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$4d$ $\mathcal{N}=1$ dualities from $5d$ dualities

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Abstract: We consider $5d$ KK dualities, that is multiple $5d$ gauge theories with the same $6d$ infinite coupling limit. We provide a prescription to associate $4d$ $\mathcal{N}=1$ quivers to the $5d$ dual quivers, such that the $4d$ quivers are also dual to each other. The $4d$ dualities are infrared dualities which can be checked matching global symmetry anomalies and in certain cases can be proven using basic Seiberg dualities sequentially. We also consider dualities obtained by Higgsing in two different ways the same $5d$ theory, in some simple examples.

Keywords: Brane Dynamics in Gauge Theories, Supersymmetry and Duality, Duality in Gauge Field Theories, Field Theories in Higher Dimensions

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1 Introduction and summary

Gauge theories living in space-time dimension greater than 4 are infrared free, nevertheless, in many supersymmetric cases, it is possible to argue that at strong coupling lives a ultraviolet superconformal field theory (SCFT) [1–3]. In this paper we are interested in 5d quiver gauge theories whose UV completion is actually a 6d SCFT. Such models go under the name of Kaluza-Klein (KK) theories.

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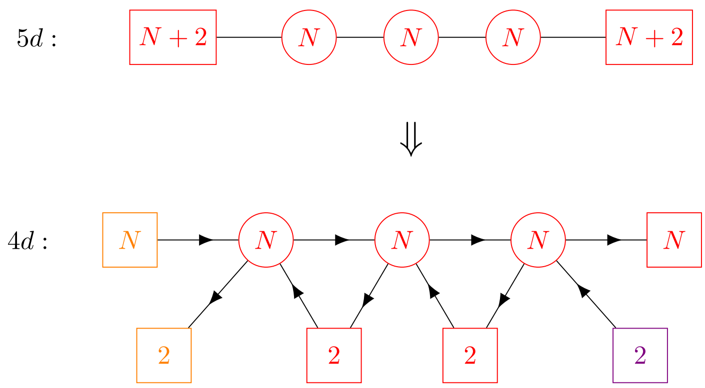
In many instances, there are more than one $5d$ gauge theories with the same infinite coupling SCFT ($5d$ or $6d$). This phenomenon goes under the name of *5d dualities*, even if the language is slightly improper, since the physical picture is really that the UV SCFT can be relevantly deformed in various different ways, triggering RG flows to different IR gauge theories.

A powerful tool to analyze the strong coupling behavior of $5d$ $\mathcal{N}=1$ gauge theories is given by Hanany-Witten branes setups [4], which in this case involve webs of 5-branes, a.k.a. pq-webs [5–7]. Pq-webs were used to study $5d$ dualities in [8–11]. Later, the pq-web technology to deal with KK theories was developed: [12–17] discuss many examples of different $5d$ $\mathcal{N}=1$ quiver gauge theories with the same $6d$ SCFT in the infinite coupling limit, described by Type IIA brane systems [18–20].

In this paper, we take $5d$ KK dualities and associate to them $4d$ $\mathcal{N}=1$ dualities. Starting from a $5d$ KK quiver with 8 supercharges, the $4d$ quiver has the same gauge structure (but in $4d$ the nodes are $\mathcal{N}=1$ 4 supercharges nodes), the same matter fields (but in $4d$ there are chiral multiplets instead of hyper multiplets). Morevoer,, for each bifundamental (a link connecting two gauge nodes) we add a “triangle”. A “triangle” means that if in $5d$ there is a bifundamental hyper connecting node A with node B, in $4d$ there is a chiral bifundamental going from node A to node B, a fundamental going from node B to a global SU(2) node, and a fundamental going from the global SU(2) node to node A. We also add a cubic SU(2) invariant superpotential term.

The previous prescription is illustrated by the following example, where round red nodes are SU gauge groups[1]



(1.1)

$$\mathcal{W} = 2\,\text{Triangles} + \textit{Flips}$$

Adding the triangles has the effect of reproducing the $5d$ *axial* symmetries (which are anomalous in $4d$ but not in $5d$, there is one U(1) axial symmetry for each link in the $5d$ quiver) and the $5d$ *instantonic* symmetries (which do not exist in $4d$, there is one U(1) instantonic symmetry for each node in the $5d$ quiver). With this prescription we are able to associate a $4d$ quiver to $5d$ quivers, in such a way that the rank of the global $4d$ symmetry is equal to the rank of the global $5d$ symmetry minus 2. We only consider quivers such that this prescription yields a $4d$ quivers without gauge anomalies.

[1] Our notation is explained at the end of this section.

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One remark about the prescription is that when we go from a hyper in $5d$ to a chiral in $4d$ we are free to choose it in the fundamental or anti-fundamental representation of the gauge group. The constraint on gauge anomaly cancellation fixes the choice of the representation. In the example (1.1) the $N+2$ hypers on the left in $5d$ have been split in N fundamentals and 2 anti-fundamentals. Our prescription produces a non-anomalous $4d$ $\mathcal{N}=1$ theory only if the rank of the SU nodes is the same for all nodes (we call this property *constant rank*). It would be interesting to generalize it to more general quivers, e.g. unitary tails with non-constant rank or ortho-symplectic quivers.

The $4d$ dualities we propose involve flipping fields, that is gauge singlets that enter the superpotential once and linearly, multiplying a gauge invariant mesonic or baryonic operator. Such flipping fields can be *moved* from one side of the other of the duality. Moving the flippers changes the global symmetry of the infrared SCFT. We do not have a preferred distribution for the flippers, hence we are not necessarily interested at the precise infrared global symmetry of our proposed $4d$ dualities.[2]

The main point of this paper is that starting from two $5d$ dual KK quivers, hence with the same $6d$ SCFT UV completion, the two $4d$ quivers constructed with the above prescription are infrared dual.

We discuss two classes of theories. In the first class, $R_{N,k}$, section 2 and 3, we are able to prove the $4d$ $\mathcal{N}=1$ dualities using basic Seiberg dualities [21, 22], that is we use deconfinement and basic dualities sequentially, in the same spirit of [23–32]. In the second class, section 4 and 5, we do not have such a proof and the proposed dualities are tested by matching the t'Hooft anomalies and the central charges and a few superconformal indexes. We conclude with section 6, where we discuss a set of theories obtained by Higgsing the class $R_{N,k}$.

Possible interpretation

In this paper we provide a prescription to obtain $4d$ duality from $5d$ dualities, but we do not investigate *why* our prescription works, that is why the $5d$ UV KK duality is transferred to a $4d$ IR duality. This is obviously an important question, so let us close this introduction with some speculations about a possible explanation.

There should be a connection between our prescription and the compactfication of $6d$ $(1,0)$ SCFT's on Riemann surfaces. Such compactification is usually done in two steps: first, one compactified the $6d$ brane system on a circle, getting a pq-web and the associated infrared $5d$ KK gauge theory (this is exactly what we are doing in this paper). Second, one constructs a $4d$ $\mathcal{N}=1$ supersymmetric duality wall [33–46]. This second step is very similar to our prescription, the difference is that we are adding the triangle terms and we are gauging the $5d$ gauge groups also in $4d$. Gauging such puncture symmetry should be related to gluing the two boundaries of the tube into a torus.

[2]Determining the exact infrared symmetry for various distribution of the flippers goes beyond the scope of this paper. However, in many cases there is a duality frame with vanishing superpotential, and no symmetry enhancement in the infrared. So it is easy to read off the global symmetry. For instance the models of section 3 are dual to the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ quiver on the r.h.s. of (3.18), which is an infrared free gauge theory, except for a few small values of k and N. One example with infrared symmetry enhancement is discussed at the end of section 4.

This suggests that our $4d$ gauge theories are related to their *mother* $6d$ SCFT on a Riemann surface with flux, but no punctures (a puncture would reveal itself as some global symmetry descending from a $5d$ gauge symmetry). More precisely, since the rank of the $4d$ global symmetry for our theories is the rank of the $6d$ global symmetry minus one, one can expect them to be a relevant superpotential deformation of the $4d$ SCFTS obtained by $6d$ SCFT on a Riemann surface with flux (which instead have the rank of the $4d$ global symmetry equal to the rank of the $6d$ global symmetry).

We leave an investigation of these issues to future work.

Notations

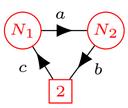
In this paper we use the quiver notation to denote the theories we are studying. The $4d$ quivers that we are going to study denote theories with 4-supercharges. Let us summarize here the notation that we will use.

Quiver diagrams.

- a circle node denotes a gauge group and the colour will specify which kind
  - a red node $\textcircled{N}$ denotes $\mathrm{SU}(N)$
  - a blue node $\textcircled{2N}$ denotes $\mathrm{USp}(2N)$
- a square node $\boxed{N}$ denotes a $\mathrm{SU}(N)$ flavor group[3]
- An oriented link between two nodes $\textcircled{N_1} \to \textcircled{N_2}$ denotes a chiral field in the fundamental representation of $\mathrm{SU}(N_2)$ and in the anti-fundamental representation of $\mathrm{SU}(N_1)$
- An arc on a node $\textcircled{N}$ denotes a chiral field in the antisymmetric representation

Flips. In this paper, an important role will be played by a class of gauge singlet chiral field σ called flippers. We say that σ *flips an operator* $\mathcal{O}$ when it enters the superpotential through the term $\sigma \cdot \mathcal{O}$. Most of the time, we will not draw these flippers in the quiver. Their presence can be inferred looking at the superpotential.

Superpotential. In theories with 4-supercharges, the holomorphic function $\mathcal{W}$ called the superpotential plays a really important role in the dynamics.

- A term in the superpotential is represented by a closed loop in the quiver notation. Often we will denote these terms by the geometrical shape and not by the actual names of the fields. For example, for a cubic term represented by the following quiver
  

  we will either write $\mathcal{W} = abc$ or $\mathcal{W} =$ Triangle

[3]Sometimes we will use a different colour for the flavor group. It happens when inside a quiver we have two (or more) identical nodes and we want an easier way to distinguish what symmetry we are talking about.

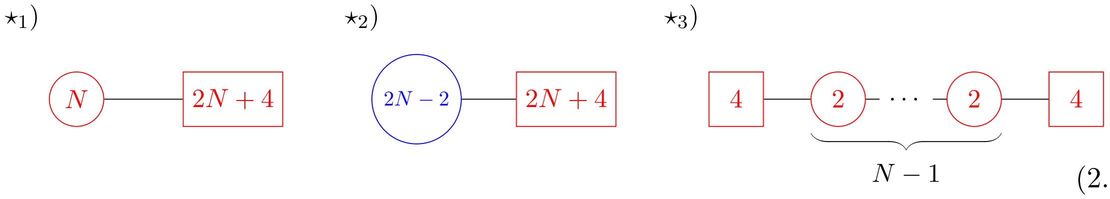
- Concerning the flippers interaction, instead of writing $\mathcal{W} = \sigma \cdot \mathcal{O}$ we will often use the following notation $\mathcal{W} = \text{Flip}[\mathcal{O}]$. Using this notation, we could avoid giving a name to the flipper σ. When we want to refer to a specific flipper we will use the notation Flipper$[\mathcal{O}]$ (or an explicit name if we gave one).

2 A simple class: $R_{N,2}$ and its two duals

In this section we consider a simple class of theories, which are special cases of the more general class studied in section 3.

2.1 5d triality

The first $5d$ dualities that we are studying combine into a triality:

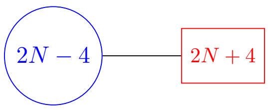


(2.1)

Throughout the paper, we denote SU nodes with red circles, Usp nodes with blue circles and global SU symmetries with square. To understand why the three theories in (2.1) are dual to each other, we recall the analysis done in [13–15]. We start from the $6d$ Type IIA brane setup figure 4. Then, we do a circle compactification and perfom T-duality along the compactified direction. We obtain a Type IIB brane setup. The $O8^-$ plane becomes two $O7^-$ planes and the $D8$ become $D7$. The resulting brane web, for $N = 3$, is shown on the left in figure 1. Then in order to read the gauge theory we have to resolve the $O7^-$ plane by 7-branes [47]. We have the choice to resolve the two $O7^-$ or just one. If we resolve the two $O7^-$ we get the brane web in the middle of figure 1. After pulling-out the 7-branes we obtain the SU(3) gauge theory with 10 fundamental hypers shown in the right of figure 1. The general N case corresponds to the left theory in (2.1) and we call it $R_{N,2}$. This name will be clear when we consider the generalization in the next section. Now, if we resolve only one $O7^-$ plane we obtain, after pulling out the 7-branes, the USp(4) gauge theory on the right of figure 2. It correponds to the middle theory in (2.1). If we perform an S-duality on the right figure of figure 1 (which amounts to a 90° rotation of the pq-web), we obtain figure 3 which describes $4F + \text{SU}(2) - \text{SU}(2) + 4F$,[4] quiver theory. It corresponds to the right theory in (2.1). Since the theories in (2.1) are either coming from the same brane system or are related by S-duality, it is clear that they are UV dual in the sense of completed by the same theory.

2.2 6d UV completion: (D_{N+2}, D_{N+2}) Minimal Conformal Matter

The $6d$ UV completion of the $5d$ theories in (2.1) is given by the following Type IIA brane setup [13–15]: This theory is called the (D_{N+2}, D_{N+2}) Minimal Conformal Matter. On the tensor branch, the system flows to the following gauge theory:



(2.2)

[4]The notation means that the first and the second SU(2) are coupled to 4 fundamental hypers and the horizontal bar represents a bifundamental hyper between the two gauge nodes.

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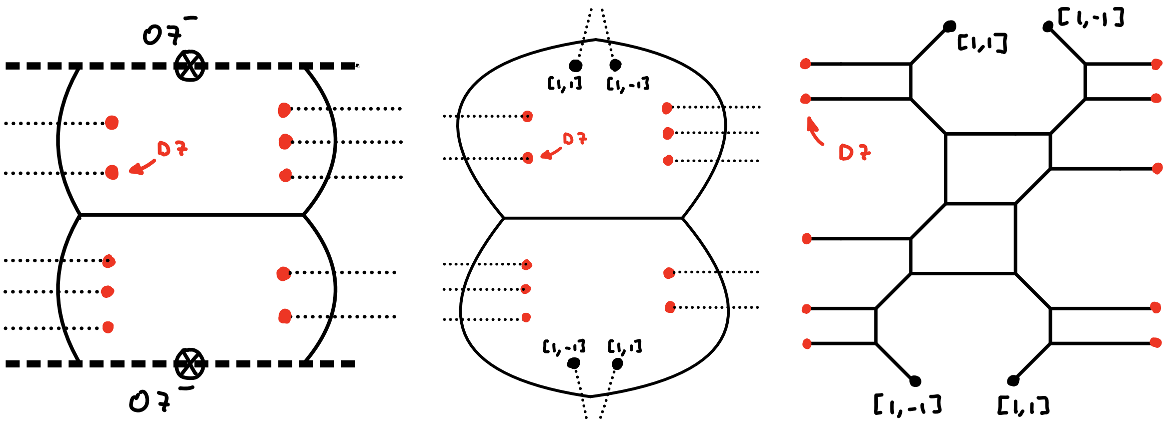


Figure 1. Resolution of the two $O7^-$ planes leading to $R_{3,2}$: a SU(3) gauge theory with 10 fundamentals.

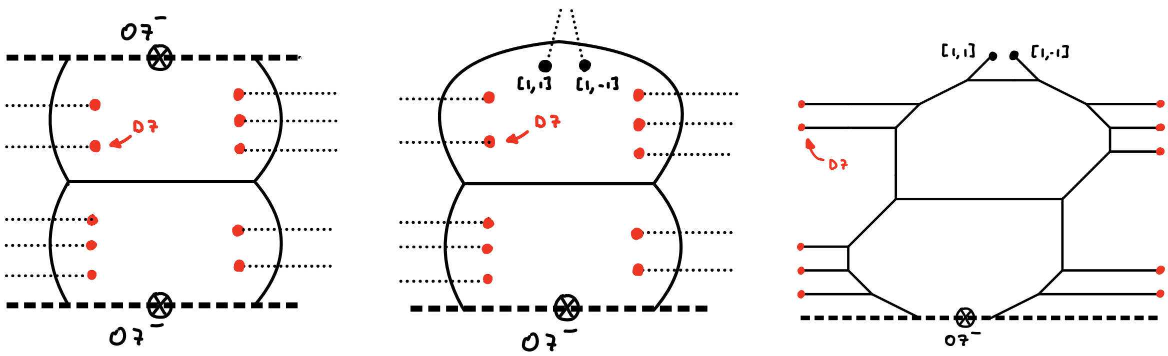


Figure 2. Resolution of one of the two $O7^-$ planes leading to the USp(4) with 10 fundamental hypers gauge theory.

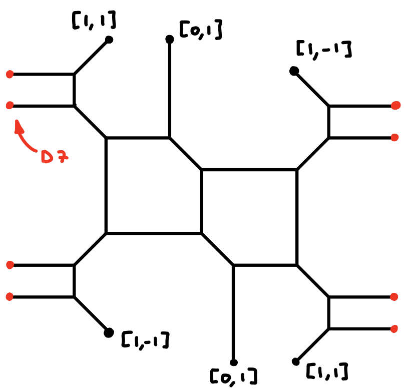


Figure 3. $4F + \mathrm{SU}(2) - \mathrm{SU}(2) + 4F$.

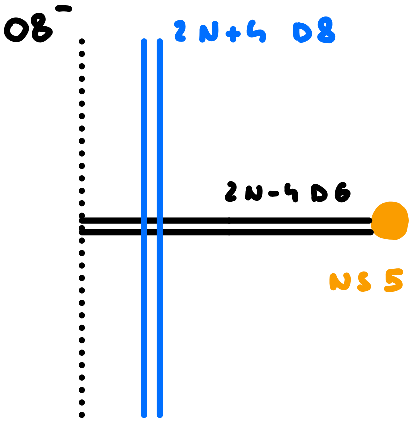
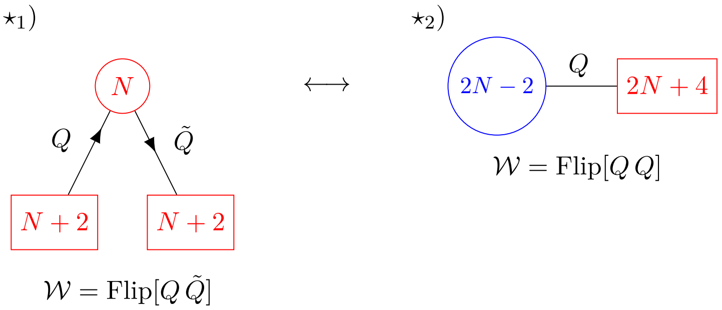


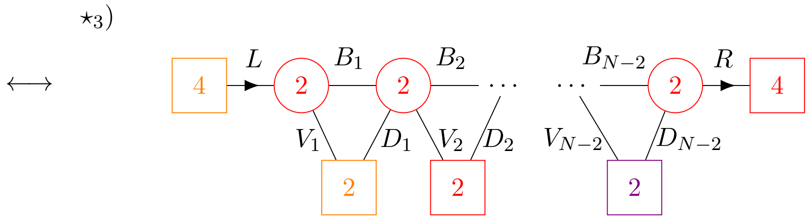
Figure 4. Type IIA brane setup corresponding to the $6d$ UV completion of $R_{N,2}$.

2.3 4d triality

We can now apply our prescription, already described in section 1. Starting from the left theory in (2.1), we replace the hypers in $5d$ by a chiral field in $4d$. We also have to split the $2N+4$ hypers into $N+2$ chirals and $N+2$ anti-chirals for the theory to be non-anomalous. We get the theory $\star_1$) in (2.3). We have also added a gauge singlet in the bifundamental of the $\mathrm{SU}(N+2)_Q \times \mathrm{SU}(N+2)_{\tilde{Q}}$ flavor symmetry and a flipping type superpotential. The role of this flipper is essential for the duality to be true as we will see in the following. Our procedure applied to the theory in the middle of (2.1) produces the theory $\star_2$) in (2.3). Finally, we focus to the right theory of (2.1). In this case, since we have a quiver in $5d$ our procedure tells us that for each bifundamental we have to associate a "triangle" with an explicit SU(2) symmetry. We obtain the following $4d$ quiver $\star_3$) in (2.4) with the correct set of flippers.



(2.3)



$$\mathcal{W} = (N-2)\,\mathrm{Triangles} + \mathrm{Flip}[L\,L] + \mathrm{Flip}[R\,R] + \sum_{i=1}^{N-2} \mathrm{Flip}[B_i\,B_i] \tag{2.4}$$

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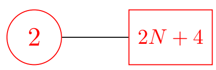
The mapping of the chiral ring generators between the different frames is

$$\begin{array}{ccccc} \star_1) & & \star_2) & & \star_3) \\ \begin{cases} \text{Flipper}[Q\,\tilde{Q}] \\ Q^N \\ \tilde{Q}^N \end{cases} & \Longleftrightarrow & \text{Flipper}[Q\,Q] & \Longleftrightarrow & \begin{cases} \text{Flipper}[L\,L] & \\ \text{Flipper}[R\,R] & \\ LB_1\dots B_{N-2}R & \\ LB_1\dots B_iV_{i+1} & i=0,\dots,N-3 \\ D_jB_{j+1}\dots B_{N-2}R & j=1,\dots,N-2 \\ \text{Flipper}[B_k\,B_k] & k=1,\dots N-2 \\ D_iB_{i+1}\dots B_jV_j & i=1,\dots,N-3\;\& \\ & j=i+1,\dots,N-2 \end{cases} \end{array} \tag{2.5}$$

We have to understand the mapping (2.5) in the following way. In the UV, the manifest global symmetries in $\star_1)$, $\star_2)$ and $\star_3)$ are different. In the IR, there is the emergence of the global symmetry. Therefore some operators in the UV will combined into an operator transforming into the bigger symmetry group. In our case the global symmetry group[5] in the IR is $SU(2N+4)$. Then we claim that in the frame $\star_1)$ the three operators Flipper$[Q\,\tilde{Q}]$, Q^N and $\tilde{Q}^N$ will combine into an operator that transforms into an antisymmetric representation of the emergent $SU(2N+4)$ global symmetry group. One necessary condition to make sense is that the number of degrees of freedom (d.o.f) corresponds to the dimension of the representation. In this case Flipper$[Q\,\tilde{Q}]$ contains N^2+4N+4 d.o.f, Q^N and $\tilde{Q}^N$ $\frac{1}{2}(N+2)(N+1)$ each. The sum equals to $2N^2+7N+6$ which indeed correspond to the dimension of the antisymmetric representation of $SU(2N+4)$. The same kind of counting works for the frame $\star_3)$.

2.4 Proof of the 4d dualities

In this subsection, we provide a "proof" of the $4d$ triality (2.3)–(2.4). By "proof", we mean the use of a sequence of well-established dualities as in [28, 30]. Starting from $\star_1)$ and apply the Seiberg duality [21], we obtain

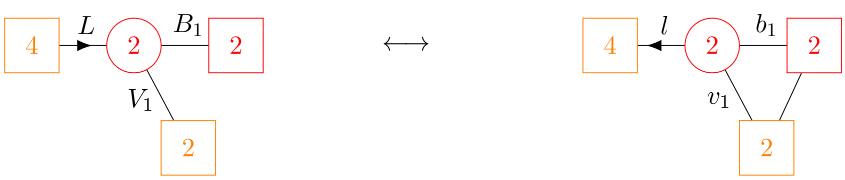


$$\mathcal{W}=0 \tag{2.6}$$

We see that the role of the flipper in $\star_1)$ in (2.3) is to give a mass to the singlet present in the Seiberg duality and therefore get $\mathcal{W}=0$ in (2.6).

Now starting from $\star_2)$ in (2.3) and applying the Intriligator-Pouliot (IP) duality [22], we once again get (2.6). This implies that also $\star_1)$ and $\star_2)$ are dual.

More work has to be done in order to prove that also $\star_3)$ is dual. It goes as follows. We first apply the Csaki-Schmaltz-Skiba-Terning (CSST) duality [48] to the left SU(2). The form of this duality that is useful for our purpose is the following



$$\mathcal{W}=0 \qquad\qquad \mathcal{W}=\text{Flip}[l\,l]+\text{Flip}[b_1b_1]+\text{Flip}[v_1v_1]+\text{Triangle} \tag{2.7}$$

[5] In this article, we don't pay attention to the global structure of the global symmetry.

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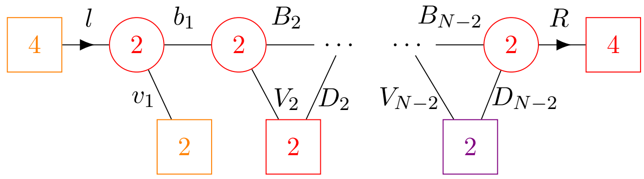
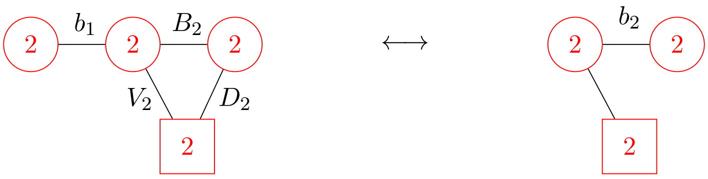
The important effect of this duality is to give a mass to the field D_1 in (2.4). Indeed, we are left with



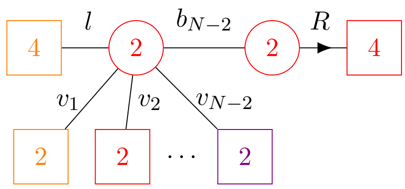
$$\mathcal{W} = (N-3)\,\text{Triangles} + \text{Flip}[R\,R] + \sum_{i=2}^{N-2} \text{Flip}[B_i\,B_i] \tag{2.8}$$

Now we realize that the second SU(2) is coupled to 6 chirals and therefore we can use the IP confinement for this SU(2) [22]. The form useful of this confinement is the following



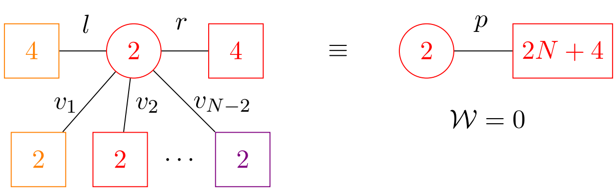
$$\mathcal{W} = \text{Flip}[b_1\,b_1] + \text{Flip}[B_2\,B_2] \qquad\qquad \mathcal{W} = \text{Flip}[b_2\,b_2] \tag{2.9}$$

After the confinement of the second SU(2) we can see that the next one on the right is also coupled to 6 chirals and therefore we can iterate the use of (2.9). We can do $(N-4)$ more s-confining (2.9). We get



$$\mathcal{W} = \text{Flip}[b_{N-2}\,b_{N-2}] + \text{Flip}[R\,R] \tag{2.10}$$

The last SU(2) is once again coupled to 6 chirals and therefore we can use for the last time the confinement [49]. We end up with



$$\mathcal{W} = 0 \tag{2.11}$$

To summarize, starting from $\star_3$) in (2.4) and doing the CSST duality followed by $(N-2)$ s-confining duality we get (2.6) which proves the $4d$ triality (2.3)–(2.4).

Notice that the duality between $\star_3$) and (2.11) is one of the simplest instances of the $4d$ *mirror symmetry* of [50–52], and it uplifts the $3d$ mirror symmetry between U(1) with N flavors and the linear Abelian quiver $\mathrm{U}(1)^{N-1}$.

Now using the proof we can justify the mapping (2.5). Indeed we can obtain the mapping from the frame (2.4) to the frame (2.11) by following the mapping of the basic dualities (CSST and the IP confinement). We get

$$\begin{array}{ccll} (2.4) & & (2.11) & \\ \begin{cases} \mathrm{Flipper}[L\,L] \\ \mathrm{Flipper}[R\,R] \\ LB_1 \dots B_{N-2}R \\ LB_1 \dots B_i V_{i+1} \\ D_j B_{j+1} \dots B_{N-2}R \\ \mathrm{Flipper}[B_k\, B_k] \\ D_i B_{i+1} \dots B_j V_j \end{cases} & \Longleftrightarrow & \begin{cases} l\,l \\ r\,r \\ l\,r \\ l\,v_{i+1} \\ r\,v_j \\ v_k^2 \\ v_i\,v_j \end{cases} & \begin{array}{l} \\ \\ \\ i = 0, \dots, N-3 \\ j = 1, \dots, N-2 \\ k = 1, \dots N-2 \\ i = 1, \dots, N-3\,\& \\ j = i+1, \dots, N-2 \end{array} \end{array} \tag{2.12}$$

Then since there is no superpotential in (2.11) all the operators in the r.h.s. of the mapping (2.12) combine into pp which transforms in the antisymmetric representation of the $\mathrm{SU}(2N+4)$ global symmetry as previously claimed.

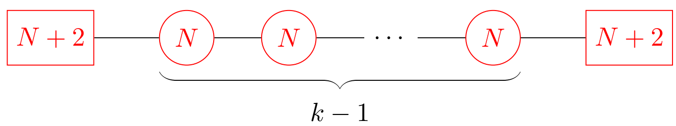
3 Rectangular pq-webs: the $R_{N,k}$ theories

3.1 5d theories and duality $R_{N,k} \leftrightarrow R_{k,N}$

$R_{N,k}$ theories

In this section, we generalize the discussion of section 2 by considering the following two-parameter family of $5d$ theories, that we call $R_{N,k}$:

$R_{N,k}$:



(3.1)

The brane web associated to the $R_{N,k}$ is shown on the left of figure 5. We can perform S-duality on this brane system and we obtain the web on the right of figure 5. It is not completely obvious how to read off the gauge theory for the S-dual theory. Let us illustrate the case of $N = 2$ and $k = 3$, figure 6: First, we pull out the $[0, 1]$ 7-branes through the D5 branes. Due to the Hanany-Witten effect, we get The second step is to pull out the $[1, 1]$ and $[1, -1]$ 7-branes through the D5 branes. We get The final step is to pull out the $[1, 1]$ and $[1, -1]$ 7-branes through the NS5 brane. We get It is easy to generalize the previous discussion and we find that the brane system on the right of figure 5 describes $(k+2)F + \mathrm{SU}(k)^{N-1} + (k+2)F$ gauge theory which corresponds to $R_{k,N}$. This result is valid for arbitrary N and k. Therefore we have shown that $R_{N,k}$ and $R_{k,N}$ are UV duals.

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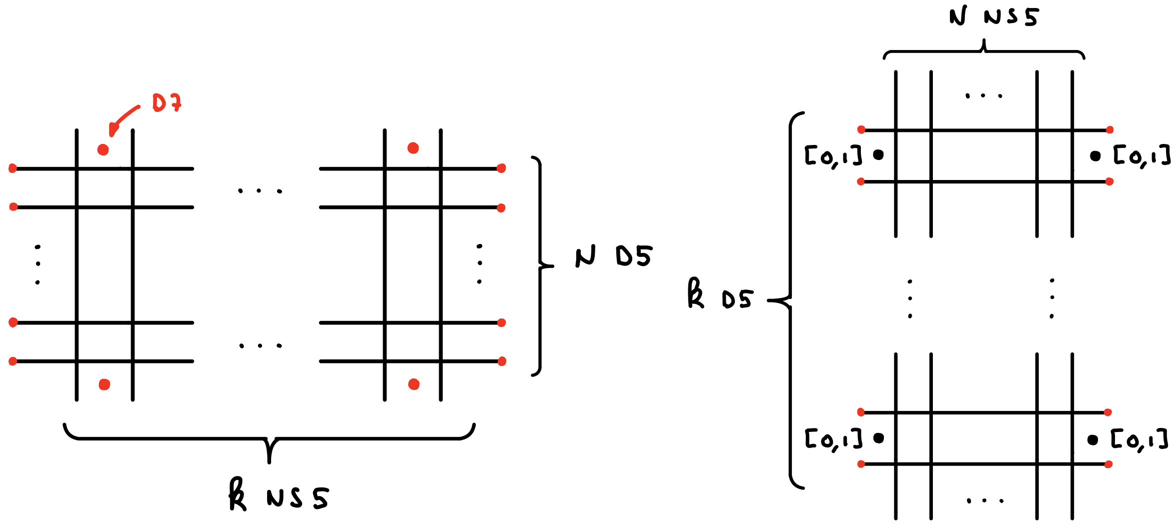


Figure 5. Brane setup for $(N+2)F+\mathrm{SU}(N)^{k-1}+(N+2)F$ on the left and for $(k+2)F+\mathrm{SU}(k)^{N-1}+(k+2)F$ for the right. The two brane systems are S-dual.

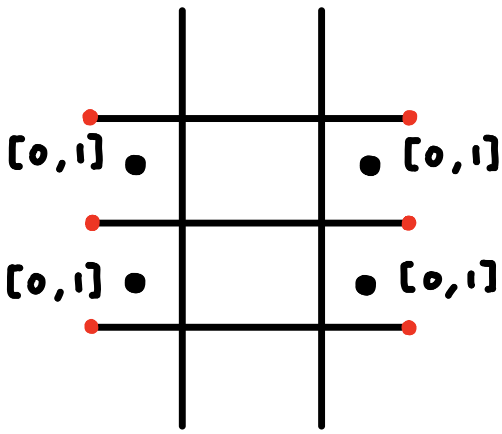


Figure 6. The S-dual brane system of $4F+\mathrm{SU}(2)^2+4F$.

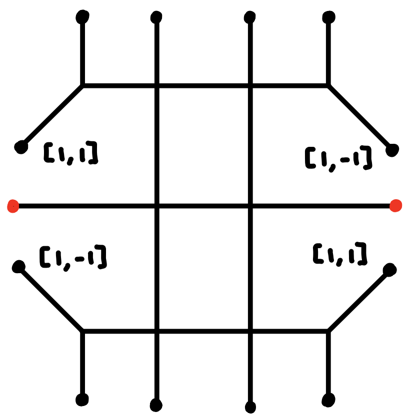


Figure 7. Brane system after pulling out the $[0,1]$ 7-branes of figure 6.

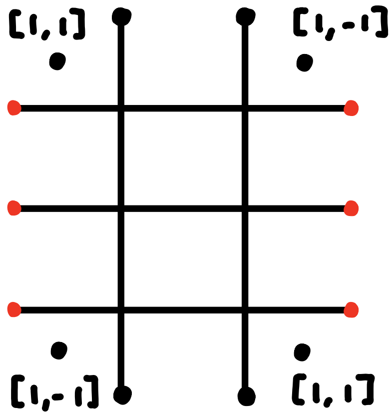


Figure 8. Brane system after pulling out the $[1,1]$ and $[1,-1]$ 7-branes of figure 7 through the D5.

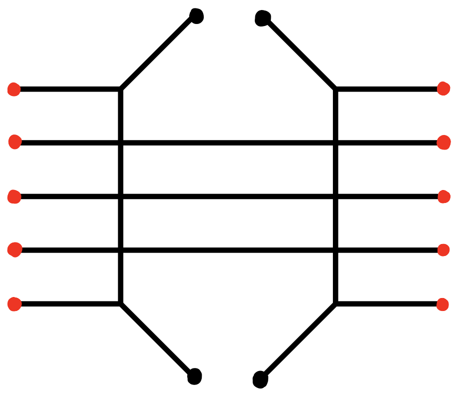


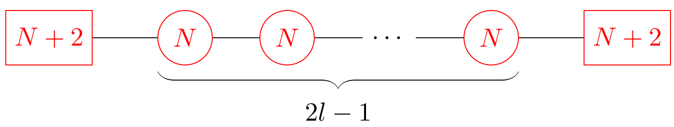
Figure 9. Brane system after pulling out the $[1,1]$ and $[1,-1]$ 7-branes of figure 8 through the NS5. In this frame, it is easy to read off the gauge theory that is $\mathrm{SU}(3)+10F$.

As in the previous section, for general k and N, there is a third dual frame involving an Usp gauge group or an antisymmetric field. While this dual frame will not play a role in $4d$, let us discuss it for completeness. In order to get this $5d$ UV dual, we assume $N \geq k$ and distinguish between the case k even and k odd (as we will discuss later, also the $6d$ UV completion depends on the parity of this parameter).

k even: $k = 2l$

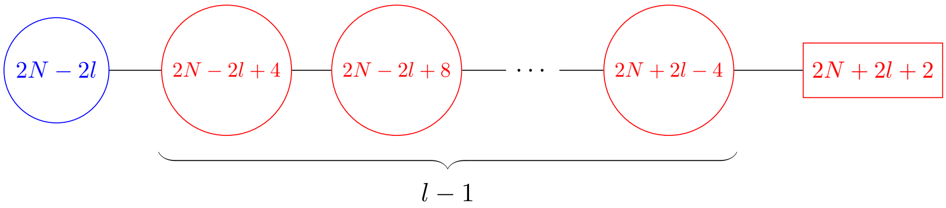
The $5d$ triality reads

$\star_1$)



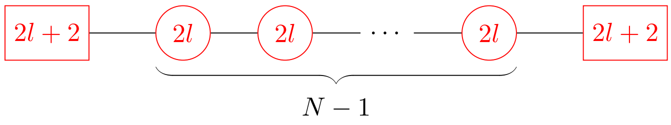
(3.2)

$\star_2$)



(3.3)

$\star_3$)



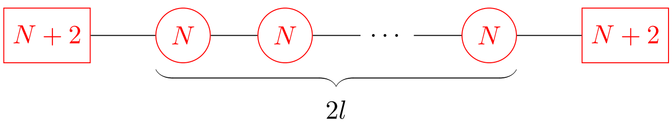
(3.4)

First remark, if we put $l = 1$ we recover the triality studied in section 2.1. The logic to understand why these theories are UV duals is the same as before. We start from the brane system in figure 10 describing a 6d theory. Then we compactify this system into an S^1 and we perform T-duality along the compactified dimension. The $O8^-$ plane becomes two $O7^-$. Then we have the choice to resolve one or two $O7$'s. If we resolve two, we get the theory $\star_1$) and if we resolve only one, we get $\star_2$). Finally, as we have seen, $\star_1$) and $\star_3$) are S-dual one to each other. We have been very brief about the derivation because all the details can be found in [14, 15].

k odd: $k = 2l + 1$

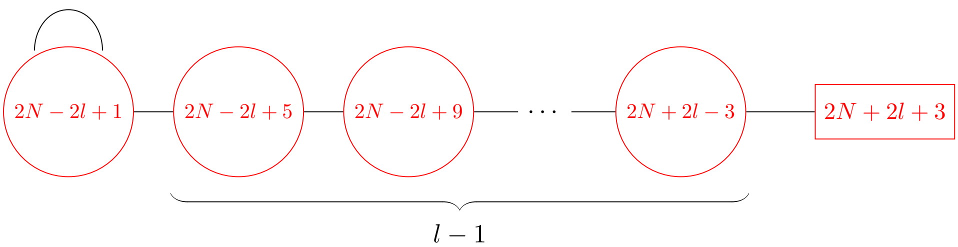
Similar arguments [14, 15] show that the triality reads

$\star_1$)



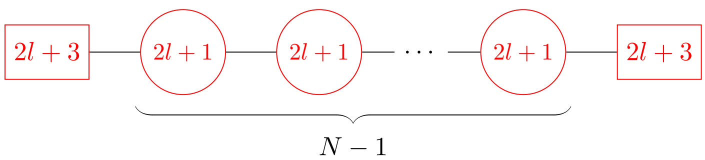
(3.5)

$\star_2$)



(3.6)

$\star_3$)



(3.7)

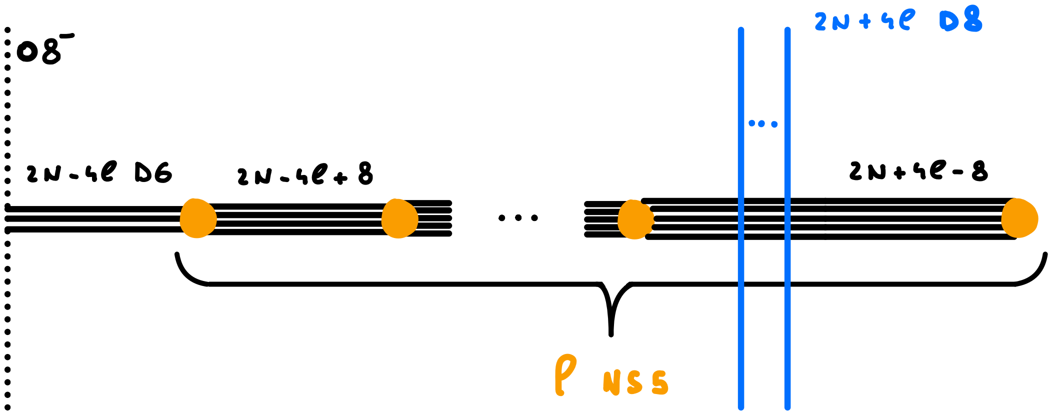


Figure 10. Type IIA brane setup corresponding to the 6d UV completion of $R_{N,2l}$ theory.

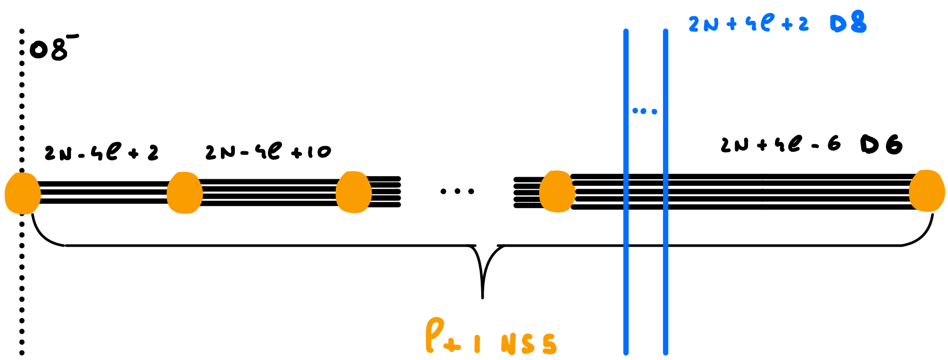


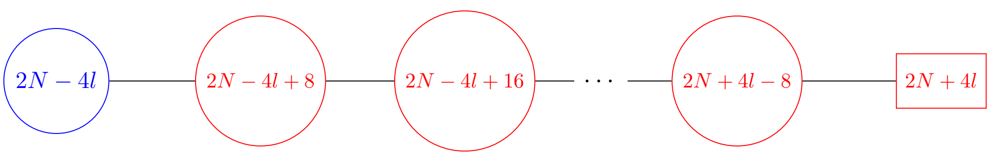
Figure 11. Type IIA brane setup corresponding to the 6d UV completion of $R_{N,2l+1}$ theory.

3.2 6d UV completion

The 6d UV completion of the theory $R_{N,k}$, $N \geq k$, theory depends on the parity of k.

k even: $k = 2l$

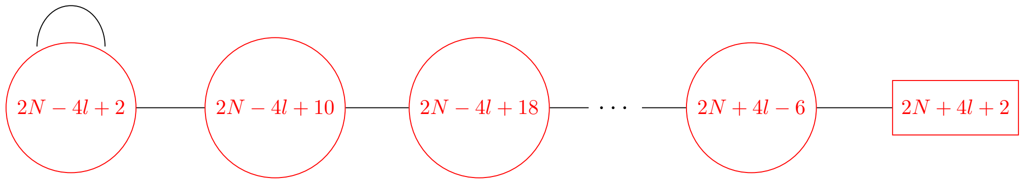
The 6d completion is given by the following Type IIA brane setup [14, 15]: The gauge theory corresponding to this brane system is a linear quiver with one USp gauge node and $l-1$ SU gauge nodes:



(3.8)

k odd: $k = 2l+1$

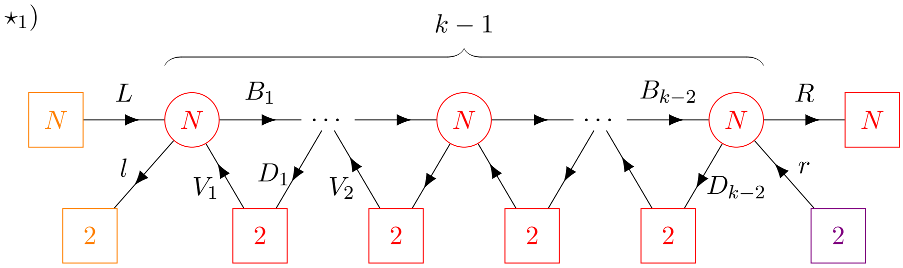
The 6d completion is given by the following Type IIA brane setup [14, 15]: The gauge theory corresponding to this brane system is a linear quiver with l SU gauge nodes and an antisymmetric hyper attached to the first node:



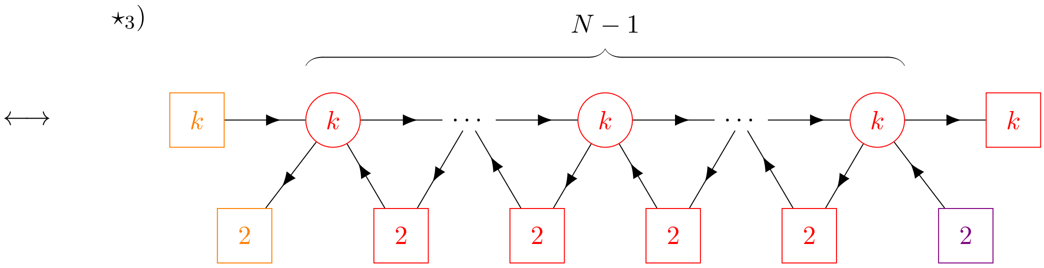
(3.9)

3.3 4d duality

Having recalled the two $5d$ UV trialities (3.2)–(3.4) and (3.5)–(3.7) we can run our prescription of section 1. We quickly realize that for generic l the theories $\star_2$) ((3.3) and (3.6)), that is the ones involving an Usp node or an antisymmetric, cannot be made non-anomalous in $4d$, this is because the ranks of the chain of SU nodes are not constant.[6] Therefore, we do not consider these theories and treat uniformly the case k even and k odd. The proposed $4d$ IR duality that we obtain using our prescription is the following:



$$\mathcal{W} = (k-2)\,\text{Triangles} + \text{Flip}[L\,l; R\,r; L^N; R^N; L\,B_1\dots B_{k-2}\,R; l^2\,B_1^{N-2}\dots B_{k-2}^{N-2}\,r^2] + \sum_{i=1}^{k-2}\text{Flip}[B_i^N] \tag{3.10}$$



$$\mathcal{W} = (N-2)\,\text{Triangles} + \text{Flip}[\tilde{L}\,\tilde{l}; \tilde{R}\,\tilde{r}; \tilde{L}^k; \tilde{R}^k; \tilde{L}\,\tilde{B}_1\dots\tilde{B}_{N-2}\,\tilde{R}; \tilde{l}^2\,\tilde{B}_1^{k-2}\dots\tilde{B}_{N-2}^{k-2}\,\tilde{r}^2] + \sum_{i=1}^{N-2}\text{Flip}[\tilde{B}_i^k] \tag{3.11}$$

We have denoted the fields appearing in (3.11) with a tilde. We remark that in order to get a non-anomalous $4d$ quiver we have to split the flavor symmetries. For example, SU$(N+2)$ is split into SU(2) and SU(N). The expression of the superpotential in (3.10) and (3.11) will

[6]It is an interesting question if the prescription can be generalized to include quivers with non constant ranks for the SU nodes. See section 6 for a first step in this direction.

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be justified in the next section. The mapping of the chiral ring generators is

$$\star_1) \qquad\qquad\qquad\qquad \star_3)$$

$$\begin{cases} \text{Flipper}[LB_1\dots B_{k-2}R] \\ L^{N-2}\,B_1^{N-2}\dots B_{k-2}^{N-2}\,r^2 \\ l^2\,B_1^{N-2}\dots B_{k-2}^{N-2}\,R^{N-2} \end{cases} \Longleftrightarrow \begin{cases} \tilde{D}_i\,\tilde{B}_{i+1}^{k-1}\dots\tilde{B}_j^{k-1}\,\tilde{V}_{j+1} \quad i=1,\dots,N-4\,\&\,j=i+1,\dots N-3 \\ \tilde{l}\,\tilde{B}_1^{k-1}\dots\tilde{B}_i^{k-1}\,\tilde{V}_{i+1} \quad i=1,\dots,N-3 \\ \tilde{D}_{j+1}\,\tilde{B}_{i+2}^{k-1}\dots\tilde{B}_{N-2}^{k-1}\,\tilde{r} \quad j=0,\dots,N-4 \\ \tilde{l}\tilde{V}_1;\,\tilde{D}_1\tilde{V}_2;\,\tilde{D}_2\tilde{V}_3;\dots;\,\tilde{D}_{N-3}\tilde{V}_{N-2};\,\tilde{D}_{N-2}\tilde{r} \\ \text{Flipper}[\tilde{B}_i^k] \quad i=1,\dots,N-2 \\ \tilde{l}\,\tilde{B}_1^{k-1}\dots\tilde{B}_{N-2}^{k-1}\,\tilde{r} \\ \text{Flipper}[\tilde{L}^k];\,\text{Flipper}[\tilde{R}^k] \end{cases} \tag{3.12}$$

The total number of operators is $2N^2 - N$ on both sides.

$$\begin{cases} D_i\,B_{i+1}^{N-1}\dots B_j^{N-1}\,V_{j+1} \quad i=1,\dots,k-4\,\&\,j=i+1,\dots k-3 \\ l\,B_1^{N-1}\dots B_i^{N-1}\,V_{i+1} \quad i=1,\dots,k-3 \\ D_{j+1}\,B_{i+2}^{N-1}\dots B_{k-2}^{N-1}\,r \quad j=0,\dots,k-4 \\ lV_1;\,D_1V_2;\,D_2V_3;\dots;\,D_{k-3}V_{k-2};\,D_{k-2}r \\ \text{Flipper}[B_i^N] \quad i=1,\dots,k-2 \\ l\,B_1^{N-1}\dots B_{N-2}^{N-1}\,r \\ \text{Flipper}[L^N];\,\text{Flipper}[R^N] \end{cases} \Longleftrightarrow \begin{cases} \text{Flipper}[\tilde{L}\tilde{B}_1\dots\tilde{B}_{N-2}\tilde{R}] \\ \tilde{L}^{k-2}\,\tilde{B}_1^{k-2}\dots\tilde{B}_{N-2}^{k-2}\,\tilde{r}^2 \\ \tilde{l}^2\,\tilde{B}_1^{k-2}\dots\tilde{B}_{N-2}^{k-2}\,\tilde{R}^{k-2} \end{cases} \tag{3.13}$$

The total number of operators is $2k^2 - k$ on both sides.

$$\begin{cases} \text{Flipper}[L\,l];\,\text{Flipper}[R\,r] \\ L^{N-1}\,B_1^{N-1}\dots B_i^{N-1}\,V_{i+1} \quad i=0,\dots,k-3 \\ D_{k-2-i}\,B_{k-2-i+1}^{N-1}\dots B_{k-2}^{N-1}\,R^{N-1} \quad i=0,\dots,k-3 \\ L^{N-1}\,B_1^{N-1}\dots B_{k-2}^{N-1}\,r \\ l\,B_1^{N-1}\dots B_{k-2}^{N-1}\,R^{N-1} \end{cases} \Longleftrightarrow \begin{cases} \text{Flipper}[\tilde{L}\,\tilde{l}];\,\text{Flipper}[\tilde{R}\,\tilde{r}] \\ \tilde{L}^{k-1}\,\tilde{B}_1^{k-1}\dots\tilde{B}_i^{k-1}\,\tilde{V}_{i+1} \quad i=0,\dots,N-3 \\ \tilde{D}_{N-2-i}\,\tilde{B}_{N-2-i+1}^{k-1}\dots\tilde{B}_{N-2}^{k-1}\,\tilde{R}^{k-1} \quad i=0,\dots,N-3 \\ \tilde{L}^{k-1}\,\tilde{B}_1^{k-1}\dots\tilde{B}_{N-2}^{k-1}\,\tilde{r} \\ \tilde{l}\,\tilde{B}_1^{k-1}\dots\tilde{B}_{N-2}^{k-1}\,\tilde{R}^{k-1} \end{cases} \tag{3.14}$$

The total number of operators is $4kN$ on both sides.

$$\text{Flipper}[l^2\,B_1^{N-1}\dots B_{k-2}^{N-1}\,r^2] \Longleftrightarrow \text{Flipper}[\tilde{l}^2\,\tilde{B}_1^{k-1}\dots\tilde{B}_{N-2}^{k-1}\,\tilde{r}^2] \tag{3.15}$$

The total number of operators is 1 on both sides.

The way to read this mapping is the same as in (2.5). In the IR there is an enhancement of the global symmetry. The claim is that all the operators inside a bracket will combine, in the IR, into an operator transforming in a specific representation of the emergent global symmetry. The justification on the mapping will be clearer with the proof of the duality.

3.4 Proof of the 4d duality

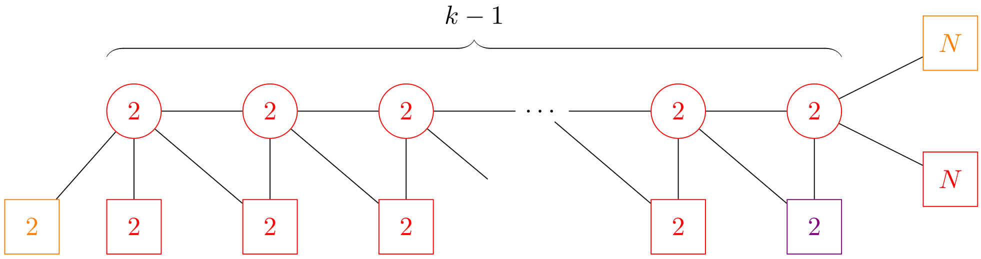
Start with $\star_1$) and do the following operations:

- k-1 Seiberg dualities on the SU nodes from left to right. This step transforms all the SU(N) nodes into SU(2) and the flavor SU(N) is moved to the right. We get (3.16).
- CSST duality (2.7) on the left SU(2) that will give a mass to the adjacent vertical field as in (2.8), we obtain (3.17).

- $k-3$ confinements (2.9). We end up with (3.18).

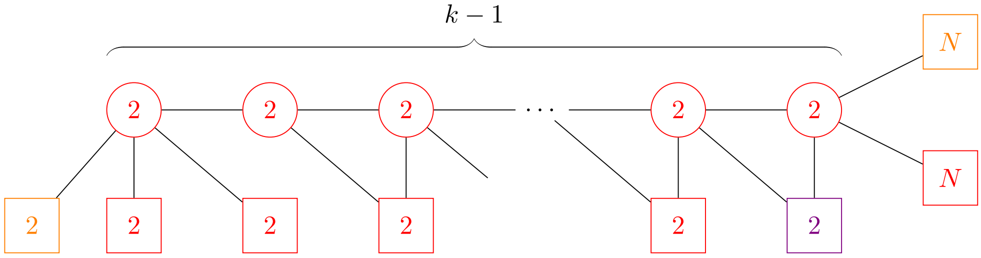
In terms of the quiver, we get the following sequence

$\star_1$)



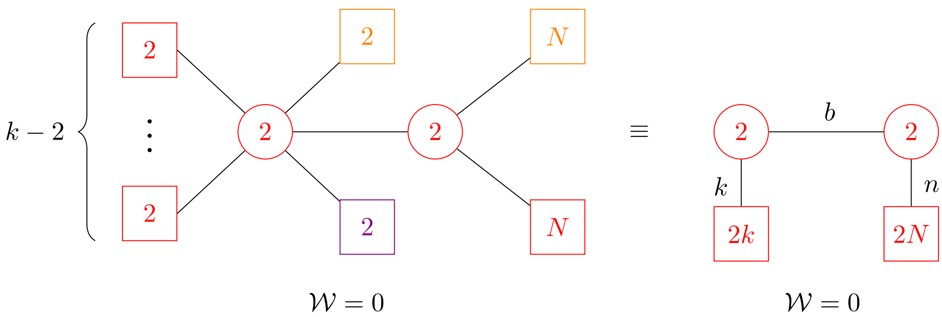
(3.16)

$\star_1$)



(3.17)

$\star_1$)



(3.18)

Once again following the mapping of the basic dualities we can see that the operators in the l.h.s. of (3.12)–(3.15) are mapped in the frame (3.18) in the following way

$$
\begin{array}{ccc}
c\star_1) & & (3.18) \\
\left\{\begin{array}{l} \text{Flipper}[LB_1\dots B_{k-2}R] \\ L^{N-2}\,B_1^{N-2}\dots B_{k-2}^{N-2}\,r^2 \\ l^2\,B_1^{N-2}\dots B_{k-2}^{N-2}\,R^{N-2} \end{array}\right. & \Longleftrightarrow & nn
\end{array}
\tag{3.19}
$$

The total number of operators on both sides is $2N^2 - N$.

$$\begin{cases} D_i\, B_{i+1}^{N-1} \dots B_j^{N-1}\, V_{j+1} \quad i = 1, \dots, k-4\,\&\, j = i+1, \dots k-3 \\ l\, B_1^{N-1} \dots B_i^{N-1}\, V_{i+1} \quad i = 1, \dots, k-3 \\ D_{j+1}\, B_{i+2}^{N-1} \dots B_{k-2}^{N-1}\, r \quad j = 0, \dots, k-4 \\ lV_1;\, D_1V_2;\, D_2V_3; \dots;\, D_{k-3}V_{k-2};\, D_{k-2}r \\ \text{Flipper}[B_i^N] \quad i = 1, \dots, k-2 \\ l\, B_1^{N-1} \dots B_{N-2}^{N-1}\, r \\ \text{Flipper}[L^N];\, \text{Flipper}[R^N] \end{cases} \qquad \Longleftrightarrow \qquad kk \tag{3.20}$$

The total number of operators on both sides is $2k^2 - k$.

$$\begin{cases} \text{Flipper}[L\, l];\, \text{Flipper}[R\, r] \\ L^{N-1}\, B_1^{N-1} \dots B_i^{N-1}\, V_{i+1} \quad i = 0, \dots, k-3 \\ D_{k-2-i}\, B_{k-2-i+1}^{N-1} \dots B_{k-2}^{N-1}\, R^{N-1} \quad i = 0, \dots, k-3 \\ L^{N-1}\, B_1^{N-1} \dots B_{k-2}^{N-1}\, r \\ l\, B_1^{N-1} \dots B_{k-2}^{N-1}\, R^{N-1} \end{cases} \qquad \Longleftrightarrow \qquad kbn \tag{3.21}$$

The total number of operators on both sides is $4kN$.

$$\text{Flipper}[l^2\, B_1^{N-1} \dots B_{k-2}^{N-1}\, r^2] \Longleftrightarrow \quad bb \tag{3.22}$$

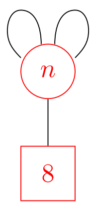
The total number of operators on both sides is 1. Since in (3.18) we reach a frame where N and k enter symmetrically, it proves the duality $T_{N,k} \leftrightarrow T_{k,N}$ in 4d, that is (3.10) $\leftrightarrow$ (3.11) and the mapping (3.12)–(3.15).

4 Systems with two $O7$ planes, a simple class: $A_{n,1}$

4.1 5d duality

In this section and the next one, we want now to test our prescription with another family of theories. We consider theories which involve two $O7^-$ planes in the Type IIB brane setup. For each $O7$, the 5d quivers contain either an SU gauge group with antisymmetric or an Usp gauge node, depending on whether a $NS5$ is stuck at the orientifold plane or not. We are going to see that also in this case our prescription works and leads to 4d dualities. Contrary to the previous family, we are able to prove the 4d dualities using basic Seiberg dualities. The 4d dualities that we obtain are more complicated, but are still a rather non-trivial check of the 5d-to-4d prescription.

Concretely, in this section we study the following 5d KK theory, that we call $A_{n,1}$.



(4.1)

$A_{n,1}$ has a 5d dual theory. The form of the dual depends on the parity of n.

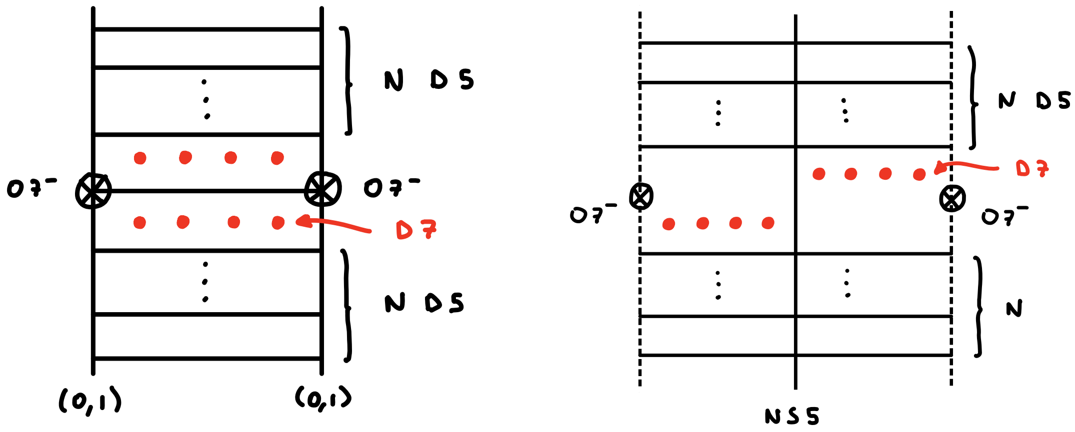
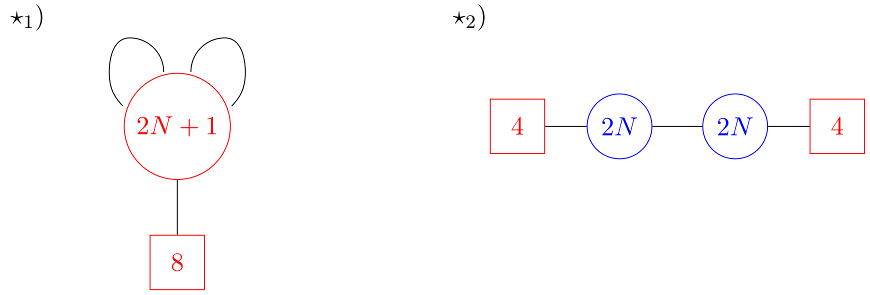


Figure 12. Brane setup for $2A + \mathrm{SU}(2N+1) + 8F$ on the left with an $NS5$ stuck on each $O7^-$ plane and for $4F + \mathrm{USp}(2N)^2 + 4F$ on the right.

n odd: 5d duality $A_{2N+1,1} \leftrightarrow U_{2N+1,1}$

We call $U_{2N+1,1}$ the dual of $A_{2N+1,1}$, the statement is that the following two theories are UV dual

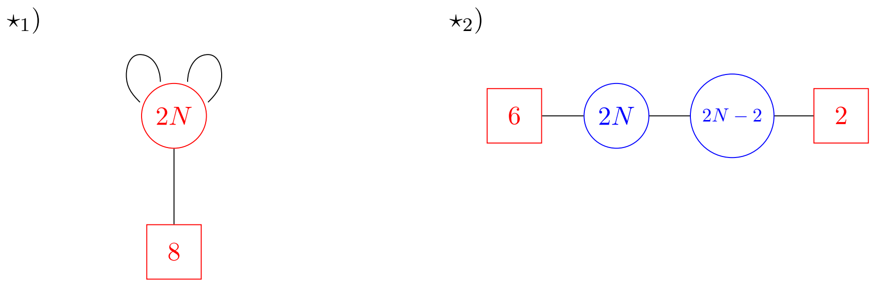


(4.2)

The analysis showing the UV duality (4.2) is morally the same as in the previous family. We have to start with the 6d type IIA brane setup shown in figure 14, do the circle reduction, T-duality and the resolution of the $O7$-planes. Then, we have a choice on how to resolve the $O7$'s. Depending on this choice, we get two different Type IIB brane setups, see figure 12, which justifies the duality (4.2). The details can be found in [14] and will not be reproduced here.

n even: 5d duality $A_{2N,1} \leftrightarrow U_{2N,1}$

We call $U_{2N,1}$ the dual of $A_{2N,1}$. This duality appears in [53] and corresponds to



(4.3)

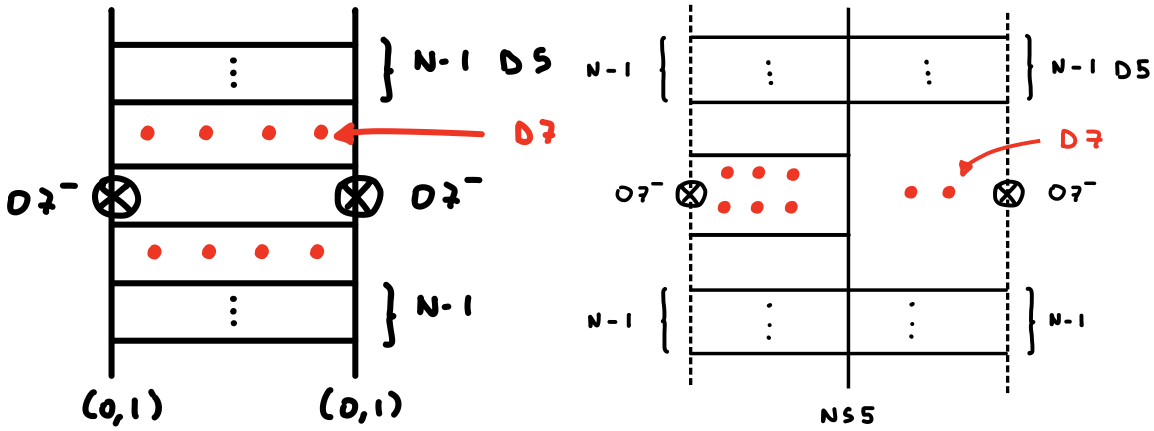


Figure 13. Brane setup for $2A + \mathrm{SU}(2N) + 8F$ on the left with an $NS5$ stuck on each $O7^-$ plane and for $4F + \mathrm{USp}(2N) - Usp(2N-2) + 2F$ on the right.

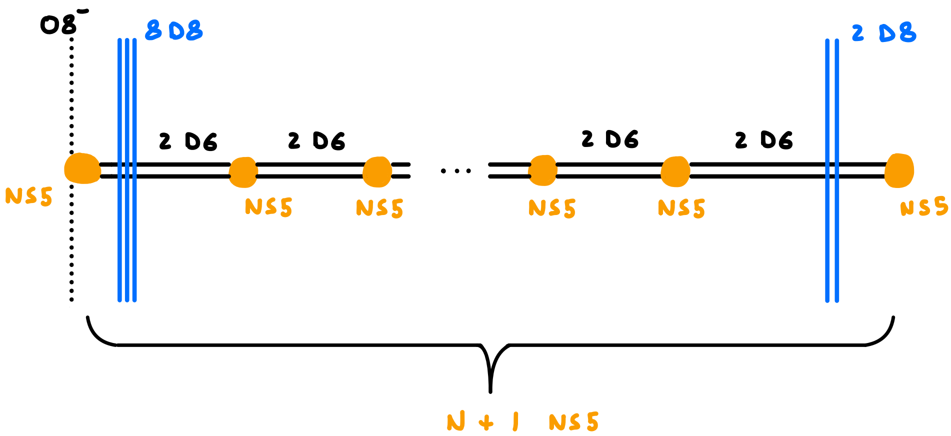
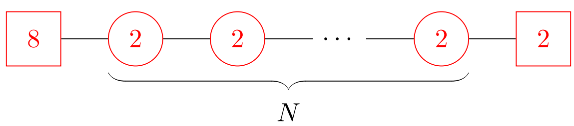


Figure 14. Type IIA brane setup corresponding to the 6d UV completion of $A_{2N+1,1}$.

4.2 6d UV completion

n odd: $n = 2N + 1$

The UV completion of the 5d theories in (4.2) is a 6d given by the following Type IIA brane setup [14, 15]: On the tensor branch, the theory is given by the following gauge theory:



(4.4)

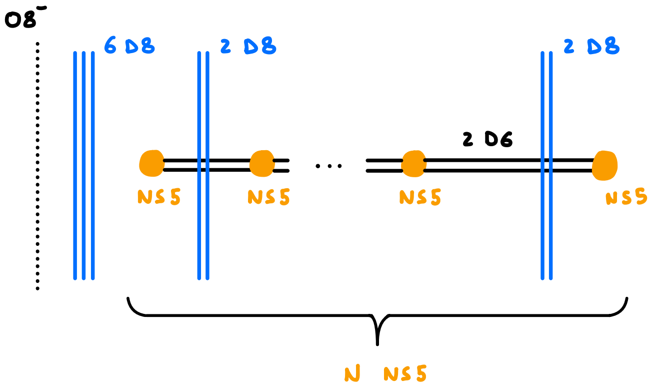
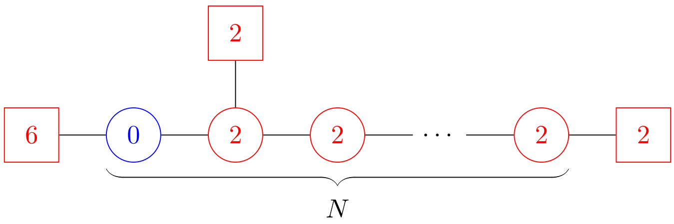


Figure 15. Type IIA brane setup corresponding to the $6d$ UV completion of $A_{2N,1}$.

n even: $n = 2N$

The UV completion of the $5d$ theories in (4.3) is a $6d$ given by the following Type IIA brane setup [14, 15]: On the tensor branch, the theory is given by the following gauge theory:

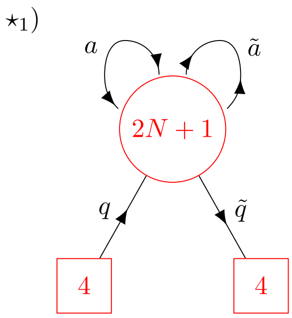


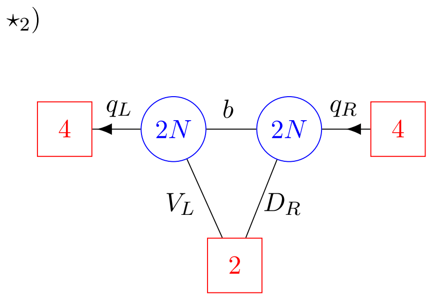
(4.5)

4.3 4d duality and some superconformal indexes

n odd: $n = 2N + 1$

Applying our prescription of section 1 to the KK duality (4.2) leads to the following $4d$ theories that we claim are IR dual





$$\mathcal{W} = \text{Flip}[a^N\, q; \tilde{a}^N\, \tilde{q}] \qquad \mathcal{W} = 1\,\text{Triangle} + \sum_{i=0}^{N-1} \text{Flip}[q_L(bb)^i q_L; q_R(bb)^i q_R; q_L b(bb)^i q_R]$$

(4.6)

Of course, our prescription does not tell us the precise flippers, which are crucial in order for the duality to be correct. In section 5.3 we provide a strategy to obtain such flippers, and we apply it to a quiver duality that generalizes (4.6).

The mapping of the chiral ring generators is given by

$$\begin{array}{ccccc}
\star_1) & & \star_2) & & \\
q\,\tilde{q}\,(a\,\tilde{a})^i & & \text{Flipper}[q_L\, b(bb)^{N-1-i}\, q_R] & & i=0,\dots,N-1 \\
q\,q\,\tilde{a}\,(a\,\tilde{a})^i & & \text{Flipper}[q_L\,(bb)^{N-1-i}\,q_L] & & i=0,\dots,N-1 \\
\tilde{q}\,\tilde{q}\,a\,(a\,\tilde{a})^i & & \text{Flipper}[q_R\,(bb)^{N-1-i}\,q_R] & & i=0,\dots,N-1 \\
(a\,\tilde{a})^j & \Longleftrightarrow & (bb)^j & & j=1,\dots,N \\
\begin{cases}\text{Flipper}[a^N\,q] \\ a^{N-1}\,q^3\end{cases} & & q_L\,V_L & & \\
\begin{cases}\text{Flipper}[\tilde{a}^N\,\tilde{q}] \\ \tilde{a}^{N-1}\,\tilde{q}^3\end{cases} & & q_R\,D_R & &
\end{array} \tag{4.7}$$

For $N=1$, we go back to the situation (2.3)–(2.4). For generic N, we don't have a proof of the duality (4.6) involving more basic Seiberg dualities. The non-trivial check of this duality is the matching of the 't Hooft anomalies and of the central charges with a-maximization [54–56]. We also compute the superconformal index for the case $N=1$.

The superconformal index for the case $N = 1$

The supersymmetric index, which coincides with the superconformal index [57–59] when computed with the superconformal R-symmetry (see [60, 61] for a review), is a valuable tool to test $4d\,\mathcal{N}=1$ duality.

The index of a $4d\,\mathcal{N}=1$ SCFT is a refined Witten index of the theory quantized on $S^3\times\mathbb{R}$

$$\mathcal{I}_{\mathcal{N}=1} = \text{Tr}_{\delta=0}(-1)^F \left(\frac{p}{q}\right)^{j_1} (p\,q)^{j_2+\frac{R}{2}} \prod_i f_i^{T_i}\,, \tag{4.8}$$

where

$$\delta = \frac{1}{2}\left\{\mathcal{Q},\mathcal{Q}^\dagger\right\} = 2j_2 + \frac{3}{2}R \tag{4.9}$$

with $\mathcal{Q}=\widetilde{\mathcal{Q}}_{\dot{-}}$ one of the Poincaré supercharges and $\mathcal{Q}^\dagger=\mathcal{S}$ the conjugate conformal supercharge, while j_1, j_2 are the Cartan generators of the isometry group $\text{SO}(4)=\text{SU}(2)_1\times\text{SU}(2)_2$ of S^3, R is the generator of the IR superconformal R-symmetry and T_i are $\mathcal{Q}$-closed generators of additional global symmetries of the theory. The parameters p and q are fugacities associated with the supersymmetry preserving squashing of the S^3 [59], while f_i are fugacities for the symmetries associated with the generators T_i.

The index counts gauge invariant operators ("words") that can be constructed from modes of the fields ("letters"). The single letter indices for a vector multiplet and a chiral multiplet transforming in the representation $\mathbf{R}$ of the gauge and flavor group and with R-charge R are

$$\begin{aligned}
i_{\text{vec}}(p,q,U) &= \frac{2pq-p-q}{(1-p)(1-q)}\chi_{adj}(U)\,, \\
i_{\text{chir}}^{\mathbf{R}}(p,q,U,V,R) &= \frac{(pq)^{\frac{R}{2}}\chi_{\mathbf{R}}(U,V)-(pq)^{\frac{2-R}{2}}\chi_{\overline{\mathbf{R}}}(U,V)}{(1-p)(1-q)}\,,
\end{aligned} \tag{4.10}$$

where $\chi_{\mathbf{R}}(U,V)$ and $\chi_{\overline{\mathbf{R}}}(U,V)$ are the characters of the representation $\mathbf{R}$ and the conjugate representation $\overline{\mathbf{R}}$, with U and V gauge and flavor group matrices, respectively.

The index is obtained symmetrizing all of such letters into words and then projecting to the gauge invariant, integrating over the Haar measure of the gauge group. It takes the general form

$$\mathcal{I}_{\mathcal{N}=1}\left(p,q,V\right)=\int\left[\mathrm{d}U\right]\prod_{k}\mathrm{PE}\left[i_{k}\left(p,q,U,V\right)\right]\,, \tag{4.11}$$

where k labels the different multiplets in the theory and $\mathrm{PE}[i_k]$ is the plethystic exponential of the single letter index of the k-th multiplet

$$\mathrm{PE}\left[i_{k}\left(p,q,U,V\right)\right]=\exp\left[\sum_{m=1}^{\infty}\frac{1}{m}i_{k}\left(p^{m},q^{m},U^{m},V^{m}\right)\right]\,, \tag{4.12}$$

which implements the symmetrization of the letters.

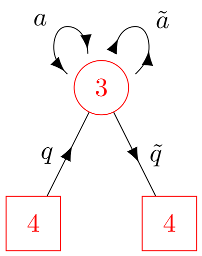
In the following, we do the following change of variables in the index:

$$\begin{aligned}p&=ty\\ q&=ty^{-1}\end{aligned} \tag{4.13}$$

For simplicity, we also put the flavor fugacities to 1. The superconformal index is then a function of the variables t and y and we are interested in the expansion in t.

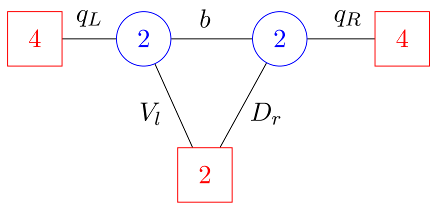
We now compute the expansion of superconformal index for the duality (4.6), setting $N=1$:

$\star_1$)



$\mathcal{W}=\ \mathrm{Flip}[aq;\tilde{a}\tilde{q}]$

$\star_2$)



$\mathcal{W}=1\,\mathrm{Triangle}\,+\mathrm{Flip}[q_L\,q_L,q_Rq_R;q_Lbq_R]$

(4.14)

On the SU(3) side, the superconformal R-charges of the elementary fields are

$$R[q,\tilde{q}]=\frac{2\sqrt{3}}{9},\qquad R[a,\tilde{a}]=2-\frac{8\sqrt{3}}{9} \tag{4.15}$$

On the USp(2) $\times$ USp(2) side, the superconformal R-charges are

$$R[b]=2-\frac{8\sqrt{3}}{9},\qquad R[q_L,q_R]=\frac{2\sqrt{3}}{9},\qquad R[V_l,D_r]=\frac{4\sqrt{3}}{9} \tag{4.16}$$

The superconformal index of both UV theories, up to order t^2, reads

$$\begin{aligned}\mathcal{I}_{\mathcal{N}=1}(t,y)=&1+16t^{4\sqrt{3}/9}+12t^{2-4\sqrt{3}/9}+16t^{2\sqrt{3}/3}+t^{4-16\sqrt{3}/9}+136t^{8\sqrt{3}/9}+16t^{4-4\sqrt{3}/3}+\\&+t^{8-32\sqrt{3}/9}+240t^{10\sqrt{3}/9}+16(y+y^{-1})t^{1+4\sqrt{3}/9}+(y+y^{-1})t^{5-16\sqrt{3}/9}+(160-35)t^2+\ldots\end{aligned} \tag{4.17}$$

We can recognize the contributions of the chiral ring generators and the currents:

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- $16t^{4\sqrt{3}/9}$: $\{q\tilde{q}\} \leftrightarrow \{\text{Flipper}[q_L b q_R]\}$
- $12t^{2-4\sqrt{3}/9}$: $\{qq\tilde{a}, \tilde{q}\tilde{q}a\} \leftrightarrow \{\text{Flipper}[q_L q_L, q_R q_R]\}$
- $16t^{2\sqrt{3}/3}$: $\{\text{Flipper}[aq], q^3, \text{Flipper}[\tilde{a}\tilde{q}], \tilde{q}^3\} \leftrightarrow \{q_L V_l, q_R D_r\}$
- $t^{4-16\sqrt{3}/9}$: $a\tilde{a} \leftrightarrow bb$
- $-35t^2$: the currents[7] of the IR symmetry $su(4) \times su(4) \times su(2) \times u(1)^2$.

From this result we claim that the IR symmetry of the SCFT is $su(4) \times su(4) \times su(2) \times u(1)^2$ (while the UV symmetry of the l.h.s. is $su(4) \times su(4) \times u(1)^3$).

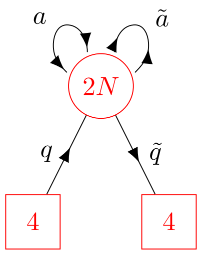
The 16 dimensional representation is $(4,1,2) + (1,\bar{4},2)$ of $su(4) \times su(4) \times su(2)$ and it breaks to $(4,1) + (4,1) + (1,\bar{4}) + (1,\bar{4})$ under $su(2) \to u(1)$. It corresponds to the operators $\{\text{Flipper}[aq], q^3, \text{Flipper}[\tilde{a}\tilde{q}], \tilde{q}^3\}$. This is in agreement with the above assignements of chiral ring operators.

The agreement of the superconformal index on the two sides of the duality gives us an additional check of the validity of the dualities discussed in this section.

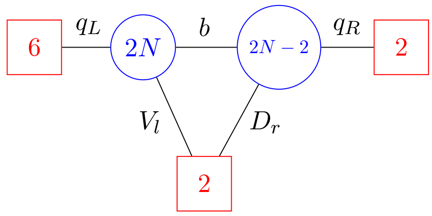
n even: $n = 2N$

In this case the $4d$ duality constructed from the $5d$ duality (4.3) is

$\star_1$)



$\star_2$)



$$\mathcal{W} = \text{Flip}[a^N; \tilde{a}^N] + \sum_{j=0}^{\lfloor \frac{N-1}{2} \rfloor} \text{Flip}[q\,(a\tilde{a})^j\,\tilde{q}] + \sum_{i=0}^{\lfloor \frac{N-2}{2} \rfloor} \text{Flip}[\tilde{a}\,(a\tilde{a})^i\,q^2; a\,(a\tilde{a})^i\,\tilde{q}^2]$$

$$\mathcal{W} = 1\,\text{Triangle} + \sum_{i=0}^{\lfloor \frac{N-2}{2} \rfloor} \text{Flip}[q_L\,(bb)^i\,q_R; q_R(bb)^i q_R; q_L b (bb)^i q_R] \tag{4.18}$$

To write the mapping of the chiral ring generators we have to distinguish between N even and odd.

[7]The counting for the currents goes as follows. There are $6 * 16 = 96$ marginal operators of the form $(q\tilde{q})(qq\tilde{a})$ but 16 of them are of the type $q^3(\tilde{q}\tilde{a})$ and are therefore 0 on the chiral ring because they are flipped (the e.o.m of the Flipper$[\tilde{q}at]$ sets them to 0). Only 80 non-trivial marginal operators are left. Same conclusion for the charge conjugate operators $(q\tilde{q})(\tilde{q}\tilde{q}a)$. Therefore, there is a total of 160 marginal operators. The t^2 coefficient is of the form (# marginal operators − dimension of the adjoint representation of the IR symmetry group) and turns out to be 125 so we conclude that the dimension of the global symmetry group is 35.

***N* even:**

$$\begin{array}{ccc} c\star_1) & & c\star_2) \\ \begin{cases} \text{Flipper}[q\tilde{q}] \\ q\,(a\tilde{a})^{N-1}\,\tilde{q} \end{cases} & \Longleftrightarrow & \begin{cases} \text{Flipper}[q_L q_L] \\ q_L\,(bb)^{N-1}\,q_L \\ \text{Flipper}[q_R q_R] \\ V_l^2 \end{cases} \end{array} \tag{4.19}$$

The total number of operators on both sides is 32.

$$\begin{cases} \text{Flipper}[q\,(a\tilde{a})^i\,\tilde{q}] \\ q\,(a\tilde{a})^{N-i-1}\,\tilde{q} \end{cases} \Longleftrightarrow \begin{cases} \text{Flipper}[q_L\,(bb)^i\,q_L] \\ q_L\,(bb)^{N-i-1}\,q_L \\ \text{Flipper}[q_R\,(bb)^i\,q_R] \\ q_R\,(bb)^{N-i-1}\,q_R \end{cases} \qquad i=1,\ldots,\frac{N-2}{2} \tag{4.20}$$

The total number of operators on both sides is $16(N-2)$.

$$\begin{cases} \text{Flipper}[\tilde{a}\,(a\tilde{a})^j\,q^2] \\ \tilde{a}\,(a\tilde{a})^{N-j-2}\,q^2 \\ \text{Flipper}[a\,(a\tilde{a})^j\,\tilde{q}^2] \\ a\,(a\tilde{a})^{N-j-2}\,\tilde{q}^2 \end{cases} \Longleftrightarrow \begin{cases} \text{Flipper}[q_L\,b\,(bb)^j\,q_R] \\ q_L\,b\,(bb)^{N-j-2}\,q_R \end{cases} \qquad j=1,\ldots,\frac{N-2}{2}-1 \tag{4.21}$$

The total number of operators on both sides is $12(N-2)-24$.

$$\begin{cases} \text{Flipper}[\tilde{a}\,(a\tilde{a})^{(N-2)/2}\,q^2] \\ a^{N-1}\,q^2 \\ \text{Flipper}[a\,(a\tilde{a})^{(N-2)/2}\,\tilde{q}^2] \\ \tilde{a}^{N-1}\,\tilde{q}^2 \end{cases} \Longleftrightarrow \begin{cases} \text{Flipper}[q_L\,b\,(bb)^{(N-2)/2}\,q_R] \\ q_L\,V_l \end{cases} \tag{4.22}$$

The total number of operators on both sides is 24.

$$\begin{cases} \text{Flipper}[a^N] \\ \text{Flipper}[\tilde{a}^N] \\ a^{N-2}\,q^4 \\ \tilde{a}^{N-2}\,\tilde{q}^4 \end{cases} \Longleftrightarrow D_r\,q_R \tag{4.23}$$

The total number of operators on both sides is 4.

$$(a\tilde{a})^m \Longleftrightarrow (bb)^m m=1,\ldots,N-1 \tag{4.24}$$

The total number of operators on both sides is $N-1$.

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***N* odd:**

$$c\star_1) \qquad\qquad\qquad c\star_2)$$

$$\begin{cases} \text{Flipper}[q\tilde{q}] \\ q\,(a\tilde{a})^{N-1}\,\tilde{q} \end{cases} \Longleftrightarrow \begin{cases} \text{Flipper}[q_L q_L] \\ q_L\,(bb)^{N-1}\,q_L \\ \text{Flipper}[q_R q_R] \\ V_l^2 \end{cases} \tag{4.25}$$

The total number of operators on both sides is 32.

$$\begin{cases} \text{Flipper}[q\,(a\tilde{a})^i\,\tilde{q}] \\ q\,(a\tilde{a})^{N-i-1}\,\tilde{q} \end{cases} \Longleftrightarrow \begin{cases} \text{Flipper}[q_L\,(bb)^i\,q_L] \\ q_L\,(bb)^{N-i-1}\,q_L \\ \text{Flipper}[q_R\,(bb)^i\,q_R] \\ q_R\,(bb)^{N-i-1}\,q_R \end{cases} \quad i = 1, \dots, \frac{N-3}{2} \tag{4.26}$$

The total number of operators on both sides is $16(N-3)$.

$$\text{Flipper}[q\,(a\tilde{a})^{(N-1)/2}\,\tilde{q}] \Longleftrightarrow \begin{cases} q_L\,(bb)^{(N-1)/2}\,q_L \\ q_R\,(bb)^{(N-1)/2}\,q_R \end{cases} \tag{4.27}$$

The total number of operators on both sides is 16.

$$\begin{cases} \text{Flipper}[\tilde{a}\,(a\tilde{a})^j\,q^2] \\ \tilde{a}\,(a\tilde{a})^{N-j-2}\,q^2 \\ \text{Flipper}[a\,(a\tilde{a})^j\,\tilde{q}^2] \\ a\,(a\tilde{a})^{N-j-2}\,\tilde{q}^2 \end{cases} \Longleftrightarrow \begin{cases} \text{Flipper}[q_L\,b\,(bb)^j\,q_R] \\ q_L\,b\,(bb)^{N-j-2}\,q_R \end{cases} \quad j = 0, \dots, \frac{N-3}{2} \tag{4.28}$$

The total number of operators on both sides is $12(N-1)$.

$$\begin{cases} a^{N-1}\,q^2 \\ \tilde{a}^{N-1}\,\tilde{q}^2 \end{cases} \Longleftrightarrow q_L\,V_l \tag{4.29}$$

The total number of operators on both sides is 12.

$$\begin{cases} \text{Flipper}[a^N] \\ \text{Flipper}[\tilde{a}^N] \\ a^{N-2}\,q^4 \\ \tilde{a}^{N-2}\,\tilde{q}^4 \end{cases} \Longleftrightarrow D_r\,q_R \tag{4.30}$$

The total number of operators on both sides is 4.

$$(a\tilde{a})^m \Longleftrightarrow (bb)^m m = 1, \dots, N-1 \tag{4.31}$$

The total number of operators on both sides is $N-1$.

Also for this case we don't have a proof of this duality (4.18). Non-trivial checks of this duality are the matching of the central charges with a-maximization, and of the superconformal index in a simpe but non-trivial case.

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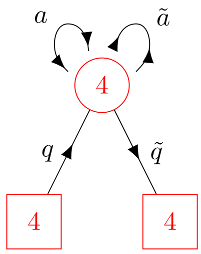
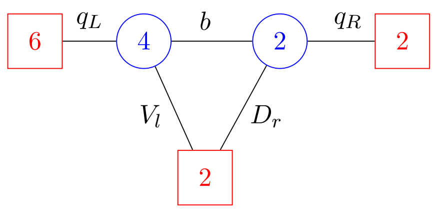
The superconformal index for the case $N = 2$

We now compute the superconformal index for the duality (4.18), setting $N = 2$:

$\star_1$)



$\star_2$)



$\mathcal{W} = \text{Flip}[a^2; \tilde{a}^2] + \text{Flip}[q\tilde{q}, \tilde{a}q^2; a\tilde{q}^2]$ $\qquad$ $\mathcal{W} = 1\,\text{Triangle} + \text{Flip}[q_L\, q_L, q_R q_R; q_L b q_R]$ $\qquad$ (4.32)

On the SU(4) side, the superconformal R-charges of the elementary fields are

$$R[q,\tilde{q}] = \frac{\sqrt{22}}{12}, \qquad R[a,\tilde{a}] = 1 - \frac{\sqrt{22}}{6} \tag{4.33}$$

On the USp(4) $\times$ USp(2) side, the superconformal R-charges are

$$R[b] = 1 - \frac{\sqrt{22}}{6}, \qquad R[q_L, q_R] = \frac{\sqrt{22}}{12}, \qquad R[V_l] = 1 - \frac{\sqrt{22}}{12}, \qquad R[D_r] = \frac{\sqrt{22}}{4} \tag{4.34}$$

The superconformal index of both UV theories, up to order t^2, reads

$$\begin{aligned} \mathcal{I}_{\mathcal{N}=1}(t,y) = {} & 1+24t+4t^{\sqrt{22}/3}+32t^{4-\sqrt{22}/2}+32t^{2-\sqrt{22}/6}+t^{8-4\sqrt{22}/3}+t^{6-\sqrt{22}}+t^{4-2\sqrt{22}/3}+ \\ & +t^{2-\sqrt{22}/3}+(24+y+y^{-1})t^{5-2\sqrt{22}/3}+(24+y+y^{-1})t^{3-\sqrt{22}/3}+ \\ & +(\frac{24\times 25}{2}+24(y+y^{-1})-73)t^2+\ldots \end{aligned} \tag{4.35}$$

We can recognize the contributions of the chiral ring generators and the currents:

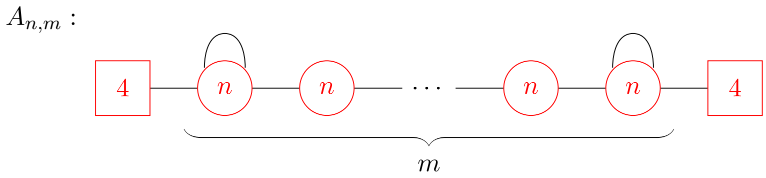
- $24t$: $\{\text{Flipper}[\tilde{a}q^2, a\tilde{q}^2], aq^2, \tilde{a}\tilde{q}^2\} \leftrightarrow \{\text{Flipper}[q_L b q_R], q_L V_l\}$
- $4t^{\sqrt{22}/3}$: $\{\text{Flipper}[a^2, \tilde{a}^2], q^4, \tilde{q}^4\} \leftrightarrow D_r q_R$
- $32' t^{2-\sqrt{22}/6}$: $\{\text{Flipper}[q\tilde{q}], qa\tilde{a}\tilde{q}\} \leftrightarrow \{\text{Flipper}[q_L q_L, q_R q_R], q_L b b q_L, V_l^2\}$
- $t^{2-\sqrt{22}/3}$: $a\tilde{a} \leftrightarrow bb$
- $-73t^2$: the currents of the IR symmetry $so(12) \times su(2) \times su(2) \times u(1)$.

From this result we claim that the IR symmetry of the SCFT is $so(12) \times su(2) \times su(2) \times u(1)$ (while the UV symmetries are $su(4)^2 \times u(1)^3$ and $su(6) \times su(2)^2 \times u(1)^2$. The $32'$ of $so(12)$ breaks to $(4, \bar{4}) + (\bar{4}, 4)$ under $so(12) \to su(4) \times su(4)$, and to $15 + \bar{15} + 1 + 1$ under $so(12) \to su(6)$. This is in agreement with the above assignments of chiral ring operators.

The agreement of the superconformal index on the two sides of the duality gives us an additional check of the validity of the dualities discussed in this section.

5 Systems with two $O7$ planes: $A_{n,m}$ and its dual

$A_{n,m}$ theories. In this section, we generalize the discussion of section 4 considering the following two-parameter family of 5d theories, that we call $A_{n,m}$:

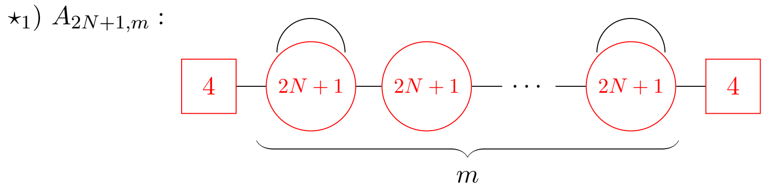


(5.1)

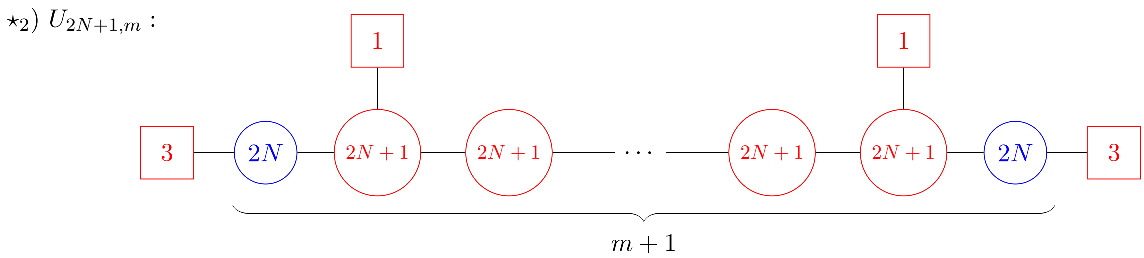
The duality statement will depend on the parity of the parameter n as in the last subsection.

5.1 n odd: 5d duality $A_{2N+1,m} \leftrightarrow U_{2N+1,m}$

In this section, we generalize the duality (4.2). We call the dual of $A_{2N+1,m}$, $U_{2N+1,m}$. $A_{2N+1,m}$ respectively $U_{2N+1,m}$ contains a hyper in the antisymmetric representation of the gauge group respectively a USp($2N$) gauge node at each end of the quiver. The quiver for $A_{2N+1,m}/U_{2N+1,m}$ is shown in (5.2)/(5.3). We have also depicted the brane systems in figure 16. The claim is that $A_{2N+1,m}$ and $U_{2N+1,m}$ are UV dual. The analysis of the brane systems that lead to this duality can be found in [14].



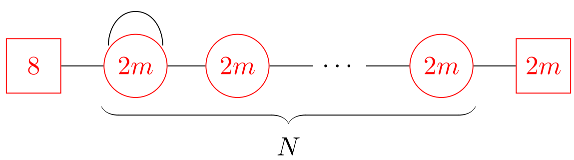
(5.2)



(5.3)

5.2 6d UV completion

The UV completion of the 5d theories in (5.2)–(5.3) is a 6d given by the following Type IIA brane setup [14, 15]: On the tensor branch, the system flows to the following gauge theory:



(5.4)

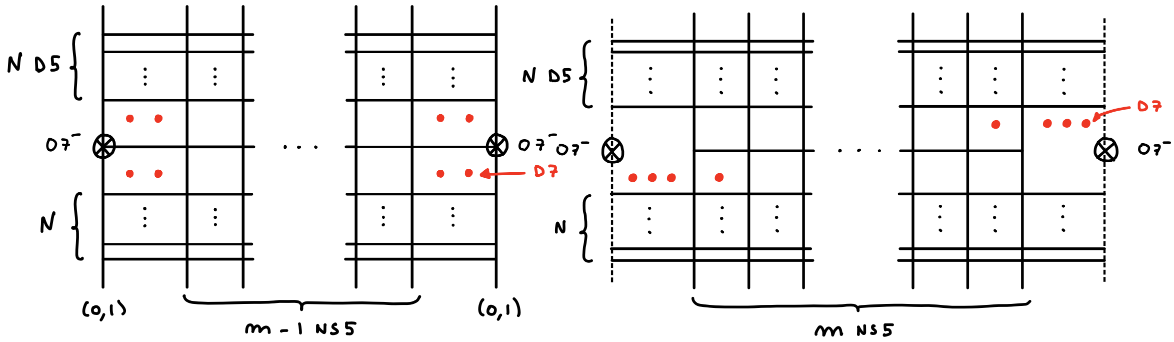


Figure 16. Brane setup for $A_{2N+1,m}$ on the left with an $NS5$ stuck on each $O7^-$ plane and for $U_{2N+1,m}$ on the right.

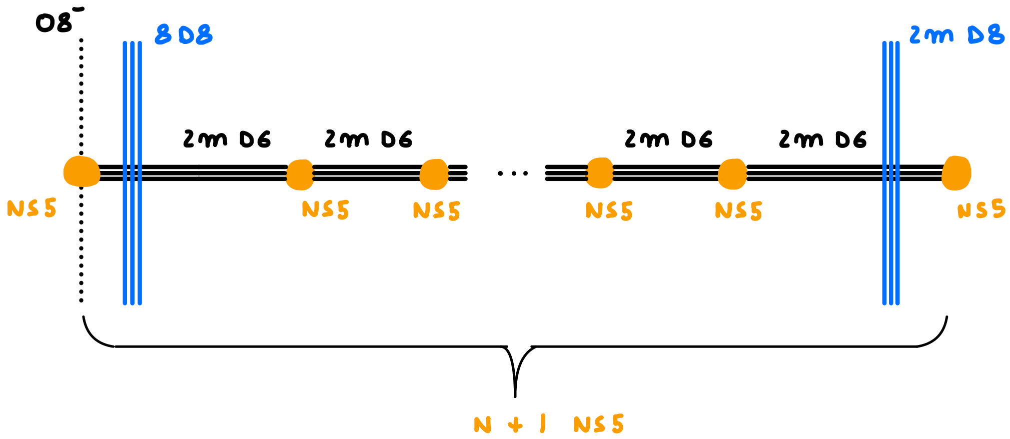
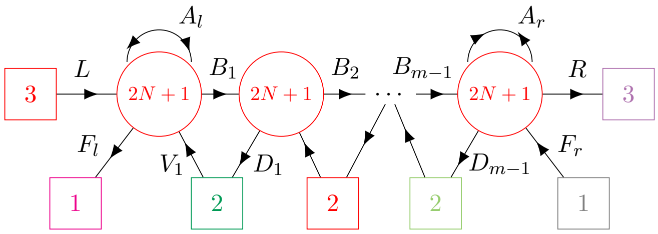


Figure 17. Type IIA brane setup corresponding to the 6d UV completion of $A_{2N+1,m}$.

5.3 4d duality

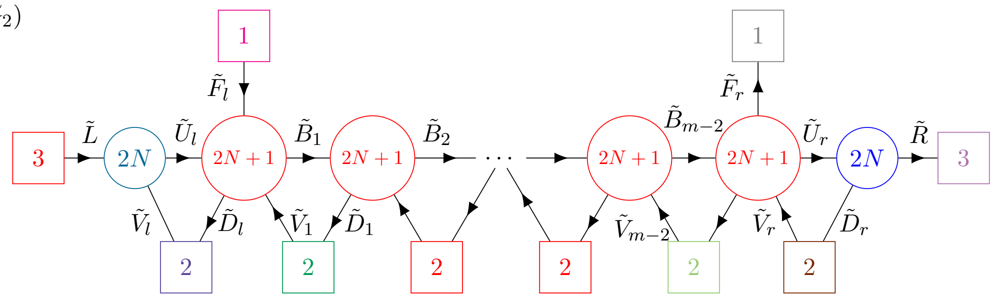
Our prescription of section 1 applied to the 5d duality (5.2)–(5.3) gives the following 4d duality

$\tilde{\star}_1$)



(5.5)

$\tilde{\star}_2$)



(5.6)

Without the flippers, these two theories are *not* dual to each other.

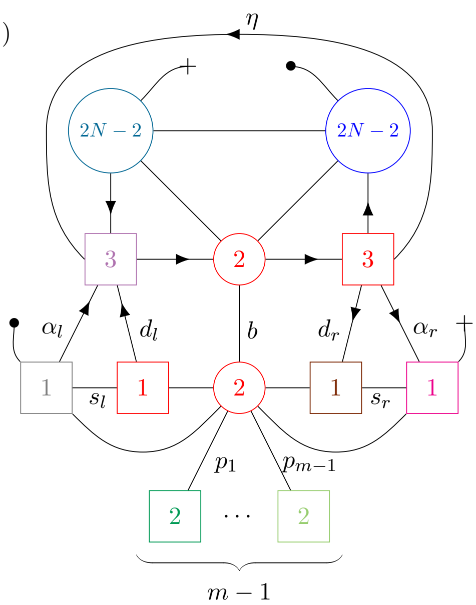
Strategy to get the set of flippers

In order to obtain the correct set of flippers to make $\widetilde{\star}_1)$ and $\widetilde{\star}_2)$ dual, we did the following procedure.

Starting with $\widetilde{\star}_1)$ and do the following operations:

- deconfinement of the two antisymmetric[8]
- m Seiberg dualities on the m SU nodes
- CSST duality on the left SU(2)
- m-2 confinements

We get $\widetilde{\star}_1)$



$$\mathcal{W} = 6\,\text{Quartic} + 6\,\text{Triangles} + \text{Flip}[b\,b] + \sum_{i=1}^{m-1} \text{Flip}[p_i\,p_i] \tag{5.7}$$

Then, we start with $\widetilde{\star}_2)$ and do the following operations:

- m-1 Seiberg dualities on the m-1 SU nodes
- CSST duality on the left SU(2)
- m-3 confinements

[8]The deconfinement is a name of a technique that replaces an antisymmetric field by a confining USp gauge node, see [28–30] for details.

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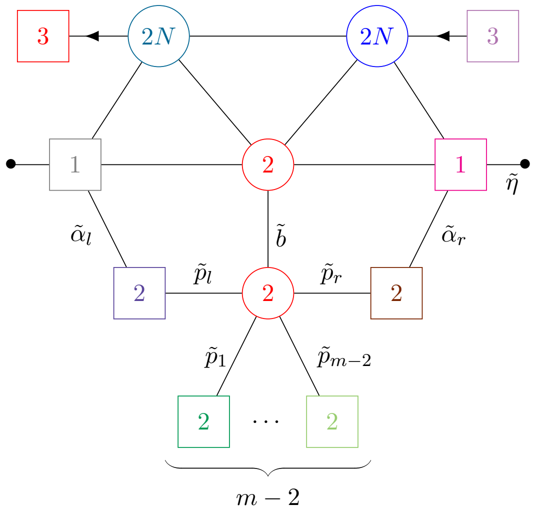
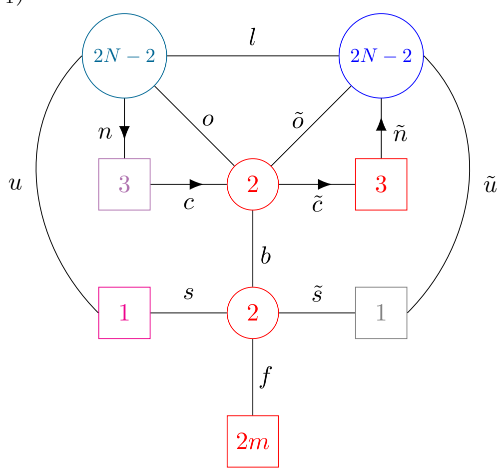
We get $\tilde{\tilde{\star}}_2$)



$$\mathcal{W} = 2\,\text{Quartic} + 4\,\text{Triangles} + \text{Flip}[\tilde{b}\,\tilde{b};\tilde{p}_l\,\tilde{p}_l;\tilde{p}_r\,\tilde{p}_r] + \sum_{i=1}^{m-2} \text{Flip}[\tilde{p}_i\,\tilde{p}_i] \tag{5.8}$$

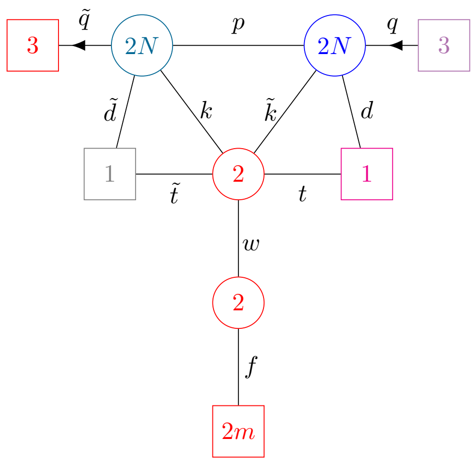
Then we can play with (5.7) and (5.8) to make manifest a bigger flavor symmetry group. Concretely, we flip the operators ($\alpha_l; \alpha_r; d_l; d_r; s_l; s_r; \eta;$ Flipper$[p_i\,p_i]$) in (5.7) and ($\tilde{\alpha}_l; \tilde{\alpha}_r; \tilde{\eta};$ Flipper$[\tilde{p}_i\,\tilde{p}_i, \tilde{p}_l\,\tilde{p}_l, \tilde{p}_r\,\tilde{p}_r]$) in (5.8). We therefore consider the following theories

$\tilde{\tilde{\star}}_1$)



$\mathcal{W} = 2\,\text{Quartic} + 3\,\text{Triangles}$

$\tilde{\tilde{\star}}_2$)



$\mathcal{W} = 3\,\text{Triangles}$ (5.9)

Once again, at this stage $\tilde{\tilde{\star}}_1$) and $\tilde{\tilde{\star}}_2$) are *not* dual. Now to make progress, we will focus on the case $N = 1$ and generic m. Moreover we saw that in the $m = 1$ case we had to flip the

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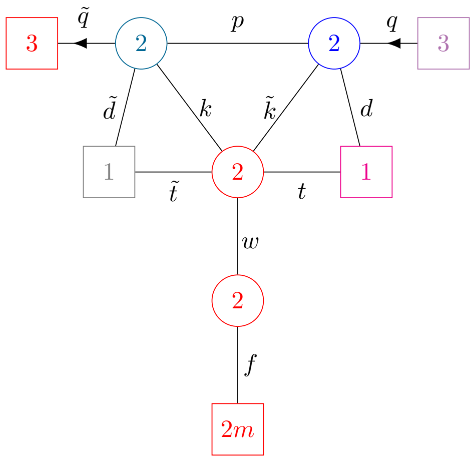
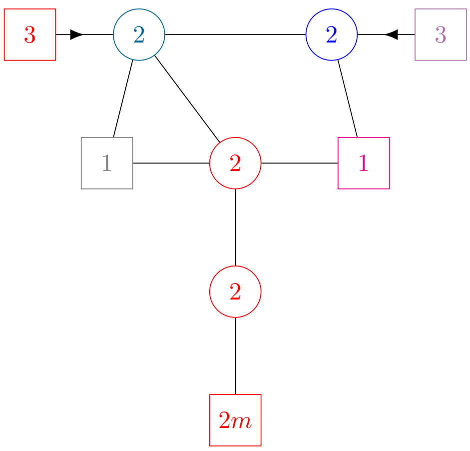
whole towers in the frame $\star_2$) (4.6). Therefore we decide to study the following theory

$\star_2$)



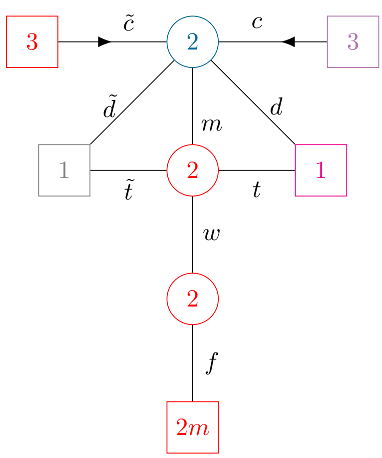
$$\mathcal{W} = 3\,\text{Triangles} + \text{Flip}[ww; qq; qd; \tilde{q}\tilde{q}; \tilde{q}\tilde{d}; qp\tilde{q}; qp\tilde{d}; \tilde{q}pd; dp\tilde{d}] \tag{5.10}$$

Now we start by doing a CSST duality on the $\mathrm{USp}(2) \equiv \mathrm{SU}(2)$. We get



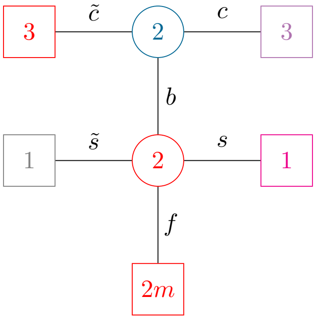
$$\mathcal{W} = 1\,\text{Triangle} + 1\,\text{Quartic} + \text{Flip}[pp; kk; ww; qq; qd; qp\tilde{q}; \tilde{q}pd; qp\tilde{d}; dp\tilde{d}] \tag{5.11}$$

Then we use the IP confinement [22] on the $\mathrm{USp}(2)$. We obtain



$$\mathcal{W} = 2\,\text{Triangles} + \text{Flip}[mm; ww; \tilde{c}\tilde{d}; cd; c\tilde{c}; d\tilde{d}] \tag{5.12}$$

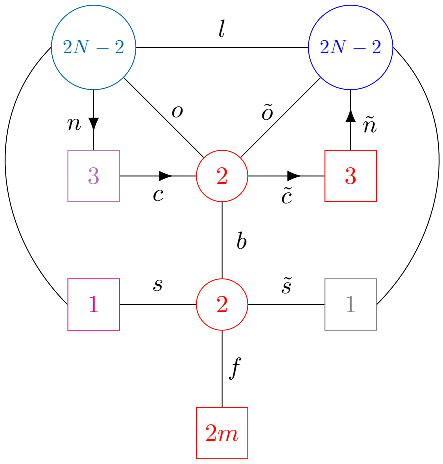
The last step is the IP confinement on the middle SU(2) to get



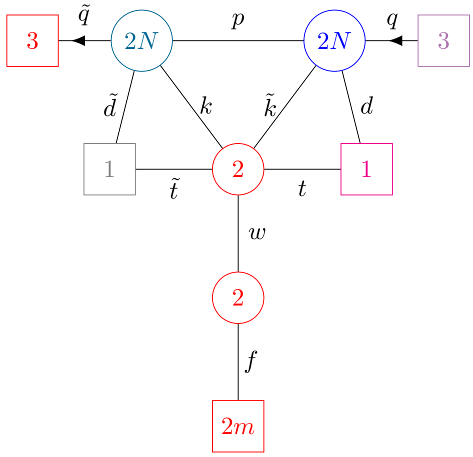
$$\mathcal{W} = \text{Flip}[bb; c\tilde{c}; cb\tilde{s}; \tilde{c}bs; bsb\tilde{s}] \tag{5.13}$$

Which is of the form $\tilde{\tilde{\star}}_1$) in (5.9) specified to the case $N = 1$. This result motivates the following educated guess for generic N:

$\star_1$)



$\star_2$)



$$\mathcal{W} = 2\,\text{Quartic} + 3\,\text{Triangles}$$
$$+\,\text{Flip}[bb, c\tilde{c}, cb\tilde{s}, \tilde{c}bs, bsb\tilde{s}]$$

$$\mathcal{W} = 3\,\text{Triangles} + \text{Flip}[ww]$$
$$+ \sum_{i=o}^{N-1} \text{Flip}[q(pp)^i q; \tilde{q}(pp)^i \tilde{q}; q(pp)^i d; \tilde{q}(pp)^i \tilde{d};$$
$$qp(pp)^i \tilde{q}; qp(pp)^i \tilde{d}; \tilde{q}p(pp)^i d; dp(pp)^i \tilde{d}] \tag{5.14}$$

We claim that $\star_1$) and $\star_2$) in (5.14) are dual. For generic N and m, we don't have a proof of this statement. However we provided a proof for the special case of $N = 1$ and generic m. The first non-trivial test of this duality is the matching of the central charges for generic N and m. Then we can match 't Hooft anomalies. We have reported the computation in the appendix A.

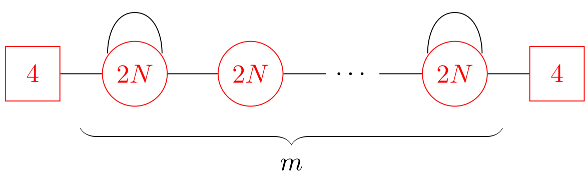
The mapping of the chiral ring generators is given by

$$
\begin{array}{ccc}
\star_1) & & \star_2) & \\
n\,(l\,l)^i\,n & & \text{Flipper}[q\,(pp)^{N-1-i}\,q] & i=0,\dots,N-2 \\
\tilde{n}\,(l\,l)^i\,\tilde{n} & & \text{Flipper}[\tilde{q}\,(pp)^{N-1-i}\,\tilde{q}] & i=0,\dots,N-2 \\
n\,(l\,l)^i\,u & & \text{Flipper}[q\,(pp)^{N-1-i}\,d] & i=0,\dots,N-2 \\
\tilde{n}\,(l\,l)^i\,\tilde{u} & & \text{Flipper}[\tilde{q}\,(pp)^{N-1-i}\,\tilde{d}] & i=0,\dots,N-2 \\
n\,l\,(l\,l)^i\,\tilde{n} & & \text{Flipper}[q\,p\,(pp)^{N-2-i}\,\tilde{q}] & i=0,\dots,N-2 \\
n\,l\,(l\,l)^i\,\tilde{u} & & \text{Flipper}[q\,p\,(pp)^{N-2-i}\,\tilde{d}] & i=0,\dots,N-2 \\
\tilde{n}\,l\,(l\,l)^i\,u & & \text{Flipper}[\tilde{q}\,p\,(pp)^{N-2-i}\,d] & i=0,\dots,N-2 \\
u\,l\,(l\,l)^i\,\tilde{u} & & \text{Flipper}[d\,p\,(pp)^{N-2-i}\,\tilde{d}] & i=0,\dots,N-2 \\
(l\,l)^j & & (pp)^j & j=1,\dots N-1 \\
cbs & & \text{Flipper}[q\,q] & \\
\tilde{c}b\tilde{s} & & \text{Flipper}[\tilde{q}\,\tilde{q}] & \\
cc & & \text{Flipper}[q\,d] & \\
\tilde{c}\tilde{c} & & \text{Flipper}[\tilde{q}\,\tilde{d}] & \\
\text{Flipper}[c\tilde{c}] & & \text{Flipper}[q\,p\,(pp)^{N-1}\,\tilde{q}] & \\
\text{Flipper}[cb\tilde{s}] & \Longleftrightarrow & \text{Flipper}[q\,p\,(pp)^{N-1}\,\tilde{d}] & \\
\text{Flipper}[\tilde{c}bs] & & \text{Flipper}[\tilde{q}\,p\,(pp)^{N-1}\,d] & \\
\text{Flipper}[bsb\tilde{s}] & & \text{Flipper}[d\,p\,(pp)^{N-1}\,\tilde{d}] & \\
s\tilde{s} & & (pp)^N & \\
c\tilde{o}\tilde{u} & & q\tilde{k}\tilde{t} & \\
\tilde{c}ou & & \tilde{q}kt & \\
cbf & & q\tilde{k}wf & \\
\tilde{c}bf & & \tilde{q}kwf & \\
oo & & kk & \\
\tilde{o}\tilde{o} & & \tilde{k}\tilde{k} & \\
\text{Flipper}[bb] & & t\tilde{t} & \\
sf & & twf & \\
\tilde{s}f & & \tilde{t}wf & \\
ff & & ff &
\end{array}
\tag{5.15}
$$

5.4 n even: 5d duality $A_{2N,m} \leftrightarrow U_{2N,m}$

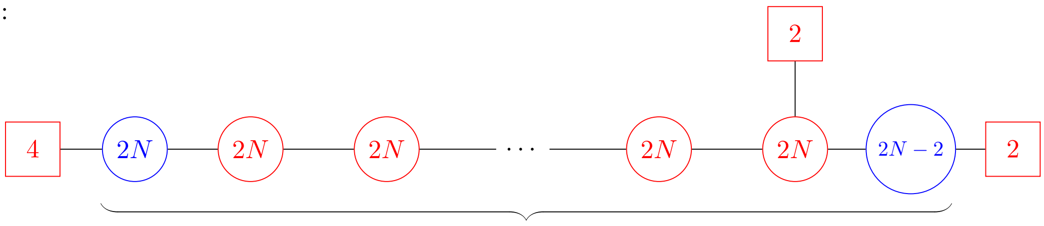
In this section, we generalize the duality (4.3). We name $U_{2N,m}$ the dual of $A_{2N,m}$. $A_{2N,m}$ ($U_{2N,m}$) contains a hyper in the antisymmetric representation of the gauge group (a USp gauge node) at each end of the quiver. The quiver for $A_{2N,m}/U_{2N,m}$ is shown in (5.16)/(5.17). We have also depicted the brane systems in figure 18. The claim is that $A_{2N,m}$ and $U_{2N,m}$ are UV dual. The analysis of the brane systems that leads to this duality can be found in [14].

$\star_1)$ $A_{2N,m}$:



(5.16)

$\star_2)$ $U_{2N,m}$:



(5.17)

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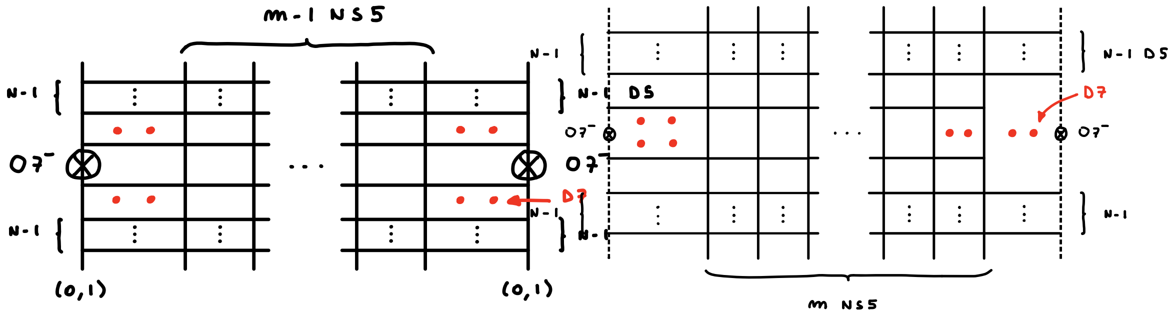


Figure 18. Brane setup for $A_{2N,m}$ on the left with an $NS5$ stuck on each $O7^-$ plane and for $U_{2N,m}$ on the right.

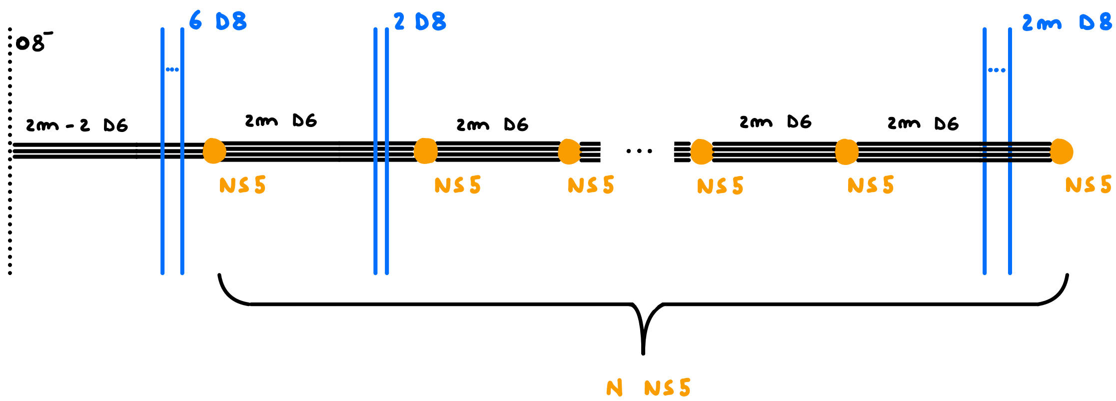
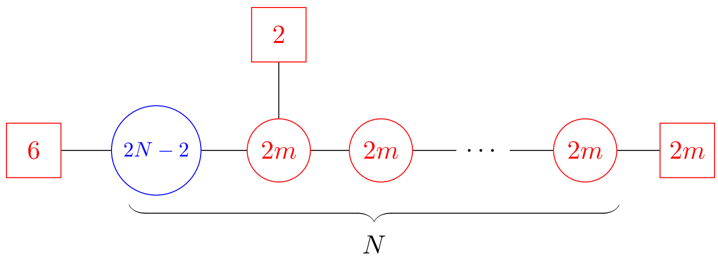


Figure 19. Type IIA brane setup corresponding to the $6d$ UV completion of $A_{2N,m}$.

5.5 6d UV completion

The UV completion of the $5d$ theories in (5.16)–(5.17) is a $6d$ given by the following Type IIA brane setup [14, 15]: On the tensor branch, the system flows to the following gauge theory:

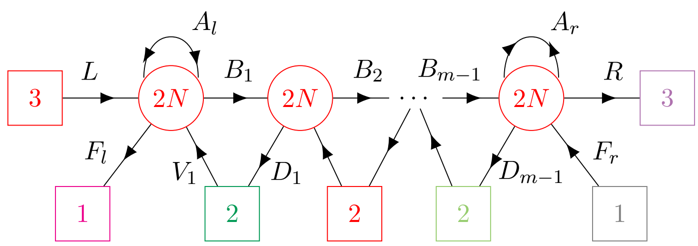


(5.18)

5.6 4d duality

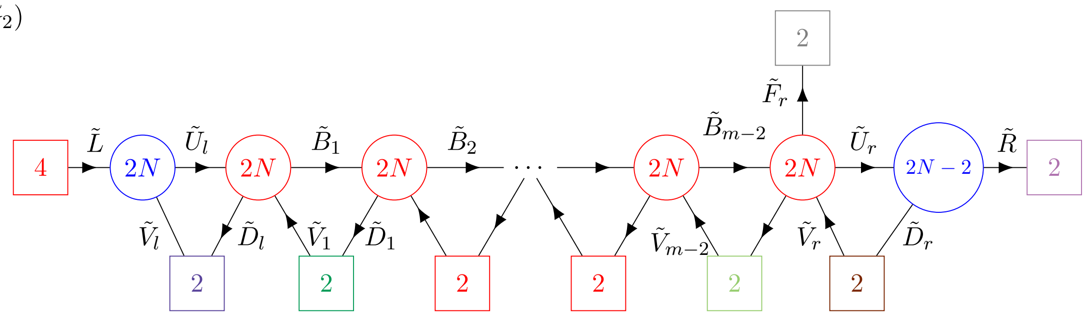
Our prescription of section 1 applied to the $5d$ duality (5.16)–(5.17) gives the following $4d$ duality

$\tilde{\star}_1$)



(5.19)

$\tilde{\star}_2$)



(5.20)

Without the flippers, these two theories are *not* dual to each other.

Strategy to get the set of flippers

Once again in order to find the correct set of flippers we do a similar procedure as in the odd case. We first put the two theories $\tilde{\star}_1$) and $\tilde{\star}_2$) in a simpler form. It means that we do to each theories the following set of manipulations.

Starting with $\tilde{\star}_1$):

- deconfinement of the two antisymmetric
- m Seiberg dualities on the m SU nodes
- CSST duality on the left SU(2)
- m-2 confinements

We end up with a frame similar to (5.7).

Starting with $\tilde{\star}_2$):

- m-1 Seiberg dualities on the m-1 SU nodes
- CSST duality on the left SU(2)
- m-3 confinements

We end up with a frame similar to (5.8).

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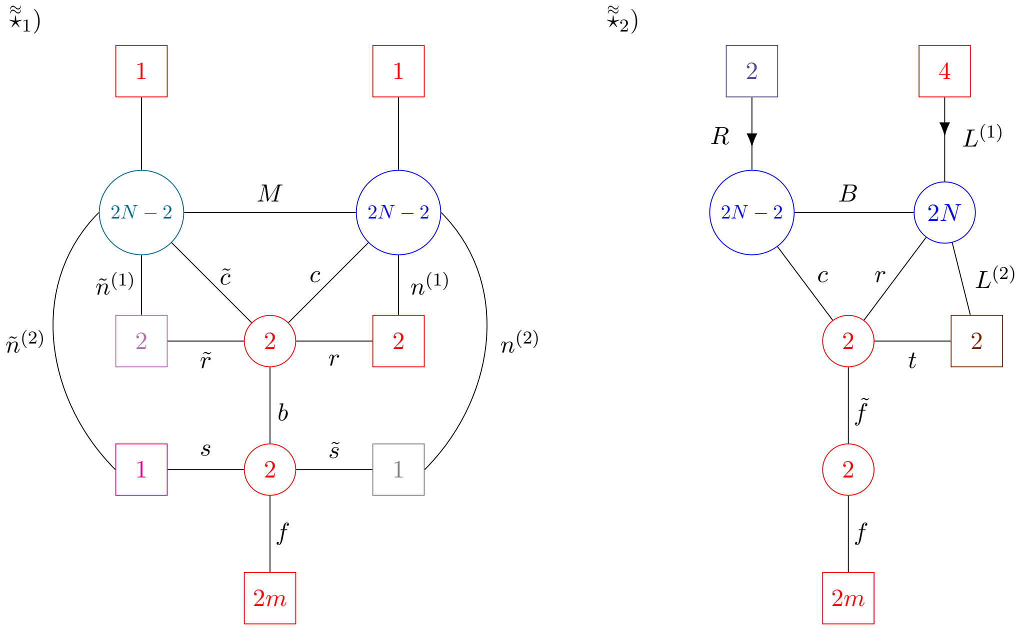
Then we arrange the two resulting theories by a flipping procedure equivalent to the one after (5.8). We are lead to consider the following theories



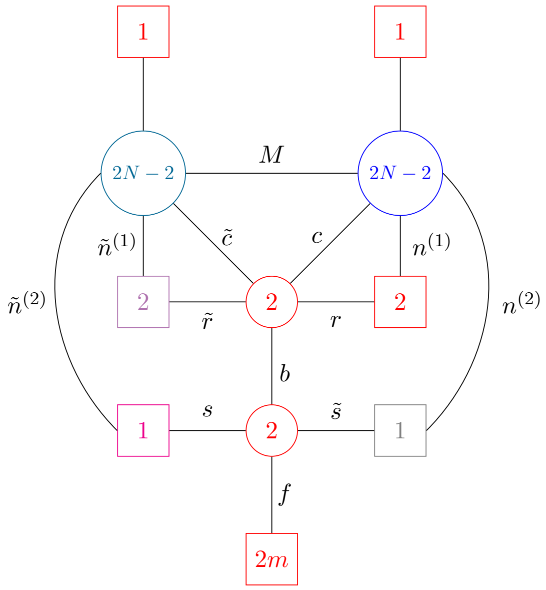
$$\mathcal{W} = 2\,\text{Quartic} + 3\,\text{Triangles} \qquad\qquad \mathcal{W} = 3\,\text{Triangles} \tag{5.21}$$

Once again at this stage $\widetilde{\widetilde{\star}}_1)$ and $\widetilde{\widetilde{\star}}_2)$ are *not* dual, it misses the set of flippers in both sides. In the odd case, in order to make progress at this stage we studied the $N = 1$ case. It allowed us to come up with the educated guess (5.14) for generic N. This educated guess turned out to be correct because it passes the non-trivial checks of matching the central charges as well as 't Hooft anomalies. Now, for the even case we consider a different procedure to obtain an educated guess. We do the following steps:

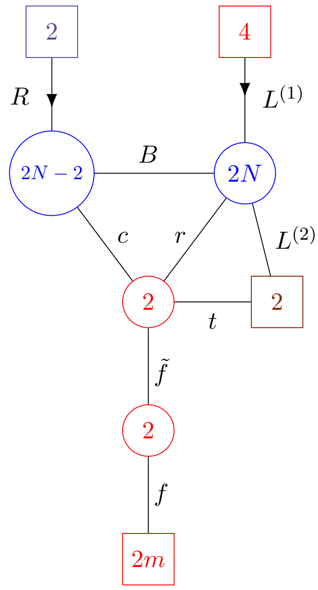
- Start with the theory with no flipper
- Compute the R-charges of all the chiral ring generators
- Flip all operators with R-charge *less* than 1
- Compute again all R-charges
- Flip additional chiral ring generators with R-charge *less* than 1 if present
- Repeat this procedure until reaching a frame with only chiral ring generators with R-charge *bigger* than 1

After applying these algorithm to (5.21) we obtain

$\star_1$)



$\star_2$)



$$
\begin{aligned}
\mathcal{W} &= 2\,\text{Quartic} + 3\,\text{Triangles} + \text{Flip}[b\,b] \\
&+ \sum_{a=0}^{\lfloor\frac{N-4}{2}\rfloor} \text{Flip}[n^{(1)}\,(M\,M)^a\,\tilde{n}^{(1)}] \\
&+ \sum_{b=0}^{\lfloor\frac{N-3}{2}\rfloor} \text{Flip}[n^{(1)}\,(M\,M)^b\,n^{(1)}; \tilde{n}^{(1)}\,(M\,M)^b\,\tilde{n}^{(1)}; \\
&l\,M(M\,M)^b\,n^{(1)}; \tilde{l}\,M(M\,M)^b\,\tilde{n}^{(1)}; n^{(1)}\,M(M\,M)^b\,\tilde{n}^{(2)}; \\
&\tilde{n}^{(1)}\,M(M\,M)^b\,n^{(2)}] + \sum_{c=0}^{\lfloor\frac{N-2}{2}\rfloor} \text{Flip}[n^{(1)}\,(M\,M)^c\,n^{(2)}; \\
&\tilde{n}^{(1)}\,(M\,M)^c\,\tilde{n}^{(2)}; l\,M(M\,M)^c\,\tilde{n}^{(2)}; \tilde{l}\,M(M\,M)^c\,n^{(2)}; \\
&n^{(1)}\,(M\,M)^c\,\tilde{l}; \tilde{n}^{(1)}\,(M\,M)^c\,l; l\,M(M\,M)^c\,\tilde{l}; \\
&n^{(2)}\,M(M\,M)^c\,\tilde{n}^{(2)}] + \sum_{d=0}^{\lfloor\frac{N-1}{2}\rfloor} \text{Flip}[n^{(2)}\,(M\,M)^c\,\tilde{l}; \\
&\tilde{n}^{(2)}\,(M\,M)^c\,l]
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{W} &= 3\,\text{Triangles} + \text{Flip}[\tilde{f}\,\tilde{f}] \\
&+ \sum_{a=0}^{\lfloor\frac{N-3}{2}\rfloor} \text{Flip}[L^{(2)}\,(B\,B)^a\,L^{(2)}; \\
&\quad L^{(2)}\,B(B\,B)^a\,R] + \\
&\sum_{b=0}^{\lfloor\frac{N-2}{2}\rfloor} \text{Flip}[L^{(1)}\,(B\,B)^b\,L^{(2)}; \\
&\quad L^{(1)}\,B(B\,B)^b\,R] + \\
&\sum_{c=0}^{\lfloor\frac{N-1}{2}\rfloor} \text{Flip}[L^{(1)}\,(B\,B)^c\,L^{(1)}; \\
&\quad R\,(B\,B)^c\,R]
\end{aligned}
\tag{5.22}
$$

We claim that $\star_1$) and $\star_2$) in (5.22) are dual. For generic N and m, we do not have a proof of this statement. The non-trivial check of the claim is the matching of the central charges with a-maximization.

6 Higgsing $R_{N,k}$

In the last section, we start the study of Higgsing of the 5d $R_{N,k}$ theories (3.1).

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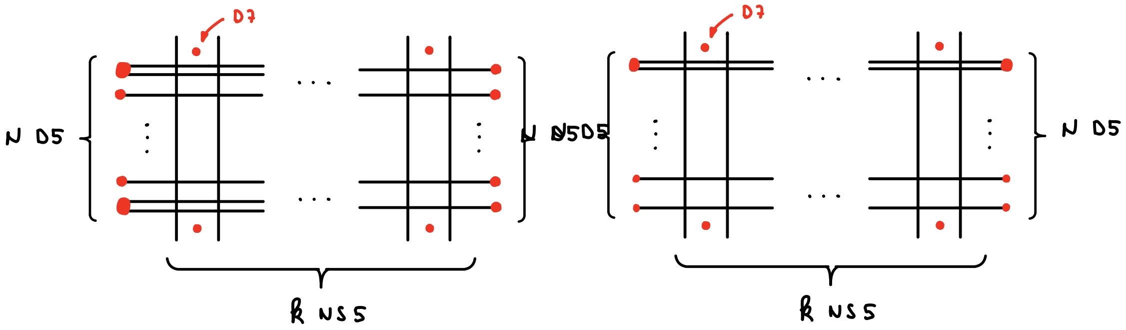
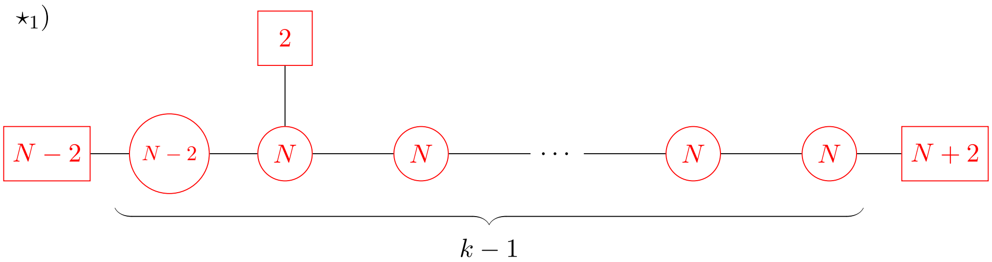


Figure 20. Brane setup for (6.1) on the left and for (6.2) on the right.

6.1 5d UV duality

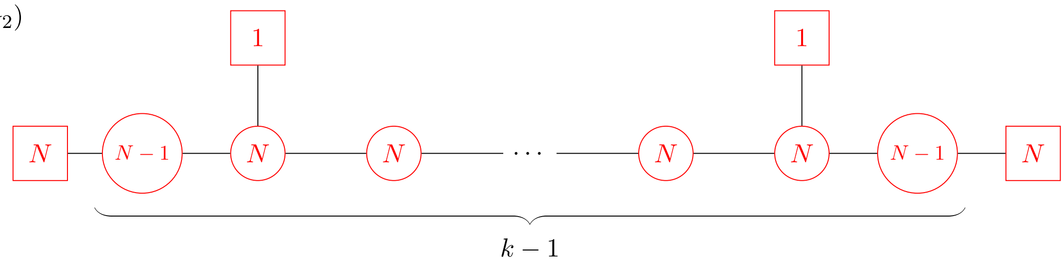
Concretely we will study two different Higgsing in 5d that is mapped to the same deformation of the 6d SCFT. Therefore we are left with another example of 5d UV duality. The question that we can ask: does the 5d UV duality that we obtain after the Higgsing procedure leads to another 4d IR duality? We do not have a general answer to this question but we will study the simplest Higgsing and the answer will turn out to be positive. At the level of the brane systems, the Higgsing is manifested by breaking 5-branes on the same 7-brane [7]. The example of Higgsing that we consider is the following. We start with the brane web on the left of figure 5 and force two pairs of 5-branes to end on the same 7-brane. We have the choice to take the two pairs either on the same side of the brane web or the opposite side. We obtain the brane systems of figure 20 and the gauge theories associated are shown in (6.1)–(6.2). The details of this example can be found in [15].

$\star_1$)



(6.1)

$\star_2$)



(6.2)

6.2 6d UV completion

The 6d UV completion of the theories (6.1)–(6.2) depends on the parity of k and can be obtain by doing the Higgsing at the level of the Type IIA brane setup corresponding to the 6d UV completion of $R_{N,k}$.

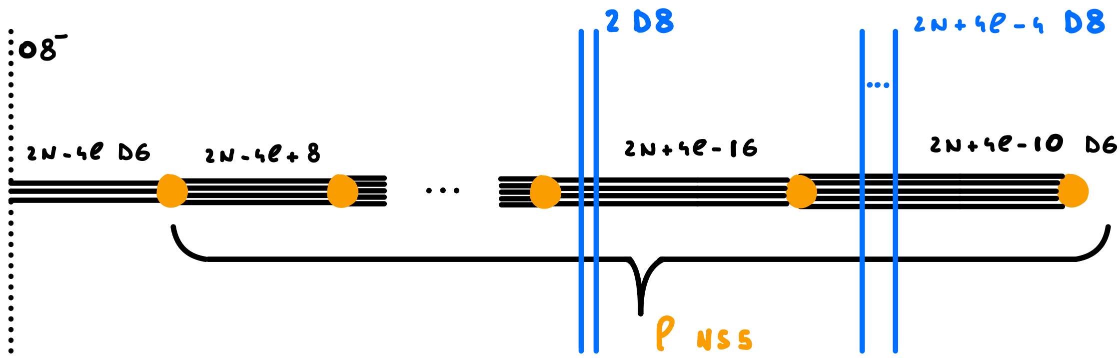


Figure 21. Type IIA brane setup corresponding to the 6d UV completion of (6.1)–(6.2) for $k = 2l$. It is obtained by Higgsing the Type IIA brane system (10) corresponding to the UV completion of $R_{N,2l}$.

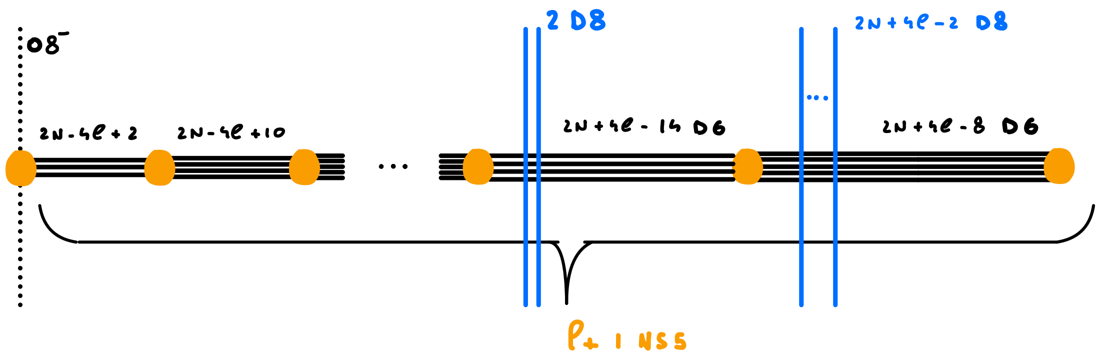
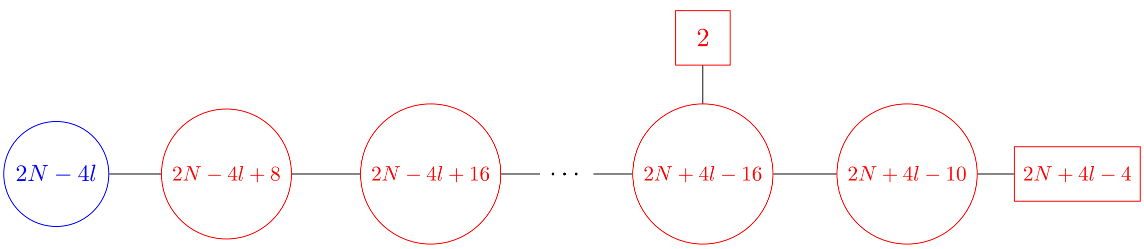


Figure 22. Type IIA brane setup corresponding to the 6d UV completion of (6.1)–(6.2) for $k = 2l+1$. It is obtained by Higgsing the Type IIA brane system (11) corresponding to the UV completion of $R_{N,2l+1}$.

k even: $k = 2l$

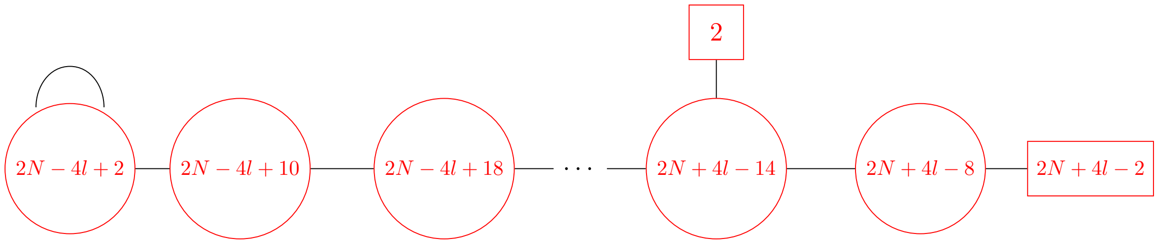
In this case, the 6d completion is given by the following Type IIA brane setup [14, 15]: The gauge theory corresponding to this brane system is a linear quiver with one USp gauge node and $l-1$ SU gauge nodes:



(6.3)

k odd: $k = 2l+1$

The 6d completion is given by the following Type IIA brane setup [14, 15]: The gauge theory corresponding to this brane system is a linear quiver with l SU gauge nodes and an antisymmetric hyper attached to the first node:

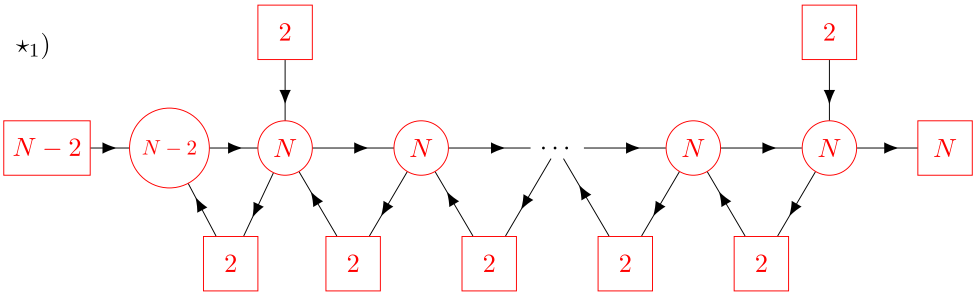


(6.4)

6.3 4d duality

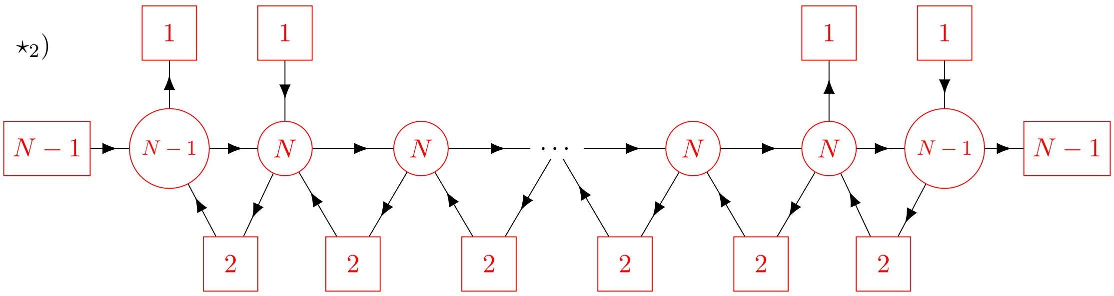
Applying our procedure to the 5d UV duality (6.1)–(6.1) we produce the following 4d $\mathcal{N} = 1$ theories

$\star_1$)



(6.5)

$\star_2$)



(6.6)

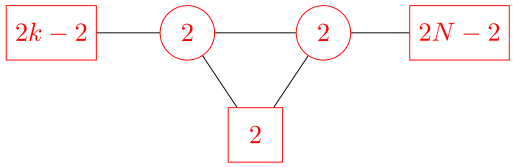
Let us remark that in order to get non-anomalous theories we were forced to split the flavor nodes at the edge of the quiver compared to the 5d avatars.

6.4 Proof of the 4d duality

The proof is really similar to the section 3.4 so we will be brief. We start from either $\star_1$) or $\star_2$), and do the following operations:

- $k - 1$ Seiberg dualities on the SU nodes from left to right
- CSST duality on the left SU(2)
- $k - 3$ confinements

Finally, we introduce some flippers and both $\star_1$) and $\star_2$) take the same following form



$\mathcal{W}$ = Triangle (6.7)

The fact that both $\star_1$) and $\star_2$) are dual (modulo flips) to the same theory (6.7) proves the duality.

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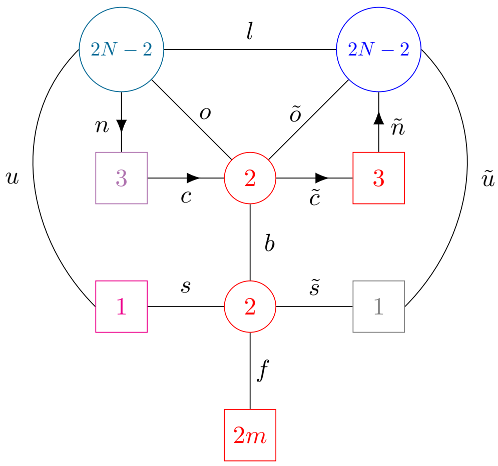
A 't Hooft anomaly matching for the duality of section 5

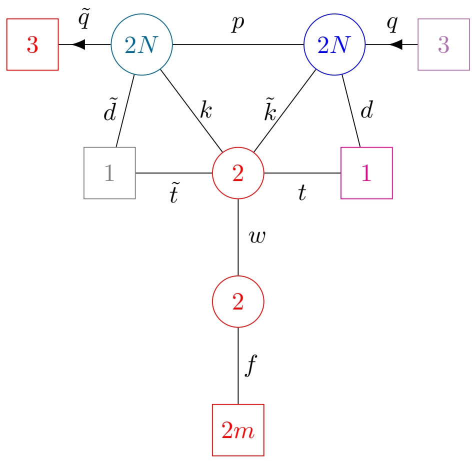
In this appendix we present the matching of the t'Hooft anomalies for the duality (5.14) that we report in (A.1) for convenience. On both l.h.s. and r.h.s. of the duality, the theories have 4 U(1)'s global symmetries. The following charges assignments respect the constraints coming from ABJ anomalies and the superpotential terms. The matching of the t'Hooft anomalies are really non-trivial, especially the ones involving the U(1)'s symmetries, and relies on having the correct set of flipper fields.

$\star_1$)



$$\mathcal{W} = 2\,\text{Quartic} + 3\,\text{Triangles}$$
$$+\,\text{Flip}[bb, c\tilde{c}, cb\tilde{s}, \tilde{c}bs, bsb\tilde{s}]$$

$\star_2$)



$$\mathcal{W} = 3\,\text{Triangles} + \text{Flip}[ww]$$
$$+ \sum_{i=o}^{N-1} \text{Flip}[q(pp)^i q; \tilde{q}(pp)^i \tilde{q}; q(pp)^i d; \tilde{q}(pp)^i \tilde{d};$$
$$qp(pp)^i \tilde{q}; qp(pp)^i \tilde{d}; \tilde{q}p(pp)^i d; dp(pp)^i \tilde{d}] \tag{A.1}$$

't Hooft anomalies involving non-abelian symmetries:

- **l.h.s.:**

$$\text{tr}\,(\text{SU}(3)^3) = (2N-2)A(\Box) + 2A(\bar{\Box}) + 3A(\Box) + A(\Box) = 2N \tag{A.2}$$
$$\text{tr}\,(\text{SU}(3)^3) = -2N \tag{A.3}$$
$$\text{tr}\,(\text{SU}(2m)^3) = 2 \tag{A.4}$$

- **r.h.s.:**

$$\text{tr}\,(\text{SU}(3)^3) = 2NA(\bar{\Box}) + N\left(A\left(\substack{\Box\\\Box}\right) + A(\Box) + 3A(\Box) + A(\Box)\right) = 2N \tag{A.5}$$
$$\text{tr}\,(\text{SU}(3)^3) = -2N \tag{A.6}$$
$$\text{tr}\,(\text{SU}(2m)^3) = 2 \tag{A.7}$$

In the previous equations, A corresponds to the anomaly coefficient. In our normalization, it takes the following value for SU(N): $A(\Box) = 1 = -A(\bar{\Box})$ and $A\left(\substack{\Box\\\Box}\right) = N - 4$.

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Fields l.h.s.	$\mathrm{U}(1)_1$	$\mathrm{U}(1)_2$	$\mathrm{U}(1)_3$	$\mathrm{U}(1)_4$
f	0	0	$\frac{1}{2m}$	0
s	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$
$\tilde{s}$	0	-1	$-\frac{1}{2}$	$-\frac{1}{2}$
b	0	0	0	$\frac{1}{2}$
u	-1	-2	$\frac{2N-1}{4N}$	$-\frac{1}{4N}$
$\tilde{u}$	1	2	$\frac{2N-1}{4N}$	$-\frac{1}{4N}$
c	$-\frac{2}{3}$	-1	$-\frac{N-1}{6N}$	$-\frac{2N-1}{6N}$
n	$-\frac{1}{3}$	0	$\frac{1}{3}\frac{2N-5}{4N}$	$\frac{4N-5}{12N}$
$\tilde{n}$	$\frac{1}{3}$	0	$\frac{1}{3}\frac{2N-5}{4N}$	$\frac{4N-5}{12N}$
o	1	1	$\frac{1}{4N}$	$\frac{1}{4N}$
$\tilde{o}$	-1	-1	$\frac{1}{4N}$	$\frac{1}{4N}$
l	0	0	$-\frac{1}{2N}$	$-\frac{1}{2N}$

Table 1. U(1)'s charges of the fields in l.h.s. of (A.1).

Fields r.h.s.	$\mathrm{U}(1)_1$	$\mathrm{U}(1)_2$	$\mathrm{U}(1)_3$	$\mathrm{U}(1)_4$
f	0	0	$\frac{1}{2m}$	0
w	0	0	$-\frac{1}{2}$	0
t	0	-1	0	$-\frac{1}{2}$
$\tilde{t}$	0	1	0	$-\frac{1}{2}$
d	1	2	$-\frac{1}{4N}$	$\frac{2N-1}{4N}$
$\tilde{d}$	-1	-2	$-\frac{1}{4N}$	$\frac{2N-1}{4N}$
k	1	1	$\frac{1}{4N}$	$\frac{1}{4N}$
$\tilde{k}$	-1	-1	$\frac{1}{4N}$	$\frac{1}{4N}$
q	$\frac{1}{3}$	0	$\frac{4N-1}{12N}$	$\frac{2N-1}{12N}$
$\tilde{q}$	$-\frac{1}{3}$	0	$\frac{4N-1}{12N}$	$\frac{2N-1}{12N}$
p	0	0	$-\frac{1}{2N}$	$-\frac{1}{2N}$

Table 2. U(1)'s charges of the fields in r.h.s. of (A.1).

't Hooft anomalies involving abelian symmetries:

- **l.h.s.:**

$$\begin{aligned}\mathrm{tr}\,(\mathrm{SU}(3)^2\mathrm{U}(1)_i) &= (2N-2)\,q_n^i\,\mu(\bar{\square}) + 2\,q_c^i\,\mu(\square) - 3(q_c^i + q_{\tilde{c}}^i)\,\mu(\bar{\square}) - (q_c^i + q_b^i + q_{\tilde{s}}^i)\,\mu(\bar{\square})\\ i=1 :&= -\frac{2N}{3}\\ i=2 :&= 0\\ i=3 :&= \frac{2N+1}{6}\\ i=4 :&= \frac{4N+1}{6}\end{aligned}$$

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$$\begin{aligned}
\mathrm{tr}\,(\mathrm{SU}(3)^2\mathrm{U}(1)_i) &= (2N-2)\, q^i_{\tilde{n}}\, \mu(\bar{\square}) + 2\, q^i_{\tilde{c}}\, \mu(\square) - 3(q^i_c + q^i_{\tilde{c}})\, \mu(\bar{\square}) - (q^i_{\tilde{c}} + q^i_b + q^i_s)\, \mu(\bar{\square}) \\
i = 1 : &= \frac{2N}{3} \\
i = 2 : &= 0 \\
i = 3 : &= \frac{2N+1}{6} \\
i = 4 : &= \frac{4N+1}{6}
\end{aligned}$$

In the previous equations, μ corresponds to the Dynkin index of the representation. In our normalization, it takes the following value for $\mathrm{SU}(N)$: $\mu(\square) = 1 = \mu(\bar{\square})$ and $\mu(\begin{smallmatrix}\square\\\square\end{smallmatrix}) = N-2 = \mu(\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}})$.

Same kind of computations give for the linear anomalies:

$$\begin{aligned}
&\mathrm{tr}\,(\mathrm{U}(1)_1) = 0 \\
&\mathrm{tr}\,(\mathrm{U}(1)_2) = 0 \\
&\mathrm{tr}\,(\mathrm{U}(1)_3) = 2N+2 \\
&\mathrm{tr}\,(\mathrm{U}(1)_4) = 2N-1
\end{aligned}$$

- **r.h.s.:**

$$\begin{aligned}
\mathrm{tr}\,(\mathrm{SU}(3)^2\mathrm{U}(1)_i) &= (2N)\, q^i_q\, \mu(\bar{\square}) - \sum_{j=0}^{N-1} \Big[(2\, q^j_q + 2j\, q^j_p)\, \mu(\begin{smallmatrix}\square\\\square\end{smallmatrix}) + (q^j_q + 2j\, q^j_p + q^j_d)\, \mu(\square) \\
&\quad + 3(q^j_q + q^j_{\tilde{q}} + (2j+1)\, q^j_p)\, \mu(\square) + (q^j_q + q^j_{\tilde{d}} + (2j+1)\, q^j_p)\, \mu(\square)\Big] \\
i = 1 : &= -\frac{2N}{3} \\
i = 2 : &= 0 \\
i = 3 : &= \frac{2N+1}{6} \\
i = 4 : &= \frac{4N+1}{6}
\end{aligned}$$

$$\begin{aligned}
\mathrm{tr}\,(\mathrm{SU}(3)^2\mathrm{U}(1)_i) &= (2N)\, q^i_q\, \mu(\square) - \sum_{j=0}^{N-1} \Big[(2\, q^j_{\tilde{q}} + 2j\, q^j_p)\, \mu(\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}) + (q^j_{\tilde{q}} + 2j\, q^j_p + q^j_{\tilde{d}})\, \mu(\bar{\square}) \\
&\quad + 3(q^j_q + q^j_{\tilde{q}} + (2j+1)\, q^j_p)\, \mu(\bar{\square}) + (q^j_{\tilde{q}} + q^j_d + (2j+1)\, q^j_p)\, \mu(\bar{\square})\Big] \\
i = 1 : &= \frac{2N}{3} \\
i = 2 : &= 0 \\
i = 3 : &= \frac{2N+1}{6} \\
i = 4 : &= \frac{4N+1}{6}
\end{aligned}$$

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$$\mathrm{tr}\,(\mathrm{U}(1)_1) = 0$$
$$\mathrm{tr}\,(\mathrm{U}(1)_2) = 0$$
$$\mathrm{tr}\,(\mathrm{U}(1)_3) = 2N + 2$$
$$\mathrm{tr}\,(\mathrm{U}(1)_4) = 2N - 1$$

We can indeed see the matching of the anomalies.

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