

Regge Spectroscopy of Higher-Twist States in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

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We study a family of higher-twist Regge trajectories in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory using the quantum spectral curve. We explore the many-sheeted Riemann surface connecting the different trajectories and show the interplay between the degenerate nonlocal operators known as (near-)horizontal trajectories. We resolve their degeneracy analytically by computing the first nontrivial order of the Regge intercept at weak coupling, which exhibits new behavior: It depends linearly on the coupling. This is consistent with our numerics, which interpolate all the way to strong coupling.

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Introduction.—Regge theory has been instrumental for our understanding of (strongly interacting) quantum field theory (QFT), especially of elastic high-energy forward scattering. Its greatest success is arguably in providing an explanation for the rising total cross sections of proton-proton scattering.

For massive QFTs, a Regge trajectory is the function $S(t)$ of the square momentum transfer t that encodes the behavior of the scattering amplitude for Mandelstam $s \gg 1$. In the $t > 0$ kinematic regime, the amplitude can reflect resonant production of particles with mass m_i , and $S(t = m_i^2)$ is the spin of such a resonance. The full Regge trajectory $S(t)$ is the analytic continuation of the spin in t that connects resonances with similar quantum numbers to the scattering regime $t < 0$. Combining the Regge and optical theorem, the total cross section behaves as $\sigma_{\text{tot}} \sim s^{\alpha-1}$, where $\alpha = S(0)$ is the intercept parameter. In the case of proton-proton scattering, the dominant trajectory is called the Pomeron with intercept α_p .

In quantum chromodynamics (QCD), α_p was computed at weak coupling, through the celebrated Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation that resums all leading logarithmic contributions [1–3]. It has an intriguing connection to integrable spin chains [4,5].

For conformal field theories (CFTs), Regge theory takes a special form [6] and describes the behavior of four-point functions in a specific kinematic regime with Lorentzian

signature. The role of t is taken over by the scaling dimension Δ , and the intercept is given by the minima of the $S(\Delta)$ Regge trajectory, the analytic continuation of CFT data. It was understood only recently that, to describe this continuation properly, one should use—generically nonlocal—light-ray operators [7,8]. This is because the usual operator product expansion of local operators does not account for the singularities emerging in the kinematic regime of the conformal Regge theory.

Light-ray operators can be constructed explicitly as integrals over local operators [8,9] and, in turn, can be used to build more complicated objects. An important class of operators is formed by a convolution of a product of light-ray operators with a Lorentz group Clebsch-Gordan coefficient: horizontal trajectories (HTs) [10]. In free theories, HTs constitute a family of operators with constant spin but arbitrary scaling dimension. On general grounds, one can see that such trajectories exist in virtually any CFT, play an essential role in the computation of Regge intercepts through mixing at finite coupling [10–12], and can even form the dominant contribution. Qualitatively—at least for gauge theories—they are similar to null Wilson lines [10,13,14]. Nevertheless, they are virtually unstudied [10].

One of those explicit cases is the minimal twist sector of $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM), where the leading BFKL Pomeron intercept receives contributions from a single HT mixing with the trajectories of the local twist-two operators and their shadows. Its value can be computed very efficiently using the quantum spectral curve (QSC), based on the integrability of (planar) $\mathcal{N} = 4$ SYM [15–17]. However, the minimal-twist sector enjoys many simplifications compared to the general theory, allowing for computational shortcuts that are absent when studying a generic scattering process.

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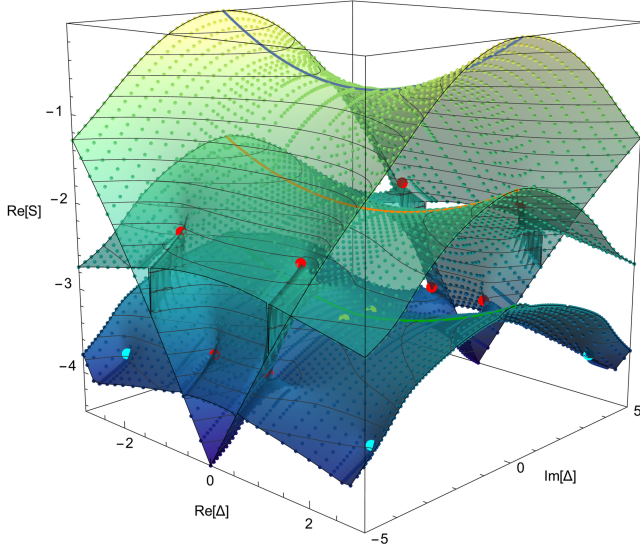


FIG. 1. Three sheets of the Riemann surface connecting the different twist trajectories at $g = 1/2$. Large colored dots indicate the branch points. The surface is generated with over 9000 points. The blue, orange, and green curves are twist-three and -five trajectories also appearing in Fig. 3 for different values of g .

In this Letter, we move beyond twist two and study a family of higher-twist states and their mixing with HTs, both perturbatively and nonperturbatively. Our numerics demonstrate how at finite coupling their Regge trajectories are connected to each other, forming a Riemann surface by complex spin continuation (see Fig. 1). Moreover, we show the degeneracy of two HTs, as $g \rightarrow 0$, explicitly for the first time, by computing the first order of the weak-coupling expansion of the intercept analytically. We resolve the mixing, caused by the contributions of the degenerate HTs, by modifying the QSC ansatz, and observe novel behavior: The whole BFKL regime, and accordingly the intercept, depends *linearly* on the coupling g . This is consistent with our numerical intercept computation, which also interpolates all the way to strong coupling.

This Letter also forms a stepping stone for the non-perturbative study of other relevant Regge trajectories, such as the odderon [18–21], the leading charge-conjugacy-odd contribution, which explains the discrepancy between hadron-hadron and hadron-antihadron scattering at high energies. Recent measurements [22–25] have confirmed the existence of this discrepancy. However, despite existing weak- [26] and strong-coupling approaches [27,28], our fundamental understanding of the odderon remains lacking and calls for a nonperturbative study into the mechanisms underlying it. With the established connections [17,29,30] between $\mathcal{N} = 4$ SYM and QCD in mind, this work forms a first step in that direction.

Twist-three states.—Our starting point are the superprimary states in planar $\mathcal{N} = 4$ SYM with classical twist $\Delta_0 - S = 3$. These are $\mathfrak{psu}(2,2|4)$ -highest weight states (in so-called ABA grading) built up from (single-trace)

operators containing three of the fundamental fields at tree level. There are infinitely many twist-three Regge trajectories, but with the data from [31] we select the lowest-lying family of such states, forming the trajectory with the expected largest intercept $\alpha = S(\Delta = 0)$ [32]. The local operators $\mathcal{O}_S$ ($S = 0, 2, 4, \dots$) on this trajectory have (Cartan) charges $(J_1, J_2, J_3, \Delta, S, S_2) = (3, 0, 0, 3 + S + \gamma, S, 0)$ with γ their anomalous dimension and are parity singlets [34]. The Regge behavior of the three-point couplings of this trajectory was studied recently in [35]. At tree level, they can be represented in the commonly used form $\mathcal{O}_S = \text{Tr}(D^S ZZZ) + \text{permutations}$ [36].

Quantum spectral curve.—The spectrum of single-trace operators in planar $\mathcal{N} = 4$ SYM is encoded in the QSC [45–47] (see also [31,48–53], and references therein). Each supermultiplet is associated to a certain solution of a set of coupled difference equations called the QQ system for eight functions $\mathbf{P}_a$ and $\mathbf{Q}_i$ (with $a, i = 1, \dots, 4$). To select the correct solution, one needs to specify boundary conditions in the form of analyticity conditions and power law behavior, depending on only the charges of the supermultiplet [36]. One can implement these conditions by using an ansatz for the $\mathbf{P}_a$:

$$\mathbf{P}_a = A_a (gx)^{-\tilde{M}_a} + (gx)^{-\Lambda} \sum_{n=1}^{\infty} \left\{ \frac{c_{1,n}}{x^{2n+1}}, \frac{c_{2,n}}{x^{2n}}, \frac{c_{3,n}}{x^{2n}}, \frac{c_{4,n}}{x^{2n-3}} \right\}, \quad (1)$$

where the $\tilde{M}_a$ and A_a are prescribed by the charges, $g = \sqrt{\lambda}/4\pi$ is the 't Hooft coupling, Λ is a convenient gauge choice, and the $c_{a,n}$ are parameters to be determined by a solution algorithm. The $\mathbf{P}_a$ have branch cuts as a function of the spectral parameter u between $\pm 2g$, encoded through the appearance of the Zhukovsky variable $x(u) = [(u + \sqrt{u^2 - 2g}\sqrt{u + 2g})/2g]$.

To solve the QSC [36], we use various equivalent reformulations: For analytic computations, we use the $\mathbf{P}\mu$ formulation [45] and compute either Δ perturbatively in the coupling g as a function of S or vice versa. For numerics, we use some of the relations from the QQ system. In both cases, we close the system by imposing the *gluing conditions* [51]. These conditions follow from analyticity properties of the $\mathbf{Q}_i$ and relate the continued function $\tilde{\mathbf{Q}}^i(u)$ to the first sheet $\mathbf{Q}_j(-u)$ through the *gluing matrix* M . For our chosen charges, they look as follows:

$$\begin{aligned} \tilde{\mathbf{Q}}^i(u) &= M^{ij}(u) \mathbf{Q}_j(-u), \\ \tilde{\mathbf{Q}}_i(u) &= -(M^{-1})_{ji}(u) \mathbf{Q}^j(-u). \end{aligned} \quad (2)$$

The minus sign in the lower equation is a consequence of the choice of $\Lambda = 3/2$ to be half-integer rather than integer and, therefore, differs from the Pomeron case [51].

To analytically continue in S , we need to include u dependence in the gluing matrix ansatz [17,51]. The minimal choice is

$$M = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 & 0 \\ \ell_2 & 0 & 0 & 0 \\ \ell_3 & 0 & \ell_4 & \ell_5 \\ 0 & 0 & \ell_5 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \ell_6 & 0 \\ 0 & 0 & 0 & 0 \\ \ell_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{2\pi u} + \begin{pmatrix} 0 & 0 & \ell_7 & 0 \\ 0 & 0 & 0 & 0 \\ \ell_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{-2\pi u}, \quad (3)$$

with ℓ_k parameters. This is corroborated by our analytic calculation for weak coupling close to the $\mathcal{O}_S$ local operators (cf. [36]).

Regge trajectories and Riemann surface.—The QSC allows for an efficient numerical algorithm to calculate the $S(\Delta)$ function [36]. Our strategy is to start from the weak-coupling data of the local operators $\mathcal{O}_2$ and $\mathcal{O}_4$ computed using [31] and then move along the Regge trajectory varying Δ to obtain S . Moreover, we can alter the value of the coupling g moving to the nonperturbative regime. This leads to the trajectories presented in the Chew-Frautschi plot in Fig. 2. These trajectories all pass through the BPS point (3,0) (A Bogomol'nyi–Prasad–Sommerfield (BPS) state preserves part of the supersymmetry in the theory.), belonging to the protected local operator $\mathcal{O}_0 = \text{Tr}(Z^3)$, and $(-3,0)$, belonging to its shadow [54].

For $g \ll 1$, Fig. 2 shows that the trajectory develops a plateau at $S = -2$ for $|\Delta| < 1$, resembling the Pomeron trajectory [15], and signals the presence of HTs [10]. To probe the mixing between the HTs and the twist-three trajectory, we map out the neighborhood of the plateau for complex Δ , finding several branch points (BPs) that

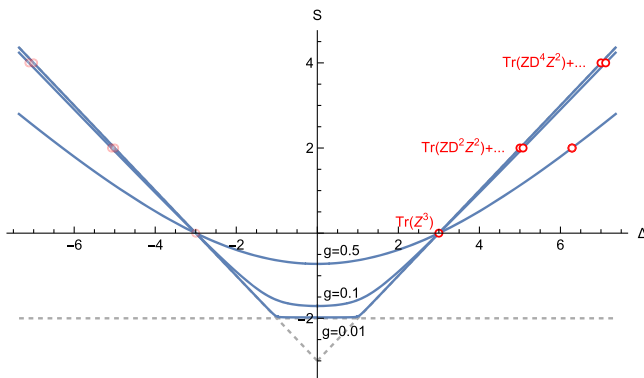


FIG. 2. Regge trajectories $S(\Delta)$ corresponding to the twist-three operators $\mathcal{O}_{0,2,4}$ (indicated by red circles, and transparent red circles for their shadows) for different values of g . Each trajectory is produced by about 400 points.

connect sheets of a nontrivial Riemann surface. The first three sheets of the surface are presented in Fig. 1 for $g = 1/2$ [56].

Numerical analysis shows second-order behavior for the monodromy around the BPs at finite g . Reducing g , the BPs approach the real (Δ, S) plane at certain integer (Δ, S) points from both sides and “cut” the Riemann surface into disconnected planes at $g = 0$. On the real (Δ, S) plane, the asymptotic lines for the trajectories are the HTs and the fixed twist lines. For finite g , these asymptotic lines mix due to the presence of the BPs and form the trajectories. This transition is presented in Fig. 3.

In particular, for $g = 0$ the red BPs sit at $(\pm 1, -2)$, the end points of the plateau. The mixing at finite g resolves two degenerate HTs and the twist-three line, being the first explicit occurrence of this phenomenon, forming the blue, orange, and green trajectories. The degeneracy of HTs plays an important role in the computation of the weak-coupling intercept and its novel behavior.

Moreover, the orange and green trajectories undergo further mixing at $(\pm 3, -2)$ including the cyan BPs [57], before asymptoting to the twist-five line. The mixing happens between at least four trajectories: the two HTs and the two twist-five trajectories. We can identify the local operators $\mathcal{O}_{\pm}$ belonging to these trajectories at $S = 2$ as being the only two twist-five parity singlet superprimaries with $g = 0$ charges $(3, 0, 0, 7, 2, 0)$ [58] (see Fig. 4 and [36]).

We expect similar behavior for mixing between the two HTs and odd twist- k trajectories around $(\pm k, -2)$. We see signs of mixing around $(0, -3)$ as well (green BPs), although preliminary calculations show that there are no HTs at $S = -3$ (and, in fact, not for any odd S).

Resolution of the HT degeneracy.—The weak-coupling intercept $\alpha = S(0)$ of the blue trajectory can be computed using QSC while considering S as a function of Δ and focusing on finding solutions close to $(\Delta, S) \sim (0, -2)$. However, we have to be careful, since for $g = 0$ also the orange trajectory passes through $(0, -2)$, and we have to account for this degeneracy. For the perturbative calculation in g , we take the ansatz

$$S = -2 + \sum_{i=1}^{\infty} I_i(\Delta) g^i \quad (4)$$

and fix the remaining charges as $(J_1, J_2, J_3, S_2) = (3, 0, 0, 0)$. We stress the novel inclusion of odd powers of g to the intercept. In [31], degeneracy observed in the $\mathbf{P}_a$ functions was resolved by including odd powers of g in the $\mathbf{P}_a$ ansatz, but this did not result in odd powers in the resulting $S(\Delta)$. In our case, the trajectories (and, consequently, the intercept) are degenerate at weak coupling for $|\Delta| < 1$ as shown in Fig. 3, and we allow for odd g dependence in all QSC quantities to give room for the description of new behavior. Let us focus on the leading order of (4). To compute I_1 , we solve the QSC [36], along

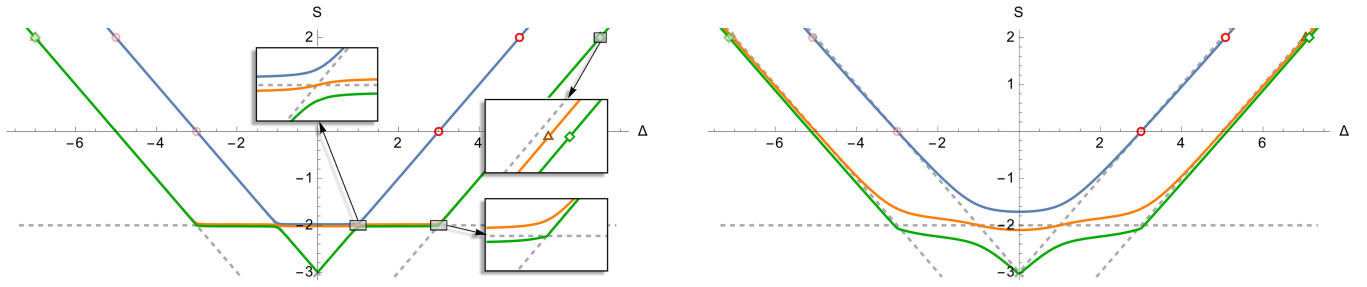


FIG. 3. The intersection of the Riemann surface $S(\Delta)$ with the real (Δ, S) plane for $g = 1/100$ (left) and $g = 1/10$ (right). The blue curve is the Regge trajectory appearing in Fig. 2; the orange and green curves belong to the twist-five trajectories accessed by continuation on the Riemann surface. The dashed lines show the zero-coupling limit of the curves. Local operators are indicated by markers, and shadows by their transparent counterparts.

similar steps as in the calculation of the Pomeron intercept [15]. First, given the ansatz (1) and the QSC equations in the $\mathbf{P}_\mu$ formulation, we fix the $\mathbf{P}_a$ solutions up to next-to-leading order (NLO) in g . Then, from these solutions, we derive a fourth-order difference equation for the $\mathbf{Q}_i$ functions, which factorizes to give a second-order one for $\mathbf{Q}_1$ and $\mathbf{Q}_3$ [59] as follows:

$$\begin{aligned} & \frac{2(u+i) + igI_1}{(u+i)^{3/2}} \mathbf{Q}_i(u+i) + \frac{2(u-i) - igI_1}{(u-i)^{3/2}} \mathbf{Q}_i(u-i) \\ & - \frac{1 + 8u^2 - \Delta^2 + 2gI_1}{2u^{5/2}} \mathbf{Q}_i(u) = 0. \end{aligned} \quad (5)$$

We solve (5) through a Mellin transformation [4,5] and with the solution of the generalized hypergeometric differential equation [37]. The solutions up to NLO for $\mathbf{Q}_{1,3}$ are analytic in the upper half plane, have the desired asymptotic behavior for large u , and have only I_1 as a free parameter. To fix I_1 , we impose the gluing conditions (2), which simplify to

$$\tilde{\mathbf{Q}}_1(u) = -\mathbf{Q}_3(-u)/\ell_2 \quad \text{and} \quad \tilde{\mathbf{Q}}_3(u) = \mathbf{Q}_1(-u)\ell_2, \quad (6)$$

as well as require a regularity condition on the real axis. Because of the charges, the $\mathbf{Q}_{1,3}$ functions have an extra $\sqrt{u}$ branch cut at small coupling in addition to the Zhukovsky cut. Hence, rather than the usual combination $\mathbf{Q}_i + \tilde{\mathbf{Q}}_i$, we must demand that

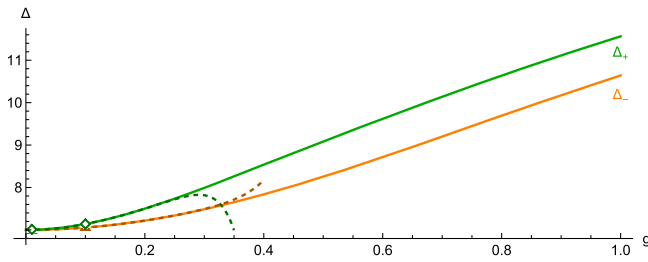


FIG. 4. The scaling dimension of $\mathcal{O}_\pm$ appearing on the orange and green trajectories in Fig. 3 (markers represent the same value of $\Delta_\pm$ in Fig. 3), with their weak-coupling predictions obtained with [31] (dashed line).

$$\mathbf{Q}_i \sqrt{x}^{-1} + \tilde{\mathbf{Q}}_i \sqrt{x} \quad (7)$$

is analytic on the real axis or, equivalently, that it is regular around $u \sim 0$ at small coupling [61]. This, combined with (6), leads to the consistency equation $I_1(\Delta)^2 = 4$, yielding two solutions with $I_1(\Delta) = \pm 2$ that, interestingly, are independent of the value of Δ . This matches our numerical observations: Each solution belongs to one of the degenerate HTs, and, as anticipated in (4), the degeneracy of the intercept is resolved at linear order in g .

Numerical intercept.—Finally, using our numerical algorithm, we test our prediction computing the nonperturbative intercept $\alpha(g)$ for a wide range of values of the coupling as presented in Fig. 5 [62]. The curve nicely interpolates between our weak-coupling one-loop computation and the six-loop strong-coupling expansion derived in [28,48] for twist $\tau = 3$. We fit a curve to our data, which has 60–80 digits of precision, and obtain

$$\begin{aligned} \alpha = & -2 + 2g + 16 \log 2 g^2 - \frac{2\pi^2}{3} g^3 \\ & - 204.773\,771\,582\,926\,61 g^4 + 136.293\,336\,388\,13 g^5 \\ & + 4733.390\,789\,74 g^6 - 6116.795\,85 g^7 + \dots, \end{aligned} \quad (8)$$

where the transcendental coefficients of the g^2 and g^3 terms fit with at least 17 significant digits to the found numerical

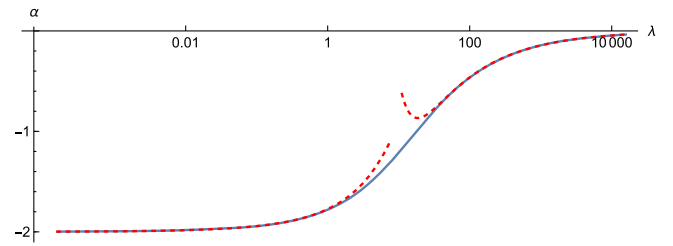


FIG. 5. The nonperturbative intercept as a function of the coupling $\lambda = 16\pi^2 g^2$ (we consider the range $g \in [10^{-3}, 10]$). The solid line is obtained by our numerical procedure (around 250 points), while the dashed lines are the perturbative predictions.

values while the other terms are presented with their significant digits.

Notice that the g^2 term coincides with the $\Delta = 0$ value of the well-known Pomeron eigenvalue [29,63,64]. Indeed, probing the expansion of $S(\Delta)$ for several values of the scaling dimension in the region $|\Delta| < 1$, we find the familiar form for the g^2 coefficient $-4(\psi[(1-\Delta)/2] + \psi[(1+\Delta)/2] - 2\psi(1))$, where ψ is the digamma function.

Furthermore, in addition to the linear term, the g^3 coefficient is also independent of Δ . Hence, it is natural to ask if all the odd powers of g have this feature, but our precision is not enough to corroborate it.

Discussion.—In this Letter, we showed that, contrary to all other known cases for $\mathcal{N} = 4$ SYM theory, the intercept (8) can have a linear dependence on the coupling g , which has important consequences.

First, it facilitates a novel, fully nonperturbative relationship between the two trajectories that are degenerate at $g = 0$: Starting from a point on the blue trajectory in the $|\Delta| < 1$ region, the analytic continuation $g \mapsto -g$ and the monodromy around the BP at $(\pm 1, -2)$ at fixed g lead to the same point on the orange trajectory. This means these two trajectories and their intercepts are related by the $g \mapsto -g$ exchange as long as $|\Delta| < 1$.

Second, in conformal Regge theory [6], the defining equation for the intercept is $\Delta(S) = 0$ [32]. Inverting this relation, one can hope to relate the perturbative Laurent expansions of the anomalous dimension and the intercept. This approach was successful for the Pomeron trajectory (see, e.g., [65]); however, it breaks down in our case. The perturbative anomalous dimensions in $\mathcal{N} = 4$ SYM are a function of g^2 ; hence, the order by order inversion does not reproduce the linear g dependence of the intercept. This shows the importance of using nonperturbative tools for Regge spectroscopy. The QSC formalism is a viable candidate for this, and this Letter also presents, through numerical optimizations and the revised regularity condition (7), a step forward in making it applicable to more general trajectories. Although the full Riemann surface is much larger than we have explored in this Letter, we observe several of its features that seem to be true more generally: Trajectories do not seem to cross or diverge, and, in the $g \rightarrow 0$ limit, all trajectories asymptote to horizontal or fixed-twist lines. We also note that all the local operators we discussed on the Riemann surface have the same J_1 , J_2 , J_3 , and S_2 charges, have even spin, and are all parity singlets, corroborating the conjecture from [10] that, in a generic CFT, *all* states with the same global symmetries (including the discrete ones) are part of the same Riemann surface.

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- [54] Around these points, one can study the near-BPS limit of the expansion, i.e., the Taylor series of Δ with respect to S : $\Delta = 3 + (1 + f_1)S + f_2S^2 + \dots$, with f_1 the slope [55] and f_2 the curvature [48] functions, both known exactly. Fitting our data in Fig. 2, we reproduce f_1 and f_2 for $g = 1/100, 1/10, 1/2$ with eight and seven digits of precision, respectively.
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- [57] In Fig. 1, the only visible cyan BPs are on the third sheet. However, they appear also in the second sheet but outside the region of our plot.
- [58] We note that at tree level $\mathcal{O}_{\pm}$ are long linear combinations of twist-five elementary fields and their derivatives and are not simply a combination of $\text{Tr}D^2ZZZZZ$ - and $\text{Tr}DZDZZZZ$ -type states [36]. The protected operator $\text{Tr}(Z^5)$ is also not on either trajectory.
- [59] Equation (5) was first derived by Alfimov [60] and independently rederived by the authors. We thank him for sharing his result.
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- [61] We thank N. Gromov for discussions on this point.
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