

High Energy Physics – Theory

Investigation of generalised uncertainty principle effects on FRW cosmology

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ABSTRACT

Based on the entropy-area relation from Nouicer's generalised uncertainty principle (GUP), we derive the GUP modified Friedmann equations from the first law of thermodynamics at apparent horizon. We find a minimum apparent horizon due to the minimal length notion of GUP. We show that the energy density of universe has a maximum and finite value at the minimum apparent horizon. Both minimum apparent horizon and maximum energy density imply the absence of the Big Bang singularity. Moreover, we investigate the GUP effects on the deceleration parameter for flat case. Finally, we examine the validity of generalised second law (GSL) of thermodynamics. We show that GSL always holds in a region enclosed by apparent horizon for the GUP effects. We also investigate the GSL in Λ CDM cosmology and find that the total entropy change of universe has a maximum value in the presence of GUP effects. To better grasp the effects of Nouicer's GUP on cosmology, we compare our results with those obtained from quadratic GUP (QGUP).

1. Introduction

Thermodynamical aspects of gravity have been an attractive research field in theoretical physics since the discovery of black hole thermodynamics [1–6]. Black hole as a thermodynamic system has entropy and temperature proportional to its horizon area and surface gravity, respectively. Considering a black hole as a thermodynamic system transforms an absolute absorber into a perfect laboratory to investigate the deep connection between the gravitation, quantum mechanics, and thermodynamics. Based on the notion of black hole thermodynamics, Jacobson obtained the Einstein field equation from thermodynamical arguments [7]. Employing the entropy-area relation with Clausius relation $\delta Q = T dS$, he derived the field equation as an equation of state. Here, δQ , T and dS are energy flux, Unruh temperature and the change in entropy, respectively. After the pioneering work of Jacobson, there have been many studies targeted the thermodynamical aspects of gravity in the literature [8–56]. The studies aimed to understand obtaining Einstein field equation from the first law of thermodynamics can be found in Refs. [8–12]. Inspired by Jacobson's paper, $(n + 1)$ -dimensional Friedmann equations were obtained from the first law of thermodynamics, $-dE = T_h dS_h$, at apparent horizon by Cai and Kim [13]. Here $-dE$ is the energy flux crossing the apparent horizon for the infinitesimal time interval at fixed horizon radius. The temperature and entropy of the apparent horizon are given by [13]

$$T_h = \frac{1}{2\pi r_A}, \quad S_h = \frac{A_h}{4}, \quad (1)$$

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where A_h and $\tilde{r}_A$ correspond to area and apparent horizon, respectively.¹ Furthermore, in the rest of their seminal paper [13], they also derived the Friedmann equations from the entropy-area relations of Gauss-Bonnet gravity and Lovelock gravity theories, where the standard entropy-area relation is break down. In Ref. [14], Akbar and Cai obtained the Friedmann equations in scalar-tensor and $f(R)$ gravity theories by following the arguments of Ref. [13]. Although the Friedmann equations can successfully be obtained in Eq. (1), the temperature is an approximation for a fixed apparent horizon. Thus the temperature is not proportional to the surface gravity at apparent horizon. Moreover, the equation of state is only limited with the vacuum energy or de Sitter spacetime for this approximation. The surface gravity of apparent horizon is given by [13,57]

$$\kappa = -\frac{1}{\tilde{r}_A} \left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A} \right), \quad (2)$$

where dot denotes the derivative with respect to time and H corresponds to Hubble parameter. Assuming temperature is proportional to surface gravity [15]

$$T_h = \frac{\kappa}{2\pi} = -\frac{1}{2\pi\tilde{r}_A} \left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A} \right), \quad (3)$$

with the standard entropy-area relation, Akbar and Cai showed that the first law of thermodynamics at apparent horizon defined as

$$dE = T_h dS_h + W dV, \quad (4)$$

where $E = \rho V$ and W correspond to the total energy in volume V enclosed by the apparent horizon and the work density, respectively. Then, many papers devoted to Friedmann equations and apparent horizon thermodynamics have been intensively studied in the Refs. [16–56].

It is widely known that the standard entropy-area relation is not valid and should be corrected in the case of various theories. For example, it is well-known that logarithmic correction to black hole entropy arises due to quantum gravity effects at the Planck scale [58–60]. The various quantum gravity approaches to the modifications of Friedmann equations such as loop quantum gravity [18,19], modified Heisenberg principle [20–25] were investigated in the literature. Other interesting modifications of entropy-area relation are considered in the context of generalised statistics such as Tsallis statistics [61] and Kaniadakis statistics [62,63]. Moreover, inspired by COVID-19, Barrow proposed that the area of horizon may be deformed in the context of quantum gravity effects and entropy is the power law function of its area [64]. Recently, the effects of fractional quantum mechanics were investigated on black hole thermodynamics in Ref. [65]. Interestingly, the fractional entropy [65] is similar to Tsallis [61] and Barrow [64] entropies although all entropies have different origins. The extensions of Refs. [61–65] to cosmological cases, namely modified Friedmann equations can be found for Tsallis entropy in Refs. [30–32], Kaniadakis entropy in Refs. [39,40], Barrow entropy in Refs. [47–54] and fractional entropy in Ref. [56].

Modifications of standard uncertainty principle may provide defining the physics both at Planck scale and large distance scales. GUP takes into account the momentum uncertainty correction while the extended uncertainty principle (EUP) takes into account the position uncertainty correction. GUP predicts a minimum measurable length at Planck scale while EUP may predict a minimum measurable momentum. Taking into account effects of GUP and EUP leads to third kind of modified uncertainty principle known as generalized and extended uncertainty principle (GEUP). It predicts the notions of minimum measurable length and minimum measurable momentum. Numerous models of modified uncertainty principle were suggested in Refs. [66–85]. Both GUP and EUP provide new insights on black thermodynamics [85–114]. For example, the black holes are prevented from total evaporation in the GUP case and a black hole remnant occurs at the final stage [86]. On the other hand, EUP implies a minimum temperature for black holes [106]. Both GUP and EUP have been widely studied in cosmological applications [20–25,115–120]. Extensions of modified uncertainty principle to cosmological scenario lead to modified Friedmann equations [20–25]. Based on the simplest form of GUP, the modified Friedmann equations were derived in Ref. [20]. They found an upper bound energy density of universe at minimal length. In Ref. [21], a cyclic universe model was defined from the GUP modified Friedmann equations. In Ref. [22], we obtained the modified Friedmann equations from a GUP model obtained in doubly special relativity [72]. Similarly, we obtained a maximum and finite energy density due to minimal length at Planck scale. We also investigated the effects of GUP on the deceleration parameter. The authors of Ref. [23] investigated the baryon asymmetry for the EUP modified Friedmann equations. They also obtained the constraints on EUP parameter from observations. Modified Friedmann equations in GEUP case can be found in Ref. [24] where the author obtained the bounds for GEUP parameters from observations. Recently, GEUP corrected Friedmann equations were investigated in Ref. [25]. The authors studied the deceleration parameters and showed that GEUP may be an alternative to dark energy at late time expansion.²

In this paper, our aim is to obtain the GUP modified Friedmann equations from the first law of thermodynamics at apparent horizon. The modified Friedmann equations reveal that the initial singularity is removed since GUP implies the minimal measurable length at Planck scale. Moreover, a detailed investigation on GSL is crucial to understand the GUP effects. Especially, we would like to understand how GUP affects the GSL in the Λ CDM cosmology. In order to modify the Friedmann equations, we use Nouicer's GUP [85] whose details are given in the next section. Nouicer proposed the following higher-order GUP, which is consistent with QGUP (A.1) up to the leading order

¹ We adopt the units $\hbar = c = G_N = L_{pl}^2 = 1$ throughout the paper.

² For a recent review on GUP, the reader may refer to Ref. [121].

$$\Delta X \Delta P \geq \frac{1}{2} e^{\alpha^2 \Delta P^2}. \quad (5)$$

Including the higher-order terms in GUP generate a quantitative correction to physics at Planck scale. Especially, higher-order terms make a significant contribution to black hole thermodynamics [85]. Besides, understanding the effects of higher-order terms is also crucial for cosmology at Planck scale. Therefore, we compare the results obtained from Nouicer's GUP with those obtained from QGUP.

The paper is organised as follows. In Section 2, we start a brief review on Nouicer's GUP. Then, we obtained the modified entropy-area relation from Nouicer's GUP. In Section 3, using modified entropy-area relation, we obtain the Friedmann equations. In Section 4, GUP effects on deceleration parameter are investigated. In Section 5, we check the validity of GSL and explore the GUP effects on GSL in the Λ CDM cosmology. Finally, we discuss our results in Section 6. The details of QGUP modified Friedmann equations are given in Appendix A.

2. Nouicer's GUP and entropy-area relation

In this section, we first begin to review Nouicer's GUP [85]. Then, we obtain the modified entropy-area relation based on the methods of Xiang and Wen [98]. We start to consider the nonlinear relation $p = f(k)$ between the momentum p and the wave vector k of particle [122,123]. This relation must satisfy the following conditions:

- The relation $p = f(k)$ reduces to usual relation $p = k$ for the lower energies than the Planck energy.
- The relation $p = f(k)$ approaches to maximum value at Planck scale for the higher energies.

Following the above conditions, Nouicer proposed the modified position and momentum operators

$$X = i e^{\alpha^2 P^2} \frac{\partial}{\partial p}, \quad P = p, \quad (6)$$

where α is a dimensionless positive GUP parameter. These operators lead to modified commutator

$$[X, P] = i e^{\alpha^2 P^2}. \quad (7)$$

Using the relations $\langle P^{2n} \rangle \geq \langle P^2 \rangle^n$, $(\Delta P)^2 = \langle P^2 \rangle - \langle P \rangle^2$ with the above equation, one can obtain the GUP in Eq. (5) for the physical state $\langle P \rangle = 0$. The square of Eq. (5) can be written by

$$W(u) e^{W(u)} = u, \quad (8)$$

where W is the Lambert function [124] and we define $W(u) = -2\alpha^2 \Delta P^2$ and $u = -\frac{\alpha^2}{2\Delta X^2}$. For $0 \geq u \geq -1/e$, using Eq. (5), the momentum uncertainty is given

$$\Delta P = \frac{1}{2\Delta X} \exp \left\{ -\frac{1}{2} W \left(-\frac{1}{e} \frac{\Delta X_0^2}{\Delta X^2} \right) \right\}, \quad (9)$$

where ΔX_0 is the minimum position uncertainty and is given by

$$\Delta X_0 = \sqrt{\frac{e}{2}} \alpha. \quad (10)$$

It is obtained from the condition $u \geq -1/e$.

Following the arguments of Ref. [98], let us start to obtain the modified entropy-area relation. When a black hole absorbs a particle, the change in area is defined by [2]

$$\Delta A \sim bm, \quad (11)$$

where b and m are the particle's size and mass. We consider two limitations on particle's size and mass. The particle is defined by a wave packet in quantum mechanics. We have $b \sim \Delta X$ since the width of the wave packet is defined by the particle's size. The second limitation comes from the fact that the momentum uncertainty is not bigger than the particle size. Thus we have $m \geq \Delta P$. With these two limitations, we can write

$$\Delta A \sim bm \geq \Delta X \Delta P. \quad (12)$$

Using Eqs. (9), (10) and $\Delta X \sim 2r_h$ event horizon, the change in area is given by

$$\Delta A \geq \frac{\gamma}{2} \exp \left\{ -\frac{1}{2} W \left(-\frac{1}{8} \frac{\alpha^2}{r_h^2} \right) \right\}, \quad (13)$$

where γ is a calibration factor. Using above expression with minimum increase of entropy, $(\Delta S)_{min} = \ln 2$, the GUP modified entropy-area relation is given by

$$\frac{dS_h}{dA} = \frac{\Delta S_{min}}{\Delta A_{min}} = \frac{1}{4} \exp \left\{ \frac{1}{2} W \left(\frac{-\alpha^2}{8r_h^2} \right) \right\}, \quad (14)$$

where we find $\gamma = 8 \ln 2$ since the above equation must give $dS_h/dA = 1/4$ in the limit $\alpha \rightarrow 0$. In the next section, we use Eq. (14) to modify the Friedmann equations.

3. Modified Friedmann equations from the first law of thermodynamics

We first begin a concise review on the basic elements of Friedmann-Robertson-Walker (FRW) universe. The line element of FRW universe is defined by [13]

$$ds^2 = h_{ab} dx^a dx^b + \tilde{r}^2 d\Omega^2, \quad (15)$$

where $\tilde{r} = a(t)r$, $a(t)$ is the scale factor, $x^a = (t, r)$, and $h_{ab} = \text{diag}(-1, a^2/(1 - kr^2))$ is the two-dimensional metric. $k = -1, 0$, and 1 present to the open, flat, and closed universe, respectively. The expression of apparent horizon $\tilde{r}_A$ is given by [13]

$$\tilde{r}_A = ar = \frac{1}{\sqrt{H^2 + k/a^2}}, \quad (16)$$

where the Hubble parameter is defined by $H = \dot{a}/a$. In the following, we consider the matter and energy of universe as a perfect fluid. So we have

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \quad (17)$$

where ρ , p and u_μ correspond to energy density, pressure and four-velocity of the fluid. The conservation of energy-momentum tensor ($\nabla_\mu T^{\mu\nu} = 0$) gives the continuity equation as

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (18)$$

The work by the volume change of universe is defined by work density [57]

$$W = -\frac{1}{2} T^{ab} h_{ab} = \frac{1}{2}(\rho - p), \quad (19)$$

and the corresponding volume is given by [15]

$$V = \frac{4}{3} \pi \tilde{r}_A^3. \quad (20)$$

Employing the Eqs. (18) and (20), the differential of the total energy of universe is given by

$$dE = \rho dV + V d\rho = 4\pi \rho \tilde{r}_A^2 d\tilde{r}_A - 4\pi(\rho + p)\tilde{r}_A^3 H dt. \quad (21)$$

From Eqs. (19) and (20), we can obtain

$$W dV = 2\pi(\rho - p)\tilde{r}_A^2 d\tilde{r}_A. \quad (22)$$

Since entropy is defined by the area, $S = S(A)$, one can give the general expressions of entropy as follows [20]:

$$S_h = \frac{f(A_h)}{4}, \quad (23)$$

and its differential can be written by

$$\frac{dS_h}{dA_h} = \frac{f'(A_h)}{4}, \quad (24)$$

where prime denotes derivative with respect to the apparent horizon area $A_h = 4\pi \tilde{r}_A^2$. Comparing Eq. (14) with the above equation, we find

$$f'(A_h) = \exp \left\{ \frac{1}{2} W \left(\frac{-\alpha^2}{8\tilde{r}_A^2} \right) \right\}. \quad (25)$$

Finally, we find $T_h dS_h$ as

$$T_h dS_h = - \left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A} \right) f'(A_h) d\tilde{r}_A. \quad (26)$$

Now, we can obtain the modified Friedmann equations since we have all the necessary ingredients. Substituting Eqs. (21)-(26) into Eq. (4) and using the differential form of apparent horizon

$$d\tilde{r}_A = -H\tilde{r}_A^3 \left(\dot{H} - \frac{k}{a^2} \right) dt, \quad (27)$$

we find

$$4\pi(\rho + p)r_A^3 H dt = f'(A_h) d\tilde{r}_A. \quad (28)$$

Employing the continuity equation (18) with the above equation, the differential form of Friedmann equation is given by

$$\frac{f'(A_h)}{r_A^3} d\tilde{r}_A = -\frac{4\pi}{3} d\rho. \quad (29)$$

Integrating the above equation with Eq. (25) gives the first Friedmann equation

$$\frac{8}{9\alpha^2} e^{\frac{3}{2}W\left(\frac{-\alpha^2}{8r_A^2}\right)} \left(1 + 3W\left(\frac{-\alpha^2}{8r_A^2}\right)\right) + C = -\frac{4\pi\rho}{3}, \quad (30)$$

in the limit $r_A \rightarrow \infty$, this equation must reduce to standard Friedmann equation, i.e., $H^2 + \frac{k}{a^2} = \frac{8\pi}{3}\rho + \frac{\Lambda}{3}$, thus we set the integration constant $C = -\frac{8}{9\alpha^2} + \frac{\Lambda}{6}$. So we can write the first Friedmann equation as

$$\frac{8}{9\alpha^2} \left[e^{\frac{3}{2}W\left(\frac{-\alpha^2}{8r_A^2}\right)} \left(1 + 3W\left(\frac{-\alpha^2}{8r_A^2}\right)\right) - 1 \right] = -\frac{4\pi}{3}\rho - \frac{\Lambda}{6}. \quad (31)$$

Finally, combining Eqs. (25) and (27) with Eq. (28), the dynamical equation can be obtained

$$e^{\frac{1}{2}W\left(\frac{-\alpha^2}{8r_A^2}\right)} \left(\dot{H} - \frac{k}{a^2} \right) = -4\pi(\rho + p). \quad (32)$$

These equations reduce to standard Friedmann equations in the limit $\alpha \rightarrow 0$.

The striking feature of the first Friedmann equation (31) comes from the argument of Lambert function, i.e., the condition $\frac{\alpha^2}{8r_A^2} \leq \frac{1}{e}$ gives the minimal apparent horizon

$$r_A^{min} = \sqrt{\frac{e}{8}}\alpha. \quad (33)$$

This minimal apparent horizon appears due to minimal length notion of GUP [20–22,25]. It implies that the singularity is removed at the beginning of the Universe. Moreover, the energy density does not diverge anymore since the minimum apparent horizon has a finite value. Using Eq. (33) in the first Friedmann equation (31) and neglecting the cosmological constant, the maximum energy density is given by

$$\rho^{max} = \frac{2(2 + e^{3/2})}{3\pi e^{3/2}\alpha^2}. \quad (34)$$

It is clear that both r_A^{min} and ρ^{max} recover the standard results in the limit $\alpha \rightarrow 0$, i.e., r_A^{min} approaches zero while ρ^{max} diverges. Using Eq. (16), the Friedmann equations can be expressed in term of Hubble parameter

$$\begin{aligned} -\frac{4\pi}{3}\rho - \frac{\Lambda}{6} &= \frac{8}{9\alpha^2} \left[e^{\frac{3}{2}W\left(\frac{-\alpha^2}{8}\left(H^2 + \frac{k}{a^2}\right)\right)} \left(1 + 3W\left(\frac{-\alpha^2}{8}\left(H^2 + \frac{k}{a^2}\right)\right)\right) - 1 \right], \\ -4\pi(\rho + p) &= e^{\frac{1}{2}W\left(\frac{-\alpha^2}{8}\left(H^2 + \frac{k}{a^2}\right)\right)} \left(\dot{H} - \frac{k}{a^2} \right). \end{aligned} \quad (35)$$

When the GUP effects are tiny, these equations can be expanded in powers of α

$$\begin{aligned} \frac{8\pi}{3}\rho + \frac{\Lambda}{3} &= \left(H^2 + \frac{k}{a^2}\right) - \frac{1}{32} \left(H^2 + \frac{k}{a^2}\right)^2 \alpha^2 + \dots, \\ -4\pi(\rho + p) &= \left(\dot{H} - \frac{k}{a^2}\right) \left[1 - \frac{1}{16} \left(H^2 + \frac{k}{a^2}\right) \alpha^2 + \dots\right]. \end{aligned} \quad (36)$$

Neglecting the GUP correction, one can easily recover the usual Friedmann equations.

In order to clearly reveal the effects of Nouicer's GUP, we compare the Nouicer's GUP modified results with the results corrected by QGUP. For the QGUP case, we give minimum apparent horizon and maximum energy density in Eq. (A.6). In Table 1, we show r_A^{min} and ρ_{max} for the different values of α in Nouicer's GUP case. In Table 2, we present the same quantities for the different values of α in QGUP case. Nouicer's GUP implies that the Universe starts at the minimum apparent horizon bigger than one obtained from QGUP case while maximum energy density obtained from Nouicer's GUP is smaller than one obtained from QGUP. From Tables 1 and 2, one can see that r_A^{min} increases and ρ_{max} decreases when α increases.

Table 1

r_A^{min} , ρ_{max} and H_{max} for the different values of α in Nouicer's GUP case.

	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$	$\alpha = 1.2$	$\alpha = 1.4$
r_A^{min}	0.35	0.466	0.583	0.7	0.816
ρ_{max}	0.853	0.48	0.307	0.213	0.157
H_{max}	1.734	1.301	1.041	0.867	0.743

Table 2

r_A^{min} , ρ_{max} and H_{max} for the different values of α in QGUP case.

	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$	$\alpha = 1.2$	$\alpha = 1.4$
r_A^{min}	0.3	0.4	0.5	0.6	0.7
ρ_{max}	1.105	0.622	0.398	0.276	0.203
H_{max}	3.333	2.5	2	1.667	1.429

4. Deceleration parameter

We would like to analyse the effects of GUP on the deceleration parameter. It is defined by

$$q = -1 - \frac{\dot{H}}{H^2}. \quad (37)$$

The positivity of q means the decelerated expansion while the negativity of q means the accelerated expansion. We choose the equation of state as $p = \omega\rho$. We restrict our analysis with the flat case $k = 0$ since it is consistent with cosmological observations [125]. Combining Friedmann equations in (35) with the deceleration parameter, we can obtain

$$q = -1 - \frac{4(1+\omega)}{H^2 e^{\frac{1}{2}W\left(\frac{-\alpha^2 H^2}{8}\right)}} \left\{ \frac{2}{3\alpha^2} \left[e^{\frac{3}{2}W\left(\frac{-\alpha^2 H^2}{8}\right)} \left(1 + 3W\left(\frac{-\alpha^2 H^2}{8}\right) \right) - 1 \right] + \frac{\Lambda}{8} \right\}. \quad (38)$$

Now, we can investigate the deceleration parameter at the beginning of the Universe. Remembering the arguments of Lambert function, i.e., the condition $\frac{\alpha^2 H^2}{8} \leq \frac{1}{e}$ yields the maximum H

$$H_{max} = \frac{2\sqrt{2}}{e\alpha}, \quad (39)$$

at the initial stage. For the maximum value of Hubble parameter, the deceleration parameter at the initial stage is given by

$$q(H_{max}) = \omega + (1+\omega) \left(\frac{\exp\left\{2 - \frac{1}{2}W\left(-\frac{1}{e^2}\right)\right\}}{3} + \frac{1}{3W\left(-\frac{1}{e^2}\right)} \right), \quad (40)$$

or equivalently

$$q(H_{max}) \approx \omega + 0.564(1+\omega). \quad (41)$$

Equation of state parameter ω must satisfy the condition $\omega < -0.361$ for the acceleration at the inflationary stage. Since GUP effects are negligible for the radiation and matter dominated eras, the deceleration parameter can be expanded

$$q = \frac{1+3\omega}{2} - \frac{(1+\omega)\Lambda}{2H^2} + \frac{3H^2(1+\omega)\alpha^2}{64} - \frac{1(1+\omega)\Lambda\alpha^2}{32} + \dots \quad (42)$$

Thus, the deceleration parameters for radiation ($\omega = 1/3$) and matter ($\omega = 0$) dominated eras can be expressed as

$$q_{rad} = 1 + \frac{H^2\alpha^2}{16}, \quad q_m = \frac{1}{2} + \frac{3H^2\alpha^2}{64}, \quad (43)$$

respectively. The results imply that the universe is more decelerated for the radiation and matter dominated eras when the GUP effects are considered.

Now, let us compare our results with the results obtained from QGUP. In Eqs. (A.7), we give the QGUP corrected deceleration parameter. In Table 1 and Table 2, we present H_{max} at the initial stage for Nouicer's GUP and QGUP, respectively. In the Nouicer's GUP case, H_{max} at the initial stage is smaller than H_{max} obtained from QGUP case. From Tables 1 and 2, one can see that H_{max} decreases while α increases.

5. Generalised second law

In this section, we want to confirm the validity of GSL, which states the total entropy of matter fields and horizon cannot decrease with time, for the GUP effects. We start to reorganise Eq. (28) as follows:

$$r_A = 4\pi(\rho + p)H\tilde{r}_A^3 e^{-\frac{1}{2}W\left(\frac{-a^2}{8r_A^2}\right)}. \quad (44)$$

From Eqs. (25), (26) and (44), we can find

$$T_h \dot{S}_h = 4\pi(\rho + p)H\tilde{r}_A^3 \left[1 - 2\pi(\rho + p)\tilde{r}_A^2 e^{-\frac{1}{2}W\left(\frac{-a^2}{8r_A^2}\right)} \right]. \quad (45)$$

The second law at apparent horizon may be violated for the accelerated expansion phase. Therefore, we must also consider the entropy of matter field inside the horizon, i.e., we must check the GSL. The Gibbs equation is given by [126]

$$T_f dS_f = d(\rho V) + p dV = V d\rho + (\rho + p) dV, \quad (46)$$

where T_f and S_f are the temperature and entropy of matter fields, respectively. In order to avoid nonequilibrium thermodynamics and spontaneous energy flow between horizon and matter, thermal equilibrium condition ($T_h = T_f$) is assumed. Otherwise, the deformation of FRW geometry is unavoidable [126]. From Eqs. (18), (20), (44) and (46), the change in entropy of matter fields can be expressed by

$$T_h \dot{S}_f = -4\pi(\rho + p)H\tilde{r}_A^3 \left(1 - 4\pi(\rho + p)\tilde{r}_A^2 e^{-\frac{1}{2}W\left(\frac{-a^2}{8r_A^2}\right)} \right). \quad (47)$$

At last, combining Eqs. (45) and (47), the total entropy evolution is written by

$$T_h (\dot{S}_h + \dot{S}_f) = 8\pi^2(\rho + p)^2 H\tilde{r}_A^5 e^{-\frac{1}{2}W\left(\frac{-a^2}{8r_A^2}\right)}. \quad (48)$$

The right hand side of the above expression never decreases with respect to time. Therefore, we conclude that the GSL is always satisfied for all eras of the universe for any spatial curvature.

Now, in order to understand how GUP affects the total evolution of entropy, we focused on a more specific case, namely the Λ CDM scenario. We first begin to solve the first Friedmann equation (35). So we find

$$H^2 = \frac{8 \exp \left\{ \frac{1}{3} \left(2W \left(\frac{\sqrt{e}}{4} (2 - 3\pi\alpha^2\rho) \right) - 1 \right) \right\} \left(1 - 2W \left(\frac{\sqrt{e}}{4} (2 - 3\pi\alpha^2\rho) \right) \right)}{3\alpha^2}, \quad (49)$$

for $k = 0$. Let us simplify the above equation. The total energy density is defined by

$$\rho = \rho_m + \rho_r + \rho_\Lambda, \quad (50)$$

where the matter density ρ_m , the radiation density ρ_r and the cosmological constant energy density ρ_Λ have the dependencies $\rho_m = \frac{\rho_{m0}}{a^3}$, $\rho_r = \frac{\rho_{r0}}{a^4}$ and $\rho_\Lambda = \rho_{\Lambda0}$ [127]. Zero subscript denotes the current value. On the other hand, the matter, radiation and cosmological constant density parameters are defined by $\Omega_{m0} = \frac{\rho_{m0}}{\rho_{c0}}$, $\Omega_{r0} = \frac{\rho_{r0}}{\rho_{c0}}$, $\Omega_{\Lambda0} = 1 - \Omega_{m0} - \Omega_{r0}$, respectively. Here, the current critical density ρ_{c0} is given by $\rho_{c0} = \frac{3H_0^2}{8\pi}$ and H_0 is the current Hubble parameter. Using the above definitions with redshift parameter $1 + z = \frac{a_0}{a}$ and taking the present scale factor $a_0 = 1$, one can write

$$\frac{\rho}{\rho_{c0}} = \frac{\Omega_{m0}}{a^3} + \frac{\Omega_{r0}}{a^4} + \Omega_{\Lambda0} = \Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{\Lambda0}. \quad (51)$$

Using Eq. (51) in the arguments of Lambert function (49), we get

$$x = \frac{\sqrt{e}}{4} (2 - 3\pi\alpha^2\rho) = \frac{\sqrt{e}}{4} \left(2 - \frac{9H_0^2\alpha^2}{8} [\Omega_{r0}(1+z)^4 + \Omega_{m0}(1+z)^3 + \Omega_{\Lambda0}] \right). \quad (52)$$

Finally, we can express the GUP modified Hubble function for the Λ CDM cosmology as follows:

$$H_{\Lambda CDM}(z) = \frac{2\sqrt{2}e^{\frac{2W(x)-1}{6}}\sqrt{1-2W(x)}}{\sqrt{3}\alpha} \quad (53)$$

Substituting Eqs. (3), (16) and the second Friedmann equation (35) into Eq. (48), we find

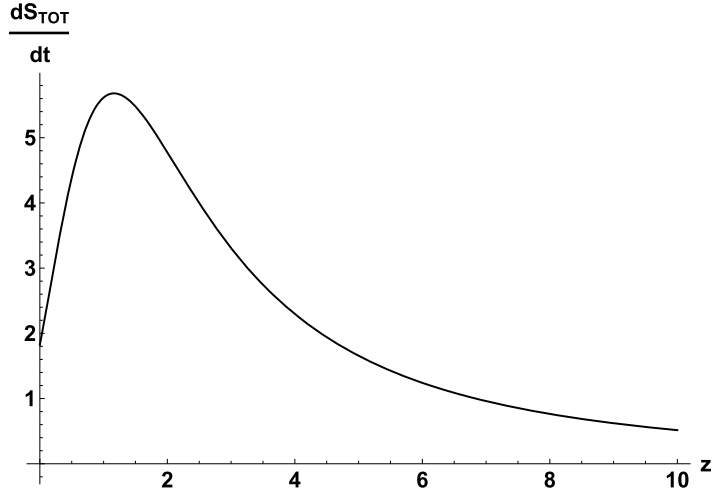


Fig. 1. Total entropy change versus the redshift parameter. We take $\Omega_m = 0.3$, $\Omega_r = 0.0001$, $H_0 = 1$ and $\alpha = 0$.

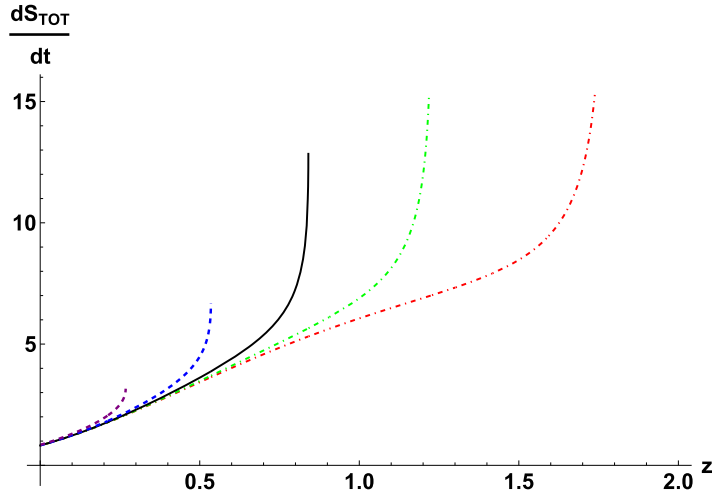


Fig. 2. Total entropy change versus the redshift parameter. From top to bottom, the curves correspond to $\alpha = 1.4, 1.2, 1, 0.8, 0.6$ for Nouicer's GUP. We take $\Omega_m = 0.3$, $\Omega_r = 0.0001$ and $H_0 = 1$.

$$\dot{S}_h + \dot{S}_f = \pi \dot{H}^2 H^{-5} \left(1 + \frac{\dot{H}}{2H^2} \right)^{-1} e^{\frac{1}{2}W\left(\frac{-a^2 H^2}{8}\right)} \quad (54)$$

Finally, using $\dot{H} = -(1+z)H(dH/dz)$, we obtain

$$\dot{S}_h + \dot{S}_f = \pi(1+z)^2 H^{-3} \left(\frac{dH}{dz} \right)^2 \left(1 - \frac{(1+z)}{2H} \frac{dH}{dz} \right)^{-1} e^{\frac{1}{2}W\left(\frac{-a^2 H^2}{8}\right)}. \quad (55)$$

Combining Eq. (53) with the above equation, we plotted total entropy change with respect to redshift. In Fig. 1, we plot the evolution of total entropy for the standard case. In Fig. 2, the total entropy change is represented for the different values of GUP parameter α . Comparing Fig. 1 with 2 reveals the dramatic effects on the evolution of total entropy. As can be seen in Fig. 2, the redshift parameter has an upper bound. In contrast to GUP case, the redshift parameter is allowed to go infinity in the standard cosmology. Since z reaches infinity, the total entropy change vanishes in the standard cosmology. On the other hand, the total entropy change has an upper bound since z has a maximum value in the presence of GUP effects.

Again, we compare our results with the results obtained from QGUP. We give the total entropy evolution in Eqs. (A.10), (A.11) and (A.12). In Fig. 3, we present the total entropy change with respect to redshift for the QGUP case. Similarly, total entropy change and redshift parameter have maximum values in the QGUP case. Comparing Fig. 2 with 3 reveals that Nouicer's GUP modified total entropy evolution has a drastic increase when z approaches to z_{max} . In Tables 3 and 4, we numerically give the maximum values of the redshift parameter, Hubble parameter and total entropy change for the various values of GUP parameter α in Nouicer's GUP and

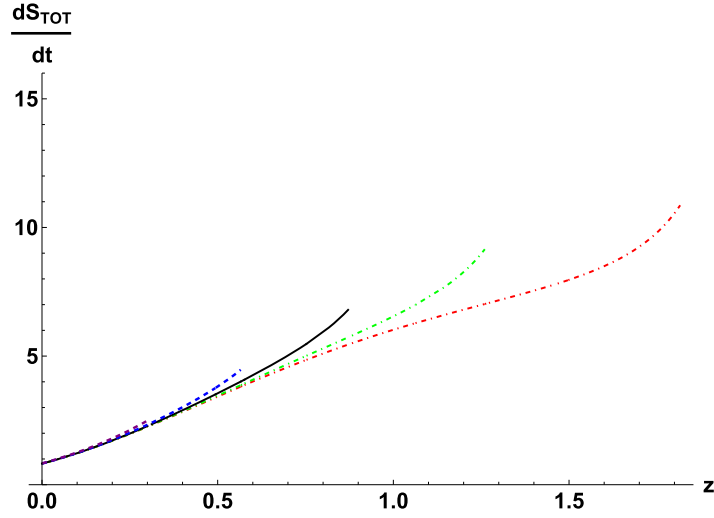


Fig. 3. Total entropy change versus the redshift parameter. From top to bottom, the curves correspond to $\alpha = 1.4, 1.2, 1.0, 0.8, 0.6$ for QGUP. We take $\Omega_m = 0.3$, $\Omega_r = 0.0001$ and $H_0 = 1$.

Table 3

The maximum values of redshift parameter, Hubble parameter and the total entropy change for the different values of α in Nouicer's GUP case.

	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$	$\alpha = 1.2$	$\alpha = 1.4$
z_{max}	1.779	1.227	0.840	0.535	0.268
$H(z_{max})$	2.859	2.114	1.716	1.430	1.125
$S_h + S_f$	100.173	26.276	12.861	6.707	3.185

Table 4

The maximum values of redshift parameter, Hubble parameter and the total entropy change for the different values of α in QGUP case.

	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$	$\alpha = 1.2$	$\alpha = 1.4$
z_{max}	1.816	1.26	0.871	0.566	0.301
$H(z_{max})$	2.887	2.165	1.732	1.443	1.237
$S_h + S_f$	10.862	9.156	6.796	4.468	2.498

QGUP cases, respectively. In the Nouicer's GUP case, z_{max} and $H(z_{max})$ are smaller than z_{max} and $H(z_{max})$ obtained from QGUP. Nouicer's GUP modified total entropy change is bigger than QGUP modified total entropy change.

Finally, we finish this section with comments on different values of GUP parameter. From Fig. 2, Fig. 3, Table 3 and Table 4, one can see that z_{max} decreases while α increases. The same behaviour can also be seen for the maximum values of H and total entropy change. Maximum values of Hubble parameter and total entropy changes decrease while α increases. It is also interesting to note that Nouicer's GUP corrected total entropy change drastically decreases for the increasing values of α .

6. Conclusions and discussions

In this section, using the entropy-area relation obtained from Nouicer's GUP [85], we obtained the GUP modified Friedmann equations from the first law of thermodynamics at apparent horizon [15]. We found a minimum apparent horizon due to the minimal length notion of GUP. We showed that the energy density of universe is finite and maximum at the minimum apparent horizon. Then, in order to see the effects of GUP, we computed the deceleration parameter for flat case and the equation of state $p = \omega\rho$. We found that ω must satisfy the condition $\omega < -0.361$ for the initial acceleration. For the radiation and matter dominated eras, the expansion of universe is more decelerated since the GUP effects give the positive contribution to deceleration parameter. Moreover, we checked the validity of GSL. We showed that the GSL is always valid for the all eras of universe in the presence of GUP effects. At last, we consider the GSL for the specific case, i.e., Λ CDM cosmology. In contrast to standard cosmology, the redshift parameter has a finite and maximum value. GUP also affects the total entropy change. The total entropy change has a finite and maximum value at maximum redshift value.

To better understand the effects of Nouicer's GUP on FRW cosmology, we compared our results with those obtained from QGUP. We showed that universe has beginning with bigger apparent horizon and less dense energy density in the Nouicer's GUP case.

Our results indicate that there is no Big Bang singularity due to the minimal apparent horizon and maximum energy density. Therefore, GUP provides more reasonable solution at the Planck scale where the classical general relativity fails. This feature is a well-known in the literature [20–22,25]. So our results are consistent with the recent studies. Interestingly, we found the total entropy change has a maximum value at maximum and finite redshift for Λ CDM cosmology. Moreover, maximum total entropy evolution has a drastic increase in Nouicer's GUP case. In fact, the maximum and finite value of z is expected since the Big Bang singularity is removed. However, the modified uncertainty effects on total entropy change in Λ CDM need further investigation. Particularly, taking into account different forms of GUP may shed light on this case. For example, linear and quadratic GUP (LQGUP) $[X, P] = i(1 + \alpha p + \beta p^2)$ includes a linear term in momentum [81–83]. Clearly, LQGUP is not consistent with QGUP and Nouicer's GUP due to the linear term in momentum. Including the linear term may affect total entropy evolution. On the other hand, a higher order GUP in the form of $[X, P] = \frac{i}{1-\beta p^2}$ is consistent with Nouicer's GUP and QGUP up to the leading order [84]. A quantitative correction at Planck scale can be determined for the GSL. We hope to report effects of various GUP models on GSL in future studies.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A. QGUP modified Friedmann equations

In this appendix, we will review the QGUP modified Friedmann equations [20]. We will compute the deceleration parameter and investigate the generalised second law for the QGUP modified Friedmann equations. We consider the QGUP given by [66,67]

$$\Delta X \Delta P \geq \frac{1}{2} (1 + \alpha^2 \Delta P^2). \quad (\text{A.1})$$

Repeating the same calculations given in Sections 2 and 3, one can obtain the QGUP modified entropy-area relation and Friedmann equations

$$\frac{dS_h}{dA_h} = \frac{\Delta S_{\min}}{\Delta A_{\min}} = \frac{\alpha^2}{32\tilde{r}_A^2} \left(1 - \sqrt{1 - \frac{\alpha^2}{4\tilde{r}_A^2}} \right)^{-1}, \quad (\text{A.2})$$

$$-\frac{8\pi\rho}{3} - \frac{\Lambda}{3} = \frac{2}{3\alpha^2} \left[\left(1 - \frac{\alpha^2}{4\tilde{r}_A^2} \right) \left(3 + 2\sqrt{1 - \frac{\alpha^2}{4\tilde{r}_A^2}} \right) - 5 \right], \quad (\text{A.3})$$

$$-4\pi(\rho + p) \left(1 - \sqrt{1 - \frac{\alpha^2}{4\tilde{r}_A^2}} \right) = \frac{\alpha^2}{8\tilde{r}_A^2} \left(\dot{H} - \frac{k}{a^2} \right), \quad (\text{A.4})$$

respectively. These equations can be expressed in term of Hubble parameter³

$$\begin{aligned} \frac{8\pi}{3} \left(\rho + \frac{\Lambda}{8\pi} \right) &= \frac{1}{2} \left(H^2 + \frac{k}{a^2} \right) + \frac{4}{3\alpha^2} \left[1 - \left(1 - \frac{\alpha^2}{4} \left(H^2 + \frac{k}{a^2} \right) \right)^{3/2} \right], \\ -4\pi(\rho + p) &= \left(\dot{H} - \frac{k}{a^2} \right) \frac{\alpha^2}{8} \frac{\left(H^2 + \frac{k}{a^2} \right)}{\left[1 - \left(1 - \frac{\alpha^2}{4} \left(H^2 + \frac{k}{a^2} \right) \right)^{1/2} \right]}. \end{aligned} \quad (\text{A.5})$$

From Eq. (A.3), the minimum apparent horizon and maximum energy density are given by

³ The reader may notice that the Friedmann equations in Eq. (A.5) are slightly different from the equations in Ref. [20]. (See Eqs. (5.5) and (5.6) in Ref. [20]). Replacing $\alpha^2 \rightarrow \alpha/\pi$ and $\Lambda \rightarrow -8\pi\Lambda$ in Eq. (A.5), one can exactly get the same QGUP modified Friedmann equation in Ref. [20]. The authors of Ref. [20] defined the GUP parameter as β while we define the GUP parameter as α^2 . Moreover, they set $\alpha = 4\pi\ell_P\beta$ to simplify the equations. In the limit $\tilde{r}_A \rightarrow \infty$, we set the integration constant to reproduce the standard Friedmann equations. Considering vacuum energy density $\rho_{\text{vac}} = \Lambda$ in the limit $\tilde{r}_A \rightarrow \infty$, the authors of Ref. [20] set the integration constant. However, these arguments lead to obtaining the cosmological constant term as $-\frac{8\pi\Lambda}{3}$ instead of $\frac{\Lambda}{3}$ in Ref. [20].

$$\tilde{r}_A^{min} = \frac{\alpha}{2}, \quad \rho_{max} = \frac{5}{4\pi\alpha^2}, \quad (\text{A.6})$$

respectively. Choosing equation of state $p = \omega\rho$ and combining the Friedman equations (A.5) with Eq. (37), one can obtain the deceleration parameter

$$q = -1 + \frac{6(1+\omega)}{H^2\alpha^2} \left[1 + \frac{8}{3H^2\alpha^2} \left(1 - \left(1 - \frac{\alpha^2 H^2}{4} \right)^{1/2} \right) - \frac{2\Lambda}{3H^2} \right] \left[1 - \left(1 - \frac{\alpha^2 H^2}{4} \right)^{1/2} \right] \quad (\text{A.7})$$

for $k = 0$. From the above equation, the maximum Hubble parameter is given by

$$H_{max} = \frac{2}{\alpha}. \quad (\text{A.8})$$

The deceleration parameter at the initial stage is given by

$$q(H_{max}) = -1 + \frac{5(1+\omega)}{2}. \quad (\text{A.9})$$

ω must satisfy the condition $\omega < -0.6$ for the acceleration at the inflationary stage. Finally, repeating the calculations defined in Section 5, the total entropy evolution is given by

$$T_h (\dot{S}_h + \dot{S}_f) = \frac{64\pi^2(\rho + p)^2 H \tilde{r}_A^7}{\alpha^2} \left(1 - \sqrt{1 - \frac{\alpha^2}{4\tilde{r}_A^2}} \right). \quad (\text{A.10})$$

We conclude that GSL is always satisfied since the above expression never decreases. In order to investigate the QGUP effects on total entropy evolution, we can write Eq. (A.10) as follows:

$$\dot{S}_h + \dot{S}_f = \frac{\pi}{8} \dot{H}^2 H^{-3} \left(1 + \frac{\dot{H}}{2H^2} \right)^{-1} \frac{\alpha^2}{1 - \sqrt{1 - \frac{\alpha^2 H^2}{4}}}. \quad (\text{A.11})$$

At last, using $\dot{H} = -(1+z)H(dH/dz)$, we get

$$\dot{S}_h + \dot{S}_f = \frac{\pi}{8} (1+z)^2 H^{-1} \left(\frac{dH}{dz} \right)^2 \left(1 - \frac{(1+z)}{2H} \frac{dH}{dz} \right)^{-1} \frac{\alpha^2}{1 - \sqrt{1 - \frac{\alpha^2 H^2}{4}}}. \quad (\text{A.12})$$

Repeating the calculations given in Section 5, one can find QGUP modified $H_{\Lambda CDM}(z)$ function. We do not give the exact expression since it is too lengthy.

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