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Topological classes of thermodynamics of the static multi-charge AdS black holes in gauged supergravities: novel temperature-dependent thermodynamic topological phase transition

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Abstract: In this paper, we investigate, in the framework of the topological approach to black hole thermodynamics, using the generalized off-shell Helmholtz free energy, the topological numbers of the static multi-charge AdS black holes in four- and five-dimensional gauged supergravities. We find that the topological number of the static-charged AdS black holes in four-dimensional Kaluza-Klein (K-K) gauged supergravity theory is $W = 0$, while that of the static-charged AdS black holes in four-dimensional gauged $-iX^0X^1$-supergravity and STU gauged supergravity theories, and five-dimensional Einstein-Maxwell-dilaton-axion (EMDA) gauged supergravity and STU gauged supergravity, and five-dimensional static-charged AdS Horowitz-Sen black hole are both $W = 1$. Furthermore, we observe a novel temperature-dependent thermodynamic topological phase transition that can happen in the four-dimensional static-charged AdS black hole in EMDA gauged supergravity theory, the four-dimensional static-charged AdS Horowitz-Sen black hole, and the five-dimensional static-charged AdS black hole in K-K gauged supergravity theory. We believe that the novel temperature-dependent thermodynamic topological phase transition could help us better understand black hole thermodynamics and, further, shed new light on the fundamental nature of gauged supergravity theories.

Keywords: Black Holes, Black Holes in String Theory

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Contents

1 Introduction

The discovery of the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [1–3] has attracted a great deal of interest in studying the thermodynamic properties of charged AdS black holes in four- and five-dimensional gauged supergravities [4–23]. In fact, the establishment of the three widely accepted thermodynamic mass formulas, i.e., the first law of black hole thermodynamics [24, 25], the Bekenstein-Smarr mass formula [26], and the Christodoulou-Ruffini squared-mass formula [27, 28], is not the only facet of the investigation of black hole thermodynamics.

Recently, topology has received considerable interest and enthusiasm as an important mathematical tool applicable to black hole physics. There are two important aspects to the topology research underway at present. One area of investigation focuses on the light rings [29–35] corresponding to some black holes, which could present more support for black hole observation in the future and has been expanded to timelike circular orbits [36, 37]. Another area of investigation focuses on the thermodynamic properties of black holes [38–59]. Extraordinarily, a new method for exploring the thermodynamic topological features of black holes has developed, as pointed out in ref. [60]. This method treats black hole solutions as topological thermodynamic defects, calculates topological numbers, and then classifies black holes into three different categories according to their topological numbers. This breakthrough work has provided novel insights into the fundamental nature of black holes and gravity. The thermodynamic topological procedure presented in ref. [60] has achieved considerable popularity because of its broad application and convenience. Therefore, it has been effectively applied to explore the topological numbers corresponding to various famous black holes [61–95]. However, the topological classes of the multi-charge static AdS black holes in four- and five-dimensional gauged supergravities are still unknown and merit further exploration, since the structure of the metric, though spherically symmetric, is notably different from that of the corresponding Reissner-Nordström-AdS (RN-AdS) cases. Hence the reason why we undertake the present paper.

In this paper, we shall investigate the topological numbers of the static multi-charge AdS black holes in four- and five-dimensional gauged supergravity theories. In the context of gauged supergravity theory, static-charged AdS black holes in four and five dimensions have four and three independent electric charge parameters, respectively. For each of these black holes, we examine various electric charge parameter configurations and explore their impact on the thermodynamic topological classification of these black holes. We find that, in four-dimensional spacetimes, for two nonzero electric charge parameters (the other two being zero), the thermodynamic topological number is temperature dependent: it is $W = 1$ for two large electric charge parameters, but for two small electric charge parameters, it can be $W = 0$ (at cold temperatures) or $W = 1$ (at high temperatures). Likewise, in five-dimensional spacetimes, we also find that the static-charged AdS black hole in Kaluza-Klein (K-K) supergravity theory has $W = 1$ if the electric charge parameter is large, but if the electric charge parameter is small, then the topological number W exhibits a similar temperature dependence, i.e., $W = 0$ (at cold temperatures) or $W = 1$ (at high temperatures). In other words, we observe a kind of novel temperature-dependent thermodynamic topological phase transition.

The remaining part of this paper is organized as follows. In section 2, we present a brief review of the thermodynamic topological method proposed in ref. [60]. In section 3, we examine the topological numbers of four-dimensional static multi-charge AdS black holes in gauged supergravity theory [4] for several different combinations of electric charge parameters, and we address each case separately in six subsections. In section 4, we investigate the topological numbers of five-dimensional static multi-charge AdS black holes in gauged supergravity theory [5] with various distinct combinations of electric charge parameters and separate our discussion of each case into five subsections. Finally, our conclusions and outlooks are given in section 5.

2 A brief review of thermodynamic topological method

In this section, we present a brief review of the novel thermodynamic topological method proposed in ref. [60]. As stated in ref. [60], we start by introducing the generalized off-shell Helmholtz free energy

$$\mathcal{F} = M - \frac{S}{\tau} \tag{2.1}$$

for the black hole thermodynamic system with mass M and Bekenstein-Hawking entropy S, the extra variable τ can be treated as the inverse temperature of the cavity enclosing the black hole. Only in the case of $\tau = 1/T$ does the generalized Helmholtz free energy exhibit on-shell features and return to the standard Helmholtz free energy $F = M - TS$ of the black hole [96–99].

In ref. [60], the essential vector ϕ is defined as

$$\phi = \left(\frac{\partial \mathcal{F}}{\partial r_h}, \quad -\cot\Theta \csc\Theta\right), \tag{2.2}$$

where r_h is the event horizon radius of the black hole, Θ is an extra factor, and $\Theta \in [0, +\infty]$. It is worth noting that the component ϕ^{Θ} diverges at $\Theta = 0$ and $\Theta = \pi$, demonstrating that the vector has an outward direction in both of these cases.

In order to build a topological current, one can employ Duan's theory [100–102] on ϕ-mapping topological currents as follows:

$$j^{\mu} = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\epsilon_{ab}\partial_{\nu}n^{a}\partial_{\rho}n^{b}, \qquad \mu, \nu, \rho = 0, 1, 2, \tag{2.3}$$

where $\partial_{\nu} = \partial/\partial x^{\nu}$ and $x^{\nu} = (\tau,\ r_h,\ \Theta)$. The normalized vector is formulated as $n = (n^r, n^{\Theta})$ with $n^r = \phi^{r_h}/||\phi||$ and $n^{\Theta} = \phi^{\Theta}/||\phi||$. It is simple to verify that this topological current is conserved

$$\partial_{\mu}j^{\mu} = 0\,. \tag{2.4}$$

Using the three-dimensional Jacobian tensor $\epsilon^{ab}J^{\mu}(\phi/x) = \epsilon^{\mu\nu\rho}\partial_{\nu}\phi^{a}\partial_{\rho}\phi^{b}$, we can describe the topological current as a δ-function of the field configuration [31, 101, 102]

$$j^{\mu} = \delta^{2}(\phi)J^{\mu}\left(\frac{\phi}{x}\right)\,. \tag{2.5}$$

This argument clearly shows that j^{μ} is nonzero only at the zero points of $\phi^{a}(x_i)$, i.e., $\phi^{a}(x_i) = 0$. Finally, the topological number at the given parameter region Σ can be determined by utilizing the following formula:

$$W = \int_{\Sigma} j^{0} d^{2}x = \sum_{i=1}^{N}\beta_i\eta_i = \sum_{i=1}^{N} w_i\,, \tag{2.6}$$

where the positive Hopf index β_i denotes the number of loops made by ϕ^{a} in the vector ϕ-space as x^{μ} moves around the zero point z_i, the Brouwer degree $\eta_i = \mathrm{sign}(J^{0}(\phi/x)_{z_i}) = \pm 1$, and w_i is the winding number for the ith zero point of ϕ. In addition, if two different closed curves, Σ_1 and Σ_2, intersect at the same zero point of ϕ, their winding numbers must be the same; if there is no zero point of ϕ within the enclosed region, then the topological number $W = 0$.

It is worth mentioning that the local winding number w_i is a key instrument for determining local thermodynamical stability. Positive w_i values indicate thermodynamically stable black holes, while negative values indicate unstable ones. The global topological number W denotes the difference between the numbers of thermodynamically stable and unstable black holes with a black hole solution at a given temperature.

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3 Four-dimensional static multi-charge AdS black holes in gauged supergravity theory

In this section, we will investigate the topological numbers of the four-dimensional static multi-charge AdS black holes in gauged supergravity theory [4]. For the general static four-charge AdS black hole in four-dimensional STU gauged supergravity theory, whose metric, Abelian gauge potentials, and scalar fields are [4]

$$
\begin{aligned}
ds_4^2 &= -\prod_{i=1}^{4} H_i^{-1/2} f dt^2 + \prod_{i=1}^{4} H_i^{1/2}\Big(f^{-1}dr^2 + r^2 d\Omega_2^2\Big)\,, \\
A^i &= \frac{\sqrt{q_i(q_i+2m)}}{2(r+q_i)}\,dt\,, \qquad X_i = H_i^{-1}\prod_{j=1}^{4} H_j^{1/4}\,,
\end{aligned}
\tag{3.1}
$$

where

$$
f = 1 - \frac{2m}{r} + \frac{r^2}{l^2}\prod_{i=1}^{4} H_i\,, \qquad H_i = 1 + \frac{q_i}{r}\,, \tag{3.2}
$$

in which l is the AdS radius, m and q_i are the mass and four independent electric charge parameters, respectively.

For the static charged AdS black hole metric described by (3.1), the most general case is that of a solution possessing four independent electric charge parameters. In addition, according to the classification of black hole solutions in figure 1 of ref. [103], there are several special truncated supergravity solutions: for example, when the electric charge parameters $q_1 = q_2$ and $q_3 = q_4$, this is known as the static charged AdS black hole solution in gauged $-iX^0X^1$-supergravity theory [104] (namely, the static pairwise-equal four-charge AdS black hole case); when $q_1 \neq q_2 \neq 0$, $q_3 = q_4 = 0$, namely the four-dimensional static charged AdS Horowitz-Sen black hole solution [105, 106]; when $q_1 = q_2 \neq 0$ and $q_3 = q_4 = 0$, i.e., the static charged AdS black hole solution in Einstein-Maxwell-dilaton-axion (EMDA) gauged supergravity theory [107]; and when $q_1 \neq 0$ and $q_2 = q_3 = q_4 = 0$, i.e., the static charged AdS black hole solution in K-K gauged supergravity; and when $q_1 = q_2 = q_3 = q_4 \neq 0$, which is the familiar RN-AdS black hole case after the coordinate transformation by $\rho = r + q_i$; and so on.

The thermodynamic quantities are given by [7]

$$
\begin{aligned}
M &= m + \frac{1}{4}\sum_{i=1}^{4} q_i\,, & Q_i &= \frac{1}{2}\sqrt{q_i(q_i+2m)}\,, & S &= \pi\prod_{i=1}^{4}(r_h+q_i)^{1/2}\,, \\
T &= \frac{f'(r_h)}{4\pi}\prod_{i=1}^{4} H_i^{-1/2}(r_h)\,, & \Phi_i &= \frac{\sqrt{q_i(q_i+2m)}}{2(r_h+q_i)}\,, & P &= \frac{3}{8\pi l^2}\,, \\
V &= \frac{\pi r_h^3}{3}\prod_{i=1}^{4} H_i(r_h)\sum_{j=1}^{4}\frac{1}{H_j(r_h)}\,.
\end{aligned}
\tag{3.3}
$$

It is easy to verify that these mentioned thermodynamic quantities simultaneously satisfy the first law and the Bekenstein-Smarr mass formula

$$
dM = TdS + \sum_{i=1}^{4}\Phi_i dQ_i + VdP\,, \tag{3.4}
$$

$$
M = 2TS + \sum_{i=1}^{4}\Phi_i Q_i - 2VP\,. \tag{3.5}
$$

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Utilizing the definition of the generalized off-shell Helmholtz free energy (2.1) and substituting the relation $l^2 = 3/(8\pi P)$ [7, 108, 109], one can easily obtain

$$\mathcal{F} = \frac{r_h}{2} + \frac{1}{4}\sum_{i=1}^{4} q_i + \frac{4\pi P}{3r_h}\prod_{i=1}^{4}(r_h + q_i) - \frac{\pi}{\tau}\prod_{i=1}^{4}\sqrt{r_h + q_i} \tag{3.6}$$

for the static four-charge AdS black hole in four-dimensional gauged supergravity. Then the components of the vector ϕ can be derived as

$$\begin{aligned}
\phi^{r_h} &= \frac{1}{2} + \frac{4\pi P}{3}\left[q_2q_3 + (q_2+q_3)q_4 + q_1(q_2+q_3+q_4) - \frac{\prod_{i=1}^4 q_i}{r_h^2} + 2r_h\sum_{i=1}^{4} q_i + 3r_h^2\right] \\
&\quad - \frac{\pi}{6\tau\prod_{i=1}^4\sqrt{r_h+q_i}}\Bigg\{12r_h^3 + 9r_h^2\sum_{i=1}^{4} q_i + 6r_h\Big[q_3q_4 + q_2(q_3+q_4) + q_1(q_2+q_3+q_4)\Big] \\
&\quad + 3\Big[q_1q_2q_3 + q_2q_3q_4 + q_1(q_2+q_3)q_4\Big]\Bigg\}, \qquad (3.7)\\
\phi^{\Theta} &= -\cot\Theta\csc\Theta\,. \qquad (3.8)
\end{aligned}$$

By solving the equation: $\phi^{r_h} = 0$, one can compute the zero point of the vector field ϕ^{r_h} as

$$\tau = \frac{3\pi r_h^2\left[q_1q_2q_3 + q_1q_2q_4 + q_1q_3q_4 + q_2q_3q_4 + 2Xr_h + 3r_h^2\sum_{i=1}^4 q_i + 4r_h^3\right]}{\prod_{i=1}^4\sqrt{r_h+q_i}\Big\{3r_h^2 + 8\pi P\left[-\prod_{i=1}^4 q_i + Xr_h^2 + 2r_h^3\sum_{i=1}^4 q_i + 3r_h^4\right]\Big\}}, \tag{3.9}$$

where

$$X = q_3q_4 + q_2(q_3+q_4) + q_1(q_2+q_3+q_4)\,.$$

Note that eq. (3.9) consistently reduces to the result obtained in the case of the four-dimensional Schwarzschild-AdS black hole [64] when the four independent electric charge parameters q_i vanish. Due to the requirement of considering four independent electric charge parameters, different values of these electric charge parameters correspond to distinct black hole solutions within various truncated supergravity theories. Therefore, we will explore the topological numbers of static charged AdS black holes in several special supergravity theories, respectively.

3.1 $q_1 \neq 0$, $q_2 = q_3 = q_4 = 0$ case (K-K gauged supergravity)

In this subsection, we focus on the case where $q_1 \neq 0$ and $q_2 = q_3 = q_4 = 0$, which corresponds to the static charged AdS black hole in four-dimensional K-K gauged supergravity theory. For the four-dimensional static charged AdS black hole in K-K gauged supergravity theory, one can plot the zero points of the component ϕ^{r_h} with $Pr_0^2 = 0.1$, $q_1/r_0 = 2$, and $q_2 = q_3 = q_4 = 0$ (the four electric charge parameters act equivalently) in figure 1, and the unit vector field n on a portion of the $\Theta - r_h$ plane in figure 2 with $\tau/r_0 = 3.5$. Here, r_0 represents an arbitrary length scale defined by the size of a cavity around the static charged AdS black hole in four-dimensional K-K gauged supergravity theory. Figure 1 illustrates that for $\tau < \tau_a = 3.68r_0$, there exist two four-dimensional static charged AdS black holes in K-K gauged supergravity: one thermodynamically stable and one thermodynamically unstable.

In figure 2, two zero points are located at $(r_h/r_0, \Theta) = (0.15, \pi/2)$, and $(0.44, \pi/2)$, respectively. The winding numbers w_i for the blue contours C_i can be characterized as

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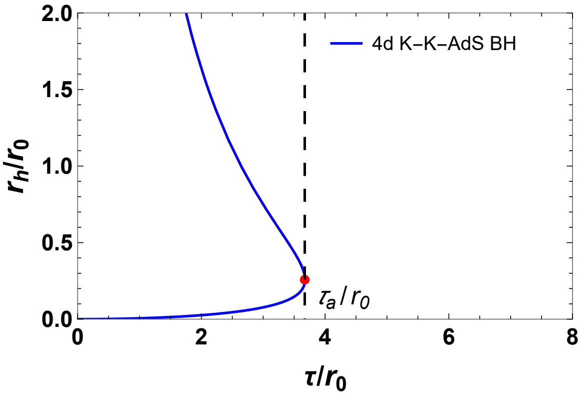

Figure 1. Zero points of the vector ϕ^{r_h} shown in the $r_h - \tau$ plane with $q_1/r_0 = 2$, $Pr_0^2 = 0.1$, and $q_2 = q_3 = q_4 = 0$. There is one thermodynamically stable and one thermodynamically unstable four-dimensional static charged AdS black hole in K-K gauged supergravity theory for $\tau < \tau_a = 3.68r_0$. Obviously, the topological number is: $W = 1 - 1 = 0$.

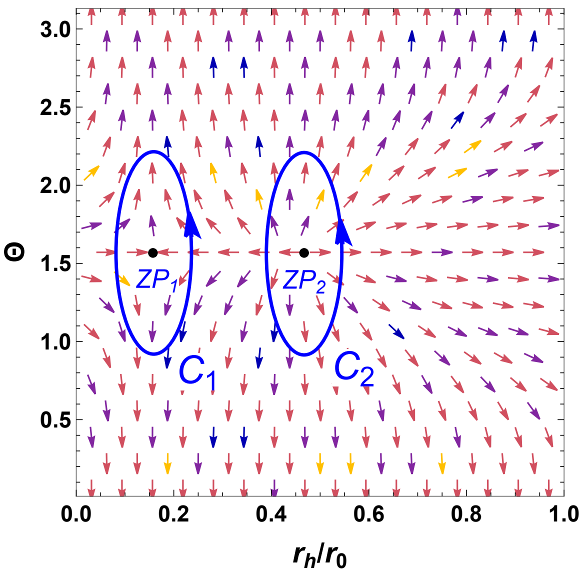

Figure 2. The arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane for the four-dimensional static charged AdS black hole in K-K gauged supergravity theory with $\tau/r_0 = 3.5$, $q_1/r_0 = 2$, $Pr_0^2 = 0.1$, and $q_2 = q_3 = q_4 = 0$. The zero points (ZPs) marked with black dots are at $(r_h/r_0, \Theta) = (0.15, \pi/2)$, and $(0.44, \pi/2)$, respectively. The blue contours C_i are closed loops enclosing the zero points.

$w_1 = -1$ and $w_2 = 1$, which differ from the four-dimensional RN-AdS black hole (which only has $w_1 = 1$). Therefore, the topological number $W = 0$ for the four-dimensional static charged AdS black hole in K-K gauged supergravity theory is easily noticed in figure 2, distinguishing it from the topological number of the four-dimensional RN-AdS black hole ($W = 1$) [60]. It implies that the topological number are significantly affected by the number of the electric charge parameters.

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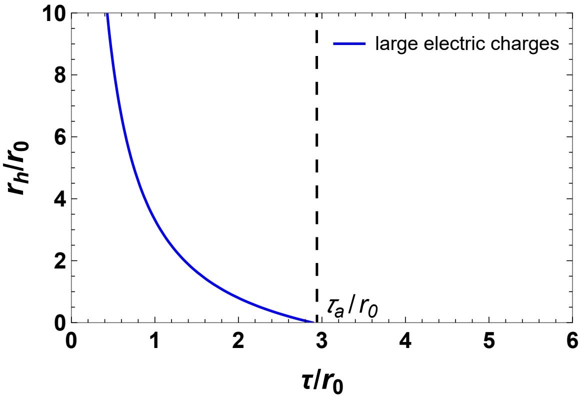

Figure 3. Zero points of the vector ϕ^{r_h} shown in the $r_h - \tau$ plane with $q_1/r_0 = q_2/r_0 = 2$, $Pr_0^2 = 0.1$, and $q_3 = q_4 = 0$. There is one thermodynamically stable four-dimensional static charged AdS black hole in EMDA gauged supergravity theory for $\tau < \tau_a = 2.89r_0$. Obviously, the topological number is: $W = 1$.

3.2 $q_1 = q_2 \neq 0$, $q_3 = q_4 = 0$ case (EMDA gauged supergravity)

In this subsection, we discuss the case where $q_1 = q_2 \neq 0$, $q_3 = q_4 = 0$, corresponding to the static charged AdS black hole in four-dimensional EMDA gauged supergravity theory. For the four-dimensional static charged AdS black hole in EMDA gauged supergravity theory, we find that different values of the two identical electric charge parameters also influence their topological numbers, which is a new property of this black hole solution (in sections 3.3 and 4.1, we will find that the four-dimensional static charged AdS Horowitz-Sen black hole solution and the five-dimensional static charged AdS black hole solution in K-K gauged supergravity theory possess similar properties). Therefore, we next discuss each of the three cases by taking larger, smaller, and critical values of two equal electric charge parameters.

3.2.1 Large values of two identical electric charge parameters

We first consider the case where two equal electric charge parameters take larger values. We plot the zero points of the component ϕ^{r_h} with $Pr_0^2 = 0.1$, $q_1/r_0 = q_2/r_0 = 2$, and $q_3 = q_4 = 0$ in figure 3, and the unit vector field n in figure 4 with $\tau/r_0 = 1$. Note that for these values of Pr_0^2, q_1/r_0 and q_2/r_0, there is one thermodynamically stable four-dimensional static charged AdS black hole in EMDA gauged supergravity theory if $\tau < \tau_a = 2.89r_0$. In figure 4, one can observe that the zero point is located at $(r_h/r_0, \Theta) = (3.32, \pi/2)$. Therefore, the topological number $W = 1$ for the above black hole can be clearly found in figures 3 and 4 by applying the local property of the zero points, which is the same as that of the four-dimensional RN-AdS black hole [60], but different from that of the four-dimensional static charged AdS black hole in K-K gauged supergravity theory in the previous subsection, which is $W = 0$.

3.2.2 Small values of two identical electric charge parameters and the temperature-dependent thermodynamic topological phase transition

Then, we consider the case where two identical electric charge parameters take smaller values. Taking $q_1/r_0 = q_2/r_0 = 0.2$, $q_3 = q_4 = 0$, and $Pr_0^2 = 0.1$, we plot the zero points of the

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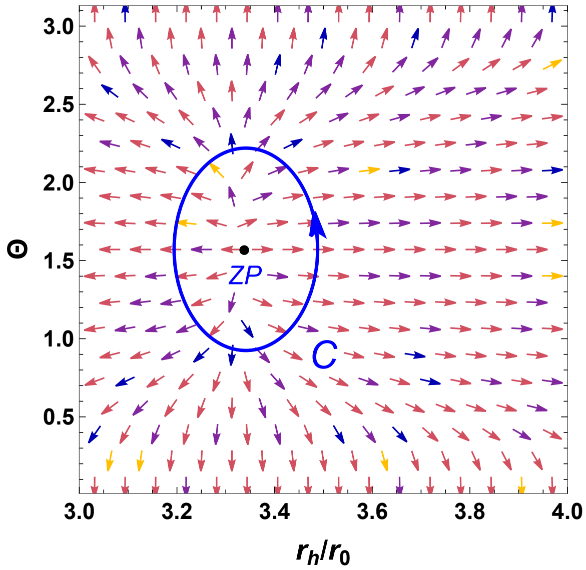


Figure 4. The arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane for the four-dimensional static charged AdS black hole in EMDA gauged supergravity theory with $\tau/r_0 = 1$, $q_1/r_0 = q_2/r_0 = 2$, $Pr_0^2 = 0.1$, and $q_3 = q_4 = 0$. The zero point (ZP) marked with a black dot is at $(r_h/r_0, \Theta) = (3.32, \pi/2)$. The blue contour C is a closed loop enclosing the zero point.

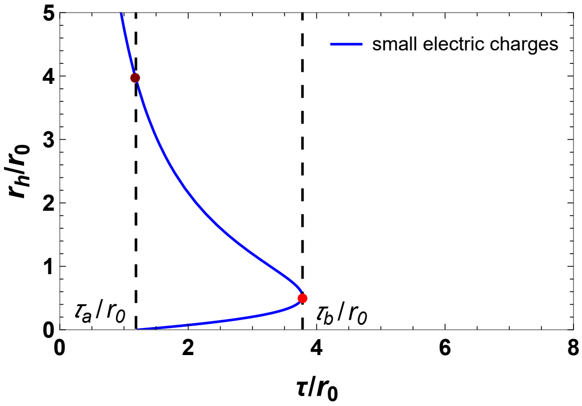
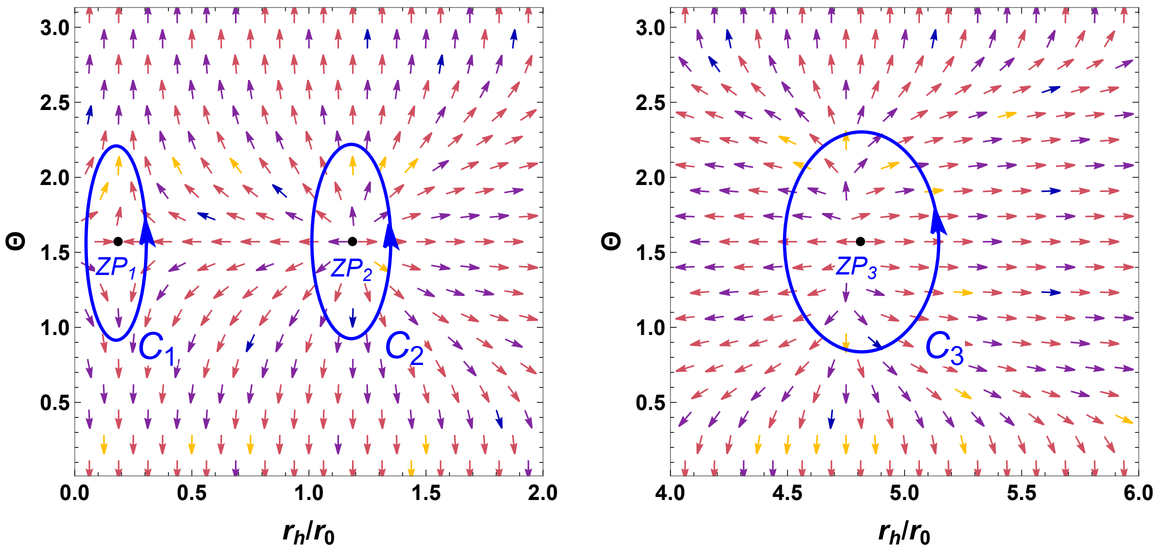


Figure 5. Zero points of the vector ϕ^{r_h} shown in the $r_h - \tau$ plane with $q_1/r_0 = q_2/r_0 = 0.2$, $q_3 = q_4 = 0$, and $Pr_0^2 = 0.1$. There are one thermodynamically stable and one thermodynamically unstable four-dimensional static charged AdS black hole in EMDA gauged supergravity theory for $1.22r_0 = \tau_a < \tau < \tau_b = 3.78r_0$, and one thermodynamically stable four-dimensional static charged AdS black hole in EMDA gauged supergravity theory for $\tau < \tau_a = 1.22r_0$.

component ϕ^{r_h} in figure 5, and the unit vector field n on a portion of the $\Theta - r_h$ plane with $\tau = 3r_0, r_0$ in figure 6, respectively. With the help of figure 5, it is easy to figure out that in four-dimensional EMDA gauged supergravity theory, for $1.22r_0 = \tau_a < \tau < \tau_b = 3.78r_0$, there are one thermodynamically stable and one thermodynamically unstable black hole branch, and one thermodynamically stable black hole branch for $\tau < \tau_a = 1.22r_0$. Therefore, the local property of the above black hole for these values of parameters is different from that of the four-dimensional RN-AdS black holes [60].



(a) The unit vector field for the four-dimensional static charged AdS black hole in EMDA gauged supergravity theory with $\tau/r_0 = 3$, $q_1/r_0 = q_2/r_0 = 0.2$, $q_3 = q_4 = 0$, and $Pr_0^2 = 0.1$.

(b) The unit vector field for the four-dimensional static charged AdS black hole in EMDA gauged supergravity theory with $\tau/r_0 = 1$, $q_1/r_0 = q_2/r_0 = 0.2$, $q_3 = q_4 = 0$, and $Pr_0^2 = 0.1$.

Figure 6. The arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane. The zero points (ZPs) marked with black dots are at $(r_h/r_0, \Theta) = (0.20, \pi/2)$, $(1.20, \pi/2)$, $(4.75, \pi/2)$, for ZP_1, ZP_2, and ZP_3, respectively. The blue contours C_i are closed loops surrounding the zero points.

Though $(r_h/r_0, \Theta) = (0.20, \pi/2)$ and $(1.20, \pi/2)$, respectively, are the locations of the zero points in figure 6(a), for the blue contours C_i, one can therefore interpret the winding numbers w_i: $w_1 = -1$, $w_2 = 1$, so that the topological number at this inverse temperature $\tau = 3r_0$ is $W = -1 + 1 = 0$; However, $(r_h/r_0, \Theta) = (4.75, \pi/2)$ is where the zero point is found in figure 6(b), hence the topological number at this inverse temperature $\tau = r_0$ is $W = 1$ since the winding number for the blue contour C_3 is $w_3 = 1$. Thus, we find that the topological number is temperature dependent: it is $W = 0$ (at inverse temperature $1.22r_0 = \tau_a < \tau < \tau_b = 3.78r_0$) or $W = 1$ (at inverse temperature $\tau < \tau_a = 1.22r_0$). At the point of the critical inversion temperature $\tau = \tau_a$, the black hole occurs a novel temperature-dependent thermodynamic topological phase transition. As can be seen from the figure 5, the function τ is smooth and continuous at the critical point τ_a, and there is no extreme point nearby τ_a, so the above temperature-dependent thermodynamic topological phase transition is supposed to be a thermodynamic topological higher-order phase transition (continuous phase transition). It is regrettable that we have not yet been able to find an effective method to analyze this higher-order thermodynamic topological phase transition.

3.2.3 Critical values of two identical electric charge parameters

The analysis conducted in the preceding two subsubsections indicates that the topological number assumes the value of unity when the magnitudes of the two equivalent electric charge parameters take larger values. Conversely, when these parameters take smaller values, the topological number is temperature-dependent: it is $W = 0$ (at cold temperatures) or $W = 1$

(at high temperatures). It is evident that a critical threshold exists for the two identical electric charge parameters, beyond which the aforementioned transition in the topological number occurs. In other words, there is a topological thermodynamic phase transition at the critical point. In the following, we will investigate the critical value for the two equal electric charge parameters.

When the electric charge parameters $q_1 = q_2 = q$ and $q_3 = q_4 = 0$, the inverse temperature τ in eq. (3.9) becomes

$$\tau = \frac{6\pi(2r_h + q)}{8\pi P(q + 3r_h)(q + r_h) + 3} \,. \tag{3.10}$$

Now, according to ref. [38], one can construct a similar new vector $\varphi = (\varphi^{r_h}, \varphi^{\Theta})$

$$\varphi^{r_h} = \frac{\partial \tau}{\partial r_h} \,, \qquad \varphi^{\Theta} = -\cot\Theta \csc\Theta \,. \tag{3.11}$$

The normalized vector field can be obtained through $\hat{n} = (\hat{n}^r, \hat{n}^{\Theta})$ with $\hat{n}^r = \varphi^{r_h}/||\varphi||$ and $\hat{n}^{\Theta} = \varphi^{\Theta}/||\varphi||$. The first advantage of the Θ-term is that the direction of the introduced vector φ is vertical to the horizontal lines at $\Theta = 0$ and π, which can be treated as two boundaries in the parameter space. A further advantage is that the zero point of φ is always located at $\Theta = \pi/2$. In addition, it is simple to check that the critical point is located exactly at the zero point of the φ. Then the components of the vector φ can be computed as

$$\varphi^{r_h} = -\frac{12\pi P[8\pi(3r_h^2 + 3qr_h + q^2) - 3]}{[8\pi P(q + 3r_h)(q + r_h) + 3]^2} \,, \qquad \varphi^{\Theta} = -\cot\Theta \csc\Theta \,. \tag{3.12}$$

Therefore, when $r_h \to 0$, the expression for the critical value of the electric charge parameter can be obtained by solving the equation $\varphi^{r_h} = 0$, which is given by

$$q_c = \frac{\sqrt{6}}{4\sqrt{\pi P}} \,. \tag{3.13}$$

Thus, if $q \geq q_c$, the topological number of the static charged AdS black hole in four-dimensional EMDA gauged supergravity theory is $W = 1$; and when $0 < q < q_c$, the topological number is $W = 0$ (at cold temperatures) or $W = 1$ (at high temperatures).

Taking $q_1/r_0 = q_2/r_0 = q_c/r_0 = 1.09$, $q_3 = q_4 = 0$, and $Pr_0^2 = 0.1$, we plot the zero points of the component ϕ^{r_h} in figure 7, and the unit vector field $\hat{n}$ on a portion of the $\Theta - r_h$ plane in figure 8, respectively. In figure 7, one can observe that there are one thermodynamically stable four-dimensional static charged AdS black hole in gauged EMDA supergravity theory for $\tau < \tau_c = 3.43r_0$. In figure 8, the critical point (CP) is located at $(r_h/r_0, \Theta) = (0, \pi/2)$, and the topological charge of this critical point is $\hat{W} = -1$, thus it is a conventional critical point [38].

3.3 $q_1 \neq q_2 \neq 0$, $q_3 = q_4 = 0$ case ($D = 4$ AdS Horowitz-Sen solution)

In this subsection, we explore a more general solution to the previous subsection, focusing on the case in which the electric charge parameters are $q_1 \neq q_2 \neq 0$ and $q_3 = q_4 = 0$. This specific case corresponds to the four-dimensional static charged AdS Horowitz-Sen black hole solution [110]. In the following, we first explore whether there is a critical relationship between two different electric charge parameters similar to that of eq. (3.13).

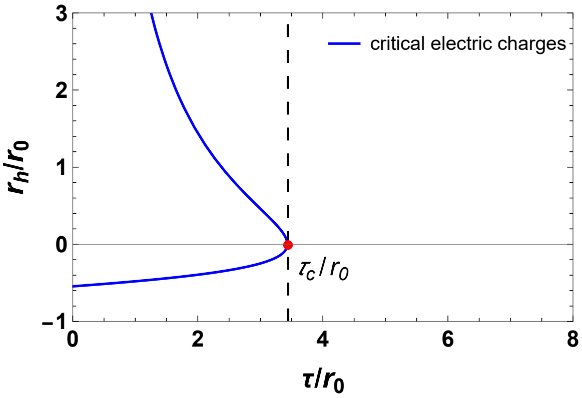


Figure 7. Zero points of the vector ϕ^{r_h} shown in the $r_h - \tau$ plane with $q_1/r_0 = q_2/r_0 = 1.09$, $q_3 = q_4 = 0$, and $Pr_0^2 = 0.1$. There is one thermodynamically stable four-dimensional static charged AdS black hole in gauged EMDA supergravity theory for $\tau < \tau_c = 3.43r_0$.

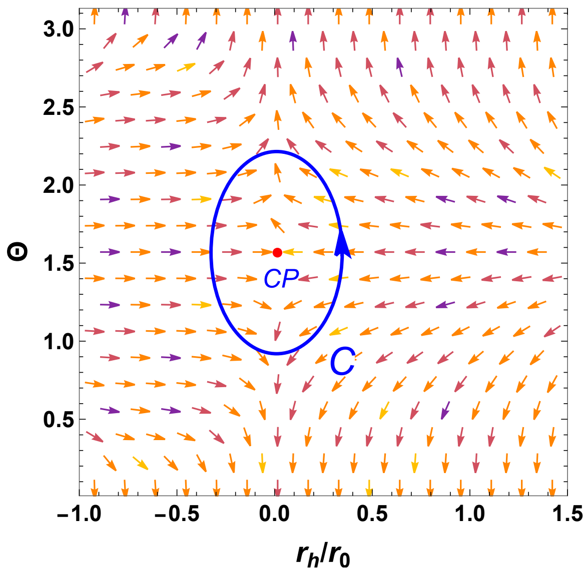


Figure 8. The arrows represent the unit vector field $\hat{n}$ on a portion of the $r_h - \Theta$ plane with $q_1/r_0 = q_2/r_0 = 1.09$, $q_3 = q_4 = 0$, and $Pr_0^2 = 0.1$. The critical point (CP) marked with a red dot is at $(r_h/r_0, \Theta) = (0, \pi/2)$. The blue contour C is a closed loop enclosing the critical point.

When the electric charge parameters $q_1 \neq q_2 \neq 0$ and $q_3 = q_4 = 0$, the inverse temperature τ in eq. (3.9) becomes

$$\tau = \frac{3\pi[4r_h^2 + 3r_h(q_1 + q_2) + 2q_1q_2]}{\sqrt{r_h + q_1}\sqrt{r_h + q_2}\{8\pi P[3r_h^2 + 2r_h(q_1 + q_2) + q_1q_2] + 3\}} . \tag{3.14}$$

According to the definition of vector φ in eq. (3.11), solving the equation $\varphi^{r_h} = 0$ and taking the limit $r_h \to 0$, one can obtain the following critical relationship as

$$q_{1c} = \frac{3}{8\pi P q_2} . \tag{3.15}$$

Therefore, for a fixed electric charge parameter q_2 and a fixed pressure P, if the electric charge parameter $q_1 \geq q_{1c}$, the topological number of the four-dimensional static charged

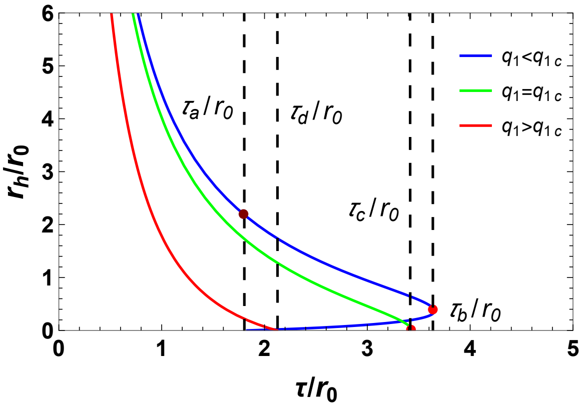


Figure 9. Zero points of the vector ϕ^{r_h} are depicted in the $r_h - \tau$ plane for the parameters $q_2/r_0 = 1$, $q_3 = q_4 = 0$, and $Pr_0^2 = 0.1$, with three distinct values of q_1/r_0: (1) $q_1/r_0 = 0.1$, which is less than the critical value q_{1c}/r_0 (represented by the blue solid line); (2) $q_1/r_0 = 15/(4\pi)$, equal to the critical value q_{1c}/r_0 (depicted by the green solid line); and (3) $q_1/r_0 = 10$, which exceeds the critical value q_{1c}/r_0 (illustrated by the red solid line).

AdS Horowitz-Sen black hole is $W = 1$; and when $0 < q_1 < q_{1c}$, the topological number is $W = 0$ (at cold temperatures) or $W = 1$ (at high temperatures), which is exhibited in figure 9 with a fixed electric charge parameter $q_2/r_0 = 1$ and a fixed pressure $Pr_0^2 = 0.1$.

In figure 9, one can observe that when $q_1/r_0 = 10 > q_{1c}/r_0$ (the red solid line), there is one thermodynamically stable four-dimensional static charged Horowitz-Sen AdS black hole for $\tau < \tau_d = 2.12r_0$, and the topological number $W = 1$; when $q_1/r_0 = 15/(4\pi) = q_{1c}/r_0$ (the green solid line), there is one thermodynamically stable four-dimensional static charged Horowitz-Sen AdS black hole for $\tau < \tau_c = 3.41r_0$, and the topological number $W = 1$; when $q_1/r_0 = 0.1 < q_{1c}/r_0$ (the blue solid line), there are one thermodynamically stable and one thermodynamically unstable four-dimensional static charged Horowitz-Sen AdS black hole branch for $1.80r_0 = \tau_a < \tau < \tau_b = 3.64r_0$, and one thermodynamically stable four-dimensional static charged Horowitz-Sen AdS black hole branch for $\tau < \tau_a = 1.80r_0$, thus the topological number is $W = 0$ (at inverse temperatures $1.80r_0 = \tau_a < \tau < \tau_b = 3.64r_0$) or $W = 1$ (at inverse temperatures $\tau < \tau_a = 1.80r_0$). According to the analysis in the previous subsection, the critical point corresponding to the inverse temperature τ_a should be a thermodynamic topological higher-order phase transition critical point, while the critical point corresponding to the inverse temperature τ_c should be a thermodynamic topological conventional phase transition critical point.

At the end of this subsection, we address an important issue. As the smaller electric charge parameter tends to zero, the four-dimensional static charged Horowitz-Sen AdS black hole asymptotically transitions into the four-dimensional static charged AdS black hole in K-K gauged supergravity theory in section 3.1. For the latter, the topological number W consistently assumes a single value, $W = 0$. The question arises: is the vanishing of the smaller electric charge parameter a prerequisite for the emergence of a single value for the topological number, or is there a critical threshold for the smaller electric charge parameter below which a temperature-dependent thermodynamic topological phase transition does not happen? To ensure that the topological number for the four-dimensional static charged

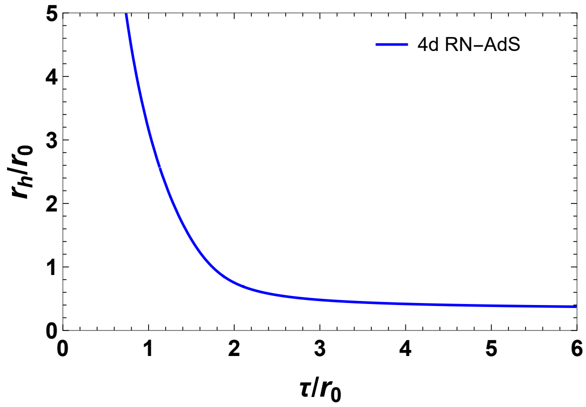


Figure 10. Zero points of the vector ϕ^{r_h} shown in the $r_h - \tau$ plane with $q_1/r_0 = q_2/r_0 = q_3/r_0 = q_4/r_0 = 1$, and $Pr_0^2 = 0.1$. There is one thermodynamically stable four-dimensional RN-AdS black hole for any value of τ.

AdS Horowitz-Sen black hole assumes a single value, it is imperative to satisfy the specific constraint stated in eq. (3.15), namely,

$$q_1 \geq q_{1c} = \frac{3}{8\pi P q_2} \,. \tag{3.16}$$

It is readily apparent that when $q_2 = 0$, the value of q_1 becomes infinite. Consequently, it is essential for one of the electric charge parameters to be zero in order to obtain a single value for the topological number.

3.4 $q_1 = q_2 = q_3 = q_4 \neq 0$ case (RN-AdS$_4$)

Considering the pressure as $Pr_0^2 = 0.1$ and the four electric charge parameters $q_1/r_0 = q_2/r_0 = q_3/r_0 = q_4/r_0 = 1$ for the four-dimensional RN-AdS black hole, we plot the zero points of ϕ^{r_h} in the $r_h - \tau$ plane in figure 10, and the unit vector field n on a portion of the $\Theta - r_h$ plane with $\tau/r_0 = 2$ in figure 11, respectively. Based upon the local property of the zero point, one can easily find that the topological number is: $W = 1$, which is consistent with the result given in ref. [60].

3.5 $q_1 = q_2 \neq 0$, $q_3 = q_4 \neq 0$ case (pairwise-equal AdS)

In this subsection, we investigate a special case of four electric charge parameters: $q_1 = q_2 \neq 0$ and $q_3 = q_4 \neq 0$, characterizing the four-dimensional static pairwise-equal four-charge AdS black hole in gauged $-iX^0X^1$-supergravity theory. In figures 12 and 13, taking $q_1/r_0 = q_2/r_0 = 2$, $q_3/r_0 = q_4/r_0 = 1$, and $Pr_0^2 = 0.1$ for the four-dimensional static pairwise-equal four-charge AdS black hole in gauged $-iX^0X^1$-supergravity theory, we plot the zero points of ϕ^{r_h} in the $r_h - \tau$ plane and the unit vector field n with $\tau = r_0$, respectively. In figure 12, one can observe that there is always one thermodynamically stable four-dimensional static pairwise-equal four-charge AdS black hole in gauged $-iX^0X^1$-supergravity theory for any value of τ. In figure 13, We have one zero point at $(r_h/r_0, \Theta) = (2.31, \pi/2)$. Based on the local property of the zero points, it is easy to find that the topological number

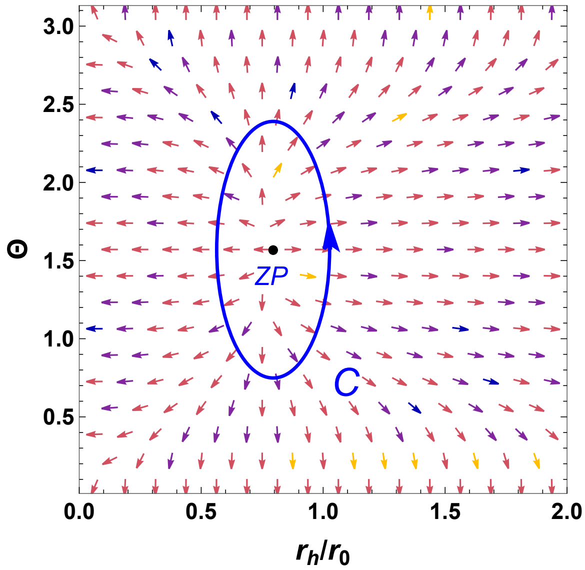


Figure 11. The arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane for the four-dimensional RN-AdS black hole with $\tau/r_0 = 2$, $q_1/r_0 = q_2/r_0 = q_3/r_0 = q_4/r_0 = 1$, $Pr_0^2 = 0.1$. The zero point (ZP) marked with a black dot is at $(r_h/r_0, \Theta) = (0.75, \pi/2)$. The blue contour C is a closed loop enclosing the zero point.

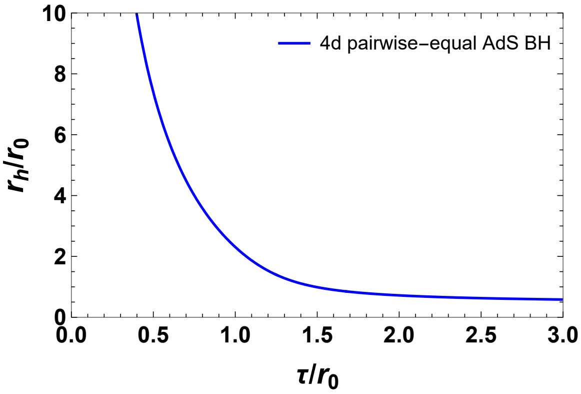


Figure 12. Zero points of the vector ϕ^{r_h} shown in the $r_h - \tau$ plane with $q_1/r_0 = q_2/r_0 = 2$, $q_3/r_0 = q_4/r_0 = 1$, and $Pr_0^2 = 0.1$. There is always one thermodynamically stable four-dimensional static pairwise-equal four-charge AdS black hole in gauged $-iX^0X^1$-supergravity theory for any value of τ. Obviously, the topological number is: $W = 1$.

$W = 1$ for the four-dimensional static pairwise-equal four-charge AdS black hole in gauged $-iX^0X^1$-supergravity theory.

3.6 $q_1 \neq q_2 \neq q_3 \neq q_4 \neq 0$ case (STU gauged supergravity)

In this subsection, we consider the most general static four-charge AdS black hole case in STU gauged supergravity theory, i.e., $q_1 \neq q_2 \neq q_3 \neq q_4 \neq 0$ case. We take $q_1/r_0 = 0.5$, $q_2/r_0 = 1$, $q_3/r_0 = 2$, $q_4/r_0 = 3$, and $Pr_0^2 = 0.1$, and then plot the zero points of the component ϕ^{r_h} in figure 14, and the unit vector field n on a portion of the $\Theta - r_h$ plane with $\tau/r_0 = 2$ in figure 15, respectively. It is easy to observe that there is always one thermodynamically

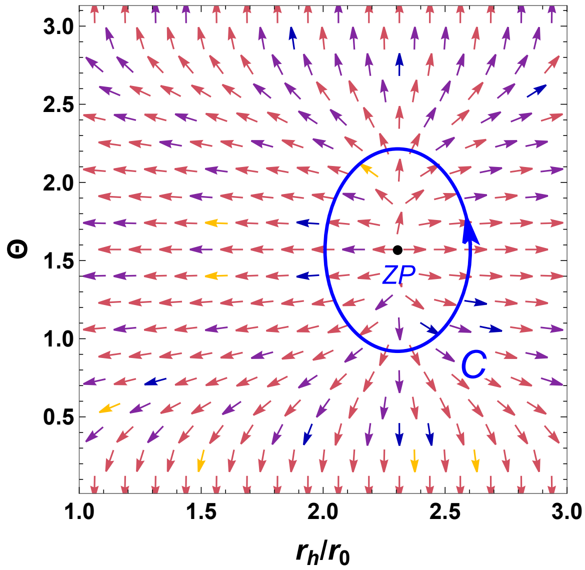


Figure 13. The arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane for the four-dimensional static pairwise-equal four-charge AdS black hole in gauged $-iX^0X^1$-supergravity theory with $\tau/r_0 = 1$, $q_1/r_0 = q_2/r_0 = 2$, $q_3/r_0 = q_4/r_0 = 1$, and $Pr_0^2 = 0.1$. The zero point (ZP) marked with a black dot is at $(r_h/r_0, \Theta) = (2.31, \pi/2)$. The blue contour C is a closed loop enclosing the zero point.

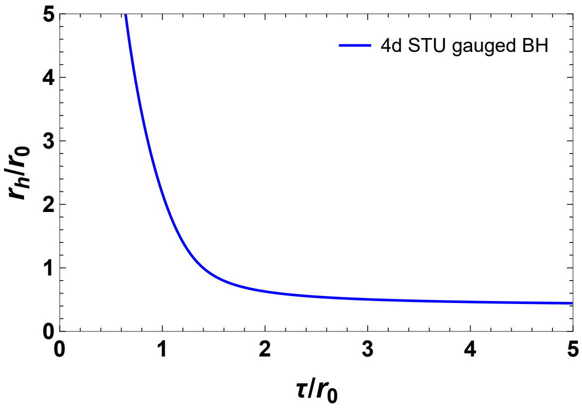


Figure 14. Zero points of the vector ϕ^{r_h} shown in the $r_h - \tau$ plane with $q_1/r_0 = 0.5$, $q_2/r_0 = 1$, $q_3/r_0 = 2$, $q_4/r_0 = 3$, and $Pr_0^2 = 0.1$. There is always one thermodynamically stable four-dimensional static four-charge AdS black hole in STU gauged supergravity theory for any value of τ. Obviously, the topological number is: $W = 1$.

stable four-dimensional static four-charge AdS black hole in STU gauged supergravity theory for any value of τ. In figure 15, we observe a zero point at $(r_h/r_0, \Theta) = (0.63, \pi/2)$. Based upon the local property of the zero points, it is simple to demonstrate that the topological number $W = 1$ for the four-dimensional static four-charge AdS black hole in STU gauged supergravity theory.

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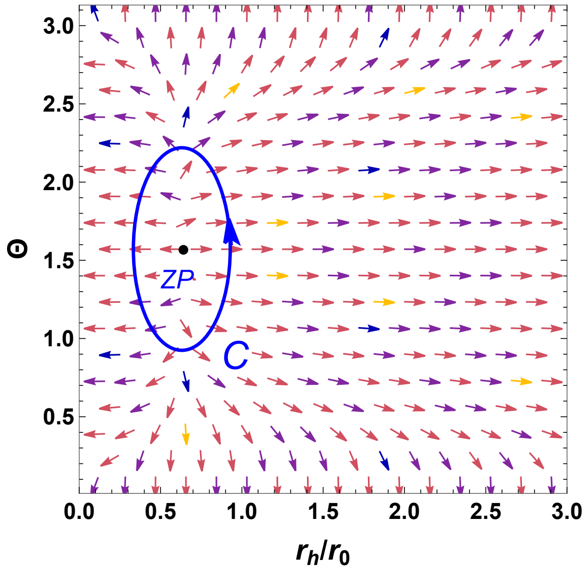

Figure 15. The arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane for the four-dimensional static four-charge AdS black hole in STU gauged supergravity theory with $\tau/r_0 = 2$, $q_1/r_0 = 0.5$, $q_2/r_0 = 1$, $q_3/r_0 = 2$, $q_4/r_0 = 3$, and $Pr_0^2 = 0.1$. The zero point (ZP) marked with a black dot is at $(r_h/r_0, \Theta) = (0.63, \pi/2)$. The blue contour C is a closed loop enclosing the zero point.

4 Five-dimensional static multi-charge black holes in gauged supergravity theory

In this section, we would like to investigate the topological numbers of the five-dimensional static multi-charge AdS black holes in gauged supergravity theory [5]. For the general static three-charge AdS black hole in five-dimensional STU gauged supergravity theory, whose metric, Abelian gauge potentials, and scalar fields are [5]

$$ds_5^2 = -\prod_{i=1}^{3} H_i^{-2/3} f dt^2 + \prod_{i=1}^{3} H_i^{1/3}\Big(f^{-1}dr^2 + r^2 d\Omega_3^2\Big)\,,$$
$$A^i = \frac{\sqrt{q_i(q_i+2m)}}{r^2+q_i}\,dt\,, \qquad X_i = H_i^{-1}\prod_{j=1}^{3} H_j^{1/3}\,, \tag{4.1}$$

where

$$f = 1 - \frac{2m}{r^2} + \frac{r^2}{l^2}\prod_{i=1}^{3} H_i\,, \qquad H_i = 1 + \frac{q_i}{r^2}\,, \tag{4.2}$$

in which l is the AdS radius, m and q_i are the mass and three independent electric charge parameters, respectively.

For the metric of a five-dimensional static, charged AdS black hole, as expressed in eq. (4.1), the most general case is represented by a solution with three independent electric charge parameters. Moreover, in line with the classification scheme for black hole solutions depicted in figure 1 of ref. [103], many specific truncated supergravity solutions are identified: for instance, when $q_1 \neq 0$ and $q_2 = q_3 = 0$, namely, the five-dimensional static charged AdS black hole solution in K-K gauged supergravity; when $q_1 = q_2 \neq 0$ and $q_3 = 0$, i.e., the five-dimensional static charged AdS black hole solution in EMDA gauged supergravity theory; when $q_1 \neq q_2 \neq 0$ and $q_3 = 0$, i.e., the five-dimensional static charged AdS Horowitz-Sen black hole solution [110]; and when $q_1 = q_2 = q_3 \neq 0$, which is the famous five-dimensional RN-AdS black hole case after the coordinate transformation by $\rho^2 = r^2 + q_i$; etc.

The thermodynamic quantities are [7]

$$\begin{aligned} M &= \frac{3\pi}{4}m + \frac{\pi}{4}\sum_{i=1}^{3} q_i\,, \qquad Q_i = \frac{1}{4}\pi\sqrt{q_i(q_i+2m)}\,, \qquad S = \frac{1}{2}\pi^2\prod_{i=1}^{3}(r_h^2+q_i)^{1/2}\,, \quad P = \frac{3}{4\pi l^2}\,, \\ T &= \frac{f'(r_h)}{4\pi}\prod_{i=1}^{3}H_i^{-1/2}(r_h)\,, \qquad \Phi_i = \frac{\sqrt{q_i(q_i+2m)}}{r_h^2+q_i}\,, \qquad V = \frac{\pi^2 r_h^4}{6}\prod_{i=1}^{3}H_i(r_h)\sum_{j=1}^{3}\frac{1}{H_j(r_h)}\,. \end{aligned} \tag{4.3}$$

Then one can verify that the above thermodynamic quantities completely obey the first law and the Bekenstein-Smarr mass formula simultaneously,

$$dM = TdS + \sum_{i=1}^{3}\Phi_i dQ_i + VdP\,, \tag{4.4}$$

$$2M = 3TS + 2\sum_{i=1}^{3}\Phi_i Q_i - 2VP\,. \tag{4.5}$$

From eq. (4.3), one can obtain the expression of the generalized Helmholtz free energy as

$$\mathcal{F} = \frac{\pi}{4}\left[\frac{2\pi P\prod_{i=1}^{3}(r_h^2+q_i)}{r_h^2} + \frac{3}{2}r_h^2 + \sum_{i=1}^{3}q_i\right] - \frac{\pi^2\prod_{i=1}^{3}\sqrt{r_h^2+q_i}}{2\tau}\,. \tag{4.6}$$

Using the definition of eq. (2.2), the components vector ϕ can be easily obtained as follows:

$$\begin{aligned} \phi^{r_h} &= -\frac{3\pi^2 r_h^5 + 2\pi^2 r_h^3\sum_{i=1}^{3}q_i + \pi^2 r_h[q_2q_3 + q_1(q_2+q_3)]}{2\tau\prod_{i=1}^{3}\sqrt{r_h^2+q_i}} \\ &\quad + \frac{\pi^2 P\left(2r_h^6 + r_h^4\sum_{i=1}^{3}q_i - \prod_{i=1}^{3}q_i\right)}{r_h^3} + \frac{3\pi r_h}{4}\,, \end{aligned} \tag{4.7}$$

$$\phi^{\Theta} = -\cot\Theta\csc\Theta\,. \tag{4.8}$$

It is simple to obtain

$$\tau = \frac{2\pi r_h^4\left(3r_h^4 + 2r_h^2\sum_{i=1}^{3}q_i + q_1q_2 + q_1q_3 + q_2q_3\right)}{3r_h^4 + 4\pi P\left(2r_h^6 + r_h^4\sum_{i=1}^{3}q_i - \prod_{i=1}^{3}q_i\right)\prod_{i=1}^{3}\sqrt{r_h^2+q_i}} \tag{4.9}$$

as the zero point of the vector field ϕ, which consistently reduces to the one obtained in the five-dimensional Schwarzschild-AdS black hole case when the three independent electric charge parameters are turned off.

Similar to section 3, varying the three independent electric charge parameters yields distinct black hole solutions within various truncated supergravity theories. In the following, we will investigate the topological numbers of static, charged AdS black holes in some famous five-dimensional supergravity theories, respectively.

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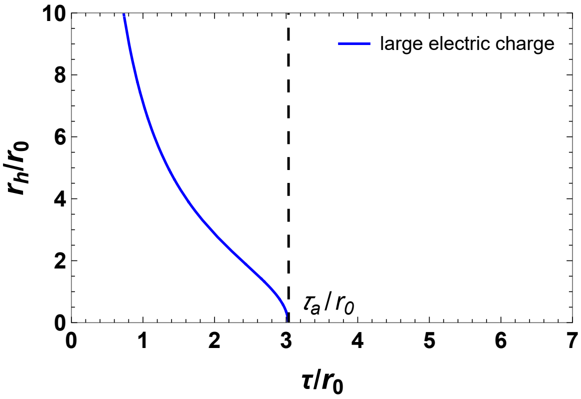

Figure 16. Zero points of the vector ϕ^{r_h} shown in the $r_h - \tau$ plane with $q_1/r_0^2 = 5$, $Pr_0^2 = 0.1$, and $q_2 = q_3 = 0$. There is one thermodynamically stable five-dimensional static charged AdS black hole in K-K gauged supergravity theory for $\tau < \tau_a = 3.03r_0$. Obviously, the topological number is: $W = 1$.

4.1 $q_1 \neq 0$, $q_2 = q_3 = 0$ case (K-K gauged supergravity)

In this subsection, we focus on the case where $q_1 \neq 0$ and $q_2 = q_3 = 0$, which corresponds to the static charged AdS black hole in five-dimensional K-K gauged supergravity theory. For the five-dimensional static charged AdS black hole in K-K gauged supergravity theory, similar to the cases of the four-dimensional static charged AdS black hole in EMDA gauged supergravity theory in section 3.2 and the four-dimensional static charged AdS Horowitz-Sen black hole in section 3.3, we find that different values of the electric charge parameter also influence its topological number. Therefore, we also discuss each of the three cases by taking the larger, smaller, and critical values of the electric charge parameter.

4.1.1 Large value of electric charge parameter

We first consider the case where the electric charge parameter takes a larger value. We plot the zero points of the component ϕ^{r_h} with $Pr_0^2 = 0.1$, $q_1/r_0^2 = 5$, and $q_2 = q_3 = 0$ in figure 16, and the unit vector field n in figure 17 with $\tau/r_0 = 2$. Note that for these values of Pr_0^2 and q_1/r_0^2, there is one thermodynamically stable five-dimensional static charged AdS black hole in K-K gauged supergravity theory for $\tau < \tau_a = 3.03r_0$. In figure 17, one can observe that the zero point is located at $(r_h/r_0, \Theta) = (2.87, \pi/2)$. Therefore, the topological number $W = 1$ for the above black hole can be clearly found in figures 16 and 17 by applying the local property of the zero point, which is the same as that of the five-dimensional RN-AdS black hole [60].

4.1.2 Small value of electric charge parameter and the temperature-dependent thermodynamic topological phase transition

Then, we consider the case where the electric charge parameter takes a smaller value. We take $q_1/r_0^2 = 1$, $q_2 = q_3 = 0$, and $Pr_0^2 = 0.1$, and then plot the zero points of the component ϕ^{r_h} in figure 18, and the unit vector field n on a portion of the $\Theta - r_h$ plane with $\tau = 3.2r_0, 2r_0$ in figure 19, respectively. From figure 18, it is a simple matter to observe that there is one thermodynamically stable and one thermodynamically unstable black hole branch for $2.95r_0 = \tau_a < \tau < \tau_b = 3.29r_0$, and one thermodynamically stable black hole branch for $\tau < \tau_a = 2.95r_0$.

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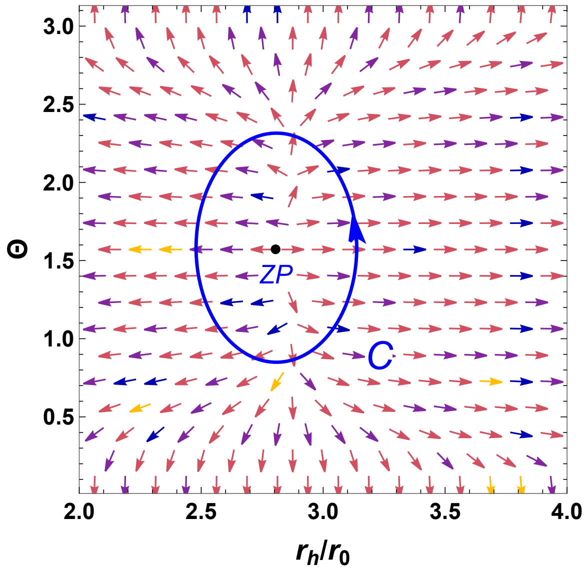


Figure 17. The arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane for the five-dimensional static charged AdS black hole in K-K gauged supergravity theory with $\tau/r_0 = 2$, $q_1/r_0^2 = 5$, $Pr_0^2 = 0.1$, and $q_2 = q_3 = 0$. The zero point (ZP) marked with a black dot is at $(r_h/r_0, \Theta) = (2.87, \pi/2)$. The blue contour C is a closed loop enclosing the zero point.

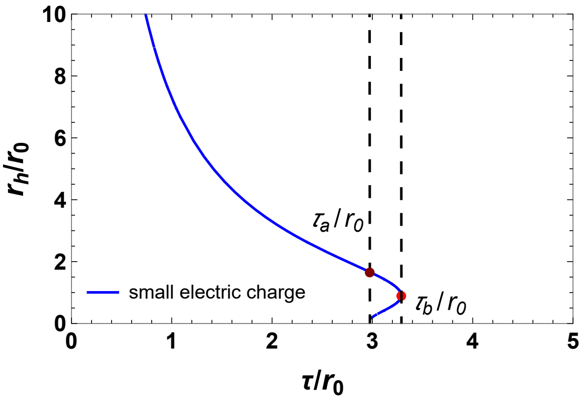
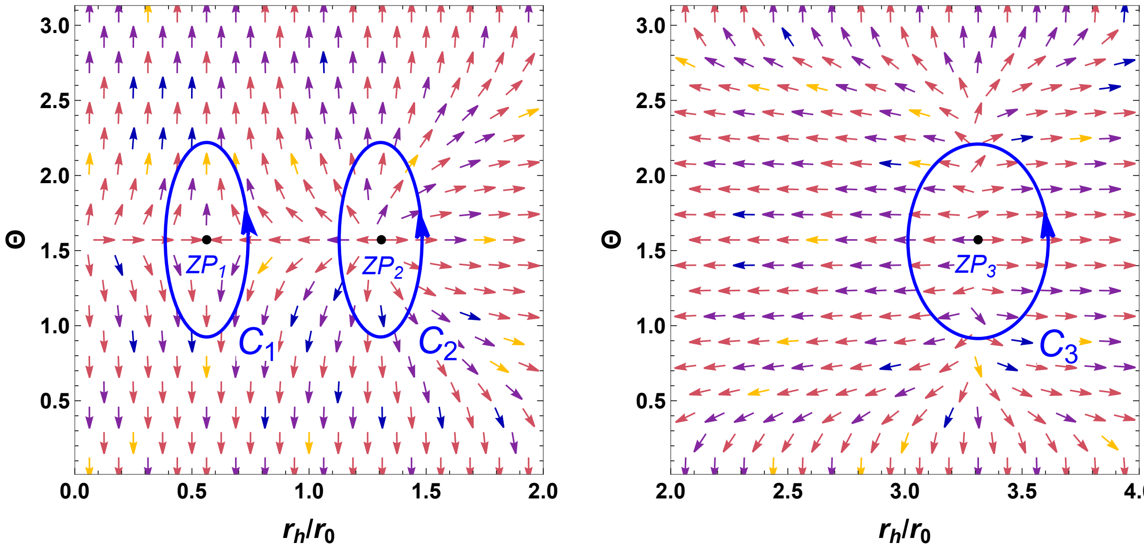


Figure 18. Zero points of the vector ϕ^{r_h} shown in the $r_h - \tau$ plane with $q_1/r_0^2 = 1$, $q_2 = q_3 = 0$, and $Pr_0^2 = 0.1$. There is one thermodynamically stable and one thermodynamically unstable five-dimensional static charged AdS black hole in K-K gauged supergravity theory for $2.95r_0 = \tau_a < \tau < \tau_b = 3.29r_0$, and one thermodynamically stable five-dimensional static charged AdS black hole in K-K gauged supergravity theory for $\tau < \tau_a = 2.95r_0$.

Although in figure 19(a), the zero points are located at $(r_h/r_0, \Theta) = (0.57, \pi/2)$, and $(1.27, \pi/2)$, respectively. Thus, one can read the winding numbers w_i for the blue contours C_i: $w_1 = -1$, $w_2 = 1$, and the topological number at this inverse temperature $\tau = 3.2r_0$ is $W = -1 + 1 = 0$; But in figure 19(b), the zero point is located at $(r_h/r_0, \Theta) = (3.29, \pi/2)$, thus the winding number for the blue contour C_3 is $w_3 = 1$, so the topological number at this inverse temperature $\tau = 2r_0$ is $W = 1$. Thus, we find that the topological number is temperature dependent: it is $W = 0$ (at inverse temperature $2.95r_0 = \tau_a < \tau < \tau_b = 3.29r_0$)



(a) The unit vector field for the five-dimensional static charged AdS black hole in K-K gauged supergravity theory with $\tau/r_0 = 3.2$, $q_1/r_0^2 = 1$, $q_2 = q_3 = 0$, and $Pr_0^2 = 0.1$.

(b) The unit vector field for the five-dimensional static charged AdS black hole in K-K gauged supergravity theory with $\tau/r_0 = 2$, $q_1/r_0^2 = 1$, $q_2 = q_3 = 0$, and $Pr_0^2 = 0.1$.

Figure 19. The arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane. The zero points (ZPs) marked with black dots are at $(r_h/r_0, \Theta) = (0.57, \pi/2)$, $(1.27, \pi/2)$, $(3.29, \pi/2)$, for ZP_1, ZP_2, and ZP_3, respectively. The blue contours C_i are closed loops surrounding the zero points.

or $W = 1$ (at inverse temperature $\tau < \tau_a = 2.95r_0$). At the point of the critical inversion temperature $\tau = \tau_a$, the black hole occurs a novel temperature-dependent thermodynamic topological phase transition. The critical point corresponding to the inverse temperature τ_a should be a thermodynamic topological higher-order phase transition critical point.

4.1.3 Critical value of electric charge parameter

In the following, we will calculate the critical value of the electric charge parameter. When the electric charge parameters $q_1 = q$ and $q_2 = q_3 = 0$, the inverse temperature τ in eq. (4.9) becomes

$$\tau = \frac{2\pi(3r_h^2 + 2q)}{\sqrt{r_h^2 + q}(8\pi Pr_h^2 + 4\pi Pq + 3)} \,. \tag{4.10}$$

Using the definition of vector φ in eq. (3.11), one can obtain the components of the vector φ as

$$\varphi^{r_h} = -\frac{2\pi r_h[4\pi P(6r_h^4 + 9r_h^2 q + 4q^2) - 9r_h^2 - 12q]}{(r_h^2 + q)^{\frac{3}{2}}[4\pi P(2r_h^2 + q) + 3]^2} \,, \qquad \varphi^{\Theta} = -\cot\Theta\csc\Theta \,. \tag{4.11}$$

Thus, as $r_h \to 0$, the critical value of the electric charge parameter q_c can be determined by solving the equation $\varphi^{r_h} = 0$, which yields:

$$q_c = \frac{3}{4\pi P} \,. \tag{4.12}$$

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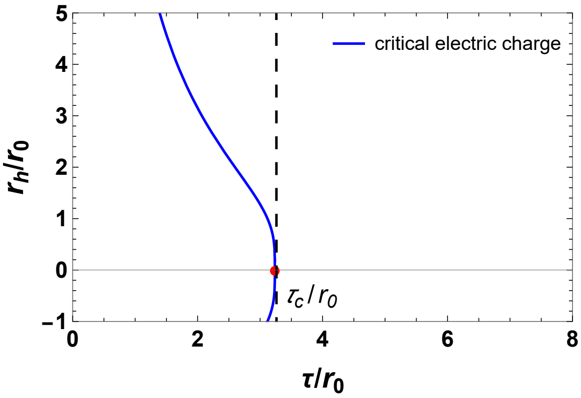

Figure 20. Zero points of the vector ϕ^{r_h} shown in the $r_h - \tau$ plane with $q_1/r_0^2 = 15/(2\pi)$, $q_2 = q_3 = 0$, and $Pr_0^2 = 0.1$. There is one thermodynamically stable five-dimensional static charged AdS black hole in K-K gauged supergravity theory for $\tau < \tau_c = 3.24r_0$.

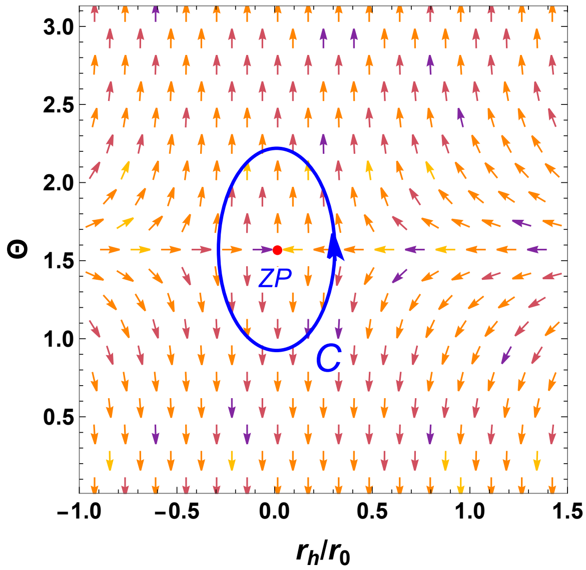

Figure 21. The arrows represent the unit vector field $\hat{n}$ on a portion of the $r_h - \Theta$ plane with $q_1/r_0^2 = 15/(2\pi)$, $q_2 = q_3 = 0$, and $Pr_0^2 = 0.1$. The critical point (CP) marked with a red dot is at $(r_h/r_0, \Theta) = (0, \pi/2)$. The blue contour C is a closed loop enclosing the critical point.

Hence, for $q \geq q_c$, the static charged AdS black hole in five-dimensional K-K gauged supergravity exhibits the topological number of $W = 1$. However, in the case where $0 < q < q_c$, the topological number transitions from $W = 0$ (at low temperatures) to $W = 1$ (at high temperatures).

Taking $q_1/r_0^2 = q_c/r_0^2 = 15/(2\pi)$, $q_2 = q_3 = 0$, and $Pr_0^2 = 0.1$, we plot the zero points of the component ϕ^{r_h} in figure 20, and the unit vector field $\hat{n}$ on a portion of the $\Theta - r_h$ plane in figure 21, respectively. In figure 20, one can observe that there are one thermodynamically stable five-dimensional static charged AdS black hole in K-K gauged supergravity theory for $\tau < \tau_c = 3.24r_0$. In figure 21, the critical point (CP) is located at $(r_h/r_0, \Theta) = (0, \pi/2)$, and the topological charge of this critical point is $\hat{W} = -1$, therefore it is a conventional critical point [38].

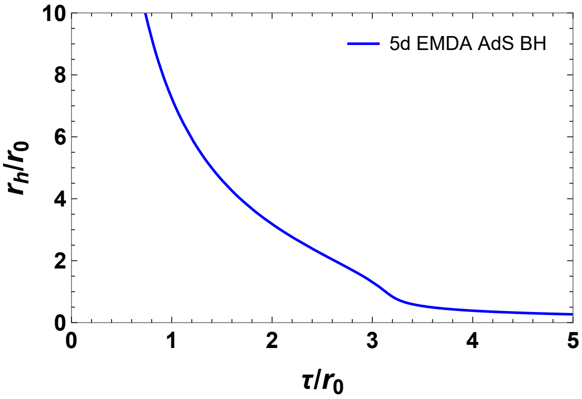

Figure 22. Zero points of the vector ϕ^{r_h} shown in the $r_h - \tau$ plane with $q_1/r_0^2 = q_2/r_0^2 = 1$, $q_3 = 0$, and $Pr_0^2 = 0.1$. There is always one thermodynamically stable static charged AdS black hole in five-dimensional EMDA gauged supergravity theory for any value of τ. Obviously, the topological number is: $W = 1$.

4.2 $q_1 = q_2 \neq 0$, $q_3 = 0$ case (EMDA gauged supergravity)

In this subsection, we investigate the case with $q_1 = q_2 \neq 0$, $q_3 = 0$, which represents the static charged AdS black hole in the five-dimensional EMDA gauged supergravity theory. Considering the pressure as $Pr_0^2 = 0.1$ and the electric charge parameters $q_1/r_0^2 = q_2/r_0^2 = 1$, $q_3 = 0$ for the static charged AdS black hole in five-dimensional EMDA gauged supergravity theory, we plot the zero points of ϕ^{r_h} in the $r_h - \tau$ plane in figure 22, and the unit vector field n on a portion of the $\Theta - r_h$ plane with $\tau/r_0 = 2$ in figure 23. Obviously, there is only one thermodynamically stable static charged AdS black hole in five-dimensional EMDA gauged supergravity theory for any value of τ. In figure 23, one can observe that the zero point is located at $(r_h/r_0, \Theta) = (3.18, \pi/2)$. Based upon the local property of the zero point, we can easily obtain the topological number $W = 1$ for the static charged AdS black hole in five-dimensional EMDA gauged supergravity theory.

4.3 $q_1 \neq q_2 \neq 0$, $q_3 = 0$ case ($D = 5$ AdS Horowitz-Sen solution)

In this subsection, we discuss a more general case to the last subsection, focusing on the case in which the electric charge parameters are $q_1 \neq q_2 \neq 0$ and $q_3 = 0$, which corresponds to the five-dimensional static charged AdS Horowitz-Sen black hole solution [110]. Here, we would like to begin by exploring an important issue. As the smaller electric charge parameter equals zero, the five-dimensional static charged AdS Horowitz-Sen black hole reduces to the five-dimensional static charged AdS black hole in K-K gauged supergravity theory in section 4.1. For the latter, the topological number W is temperature-dependent; it is $W = 1$ for large electric charge parameter but can be $W = 0$ (at low temperatures) or $W = 1$ (at high temperatures) for small electric charge parameter. This raises the question: is there a critical value for the smaller electric charge parameter below which a temperature-dependent thermodynamic topological phase transition occurs?

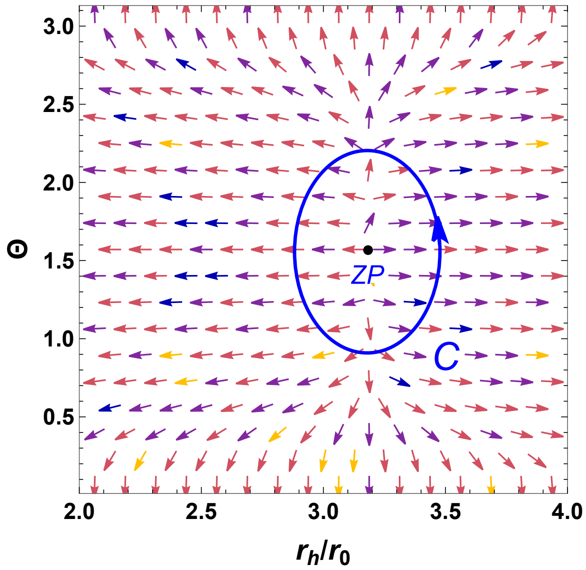

Figure 23. The arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane for the static charged AdS black hole in five-dimensional EMDA gauged supergravity theory with $\tau/r_0 = 2$, $q_1/r_0^2 = q_2/r_0^2 = 1$, $q_3 = 0$, and $Pr_0^2 = 0.1$. The zero point (ZP) marked with a black dot is at $(r_h/r_0, \Theta) = (3.18, \pi/2)$. The blue contour C is a closed loop enclosing the zero point.

When the electric charge parameters $q_1 \neq q_2 \neq 0$, $q_3 = 0$, the inverse temperature τ in eq. (4.9) becomes

$$\tau = \frac{2\pi[3r_h^4 + 2r_h^2(q_1 + q_2) + q_1 q_2]}{r_h\sqrt{r_h^2 + q_1}\sqrt{r_h^2 + q_2}[8\pi P r_h^2 + 4\pi P(q_1 + q_2) + 3]}\,. \tag{4.13}$$

By employing the definition of the vector φ given in eq. (3.11), and by solving the equation $\varphi^{r_h} = 0$, while taking the limit as $r_h \to 0$, the critical value for the smaller electric charge parameter, q_{1c}, is found to be

$$q_{1c} = 0\,. \tag{4.14}$$

Therefore, the smaller electric charge parameter does not have the critical value described above, allowing the temperature-dependent thermodynamic topological phase transitions to occur.

Taking the pressure as $Pr_0^2 = 0.1$ and the electric charge parameters $q_1/r_0^2 = 1$, $q_2/r_0^2 = 2$, and $q_3 = 0$ for the five-dimensional static charged AdS Horowitz-Sen black hole, we plot the zero points of ϕ^{r_h} in the $r_h - \tau$ plane in figure 24, and the unit vector field n on a portion of the $\Theta - r_h$ plane with $\tau/r_0 = 3$ in figure 25. It is easy to observe that there is only one thermodynamically stable five-dimensional static charged AdS Horowitz-Sen black hole for any value of τ. In figure 25, one can find that the zero point is located at $(r_h/r_0, \Theta) = (1.07, \pi/2)$. Based upon the local property of the zero point, we can straightforwardly obtain the topological number $W = 1$ for five-dimensional static charged AdS Horowitz-Sen black hole.

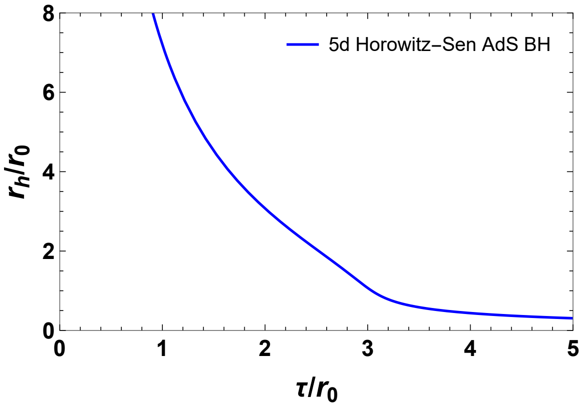

Figure 24. Zero points of the vector ϕ^{r_h} shown in the $r_h - \tau$ plane with $q_1/r_0^2 = 1$, $q_2/r_0^2 = 2$, $q_3 = 0$, and $Pr_0^2 = 0.1$. There is always one thermodynamically stable five-dimensional static charged AdS Horowitz-Sen black hole for any value of τ. Obviously, the topological number is: $W = 1$.

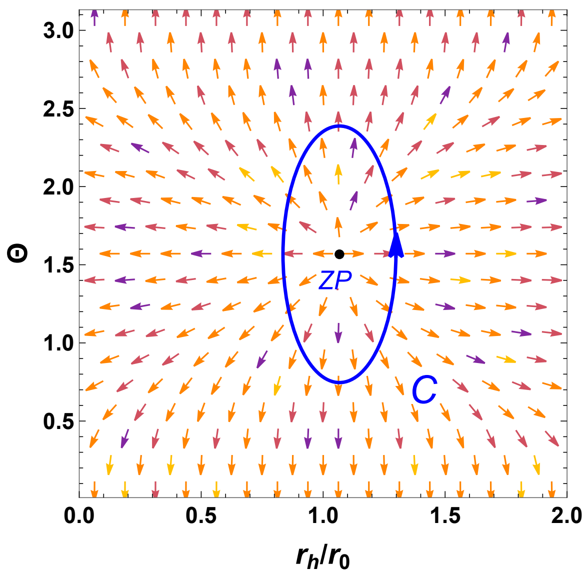

Figure 25. The arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane with $\tau/r_0 = 3$, $q_1/r_0^2 = 1$, $q_2/r_0^2 = 2$, $q_3 = 0$, and $Pr_0^2 = 0.1$. The zero point (ZP) marked with a black dot is at $(r_h/r_0, \Theta) = (1.07, \pi/2)$. The blue contour C is a closed loop enclosing the zero point.

4.4 $q_1 = q_2 = q_3 \neq 0$ case (RN-AdS$_5$)

Considering the pressure as $Pr_0^2 = 0.1$ and the three electric charge parameters $q_1/r_0^2 = q_2/r_0^2 = q_3/r_0^2 = 1$ for the five-dimensional RN-AdS black hole, we show the zero points of ϕ^{r_h} in the $r_h - \tau$ plane in figure 26, and the unit vector field n on a portion of the $\Theta - r_h$ plane with $\tau/r_0 = 3$ in figure 27, respectively. Based on the local property of the zero point, one can easily indicate that the topological number is: $W = 1$, which is consistent with the result given in ref. [60].

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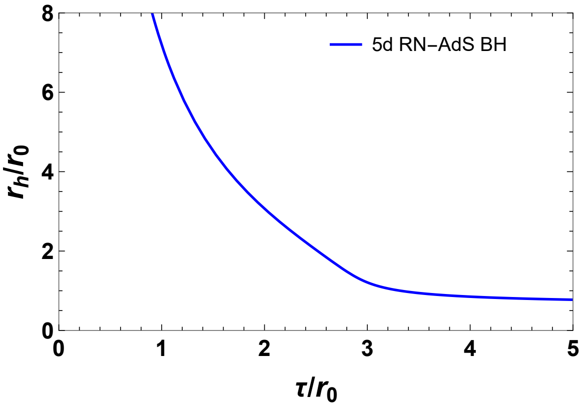

Figure 26. Zero points of the vector ϕ^{r_h} shown in the $r_h - \tau$ plane with $q_1/r_0^2 = q_2/r_0^2 = q_3/r_0^2 = 1$, and $Pr_0^2 = 0.1$. There is one thermodynamically stable five-dimensional RN-AdS black hole for any value of τ.

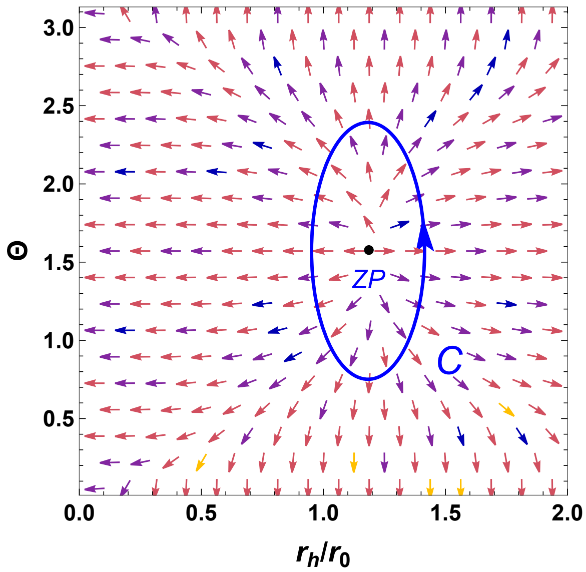

Figure 27. The arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane for the five-dimensional RN-AdS black hole with $\tau/r_0 = 3$, $q_1/r_0^2 = q_2/r_0^2 = q_3/r_0^2 = 1$, $Pr_0^2 = 0.1$. The zero point (ZP) marked with a black dot is at $(r_h/r_0, \Theta) = (1.21, \pi/2)$. The blue contour C is a closed loop enclosing the zero point.

4.5 $q_1 \neq q_2 \neq q_3 \neq 0$ case (STU gauged supergravity)

In this subsection, we investigate the most general static three-charge AdS black hole in five-dimensional STU gauged supergravity theory, namely, $q_1 \neq q_2 \neq q_3 \neq 0$ case. We take $q_1/r_0^2 = 1$, $q_2/r_0^2 = 2$, $q_3/r_0^2 = 3$, and $Pr_0^2 = 0.1$, and then plot the zero points of the component ϕ^{r_h} in figure 28, and the unit vector field n on a portion of the $\Theta - r_h$ plane with $\tau/r_0 = 3$ in figure 29, respectively. It is easy to see that there is always one thermodynamically stable five-dimensional static three-charge AdS black hole in STU gauged supergravity theory for any value of τ. In figure 29, we can observe a zero point at $(r_h/r_0, \Theta) = (1.31, \pi/2)$. Based upon the local property of the zero points, it is simple to indicate that the topological

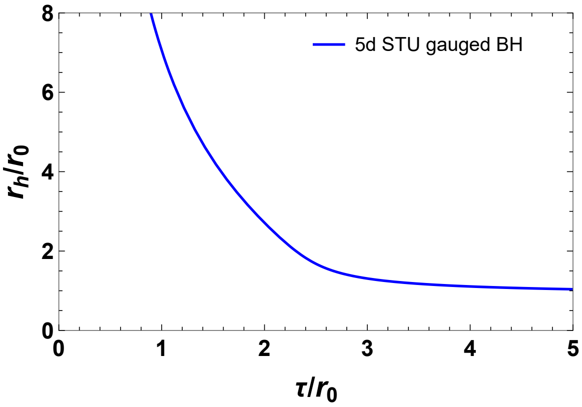

Figure 28. Zero points of the vector ϕ^{r_h} shown in the $r_h - \tau$ plane with $q_1/r_0^2 = 1$, $q_2/r_0^2 = 2$, $q_3/r_0^2 = 3$, and $Pr_0^2 = 0.1$. There is always one thermodynamically stable five-dimensional static three-charge AdS black hole in STU gauged supergravity theory for any value of τ. Obviously, the topological number is: $W = 1$.

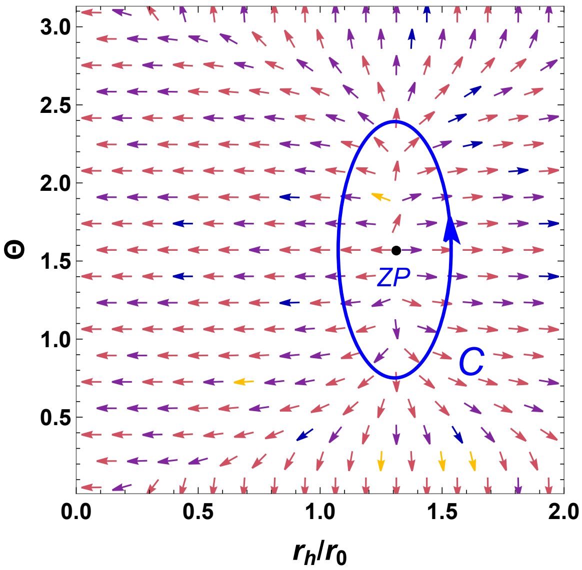

Figure 29. The arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane for the five-dimensional static three-charge AdS black hole in STU gauged supergravity theory with $\tau/r_0 = 3$, $q_1/r_0^2 = 1$, $q_2/r_0^2 = 2$, $q_3/r_0^2 = 3$, and $Pr_0^2 = 0.1$. The zero point (ZP) marked with a black dot is at $(r_h/r_0, \Theta) = (1.31, \pi/2)$. The blue contour C is a closed loop enclosing the zero point.

number $W = 1$ for the five-dimensional static three-charge AdS black hole in STU gauged supergravity theory.

5 Conclusions and outlooks

In this paper, making use of the generalized off-shell Helmholtz free energy, we investigate the topological number of the four-dimensional static multi-charge AdS black holes in gauged supergravity theory [4] and the five-dimensional static multi-charge AdS black holes in gauged supergravity theory [5]. In gauged supergravity theory, four- and five-dimensional static

BH solution	W	Generation point	Annihilation point
$q_1 = q,\ q_2 = q_3 = q_4 = 0$ case	0	0	1
$q_1 = q_2 = q,\ q_3 = q_4 = 0$ case ($q \geq \frac{\sqrt{6}}{4\sqrt{\pi P}}$)	1	0	0
$q_1 = q_2 = q,\ q_3 = q_4 = 0$ case ($q < \frac{\sqrt{6}}{4\sqrt{\pi P}}$)	0 or 1	0	1
$q_1 \neq q_2 \neq 0,\ q_3 = q_4 = 0$ case ($q_1 \geq \frac{3}{8\pi P q_2}$)	1	0	0
$q_1 \neq q_2 \neq 0,\ q_3 = q_4 = 0$ case ($q_1 < \frac{3}{8\pi P q_2}$)	0 or 1	0	1
$q_1 = q_2 = q_3 = q_4 \neq 0$ case (RN-AdS$_4$)	1	0	0
$q_1 = q_2 \neq 0,\ q_3 = q_4 \neq 0$ case	1	0	0
$q_1 \neq q_2 \neq q_3 \neq q_4 \neq 0$ case	1	0	0

Table 1. The topological number W, numbers of generation and annihilation points for the four-dimensional static multi-charge AdS black holes in gauged supergravity.

BH solution	W	Generation point	Annihilation point
$q_1 = q,\ q_2 = q_3 = 0$ case ($q \geq \frac{3}{4\pi P}$)	1	0	0
$q_1 = q,\ q_2 = q_3 = 0$ case ($q < \frac{3}{4\pi P}$)	0 or 1	0	1
$q_1 = q_2 \neq 0,\ q_3 = 0$ case	1	0	0
$q_1 = q_2 = q_3 \neq 0$ case (RN-AdS$_5$)	1	0	0
$q_1 \neq q_2 \neq q_3 \neq 0$ case	1	0	0

Table 2. The topological number W, numbers of generation and annihilation points for the five-dimensional static multi-charge AdS black holes in gauged supergravity.

charged AdS black holes have four and three independent electric charge parameters, respectively. In this study, we investigate the effect of the electric charge parameter configurations in static charged AdS black holes on the thermodynamic topological classification in the context of four- and five-dimensional gauged supergravity theories. For each black hole case, we examine various electric charge parameter configurations corresponding to several well-known truncated supergravity solutions and determine their topological numbers, respectively. The findings are summarized in tables 1–2. We find that the topological number of the static charged AdS black holes in four-dimensional K-K gauged supergravity theory is $W = 0$, while that of the static charged AdS black holes in four-dimensional gauged $-iX^0X^1$-supergravity and STU gauged supergravity theories, and five-dimensional EMDA gauged supergravity and STU gauged supergravity, as well as five-dimensional static charged AdS Horowitz-Sen black hole are both $W = 1$.

Furthermore, we observe a novel temperature-dependent thermodynamic topological phase transition that can happen in the four-dimensional static charged AdS black hole in EMDA gauged supergravity theory, the four-dimensional static charged AdS Horowitz-Sen black hole, and the five-dimensional static charged AdS black hole in K-K gauged supergravity theory. In other words, in our analysis of four-dimensional black hole cases, we demonstrate that when only two electric charge parameters are nonzero (with the other two set to zero), the thermodynamic topological number W exhibits a temperature-dependent behavior. Specifically, $W = 1$ when the two electric charge parameters are large, whereas for smaller electric charge parameters, W can either be $W = 0$ at lower temperatures or $W = 1$ at higher temperatures. This behavior is mirrored in the context of five-dimensional black hole cases within the framework of K-K gauged supergravity theory, where the static charged AdS black hole also displays a topological number ($W = 1$) for the large electric charge parameter. However, for the smaller electric charge parameters, the topological number W once again shows a temperature dependence: $W = 0$ at cold temperatures or $W = 1$ at high temperatures.

Therefore, we believe that the current studies related to the thermodynamic topological classes of black holes are still only the tip of the iceberg, and it is worthwhile to explore the nature of the topological number of black hole thermodynamics more deeply. A most related issue is to explore whether there are other black hole solutions in gauged supergravity theories that can also happen this novel temperature-dependent thermodynamic topological phase transition. As mentioned above, we only investigated the topological numbers of static charged AdS black holes in several famous four- and five-dimensional truncated supergravity models, and the static charged AdS black hole solutions in other supergravity theories can be investigated in the future, e.g., the S^3-supergravity model [110–112], the ST^2-supergravity model [113], etc.

Acknowledgments

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