5) for K and use the identity

$$\begin{aligned}\partial_\lambda \ln g &= g^{\alpha\bar{\beta}}\partial_\lambda g_{\alpha\bar{\beta}} = g^{\alpha\bar{\beta}}\partial_\alpha\partial_{\bar{\beta}}(\partial_\lambda K)\\ &= -g^{\alpha\bar{\beta}}\left(\mathcal{P}\partial_\alpha\partial_{\bar{\beta}}\mathcal{P} + \partial_\alpha\mathcal{P}\partial_{\bar{\beta}}\mathcal{P} - \partial_\alpha f\partial_{\bar{\beta}}\bar{f}\right).\end{aligned} \tag{112}$$

The first order partial derivatives of $\mathcal{P}$ in this equation are computed from the definition of $\mathcal{P}$ by

$$\mathcal{P} := ik^\alpha\partial_\alpha K = -i\bar{k}^{\bar{\alpha}}\partial_{\bar{\alpha}}K, \tag{113}$$

using which one finds

$$\begin{aligned}\partial_\alpha\mathcal{P} &= -i\bar{k}^{\bar{\beta}}g_{\alpha\bar{\beta}} =: -ik_\alpha,\\ \partial_{\bar{\beta}}\mathcal{P} &= ik^\alpha g_{\alpha\bar{\beta}} =: i\bar{k}_{\bar{\beta}}.\end{aligned} \tag{114}$$

The combination $g^{\alpha\bar{\beta}}\partial_\alpha\partial_{\bar{\beta}}\mathcal{P}$ in eq. (112) then is found using eq. (114):

$$\begin{aligned}g^{\alpha\bar{\beta}}\partial_\alpha\partial_{\bar{\beta}}\mathcal{P} &= ig^{\alpha\bar{\beta}}\partial_\alpha(k^\gamma g_{\gamma\bar{\beta}})\\ &= i(\partial_\alpha k^\alpha + k^\gamma g^{\alpha\bar{\beta}}\partial_\gamma g_{\alpha\bar{\beta}})\\ &= ik^\gamma\partial_\gamma \ln g,\end{aligned} \tag{115}$$

where the identity $\partial_\alpha g_{\gamma\bar{\beta}} \equiv \partial_\gamma g_{\alpha\bar{\beta}}$ has been used in the second equality and $\partial_\alpha k^\alpha \equiv 0$ in the third. The last term in eq. (112) is simplified using the identity $\partial_\alpha f = i\epsilon_{\alpha\gamma}k^\gamma f'$ and its conjugate to give

$$g^{\alpha\bar{\beta}}\partial_\alpha f\partial_{\bar{\beta}}\bar{f} = |f'|^2(\epsilon_{\alpha\gamma}\epsilon_{\bar{\beta}\bar{\delta}}g^{\alpha\bar{\beta}})k^\gamma\bar{k}^{\bar{\delta}} = \frac{|f'|^2}{g}g_{\gamma\bar{\delta}}k^\gamma\bar{k}^{\bar{\delta}}. \tag{116}$$

Plugging eqs. (114), (115), (116) in eq. (112) then gives the desired result for running of $\ln g$:

$$\partial_\lambda \ln\frac{g}{|f'|^2} = -\mathcal{P}ik^\alpha\partial_\alpha \ln\frac{g}{|f'|^2} - \left(1 - \frac{|f'|^2}{g}\right)g_{\alpha\bar{\beta}}k^\alpha\bar{k}^{\bar{\beta}}, \tag{117}$$

where we have used the fact that f is independent of λ and it also satisfies the identity $ik^\alpha\partial_\alpha f' \equiv 0$. From this equation it is easily seen that the $g = |f'|^2$ condition is generally conserved by the flow. This means that if the condition holds at some scale, say $g|_{\lambda=0} = |f'|^2$, then it will hold at any other scale. Indeed, by choosing this as an "initial" condition, eq. (117) implies that $(\partial_\lambda \ln g)|_{\lambda=0} = 0$. Taking

$\partial_\lambda|_{\lambda=0}$ of this equation and using the above as the initial conditions for that equation then gives $(\partial_\lambda^2 \ln g)|_{\lambda=0} = 0$. Continuing this to arbitrary orders gives $(\partial_\lambda^n \ln g)|_{\lambda=0} = 0$ for all $n \geq 1$, hence showing that $\partial_\lambda \ln(g/|f'|^2) = 0$ is valid for all λ. The same logic also applies to the Ricci flatness condition: the Kähler geometry in general runs by the RG flow in a way to preserve the Ricci flatness condition:

$$R_{\alpha\bar{\beta}}|_{\lambda=0} = 0 \ \rightarrow \ R_{\alpha\bar{\beta}}|_{\forall\lambda} = 0. \tag{118}$$

Appendix B. Periodicity as a symmetry of the RG flow equation

Let us once again consider the RG flow eq. (5) as a PDE for the unknown function $K(B, C, \bar{B}, \bar{C}, \lambda)$ written as:

$$\partial_\lambda K = -\frac{1}{2}\left[\left(B\partial_B - C\partial_C\right)K\right]^2 + |f(BC)|^2 \tag{119}$$

and try to interpret the periodicity map

$$B \rightarrow B' = \beta(\lambda)F(BC)B, \quad C \rightarrow C' = \frac{C}{\beta(\lambda)F(BC)} \tag{120}$$

as a symmetry of this equation. Although, in our case both $\beta(\lambda)$ and $F(BC)$ are very specific functions of one variable, here we present a proof for arbitrary choices of these functions. By the way, we note that the map (120) forms an abelian group of transformations for all possible choices of these functions. In particular, the choice $(\beta = 1, F = 1)$ describes the identity map and each such map has an inverse obtained by setting $(\beta, F) \rightarrow (\beta^{-1}, F^{-1})$ and exchanging the roles of primed and unprimed variables. The above change of independent variables leaves the RG equation invariant, provided K also changes in a specific manner. To see this, we first relate partial derivatives with respect to these two sets of variables in the standard way. A little algebra then gives

$$\begin{aligned} &B\partial_B - C\partial_C = B'\partial_{B'} - C'\partial_{C'}, \\ &\partial_\lambda|_{B,C,\bar{B},\bar{C}} = \partial_\lambda|_{B',C',\bar{B}',\bar{C}'} + \partial_\lambda \ln\beta\left(B'\partial_{B'} - C'\partial_{C'} + \bar{B}'\partial_{\bar{B}'} - \bar{C}'\partial_{\bar{C}'}\right). \end{aligned} \tag{121}$$

Eq. (119) in new variables then reads

$$\partial_\lambda K + \partial_\lambda \ln\beta\left(B'\partial_{B'} - C'\partial_{C'} + \bar{B}'\partial_{\bar{B}'} - \bar{C}'\partial_{\bar{C}'}\right)K = -\frac{1}{2}\left[\left(B'\partial_{B'} - C'\partial_{C'}\right)K\right]^2 + |f(B'C')|^2. \tag{122}$$

Besides, we have the gauge symmetry constraint (9) on K which preserves its form also in $(B', C', \bar{B}', \bar{C}')$ variables:

$$\left(B'\partial_{B'} - C'\partial_{C'} - \bar{B}'\partial_{\bar{B}'} + \bar{C}'\partial_{\bar{C}'}\right)K = 0, \tag{123}$$

because the map (120) preserves the $U(1)_G$ gauge charges of the variables. Using this constraint in eq. (122), after simple manipulations, one finds the equation

$$\partial_\lambda K = -\frac{1}{2}\left[\left(B'\partial_{B'} - C'\partial_{C'}\right)K + 2\partial_\lambda \ln\beta\right]^2 + 2(\partial_\lambda \ln\beta)^2 + |f(B'C')|^2. \tag{124}$$

Defining the new Kähler potential by

$$K' = K + (\partial_\lambda \ln\beta)\ln\frac{B'\bar{B}'}{C'\bar{C}'} + \gamma, \tag{125}$$

with $\gamma = \gamma(\lambda)$ defined by $\partial_\lambda\gamma = -2(\partial_\lambda \ln\beta)^2$, we see that the RG flow equation for K' in (B', C') has the same form as that for K in (B, C) variables:

$$\partial_\lambda K' = -\frac{1}{2}\left[\left(B'\partial_{B'} - C'\partial_{C'}\right)K'\right]^2 + |f(B'C')|^2. \tag{126}$$

By eq. (125), the moment maps associated to K, K' are related as:

$$\mathcal{P}' = \mathcal{P} + 2\partial_\lambda \ln\beta. \tag{127}$$

This allows to construct periodic solutions obeying the condition $\mathcal{P} \sim \mathcal{P} + \alpha$ for some constant α (e.g. $\alpha = 8\pi$ in our case). For this, all we need is to set $2\partial_\lambda \ln\beta = \alpha$, or

$$\beta(\lambda) = \beta_0 e^{\alpha\lambda/2}, \tag{128}$$

and look for a solution of eq. (119) that is also a solution of eq. (126) with (B, C) replaced by (B', C'). This amounts to finding a solution of eq. (119) that obeys a periodicity property in K of the form:

$$K(B, C, \bar{B}, \bar{C}, \lambda) = K(B', C', \bar{B}', \bar{C}', \lambda) - \alpha\ln|\frac{B'}{C'}| + \frac{1}{2}\alpha^2\lambda. \tag{129}$$

For the case of interest with $U(1)_G \times U(1)_R$ symmetry and with $F(BC) = BC$ we have indeed found general solution of this problem in section 4 (see eqs. (40)-(45)).

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