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Anti-Instability of Complex Ghost

Jisuke Kubo[1,2,⋆] and Taichiro Kugo[3,⋆]

[1] *Max-Planck-Institut für Kernphysik (MPIK), Saupfercheckweg 1, 69117 Heidelberg, Germany*
[2] *Department of Physics, University of Toyama, 3190 Gofuku, Toyama 930-8555, Japan*
[3] *Center for Gravitational Physics and Quantum Information, Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan*
⋆E-mail: kubo@mpi-hd.mpg.de (JK); kugo@yukawa.kyoto-u.ac.jp (TK)

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We argue that Lee–Wick's complex ghost appearing in any higher derivative theory is stable and its asymptotic field exists. It may be more appropriate to call it "anti-unstable", in the sense that the more the ghost "decays" into lighter ordinary particles, the larger the probability that the ghost remains as itself becomes. This is explicitly shown by analyzing the two-point functions of the ghost Heisenberg field which is obtained as an exact result in the $N \to \infty$ limit in a massive scalar ghost theory with light $O(N)$-vector scalar matter. The anti-instability is a consequence of the fact that the poles of the complex ghost propagator are located on the physical sheet in the complex plane of four-momentum squared. This should be contrasted with the case of the ordinary unstable particle, whose propagator has no pole on the physical sheet.

Subject Index B30, B37, E05

1. Introduction

The negative metric ghost mode which is contained in higher derivative quantum field theory (QFT) always acquires a complex mass by radiative corrections, thus becoming a complex ghost. Lee and Wick [1–3] claimed that such a complex ghost cannot be created by collisions of positive metric physical particles (possessing real energies) because of energy conservation law, and thus the unitarity of physical particles alone must hold.

We recently pointed out in a previous paper [4], referred to as Paper I henceforth, that their treatment of the delta function expressing energy conservation,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, \exp\Big(it\sum_i E_i\Big) = \delta_c\Big(\sum_i E_i\Big), \tag{1}$$

which appears at each vertex in a Feynman diagram, is *wrong* when any of the particles i is a complex ghost possessing complex energy E_i. This function $\delta_c(E)$ is actually divergent when the argument E is complex, but is well-defined as a distribution which was first introduced and called a complex delta function by Nakanishi [5,6] when discussing unstable particles long ago. Treating the complex delta function properly, we have shown that the complex ghost can actually be created by collisions of physical particles, hence implying the violation of S-matrix unitarity of physical particles alone, unfortunately.

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Nevertheless, some people still raise the question: "Even if the complex ghost is created by a collision of physical particles, it can decay into lighter physical particles so that it eventually disappears after a sufficiently long time. Then, the unitarity of physical particles alone is again recovered, isn't it?" (see e.g., Refs. [7–9]).

To this conceivable question, we have already given a brief but clear answer in Paper I. We should note that there is a crucial difference between a complex ghost and an ordinary unstable particle; the former has a negative norm while the latter has a positive norm. We have written in Paper I as follows:

"First of all, the ghost state created in the superposition $\varphi + \varphi^\dagger$ (i.e., ghost + conjugate-ghost) has a negative norm. Since the Dyson's S-matrix of the present system is unitary, the negative norm, say -1, of the initial ghost state must be conserved. So, whatever final states are produced from the initial ghost state, the norms of all those final states sum up to the value -1 of the initial ghost state's norm. To realize this negative value, however, ghost particles must be contained among the final states. This implies that the ghosts can never disappear by completely 'decaying out' into lower mass physical particles."

In contrast, an ordinary unstable particle has a positive norm. Its initial norm, say $+1$, can be conserved even if it completely decays into ordinary lighter particles and disappears, which is indeed the case [10].

We think this explanation is enough to prove the *stability* of the complex ghost particle. However, a skeptic might go on to say: "A variety of discussions may be possible for the complex ghost in Lee's model. However, does such an *asymptotic field of complex ghost* really exist in the fourth-order derivative theory in the first place?"

We did not directly answer this objection in Paper I. This is because we there discussed the problem solely in Lee's complex ghost model [2] in which the ghost φ and conjugate-ghost $\varphi^\dagger$ fields are prepared from the beginning. Their asymptotic fields were essentially assumed to exist in Lee's model in perturbation theory framework. One may cast doubt on the equivalence between Lee's complex ghost model and the original fourth-order derivative theory; the ghost in the fourth-order derivative theory is a single real field while the complex ghost in Lee's model is actually described by two fields, φ and $\varphi^\dagger$. Is it possible at all that two such complex conjugate fields emerge from a single real ghost field by radiative corrections?

The purpose of the present paper is to give a clear affirmative answer to this problem of existence of the complex (conjugate pair of) ghost asymptotic fields. The basic information for asymptotic fields contained in a Heisenberg field Φ is given by the two-point Green function (propagator) $\langle 0|\mathrm{T}\Phi(x)\Phi(0)|0\rangle$. We have the original fourth-order derivative field Φ as this Heisenberg field Φ which can be decomposed into a positive metric lower-mass field A and the negative metric massive ghost field ϕ; that is, $\Phi = A + \phi$. We have to analyze the system in which the massive ghost ϕ can get a complex mass by a self-energy diagram consisting of the loop of light physical particles ψ_i. However, since the ghost interaction with ψ_i occurs only through the original fourth-order derivative field $\Phi = A + \phi$, the same self-energy diagram of the ψ_i loop also contributes to the A–ϕ transition as well as to A's self-energy. For the present problem, however, this mixing between A and ϕ fields is not essential and merely introduces unnecessary complications. Thus we drop the A component field and retain only the ghost field ϕ component in Φ in this paper. Also, since we would like to consider a model in which our calculation for the ghost two-point function $\langle 0|\mathrm{T}\phi(x)\phi(0)|0\rangle$ becomes

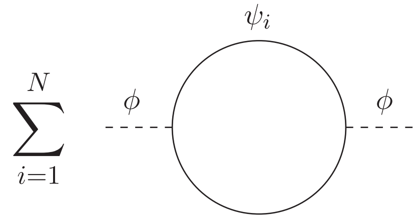

Fig. 1. Ghost self-energy diagram of ψ_i-loop in the leading order in $1/N$ expansion.

exact in a certain limit, we elaborate an $O(N)$-vector scalar matter field model given shortly in Section 2.

We easily compute the ghost two-point function in the leading order in $1/N$ expansion. It is merely a one-loop computation but is an exact result in the $N \to \infty$ limit. Rewriting the result into the form of the dispersion relation in Section 3, and comparing it with the spectral representation, we can find the asymptotic fields of the system. We also derive a sum rule for the spectral function and wave-function renormalization factor from the spectral representation in Section 4. This sum rule (Eq. 38) may be called an anti-instability relation and will further solidify the above-cited argument for the stability of the complex ghost in Paper I. Section 5 is devoted to two additional remarks on a confusing point in the narrow resonance approximation and on the reason why the ghost asymptotic fields appear in a pair of complex conjugate ghosts. In the final section, Section 6, we summarize the results and emphasize the general validity of our argument for the existence of asymptotic complex ghost states in any higher derivative theories.

2. The two-point vertex function $\Gamma_\phi^{(2)}(p)$

We consider the following system of a heavy scalar field ϕ with mass m and a lighter $O(N)$-vector multiplet of scalar fields $\psi_i (i = 1, 2, \cdots, N)$ with mass μ ($< m$), which is described by the following Lagrangian:[1]

$$\mathcal{L} = -\sum_{i=1}^{N} \frac{1}{2}(\partial_\mu \psi_i \partial^\mu \psi_i + \mu^2 \psi_i^2) - \epsilon_g \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi + m_0^2 \phi^2) + \sum_{i=1}^{N} \frac{1}{2} \frac{g}{\sqrt{N}} \phi \psi_i \psi_i \,. \tag{2}$$

Note that we are adopting a space-favored metric $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$, so that the $O(N)$-vector light fields ψ_i are of positive metric. We are interested in the case where the heavier field ϕ is a ghost, i.e., possessing negative metric $\epsilon_g = -1$, but, for comparison, we also consider the $\epsilon_g = +1$ case in which ϕ is an ordinary positive metric particle.

In the leading order in $1/N$-expansion, the two-point vertex function $\Gamma_\phi^{(2)}(p)$ of the heavy field ϕ is given by

$$\Gamma_\phi^{(2)}(p) = -\epsilon_g (p^2 + m_0^2) + \Sigma(p), \tag{3}$$

where $\Sigma(p)$ stands for the self-energy diagram of the ψ_i-loop in Fig. 1, which reads

[1] Similar model Lagrangians to this Eq. (2) were considered in Refs. [7,11].

$$\Sigma(p) = \frac{1}{2}g^2 \int \frac{d^n k}{i(2\pi)^n} \frac{1}{\mu^2 + k^2} \frac{1}{\mu^2 + (p-k)^2}$$
$$= \frac{g^2}{32\pi^2} \int_0^1 dx \left[\bar{\varepsilon}^{-1} - \ln\bigl(\mu^2 + x(1-x)p^2\bigr)\right]$$
$$= \frac{g^2}{32\pi^2} \left[\bar{\varepsilon}^{-1} + f(s)\right],$$
$$f(s) = 2 - \ln\mu^2 - 2\sqrt{\frac{4\mu^2 - s}{s}} \mathrm{Arctan} \sqrt{\frac{s}{4\mu^2 - s}}$$
$$= 2 - \ln\mu^2 + \sqrt{1 - \frac{4\mu^2}{s}} \ln\left(\frac{\sqrt{1 - 4\mu^2/s} - 1}{\sqrt{1 - 4\mu^2/s} + 1}\right) \tag{4}$$

with

$$s \equiv -p^2, \qquad \bar{\varepsilon}^{-1} \equiv \frac{2}{4-n} - \gamma + \ln 4\pi. \tag{5}$$

Here n is the space-time dimension to be eventually set equal to 4.

The function $f(s)$ in $\Sigma(p)$ has a branch point at $s = 4\mu^2$ and we take the cut to be along the real axis from $s = 4\mu^2$ to ∞. The imaginary part of $f(s)$ along the cut is given by

$$\lim_{\varepsilon \to +0} \mathrm{Im}\, f(s \pm i\varepsilon) = \pm\pi \sqrt{1 - \frac{4\mu^2}{s}} \quad \text{for real } s > 4\mu^2. \tag{6}$$

Note that the points $s \pm i\varepsilon$ ($\varepsilon > 0$) here are taken on the physical sheet of the complex s plane. If we take those points on the second sheet, the sign $\pm$ on the right-hand side (RHS) becomes the opposite, $\mp$, since the upper and lower sides of the real s-axis for $s > 4\mu^2$ on the physical sheet smoothly continue to the lower and upper sides on the second sheet, respectively.

Depending on the magnitude of the bare mass squared parameter m_0^2, the two-point function $\Gamma_\phi^{(2)}(p)$ has a zero at $s = m^2$ on the real axis of $s(\equiv -p^2)$ below the threshold $s < 4\mu^2$, or otherwise, has a complex conjugate pair of zeros at

$$s = M^2 = m^2 + i\gamma m \quad \text{and} \quad M^{*2} = m^2 - i\gamma m \tag{7}$$

for the $m^2 > 4\mu^2$ case. That is, m^2 is the squared mass of the stable particle, or the real part of the complex squared mass of the 'unstable' particle. We renormalize the mass squared parameter as $m_0^2 = m^2 + \delta m^2$ to realize

$$-\epsilon_g\, \delta m^2 + \mathrm{Re}\, \Sigma(p)\big|_{s=M^2} = 0 \tag{8}$$

(implying $M^2 = m^2$ for the case $m^2 < 4\mu^2$, as shown shortly), so that the renormalized two-point vertex function $\Gamma_\phi^{(2)}(p)$ reads

$$\Gamma_\phi^{(2)}(p) = \epsilon_g\,(s - m^2) + \frac{g^2}{32\pi^2}\bigl(f(s) - \mathrm{Re}\, f(M^2)\bigr) =: F(s). \tag{9}$$

The imaginary part $\mathrm{Im}\, M^2 = \gamma m$ is determined by the requirement that M^2 be the zero of the two-point vertex function $\Gamma_\phi^{(2)}(p)$: $F(M^2) = 0$, i.e.,

$$-\epsilon_g\,(M^2 - m^2) = \frac{g^2}{32\pi^2}\bigl(f(M^2) - \mathrm{Re}\, f(M^2)\bigr) \;\rightarrow\; -\gamma m = \epsilon_g \frac{g^2}{32\pi^2} \mathrm{Im}\, f(m^2 + i\gamma m). \tag{10}$$

If $m^2 < 4\mu^2$, since $f(s)$ is real on the real axis of s below the threshold $s < 4\mu^2$, we see that $\gamma = 0$ satisfies Eq. (10) and hence $-p^2 = m^2$ is the zero of the two-point vertex function $\Gamma^{(2)}(p)$ so that m^2 indeed represents the renormalized mass squared of the stable ϕ-particle, as announced

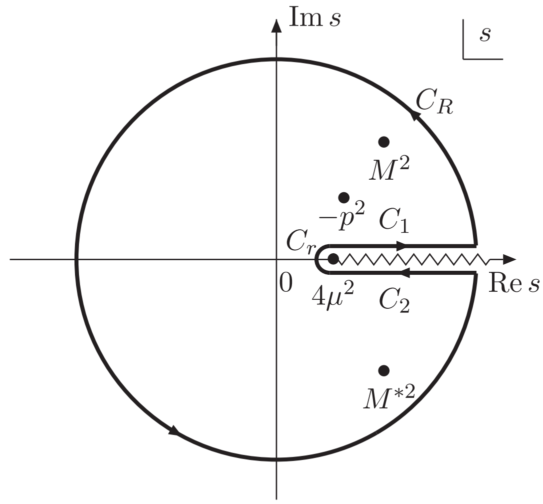

Fig. 2. Contour $C = C_1 + C_R + C_2 + C_r$ on the physical sheet.

above. However, if m^2 moves above the threshold $4\mu^2$, then $f(s)$ develops the imaginary part (Eq. 6) across the cut and hence the zero of $\Gamma^{(2)}(p)$ splits into two zeros of a complex conjugate pair, M^2 and M^{*2}, as written in Eq. (7). As already noted by Lee, Wick, and others [1–3], the direction of this splitting is the opposite for the ordinary and ghost particle cases, $\epsilon_g = \pm 1$. This can be seen explicitly in the present calculation. Eq. (6) implies that the quantity Im $f(m^2 + i\gamma m)$ on the RHS of Eq. (10) has the same sign as γm when $s = m^2 + i\gamma m$ is located on the physical sheet, and has the opposite sign to γm when $s = m^2 + i\gamma m$ is on the second sheet. This means that the solutions $M^2 = m^2 + i\gamma m$ as well as M^{*2} satisfying Eq. (10) exist on the physical sheet only for the ghost case $\epsilon_g = -1$, while, for the ordinary particle case $\epsilon_g = +1$, they move to the second sheet and disappear from the physical sheet.

We can give an approximate expression for the complex ghost zero $s = M^2$ on the upper half of the plane of the physical sheet for the case $g/m \lesssim O(1)$; then, $(g/m)^2/32\pi^2 \ll 1$ so that Eq. (10) implies $\gamma m \ll m^2 \rightarrow f(m^2 + i\gamma m) \simeq f(m^2)$, and hence Eq. (6) leads to

$$M^2 \simeq m^2 + i\frac{g^2}{32\pi}\sqrt{1 - \frac{4\mu^2}{m^2}}\ . \tag{11}$$

3. Dispersion relation for the ϕ propagator

Since we have understood the analyticity and singularity structure of the two-point function $\Gamma_\phi^{(2)}(p)$, we can now derive a dispersion relation for the ϕ propagator

$$D_\phi(s = -p^2) = \frac{i}{\Gamma_\phi^{(2)}(p)} = \frac{i}{F(s)} \tag{12}$$

following the usual procedure. Consider the following contour integration

$$I \equiv \frac{1}{2\pi i}\int_C ds\, \frac{D_\phi(s)/i}{s + p^2} \tag{13}$$

(for a general complex value of $-p^2$) along the closed contour on the physical sheet, $C = C_1 + C_R + C_2 + C_r$, depicted in Fig. 2. From the consideration above, we know that the propagator $D_\phi(s)$ is the function of s which is real on the real axis except on a branch cut starting from $s = 4\mu^2$ to ∞ and analytic everywhere on the physical sheet (i.e., outside the cut) other than the complex conjugate poles at $s = M^2$ and M^{*2}. Thus the integrand function $D_\phi(s)/i(s + p^2)$

in Eq. (13) is also analytic everywhere inside the closed contour C except for the three poles at $s = M^2$, M^{*2}, and $s = -p^2$. Applying Cauchy's residue theorem, we obtain

$$I = \frac{D_\phi(-p^2)}{i} + \frac{\epsilon_g Z}{M^2 + p^2} + \frac{\epsilon_g Z^*}{M^{*2} + p^2}\,, \tag{14}$$

$$Z^{-1} \equiv \epsilon_g \left.\frac{\partial F(s)}{\partial s}\right|_{s=M^2} = \lim_{s\to M^2} \epsilon_g \frac{F(s)}{s - M^2}\,. \tag{15}$$

On the other hand, since

$$\begin{gathered} |D_\phi(s)| \to |s|^{-1} \quad \text{as} \quad |s| \to \infty, \\ |D_\phi(s)| < \exists K \ \text{ on } C_r \quad \text{as} \quad r = |s - 4\mu^2| \to 0, \end{gathered} \tag{16}$$

where K is a finite positive constant, the contribution to the integral (Eq. 13) along C comes only from the discontinuity of the propagator across the cut:

$$I = \frac{1}{\pi}\int_{4\mu^2}^{\infty} ds\, \frac{-\rho(s)}{s + p^2}\,, \tag{17}$$

$$\rho(s) = -\mathrm{Im}\left(\lim_{\varepsilon\to+0} \frac{D_\phi(s+i\varepsilon)}{i}\right) = -\frac{1}{2i}\lim_{\varepsilon\to+0}\left(\frac{1}{F(s+i\varepsilon)} - \frac{1}{F(s-i\varepsilon)}\right) = \lim_{\varepsilon\to+0}\frac{\mathrm{Im}\,F(s+i\varepsilon)}{|F(s)|^2}\,, \tag{18}$$

where $F(s)$ is the two-point function $\Gamma_\phi^{(2)}(p)$ as the function of s defined in Eq. (9). Equating Eqs. (14) and (17), we obtain a dispersion relation for our ϕ propagator $D_\phi(-p^2)$:[2]

$$\begin{gathered} D_\phi(-p^2) = \frac{iZ}{M^2+p^2} + \frac{iZ^*}{M^{*2} + p^2} + \frac{1}{i\pi}\int_{4\mu^2}^{\infty} ds\, \frac{\rho(s)}{s + p^2} \\ \text{for the complex ghost case; } \mathrm{Re}\, M^2 = m^2 > 4\mu^2 \text{ and } \epsilon_g = -1\,. \end{gathered} \tag{19}$$

This Eq. (19) takes the form of Källen–Lehman's spectral representation for the propagator, so, if we recall the well-known method of its derivation by inserting the complete set of states as intermediate states in the operator expression of the two-point function[3]

$$\langle 0|\mathrm{T}\phi(x)\phi(0)|0\rangle \left(= \int_{C(p^0)} \frac{d^4p}{(2\pi)^4} e^{ipx} D_\phi(-p^2)\right)\,, \tag{20}$$

we can understand the meaning of each term on the RHS of Eq. (19). The last $\rho(s)$ integral term is understood, as usual, as coming from the discontinuity caused by the continuum spectrum of two physical ψ_i-particle intermediate states. Then, the first and second pole terms must be understood as coming from the two discrete one-particle states possessing complex conjugate masses M^2 and M^{*2}. Note that the poles appearing on the physical sheet mean the existence of the corresponding one-particle asymptotic states in the complete set of states of the theory.

Indeed, it is instructive to consider the same propagator $D_\phi(-p^2)$ for the other parameter-value cases in the present system (2). First consider the case $m^2 < 4\mu^2$, for which $D_\phi(s)$ has only a single pole term at $s = m^2$ on the real axis, so that the dispersion relation (19) takes the form

$$D_\phi(-p^2) = \frac{\epsilon_g}{i}\frac{Z}{m^2 + p^2} + \frac{1}{i\pi}\int_{4\mu^2}^{\infty} ds\, \frac{\rho(s)}{s + p^2} \qquad \text{for the } m^2 < 4\mu^2 \text{ case}\,. \tag{21}$$

[2]Essentially the same expression as this Eq. (19) for the complex ghost propagator was also given by Coleman [12] and Grinstein et.al. [7].

[3]Here in Eq. (20), the p^0-integration must be performed along a much-deformed contour $C(p^0)$ from the real axis on the complex p^0 plane while 3D $\boldsymbol{p}$-integration is the usual Fourier transformation along the real axis of $\boldsymbol{p}$, as will be explained at the end of this section.

This is the usual *stable particle* case if $\epsilon_g = +1$: There is a one-particle pole in the propagator $D_\phi(s)$, and the corresponding asymptotic field $\phi^{\rm as}(x)$ exists satisfying $(\Box - m^2)\phi^{\rm as}(x) = 0$. Next consider a more interesting case, $m^2 > 4\mu^2$ with $\epsilon_g = +1$ (i.e., positive norm). This is the ordinary *unstable particle* case, for which the complex conjugate poles move into the second sheet and disappear from the physical sheet, as explicitly shown above, so that the dispersion relation (19) takes the form

$$D_\phi(-p^2) = +\frac{1}{i\pi}\int_{4\mu^2}^{\infty} ds\, \frac{\rho(s)}{s+p^2} \qquad \text{for the } m^2 > 4\mu^2 \text{ case with } \epsilon_g = +1\,. \tag{22}$$

There is no one-particle pole term here, which agrees with the fact that there is no asymptotic field corresponding to an unstable particle. As everyone knows, however small the decay probability (into two ψ_i particle states here) is, any unstable particle decays out into lighter ordinary particles and eventually disappears in sufficiently long time [10]. Thus the complete set of states is spanned by stable particles alone.

We thus conclude from the dispersion relation (19) for the ϕ propagator that the Heisenberg field ϕ (massive regulator part of the fourth-order derivative Heisenberg field) has the conjugate pair of asymptotic fields of a complex ghost, φ and $\varphi^\dagger$:

$$\phi(x) \xrightarrow[x^0\to\infty]{} Z^{1/2}\varphi(x) + Z^{*1/2}\varphi^\dagger(x), \tag{23}$$

which satisfy the free field equations, $(\Box - M^2)\varphi(x) = 0$ and its complex conjugate $(\Box - M^{*2})\varphi^\dagger(x) = 0$.

The unfamiliar metric structure of these asymptotic complex fields can most easily be found simply by canonical quantization of their unique free field Lagrangian:[4]

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu\varphi\,\partial^\mu\varphi + M^2\varphi^2 + \partial_\mu\varphi^\dagger\,\partial^\mu\varphi^\dagger + M^{*2}{\varphi^\dagger}^2\right]. \tag{24}$$

However, we note that this is essentially the same Lagrangian as given by Nakanishi [13] for the BC field sector of Lee's complex ghost model [2] in which the complex ghost fields φ and $\varphi^\dagger$, or equivalently, $B = (\varphi + \varphi^\dagger)/\sqrt{2}$ and $C = i(\varphi - \varphi^\dagger)/\sqrt{2}$ fields, are not the asymptotic fields but the fields introduced in the model from the beginning. Anyway, since the property of these fields is uniquely specified by the Lagrangian (24), we can use Nakanishi's results, which we recapitulated in Paper I. The complex ghost field $\varphi(x)$ is expanded into plane waves as

$$\varphi(x) = \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}}\left(\alpha(\boldsymbol{p})e^{i\boldsymbol{px}-i\omega_{\boldsymbol{p}}x^0} + \beta^\dagger(\boldsymbol{p})e^{-i\boldsymbol{px}+i\omega_{\boldsymbol{p}}x^0}\right), \tag{25}$$

where $\omega_{\boldsymbol{p}}$ is the complex energy $\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + M^2}$ and the creation and annihilation operators satisfy the off-diagonal commutation relations:

$$\begin{aligned}
&[\alpha(\boldsymbol{p}), \beta^\dagger(\boldsymbol{q})] = [\beta(\boldsymbol{p}), \alpha^\dagger(\boldsymbol{q})] = -\delta^3(\boldsymbol{p}-\boldsymbol{q})\,,\\
&[\alpha(\boldsymbol{p}), \alpha^\dagger(\boldsymbol{q})] = [\beta(\boldsymbol{p}), \beta^\dagger(\boldsymbol{q})] = 0\,.
\end{aligned} \tag{26}$$

Then, by using these, the Feynman propagator is immediately found to be given as

$$\begin{aligned}
&\langle 0|\,\mathrm{T}\varphi(x)\,\varphi(y)\,|0\rangle\\
&= -\int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\left\{\theta(x^0-y^0)e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})-i\omega_{\boldsymbol{p}}(x^0-y^0)} + \theta(y^0-x^0)e^{-i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})+i\omega_{\boldsymbol{p}}(x^0-y^0)}\right\}.
\end{aligned} \tag{27}$$

[4]The overall sign of this action is a convention which can be changed by redefining the asymptotic field φ to $i\varphi$. We use the same sign choice as in Paper I, which is opposite to that used by Nakanishi in Ref. [13].

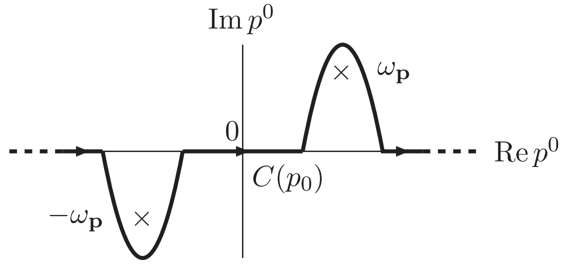

Fig. 3. p^0 integration contour $C(p^0)$ in Eq. (28).

This 3D momentum expression is rewritten into a 4D momentum expression by introducing p^0 variable as

$$= -\int_{C(p^0)} \frac{d^3\boldsymbol{p}\, dp^0}{i(2\pi)^4} \frac{e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})-ip^0(x^0-y^0)}}{M^2+p^2}\,. \tag{28}$$

In order for this 4D expression to reproduce the 3D expression (27), the p^0 integration contour $C(p^0)$ here has to be the deformed one from the real axis $R = (-\infty, +\infty)$ such that it passes below the left pole at $p^0 = -\omega_{\boldsymbol{p}}$ and above the right pole at $p^0 = +\omega_{\boldsymbol{p}}$ as shown in Fig. 3. With this understanding, we see that the asymptotic fields $Z^{1/2}\varphi + Z^{*\,1/2}\varphi^\dagger$ actually reproduce the two complex conjugate poles in the ϕ-propagator (19) in 4D momentum representation.

4. Spectral representation for the commutator

Once a spectral representation is found for one type of two-point functions, we can immediately write down those for other types of two-point functions by using the same spectral function. We first note that the dispersion relation (19) is rewritten into the spectral representation for the propagator in x space:

$$\langle 0|\, \mathrm{T}\phi(x)\phi(0)\, |0\rangle = -Z\Delta_{\mathrm{F}}(x;\, M^2) - Z^*\Delta_{\mathrm{F}}(x;\, M^{*2}) + \int_{4\mu^2}^{\infty} ds\, \frac{\rho(s)}{\pi}\Delta_{\mathrm{F}}(x;\, s)\,, \tag{29}$$

where $\Delta_{\mathrm{F}}(x; m^2)$ denotes the Feynman propagator function for the free field with mass squared m^2, including also the complex m^2 case:

$$\Delta_{\mathrm{F}}(x;\, m^2) = \int_{C(p^0)} \frac{d^4p}{i(2\pi)^4} \frac{e^{ipx}}{m^2+p^2}\,. \tag{30}$$

Knowing this form, we can immediately write down, in particular, the vacuum expectation value (VEV) of the commutation relation of the Heisenberg operator ϕ, in which we are now interested:

$$\langle 0|\, [\phi(x),\ \phi(0)]\, |0\rangle = -Zi\Delta(x;\, M^2) - Z^*i\Delta(x;\, M^{*2}) + \int_{4\mu^2}^{\infty} ds\, \frac{\rho(s)}{\pi} i\Delta(x;\, s)\,, \tag{31}$$

in terms of the commutator function $i\Delta(x; m^2)$ for the free field with (generally complex) mass squared m^2 [13]:

$$\Delta(x;\, m^2) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 E_{\boldsymbol{p}}} \sin[\boldsymbol{p}\cdot\boldsymbol{x} - E_{\boldsymbol{p}}x^0], \qquad E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2+m^2}\,. \tag{32}$$

This Eq. (31) leads to a very interesting relation. Take a time derivative $\partial/\partial x^0$ and set $x^0 = 0$ on both sides of this equation. Then, the left-hand side (LHS) is reduced to the equal time commutator between the Heisenberg field operator $\phi(0)$ and its conjugate momentum operator $\pi(x) \equiv \partial\mathcal{L}/\partial\dot{\phi}(x)$ at time $x^0 = 0$, which yields a simple value thanks to the canonical commutation relation (CCR):

$$\text{LHS} = \langle 0|\,[\dot{\phi}(\boldsymbol{x},0),\ \phi(0)]\,|0\rangle = \big\langle 0|\,[\epsilon_g\,\pi(\boldsymbol{x},0),\ \phi(0)]\,|0\big\rangle = -i\epsilon_g\,\delta^3(\boldsymbol{x})\,. \tag{33}$$

On the RHS also, since $i\Delta(x;m^2)$ is the free-field commutator function, the CCR gives the same quantity;

$$i\dot{\Delta}(\boldsymbol{x},0;m^2) = -i\delta^3(\boldsymbol{x}), \tag{34}$$

independently of the mass value m^2. Using this and dividing both sides by a common factor $-i\delta^3(\boldsymbol{x})$, we obtain

$$-1 = -(Z+Z^*) + \int_{4\mu^2}^{\infty} ds\,\frac{\rho(s)}{\pi} \qquad \text{for the ghost field case}\,. \tag{35}$$

Note that we have set $\epsilon_g = -1$ on the LHS here since this relation is derived from the dispersion relation (19) valid for the ghost field case. If we apply the same procedure to the dispersion relations, Eqs. (21) and (22), for ordinary stable and unstable particle cases, respectively, we obtain

$$+1 = Z + \textstyle\int_{4\mu^2}^{\infty} ds\,\dfrac{\rho(s)}{\pi} \qquad \text{for the stable particle case with } \epsilon_g = +1\,, \tag{36}$$

$$+1 = \textstyle\int_{4\mu^2}^{\infty} ds\,\dfrac{\rho(s)}{\pi} \qquad \text{for the unstable particle case}\,. \tag{37}$$

Eq. (36) is the well-known relation written in any field theory textbook, whose physical interpretation following from the derivation of spectral representation is as follows: Z is the probability that the state $\phi(x)|0\rangle$ (generated by acting the Heisenberg field $\phi(x)$ on the vacuum $|0\rangle$) contains the one-particle state $\big|\boldsymbol{p};m^2\big\rangle$, while

$$\int_{4\mu^2}^{\infty} ds\,\frac{\rho(s)}{\pi} =: c > 0$$

represents the probability that the state $\phi(x)|0\rangle$ contains many-particle states (only two particle states in this calculation). Thus Eq. (36) says that the total probability that $\phi(x)|0\rangle$ contains one-particle and many-particle states adds up to 1. In the same way, Eq. (37) for the unstable particle case shows that $\phi(x)|0\rangle$ contains no one-particle asymptotic state and the total probability is saturated only by the contribution c from the continuum many-particle states consisting of lighter particles produced by decays.

Now comes the relation (35), in view of which we reach the following interpretation. First, $Z+Z^*$ represents the probability that $\phi(x)|0\rangle$ contains the complex ghost asymptotic one-particle state, which is the *superposition of ghost $\varphi(x)|0\rangle$ and conjugate ghost* $\varphi^\dagger(x)|0\rangle$, as explained above. The state $\phi(x)|0\rangle$ also contains many-particle states which appear as the 'decay products' of the original ghost ϕ. The probability of the many-particle states of 'decay products', $\int_{4\mu^2}^{\infty} ds\,\rho(s)/\pi = c > 0$, is the same as the previous two cases and hence positive. Then, the relation (35) tells us a very interesting but counter-intuitive relation:[5]

$$Z + Z^* = 1 + c\,. \tag{38}$$

Surprisingly, the probability $Z+Z^*$ that the ghost remains as itself becomes even larger as c increases; that is, the more the ghost 'decays' into lighter ordinary particles, the larger the probability that the ghost remains as itself becomes. However, this strange property of

[5]Exactly the same equation as this Eq. (38) was written in Eq. (72) in Coleman's lecture [12], where he noted that it indicates the high-p^2 behavior of the original propagator in the fourth-order derivative theory remains as good as that at tree level, i.e., damps as $\sim 1/p^4$.

ghost, which we call *anti-instability*,[6] was already pointed out in Paper I, as cited in the Introduction.

We thus have completely proved the existence of asymptotic fields of complex ghost, φ and $\varphi^\dagger$, in Eq. (23). It is guaranteed by the anti-instability of the negative metric ghost.

5. Two additional remarks

Before closing this paper, we add two remarks.

One is on a possibly confusing point concerning the narrow resonance approximation given by Grinstein *et al.* in Ref. [7]. Those authors also wrote down the same form of dispersion relation as our Eq. (19) for the complex ghost field in a similar scalar field theory model. They noticed that the spectral density $\rho(s)$ is approximately given by

$$\rho(s) \simeq \frac{m\gamma}{(s-m^2)^2+m^2\gamma^2} = \frac{1}{2i}\left[\frac{1}{s-m^2-im\gamma} - \frac{1}{s-m^2+im\gamma}\right] \tag{39}$$

near the resonant energy $s = m^2$ for the narrow resonance case $\gamma \ll m$. This is generally true, since the condition $\gamma/m \ll 1$ (implying $(g/m)^2/32\pi \ll 1$ in our case) means the small coupling constant so that $Z \simeq 1$ and $1/F(s+i\varepsilon) \simeq -Z/(s-M^2) \simeq -1/(s-M^2)$ and $1/F(s-i\varepsilon) \simeq -1/(s-M^{*2})$ near the energy $s = m^2$. Noting that $\rho(s)$ is strongly peaked at $s = m^2$, they extend the s-integration in Eq. (19) over the whole real axis R of s. Then the s-integration can be done by closing the integration contour in either the upper or lower half-plane and yields

$$\frac{1}{i\pi}\int_{-\infty}^{\infty} ds\,\frac{\rho(s)}{s+p^2} = -\frac{i}{M^{*2}+p^2} \qquad \text{for } \operatorname{Im}(-p^2) > 0 \tag{40}$$

for the $-p^2 = (p^0)^2 - \boldsymbol{p}^2$ variable on the *upper half-plane*. Then, if this is substituted into the third continuum term in Eq. (19), it cancels(!) the second pole term $iZ/(M^{*2}+p^2)$ in this approximation with $Z \simeq 1$, and Eq. (19) now gives the expression for the ghost propagator:

$$D_\phi(-p^2) \simeq \frac{i}{M^2+p^2} \qquad \text{for } \operatorname{Im}(-p^2) > 0. \tag{41}$$

In the same way, for $-p^2$ on the lower half-plane, we find

$$D_\phi(-p^2) \simeq \frac{i}{M^{*2}+p^2} \qquad \text{for } \operatorname{Im}(-p^2) < 0. \tag{42}$$

These are of course valid approximate results for $-p^2$ on the upper and lower half-planes, respectively, but are nevertheless rather misleading expressions. It should never be interpreted that they imply the disappearance of the complex ghost pole at $-p^2 = M^2$ or the complex conjugate ghost pole at $-p^2 = M^{*2}$. They are merely approximate results that are numerically valid only near $-p^2 = m^2$ above the cut $\operatorname{Re}(-p^2) \geq 4\mu^2$. For instance, for real $-p^2$ below the threshold $-p^2 < 4\mu^2$, $D_\phi(-p^2)$ is real (and has no cut) since both complex conjugate poles contribute to it, but the approximate result, either Eq. (41) or Eq. (42), is complex, thus failing to reflect the analytic structure of the propagator function $D_\phi(-p^2)$. Eqs. (41) and (42) are two separate functions and either expression, (41) or (42), does not know the existence of the pole at the other half-plane.

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[6]This word is inspired by Coleman's lecture [12] in which he suggested that the complex ghosts be referred to as "antistable particles" because of *the radical difference* of situations from the ordinary unstable particles.

Another remark is on the reason why the asymptotic fields appear in a pair of complex conjugate ghosts, φ and $\varphi^\dagger$. One obvious reason is that the original Heisenberg field ϕ is a real field. The asymptotic field of the hermitian Heisenberg field should be real as a whole, so the combination

$$Z^{1/2}\varphi(x) + Z^{*1/2}\varphi^\dagger(x) \equiv |Z|^{1/2}\phi^{\rm as}(x). \tag{43}$$

Another reason is to realize the non-vanishing and negative norm of the original Heisenberg field ϕ. As is proved generally and easily, any complex energy eigenstate of a Hermitian Hamiltonian is of zero-norm and can have a non-vanishing inner product only with its conjugate energy eigenstate. The superposition state created by the combination (43) of a conjugate pair of asymptotic fields is just such a state that can carry a non-vanishing (negative) norm; indeed, using the *real* asymptotic field $\phi^{\rm as}$ in Eq. (43) and inserting the plane wave expansion (25) and its complex conjugate for the asymptotic fields φ and $\varphi^\dagger$ there, we have

$$\begin{aligned} \left|\boldsymbol{p};x^0\right\rangle &\equiv \int d^3\boldsymbol{x}\sqrt{\frac{2|\omega_{\boldsymbol{p}}|}{(2\pi)^3}}\,e^{i\boldsymbol{p}\boldsymbol{x}}\,\phi^{\rm as}(x)\,|0\rangle \\ &= \left(e^{i(\theta_Z-\theta_{\boldsymbol{p}})/2}\beta^\dagger(\boldsymbol{p})e^{i\omega_{\boldsymbol{p}}x^0} + e^{-i(\theta_Z-\theta_{\boldsymbol{p}})/2}\alpha^\dagger(\boldsymbol{p})e^{i\omega^*_{\boldsymbol{p}}x^0}\right)|0\rangle\ , \end{aligned} \tag{44}$$

where θ_Z and $\theta_{\boldsymbol{p}}$ are the phases of Z and $\omega_{\boldsymbol{p}}$; $Z=|Z|e^{i\theta_Z}$ and $\omega_{\boldsymbol{p}}=|\omega_{\boldsymbol{p}}|e^{i\theta_{\boldsymbol{p}}}$. The norm of this superposition is calculated by using the commutation relations (26) as

$$\begin{aligned} \left\langle\boldsymbol{q};x^0|\boldsymbol{p};x^0\right\rangle &= \langle 0|\,[\alpha(\boldsymbol{q}),\ \beta^\dagger(\boldsymbol{p})]e^{i(2\theta_Z-\theta_{\boldsymbol{p}}-\theta_{\boldsymbol{q}})/2}e^{i(\omega_{\boldsymbol{p}}-\omega_{\boldsymbol{q}})x^0}\,|0\rangle \\ &\quad + \langle 0|\,[\beta(\boldsymbol{q}),\ \alpha^\dagger(\boldsymbol{p})]e^{-i(2\theta_Z-\theta_{\boldsymbol{p}}-\theta_{\boldsymbol{q}})/2}e^{i(\omega^*_{\boldsymbol{p}}-\omega^*_{\boldsymbol{q}})x^0}\,|0\rangle \\ &= -\delta^3(\boldsymbol{q}-\boldsymbol{p})\,2\cos(\theta_Z-\theta_{\boldsymbol{p}})\,. \end{aligned} \tag{45}$$

This realizes the non-vanishing negative norm as announced above.[7] One should also note the fact that *this norm of the state* $|\boldsymbol{p};x^0\rangle$ *generated by the real asymptotic field* $\phi^{as}(x)$ *is independent of time* x^0. This is remarkable since the first and second states in Eq. (44) which are created by the complex asymptotic fields φ and $\varphi^\dagger$, respectively, each has a terrible time dependence, $e^{i\omega_{\boldsymbol{p}}x^0}$ or $e^{i\omega^*_{\boldsymbol{p}}x^0}$, exponentially divergent or damping as $x^0\to\pm\infty$. These time-dependent coefficients are, actually, fake, since they are not relevant to the magnitude of the state $\beta^\dagger(\boldsymbol{p})\,|0\rangle$ nor $\alpha^\dagger(\boldsymbol{p})\,|0\rangle$ because they are of zero norm. Those exponentially divergent and damping factors cancel each other between $\beta^\dagger(=\varphi)$ and $\alpha^\dagger(=\varphi^\dagger)$ states and realize an x^0-independent state. In this sense, it is very important that the asymptotic fields of ϕ always appear in the real combination (Eq. 43) which creates the superposition states (Eq. 44).

6. Conclusion

We have shown in this paper that the complex conjugate pair of ghost fields φ and $\varphi^\dagger$ are actually contained as asymptotic fields in the ghost (regulator) Heisenberg field ϕ in the higher derivative theories. Those asymptotic complex conjugate ghost particles each have zero norm but always appear in a superposition form of ghost and conjugate ghost, and carries negative norm. Owing to the negative norm, they have a peculiar stability property called anti-instability; the more they 'decay' into ordinary particles, the more the ghosts appear. This solidifies our previous

[7] It can easily be proved at least for the case $\mu^2/m^2\ll 1$ that the phase condition $0<\theta_Z-\theta_{\boldsymbol{p}}<\pi/2$ holds independently of the coupling strength g^2/m^2, so that the norm (45) always remains negative.

conclusion that the unitarity of physical particles alone is necessarily violated as far as (negative metric) ghost exists in the theory.

This anti-instability is concisely expressed in the form of Eq. (38); $Z + Z^* = 1 + c$, where $Z + Z^*$ is the probability of complex ghost one-particle state and c the probability of many ordinary particle states contained in the state $\phi(x)|0\rangle$ created by the ghost Heisenberg field $\phi(x)$. It was proved based on the dispersion relation Eq. (19) for the ghost propagator. The proof is very robust and non-perturbative in the sense that the anti-instability relation of the form in Eq. (38) can always be derived as far as the analyticity of the ghost propagator is given as shown in Fig. 2. We only needed the positivity of the discontinuity function $\rho(s)$ along the cut on the real axis.

We should, however, also note the fact that the structure of the analyticity as shown in Fig. 2 is very special. Once the intermediate states contain the complex ghost particle, their energy P^0 takes values extending over a 2D region on the complex P^0 plane, i.e., not restricted on the (1D) real axis, and so the dispersion relation or the spectral representation would become a much more complicated form whose precise expression has never been given (cf. Ref. [11]).

Nevertheless, on the other hand, this also implies the robustness of our general conclusion that the physical S-matrix unitarity is violated in higher derivative theories. The fields in those theories are always decomposed into second-order derivative fields among which some massive fields are negative metric ghosts. Our discussions in this paper can apply to those ghost fields, which necessarily become complex ghosts by the 'decay' to the ordinary lighter particles. Assume that the unitarity with ordinary positive metric particles alone could hold. Then, the total state vector space must be spanned only with those ordinary particles. If so, we have the usual form of spectral representation for the two-point functions of the ghost Heisenberg field ϕ, which is given by the integral along the real s-axis with positive definite spectral function $\rho(s)$. In particular, from the spectral representation for the function $\langle 0|\,[\phi(x),\ \phi(0)]\,|0\rangle$, we would get the anti-instability relation (38) with no complex ghost asymptotic states, $Z + Z^* = 0$, so that $1 + c = 0$. However, this contradicts the positivity assumption of $\rho(s)$, $c = \int ds\rho(s) > 0$. Thus the original unitarity assumption is wrong.

This implies that complex ghost states must exist in the total state vector space and those ghost states are contained in the state $\phi(x)|0\rangle$ of the ghost Heisenberg field $\phi(x)$. Thus we must admit the fact that the complex ghosts are necessarily created via the ghost field $\phi(x)$ contained in the original higher-derivative field. If we could have the physical unitarity despite this, therefore, the only possibility would be to find (or construct) a special higher-derivative theory possessing a certain symmetry such as BRST symmetry in gauge theories which guarantees that the complex ghosts always appear in zero-norm combinations in the physical subspace specified by the charge of the symmetry.

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