Wave functions of multiquark hadrons from representations of the symmetry groups S_n

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Construction of the wave functions of multiquark hadrons by a traditional method based on the tensor products of colors, flavors, spins (and orbital) parts becomes quite complex when quark numbers grow $n = 5, 6 \ldots 12$, as it gets difficult to satisfy the requirements of Fermi statistics. Our novel approach is focused directly on representations of the permutation symmetry generators. After showing how C_3 is manifested in the wave functions of (excited) baryons, we use it to construct the wave functions for a set of pentaquarks and hexaquarks (n $= 5, 6$). We also have some partial results for larger systems, with $n = 9$ and 12, and even beyond that as far as $n = 24$.

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I. INTRODUCTION

A. Outline

The main issue discussed in this work is a very old one: how to construct hadronic wave functions (WFs), which are totally antisymmetric under quark permutations, as Fermi statistics requires. While for mesons and some baryons it is rather simple task, in general, it is not so since there arise parts of the WFs which possess "mixed symmetries", for color, flavor, spin (and orbital part, at nonzero orbital momentum L). To construct WFs, by linear superposition of these parts, with appropriate permutation symmetries, rapidly becomes difficult, as the number of constituents grows.

The simplest baryons like Δ [and other members of $SU(3)$ decuplet] have WF factorized into a product of *antisymmetric* color and *symmetric* flavor and spin WFs as $I, S = 3/2$. Its wave function is thus represented by a single *monom* $(u^\uparrow u^\uparrow u^\uparrow)$.

Yet already the flavor-spin WF of nucleons [and other members of $SU(3)$ octet] get more complicated since their quantum numbers $(I, S = 1/2)$ prevent construction of separate permutation-symmetric WFs for spin and flavor. One can easily construct spin-1/2 WFs symmetric (or antisymmetric) under the interchange of, say, quarks 1 and 2 (P_{12}), yet they are not symmetric under other (e.g., P_{23}, P_{13}) permutations. One has to look at the superposition of $S_{\rm sym}F_{\rm sym}$ and $S_{asym}F_{asym}$ terms, symmetric under P_{12}, and check which one would be symmetric under other permutations.

The method we developed in this paper stems from our recent paper [1] in which we addressed the WFs of (negative parity) $L = 1$ and $L = 2$ nucleon resonances. Their WFs include orbital factors, on top of spin and flavor variables. Those depend on angles of the (modified) Jacobi coordinates and also have mixed permutation symmetries; thus, the problem gets even more complicated. With more parts of flavor-spin-orbital combinations, we solved the problem and obtained explicit WFs by first building "tensor cubed" representations of the generators of the symmetry group S_3. We also advocated usage of spin-tensor notations, in space of all existing "monoms", using *Mathematica*.

In this paper, we generalize this method further, to multiquark hadrons, building representations of higher symmetry groups S_n, with $n = 4, 5, 6..$ respectively; see Sec. II A. We focus mostly on $n = 6$, hexaquarks (or dibaryons) and pentaquarks. Starting from spin-tensor notations with all monoms, combining color, flavor, spin etc. indices, one finds that their dimensions rapidly become quite large—e.g., for n $= 6$ u, d quarks there are $3^6 \cdot 2^6 \cdot 2^6 = 2985984$ monoms, and the square of that for the $n = 12$ case. Those become hard to handle even using *Mathematica* due to computer memory limitations.

Yet the actually needed dimensions of pertinent vectors and matrices can be greatly reduced by using what we call a "good basis" of linearly independent and mutually orthogonal combinations, as we will detail below. For example, for hexaquark with maximal spin, the reduction of color-flavor monom space $3^6 \cdot 2^6$ get reduced to just $5 \cdot 5$ space.

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Another important idea (well known in mathematics) is that there is no need to consider all $n!$ permutations of the symmetry group C_n, but build matrix forms only for its *two* basic generators, namely $P_{12}, P_{\rm cycle}$. After diagonalization of those, it is rather easy to see if they possess *common* eigenvectors with the needed symmetry. If they do, those are the WF one is looking for.

Our approach is much more direct than standard approach, based on building subsequent representations starting from spin, flavor $SU(2)$ and color $SU(3)$ groups via tensor products, eventually reaching the (tensor product) power n (for n quarks) starts with selecting certain pairs, then pairs of pairs etc. The very first step on this road—randomly selecting the original pairs—is arbitrary and contrary to the final goal, keeping certain global permutation intact.

We instead of the usual quantum numbers focus on representations of symmetry groups S_n. It also starts with a tensor product of a certain number of its representations. Completing the historic Introduction, let us add what (we learned) from its mathematical history.

A given Young tableau is a simultaneous description of both an $SU(N)$ and an S_n representation, the S_n representations of the pieces of the wave function for each sector match the $SU(N)$ one, except instead of neglecting the N-tall stacks, we include them. So, if λ_c is the Young tableau corresponding to the $SU(3)$ representation for color, λ_f is the Young tableau corresponding to the $SU(2)$ representation for flavor, and so on, then their tensor product should be decomposed into a sum of irreducible representations,

$$\lambda_c \otimes \lambda_f \otimes \ldots = \bigoplus_{\mu \in \text{ reps of } S_n} C^{\mu}_{\lambda_c \lambda_f \ldots} \mu, \tag{1}$$

with "Clebsch-Gordon" coefficients. (In mathematics, these constants are typically called Kronecker Coefficients.) The question of whether there exists a state with the correct Fermi statistics translates to the question of whether the coefficient $C_{{\rm anti}\lambda_c\lambda_f\ldots}$ in these series is zero or not.

There has been a large amount of mathematical investigation into them, and there exist efficient online resources to find them. We suggest the website [2], which can help answering the question of existence of the WFs, even for quark numbers that are too large for our procedure based on *Mathematica* to handle and find them explicitly (see Sec. B). When these Kronecker coefficient tables return 0, it rules out the possibility of Fermi-statistics-obeying WFs for a given combination of quantum numbers. Needless to say, in all cases for which, we were able to do it explicitly, the results agree with the output of this page. For some explanations of how to use [2], see Appendix E.

Completing the general introduction, let us briefly comment on the selection of particular examples we will discuss. Since we have already discussed baryons and tetraquarks in our previous work [1], the natural extensions are *penta*− and, especially, *hexaquarks* (or dibaryons, for general reviews, see, e.g., [3,4]). In Sec. IV A, we start showing how all procedures suggested are used in the case of an $uuuddd$ hexaquark with a maximal spin $S = 3$, which is then applied to other spins, flavors, and orbital momentum in Sec. IV B. Pentaquarks are discussed in Sec. V. We also discuss challenging applications to larger systems, with nine quarks (tribaryons) and 12 quarks (tetrabaryons), in Sec. VIII.

B. Historic remarks

Hadronic spectroscopy started in 1960's from flavor $SU(3)$ symmetry, with Gell-Mann's "eightfold way" based on its octet adjoint representation. Then *decuplets* came, both algebraically and then experimentally, completed with the famous observation of $\Omega^- = sss, I = J = 3/2$ hyperon. Combining flavor with $SU(2)$ spin into the $SU(6)$ group, one was able to get "squared" and "cubed" representations of it, as needed for basic mesons and baryons. Yet already for baryons with nonzero orbital momentum $L = 1$, 2.. it gets so complicated that (e.g., in classic papers like [5]) their WFs enforcing Fermi statistics were not explicitly constructed. We did so recently in [1].

This paper deals with technical theoretical issues, aiming at explicit construction of WFs for mutiquark hadrons, with $n = 4, 5, 6 \ldots 12$. One may wonder if those are actually needed for any real-life applications. Indeed, for about 40 years (1965–2005) it was considered to be a purely academic subject, but during the last two decades, discoveries of many multiquark resonances suddenly became numerous, turning the field of hadronic spectroscopy into a real renaissance. The leader in this direction is the *LHCb* Collaboration, making good use of the multiple production of c, b quarks at LHC. Indeed, new multiquark hadrons happen to be associated with at least one or two heavy quarks.

While multiquark configurations may be heavier than the ones made of distinct baryons, they still may constitute a virtual admixture to those. We know from experiment that protons have "sea" with antiquarks, in their parton composition. Consider the alpha particle, $^4{\rm He} = ppnn$ state, quite deeply bound and compact according to nuclear physics standards. It is a "doubly magic" one, with the proton and neutron lowest shell closed. But it can be mixed, in its core, with a 12-q u^6d^6 S-shell cluster, also a "magic" state, in color-flavor-spin. We will return to this issue in Sec. VIII.

Theory of mutiquark hadrons had a rather controversial history. Some ideas, which looked quite natural at the start, turned out to be rather misleading. The first approach accounting for color confinement was represented by various "bag models", e.g., the famous one from the MIT group. Hadrons were pictured as "bubbles of perturbative vacuum" located inside the lower "true nonperturbative vacuum". The energy one has to pay for its creation was

represented by the "bag constant" times volume $B \cdot V$, accounting for the different energy density of the two vacua.

If so, the following problem arises. As certain energy BV was already used to create a bubble for a meson or baryon, putting more quarks into it should be energetically beneficial. If the perturbative vacuum inside is already "empty" of nonperturbative fields, little penalty would come from adding extra quarks. If so, why some (or all) nuclei are stable rather than collapse into multiquark bags? Indeed, bubbles do have a well-known tendency to coalesce.

Specifically, the MIT bag model predicted the 6-q dibaryon $H = u^2d^2s^2, J^P = 0^+, I = 0$, at 2150 MeV, so light that it will decay only by a double weak decays of both s quarks [6]. (There were suggestions that such a particle was even observed in cosmic rays coming from a particular star, but this was shown not to be possible [7] because even double beta decay will happen on the way.) This H resonance was not observed.

Another naive idea, which gets popular in connection with multiquark hadrons (see, e.g., [8]), is known as the *diquark* model. Indeed, two ud quarks can be combined either into a "good" diquark, with color $\underline{3}$ and spin-isospin $S = I = 0$ 'in which a perturbative spin-spin force is attractive. Nonperturbative instanton-induced forces are attractive as well, making then Cooper pairs of color-superconductivity in dense quark matter [9,10].

This induced the idea that one can construct multiquark systems out of "good diquarks" used as elementary building blocks. If Δ mass be approximated just by $3M$ (masses of constituent quarks), the mass of the nucleon as $3M - B_{qq}$ with one good diquark binding, then $B_{qq} \sim 300$ MeV. Proceeding similarly, one may similarly estimate expected masses of multiquark hadrons.

An extreme case of this ides was "quark-diquark"symmetry model [11] with a "baryon-meson symmetry" and beyond. If the diquark binding can be crudely approximated as $B_{qq} \approx M$, in which nucleon (octet) should be approximately degenerate with mesons,

$$3M - B_{qq} \approx 2M,$$

and hexaquarks made of three diquarks are degenerate with (decuplet) baryons,

$$6M - 3B_{qq} \approx 3M.$$

Proceeding further with such logic all the way to 12-q state, one may predict that states made of six "good diquarks" have mass,

$$12 \cdot M - 6 \cdot B_{qq} \approx 6M \approx 2M_\Delta,$$

which is much lower than four nucleon masses,

$$4(3 \cdot M - B_{qq}).$$

So, such "diquark models" would also predict collapse of nuclei into multiquark states.

(Yes, diquarks are not color neutral, and there are also color confining forces between them, but, like with the bag model this generates energy growing with quark number as power smaller than one, also leading to eventual collapse.)

Moreover, if "good diquarks" are treated as separate elementary objects, they are scalar bosons. Therefore, their WFs should be symmetric under permutations, unlike that of quarks. So, if all of these assumptions be true, nuclei (and ^{4}He specifically) should not exist at all. Obviously, such models are wrong, missing something very important.

A construction using "good diquarks" as building blocks can only be a reasonable approximation if these diquarks are far from each other. Yet it is completely unclear why such a configuration may be dominant. Furthermore, 6 quarks have 15 quark pairs. For 12 quarks there are 66 pairs. Using the most attractive channels in just three (or six) of them, and ignoring interactions of all other quark pairs is indeed very misleading. Experience with atomic and nuclear shell model tells us that, if possible, all fermions will sit at the same 1S shell states. Yes, one has to construct the WFs with correct Fermi symmetry (as we will do). Its energy can be calculated only by adding *all* pairwise forces (to say nothing about three-body forces and so on) between them.

An instructive case are tetraquarks, which gets discussed more lately, especially due to discovery of an all-charm $cc\bar{c}\bar{c}$ set of resonances at LHC. Those can be of two structures, either made of two "good" $\underline{3}3$ or two "bad" $6\underline{6}$ diquarks. Naive diquark ideology suggest that the former case should lead to lower energy. However in the latter case, the attractive color forces *between diquarks* are stronger (two QCD string rather than one). Including all six pairwise interactions proportional to relative color $(\vec{\lambda}_i\vec{\lambda}_j)$, one finds [12,13] that both structures lead to the *same* binding.

Another instructive example to be discussed below has been provided by [14] (KKO), for $uuuddd, I = 0, S = 3$ hexaquark state. Out of the Young tableaux, these authors constructed *five* distinct mix-symmetry WFs (we will discuss them below) and combined those into *unique* WF with total (Fermi-required) antisymmetry. Contrary to naive expectations, whether they do contain "good diquarks" or not, their energies from $(\vec{\lambda}\vec{\lambda})$ Hamiltonian turned out to be the *same*. And indeed, which quarks we decided to pair first is a completely random choice, out of many possibilities.

In this paper, we will not use pairing of some particular quarks into preselected quantum numbers but directly construct the pertinent (anti)symmetric WFs based on a representation of the permutation groups S_n. They will be shown to lead directly to explanations of the two examples just discussed.

II. MULTIQUARK WFS AND REPRESENTATIONS OF THE SYMMETRIC GROUPS

This paper is quite technical, so we try to make it as pedagogical as possible. While starting from general strategy and defining the needed steps, we also show how they work for well-known cases.

The monom basis and spin-tensors: A quark has a color index $a = 1, 2, 3$, a spin $i = 1, 2$. We will consider various options for flavor, starting from $SU(2), u, d$, then $SU(3), u, d, s$ etc. Even for a simpler former case, a single quark has $3 \cdot 2 \cdot 2 = 12$ states, and then for n quarks a complete basis of all possible "monoms" has 12^n elements,

$$|c_1 \ldots c_n f_1 \ldots f_n s_1 \ldots s_n > .$$

Even for baryons, $n = 3$, it is rather large, while the number of nonzero elements are often small enough to simply list those. But this number grows rapidly with n, so it is rather impractical to continue doing so. Standard spin-tensor notations and usage of software such as *Mathematica* make it uniform and practical for $n = 4, 5, 6 \ldots$.

Throughout this paper, we use various spin-tensor notations, in which the WF components are numerated by quantum states of each quark, but in a certain predetermined order, e.g.,

$$\Psi = \sum C(c_1 \ldots c_n f_1 \ldots f_n s_1 \ldots s_n) |c_1 \ldots c_n f_1 \ldots f_n s_1 \ldots s_n >, \tag{2}$$

with each $c_i = 1, 2, 3$ being color indices, $s_i = 1, 2$ the spin indices, and f_i corresponding to quark flavors, ud or uds etc. In *Mathematica,* one can use *Flatten* command to eliminate brackets of such tensor and reduce the WF to a single one-dimensional vector (list). Going back from it to original tensor is also relatively simple.

A. Outline of procedure

The proposed procedure for obtaining the WFs of given multiquark hadron consistent with Fermi statistics can be summarized with the following steps:

(1) Find *possible tensor structures* that WFs should take in *each* sector. Write any one tensor that obeys all the correct symmetries of the corresponding Young tableaux. For example in color sector, one can choose antisymmetric $\epsilon_{c1,c2,c3}$ for baryons, with color indices $c1, c2, c3 = 1, 2, 3$ For hexaquarks, it generalizes to two of them $\epsilon_{c1,c2,c3}\epsilon_{c4,c5,c6}$, with all permutations of indices. If antiquarks are present, there can be also be Kroneker $\delta^{c\bar{q}}_{cq}$ symbols included.

(2) Write *all possible permutations* of that object's n indices under generators of S_n. Express those $n!$ tensors as vectors over the basis of all N^n monoms spanning $\mathbb{S}^{N^n}$.

(3) *Orthogonalize* this list of vectors, using a procedure such as Gram-Schmidt. This will result in a new orthonormal basis, typically being much shorter than the original list. To predict exactly how much shorter, one can use the explicit formulas for this n' given in Eqs. (A2) and (A3).

(4) Find the matrices that correspond to each of the two *permutation group generators* acting on this basis $\mathbb{S}^{N^n}$ (see B) to obtain 2 $n' \times n'$ matrices.

(5) After steps 1–4 for each sector (color, flavor, spin, orbital...) are done, take the *Kronecker (tensor) product* of all sectors' permutation generator matrices, to find the effects of these permutation generators in the total Hilbert space. This yields two square matrices of dimension $n'_{\text{total}} = n'_{\text{color}} \cdot n'_{isospin} \cdot \ldots$

(6) *Diagonalize* both generators, looking for eigenvectors with the same required symmetry (eigenvalues 1 or -1) depending on the case (see D). If found, these eigenstates are the WFs. They can be projected back to the original monom basis, if desired.

III. BARYONS

In mathematics, groups usually are defined via geometrical settings, e.g., the symmetry group S_3 of free elements defined as a self-maps of the equilateral triangle, S_4 by that of tetrahedron, and S_6 of five-dimensional construction with six corners. (In fact, specific kinematics based on a light cone description of the corresponding wave functions in momentum representation lead precisely to WFs defined on exactly those geometrical objects.)

Starting with baryons, $n = 3$, the well-known simplification is that the *color* wave function takes the form of Levi-Civita antisymmetric symbol (*Mathematica*),

$$\epsilon_{abc} = LeviCivita[3][[a, b, c]].$$

It factorizes from the rest and is antisymmetric. Therefore, the rest of the WF—spin, flavor, and orbital parts—should be symmetric under particle interchanges.

It is simple to achieve for Δ [and other members of the $SU(3)$ decuplet] by making *each* of them (flavor and spin) WFs *symmetric* individually. Say, when isospins and spins are all up, $I_z, S_z = 3/2$ the WF of such Δ^{++} is reduced to a single flavor-spin monom $(u^\uparrow u^\uparrow u^\uparrow)$ (out of potential $2^3 * 2^3 = 64$ possible components).

Yet already the nucleons [and other members of $SU(3)$ octet] are more complicated since their quantum numbers $(I, S = 1/2)$ prevent construction of *separately* symmetric WFs for spin and flavor. Three permutations of corresponding Young tableau are shown in Fig. 1. Recall that any pair placed vertically mean antisymmetric convolution $\sim \epsilon_{ij} S1^i S2^j$, $i, j = 1, 2$ into zero spin pair, so the first diagram tells us that total spin is that of quark 3, $S3$.

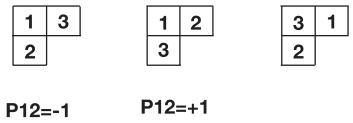

FIG. 1. Three Young symmetry diagrams for three quark spins combined into total $S = 1/2$.

The second does so for quarks 1 and 3, and the third to quarks 3,2.

[Note that convolution with epsilon is necessary, and the symmetric WF $\sim(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)$ while have total $S_z = 1/2$ correspond to total spin $S = 3/2$ and therefore, to different Young tableau, with three horizontal squares.]

The first tableau from Fig. 1 is antisymmetric in respect to P_{12} permutation; the second is symmetric under it. The third is antisymmetric over 2–3 permutation, but its transformation under 1–2 is more complicated and can in general be included as some matrix. The step 1 of our procedure leads to realization that the third term is linear combination of the first two; thus, one gets a reduction from $2^3 = 8$ monoms in spin to only 2.

Let us show the steps of our suggested program, to ensure consistency with other applications. Consider spin WF (isospin is the same). There are $2^3 = 8$ monoms. Using our *Mathematica* command one writes and obtains back the following:

```
In:  GoodBasis[TensorProduct[LeviCivitaTensor[2], u]]
Out: {{0, 0, 1/Sqrt[2], 0, -(1/Sqrt[2]), 0, 0, 0},
      {0, Sqrt[2/ 3], -(1/Sqrt[6]), 0, -(1/Sqrt[6]), 0, 0, 0}}
```

The input $TensorProduct[LeviCivitaTensor[2], u]$ includes *all* permutations of particles; $u = (1, 0)$ is the spin-up elementary vector. The *GoodBasis* command orthogonalizes these possible vectors, and tells us that all permutations produce only *two* mutually independent and orthogonal 8-d vectors. From that point on, we work only with such "good basis" states, and spin-flavor WFs will be written as $2 * 2 = 4$-dimensional, instead of $8 * 8 = 64$-dimensional. (While the corresponding simplifications does not look like much, below we will see that similar steps for other cases would reduce space of few millions monoms to those of just about a hundred dimensions.)

At this point we made a digression from our suggested procedures and remind what was done historically, starting from 1970's and widely used (e.g., in [5]). Let us for a moment consider not spin, color, and flavors, but just coordinates/momenta of three quarks. For the total momentum fixed, one needs just two Jacobi coordinates (or momenta) traditionally called

$$\vec{\rho} = (\vec{r}_1 - \vec{r}_2)/\sqrt{2}$$
$$\vec{\lambda} = (\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3)/\sqrt{6}. \tag{3}$$

Note that they happen to be exactly the same combinations of coordinates as spins in our "good basis". Note further that the first of those, ρ, is antisymmetric under P_{12} permutation, while the second is symmetric under it. Thus, *two* spin combinations,

$$S_\rho = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\uparrow \tag{4}$$

$$S_\lambda = \frac{1}{\sqrt{6}}(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow), \tag{5}$$

were introduced. The same are definitions for flavor, F_ρ, F_λ, with an obvious change from spin-up to u, and spin-down to d quark.

Complete matrices of permutations are

$$P_{12} \times \begin{pmatrix}\rho\\ \lambda\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\rho\\ \lambda\end{pmatrix}$$
$$P_{23} \times \begin{pmatrix}\rho\\ \lambda\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}\rho\\ \lambda\end{pmatrix}, \tag{6}$$

because ρ, λ doublet transformation under P_{12} is simply antisymmetric and symmetric, so the corresponding matrix is diagonal. One can easily construct spin-$1/2$ WFs symmetric (or antisymmetric) under interchange of quarks 1 and 2 (P_{12}), yet they are not symmetric under other (e.g., P_{23}) permutations. The combinations $S_\rho F_\rho$ and $S_\lambda F_\lambda$ terms are both symmetric under P_{12}, but not under other permutations. Looking for their superposition, people guessed that $S_\rho F_\rho + S_\lambda F_\lambda$ is in fact symmetric under all elements of S_3. Note that this approach has an unfortunate element of guessing, and even if a positive result for a guess is obtained, it is not yet clear if other successful solutions may exist.

In our recent paper [1], we addressed the problem a bit differently. A Kroneker product of spin and flavor combinations in basis $\rho\rho, \rho\lambda, \lambda\rho, \lambda\lambda$ is a 4×4 matrix. $P_{12} \times P_{12}$ is also diagonal, with two eigenvalues 1 and two -1. Kroneker *squares* (products of a matrix to itself) ($MM = KroneckerProduct[M, M]$ in *Mathematica* language) was diagonalized and its eigenstates with eiegenvalue 1 (which corresponds to symmetric wave function) found. Then we located eigenstate *common* to both P_{12} and P_{23}, producing therefore the required permutation-symmetric spin-flavor WF of the nucleon.

Now, returning to procedure advocated in this work, the only little modification is needed: instead of the second operator used above, P_{23}, one should use the second generator of the group, $P_{\rm cyclic}$. Its Kronecker product to itself is the following 4×4 matrix:

$$P_{\rm cyclic}=\begin{pmatrix} 1/4,\sqrt{3}/4,\sqrt{3}/4,3/4 \\ -(\sqrt{3}/4),1/4,-(3/4),\sqrt{3}/4 \\ -(\sqrt{3}/4),-(3/4),1/4,\sqrt{3}/4 \\ 3/4,-(\sqrt{3}/4),-(\sqrt{3}/4),1/4 \end{pmatrix}.$$

Diagonalizing it (using $Eigensystem[P_{\rm cyclic}]$ command), one then finds that *two* of its four eigenvalues are 1, with eigenvectors being $\{1,0,0,1\},\{0,-1,1,0\}$ One can then observe that only *one* of them (the first) is common to both permutations (and in fact, to all elements of the S_3 group).

In [1], we generalized this approach to the WFs of the (negative parity) $L=1$ N^* resonances. Those were not explicitly defined in 1970's, by Isgur and Karl (who used instead certain limits of the WFs of strange baryons instead). Excited baryons with $L\neq 0$ have orbital WFs depending on angles of Jacobi coordinates, which also may have various permutation symmetries. Those depend on quark coordinates linearly (quadratically, etc.). As we already discussed, the Jacobi coordinates ρ, λ have the same permutation symmetries and chosen spin and flavor states. So, to get WF for $L=1$ baryons, one has to find the tensor product of three copies $KronekerProduct[M,M,M]$ (then four, etc), diagonalize these $2^3\times 2^3$ matrices, look for common eigenvectors of all permutations with the eigenvalue 1 and found them, uniquely in most cases.

IV. HEXAQUARKS AND S_6 REPRESENTATIONS

A. The *uuuddd* hexaquark with the maximal spin $S=3$

Jumping now to multiquark hadrons, we start with this special case. First of all, it has been seen experimentally [15] as a resonance $d^*(2380)$ in the reaction,

$$p+n\to d+\pi^0+\pi^0.$$

Its small width $\Gamma_{d^*}\approx 70$ Mev is significantly below that of the delta baryon $\Gamma_\Delta\approx 115$ MeV. This fact was used against its interpretation as a $\Delta\Delta$ bound state. Also, the $\Delta\Delta$ binding needs then to be $\approx$84 MeV, perhaps too large for a "molecular" state.

Note further, that spin and isospin (for u, d) are the same $SU(2)$ structure; one can interchange them. So, one expects its "mirror image" with $I=3$, $S=0$ and the same mass. Similar consideration will be true for other hexaquarks.

Theory wise, as we already noted in the Introduction, it is at the moment the only hexaquark state for which full antisymmetric WF has actually been derived [16] and will be mentioned below as the KKO WF. That has been done by traditional means, starting from diquarks and then adding up representations of colors and flavors to six. The answer required a lot of work and is written explicitly.

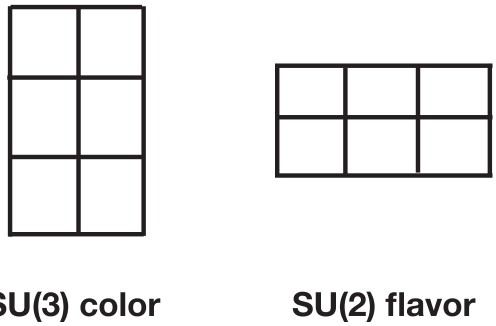

FIG. 2. Color and flavor Young tableax for hexaquarks.

Simplification in this case comes from the total spin being at its maximal value $S=3$, so that all quark spins $\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$ point in the same direction. Thus, the spin part of the WF is trivially symmetric and factorizes. What remains to deal with are the intermixed *color* and *flavor* WFs. The former make representations of the $SU(3)$ group, and the latter either $SU(2),u,d$ or $SU(3),u,d,s$ flavor groups.

The space of all color states have 3^6 monoms. Color Young tableaux shown in Fig. 2 (left) should look like two complete vertical sets of squares. In our notations, they correspond to all permutations of

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TensorProduct[LeviCivitaTensor[3],
LeviCivitaTensor[3]].
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For step 2, we considered every possible one of these rearrangements and wrote them in as vectors in $\mathbb{C}^{3^6}$, using *Mathematica*'s *Flatten* function. There are 6! permutations in symmetric S_6 group, so that is the number of 3^6-dimensional vectors in our list. (By a coincidence $6!=720$ happens to be close to $3^6=729$.)

Orthogonalization of this set of vectors was accomplished through *Mathematica*'s *Orthogonalize* procedure, which by default uses Gram-Schmidt method and generates a set of independent orthonormal vectors. From dimension 720, that reduces to "good basis" of only five linearly independent combinations. The notations below use $k=1$ is P_{12},$k=2$ is $P_{\rm cycle}$ permutation generators, defined in this basis $\{\mathbf{b}_i\}_{i=1}^5$ were then found by taking

$$(P^k_{\rm color})_{ij}=\mathbf{b}_i^T\cdot P^k_{\mathbb{C}^{729}}\cdot\mathbf{b}_j,$$

where $P^k_{\mathbb{C}^{729}}$ was generated using the technique described in Appendix B.

For the flavor sector for *uuuddd* quarks, the necessary tensor structure is three diquarks,

```
TensorProduct[LeviCivitaTensor[2],
LeviCivitaTensor[2],
LeviCivitaTensor[2]].
```

Just like for color, there are 6! quark permutations of the flavor indices f_i's, also leading to just five unique linearly

independent combinations. We then compute two $P^k_{\rm flavor}$ 5×5 matrices for generators of S_6.

The next step is to perform $KronekerProduct$ of color and flavor spaces, and representing two generators of the S_6 group, P_{12} and $P_{\rm cycle}$. Their diagonalization allows us to search for common eigenstates with total eigenvalue -1 (antisymmetry for Fermions). There is indeed one such state found.

Most of the wave functions we obtained are too large to list here, but, for instance, the color-flavor 25-dimensional wave function can be written in the ("flattened") basis $\{\mathbf{b}_{\rm color}\}\otimes\{\mathbf{b}_{\rm flavor}\}$,

$$\left(0,0,0,0,\frac{1}{\sqrt{5}},0,0,0,-\frac{1}{\sqrt{5}},0,0,0,\frac{1}{\sqrt{5}},0,0,0,\frac{1}{\sqrt{5}},0,0,0,-\frac{1}{\sqrt{5}},0,0,0,0\right). \tag{7}$$

Note that there are three positive terms and two negative terms with otherwise equal contributions: we checked that they are exactly those found by Kim *et al.* in [16].

B. Hexaquarks with arbitrary spin, flavor, and orbital momenta $L=0,1,2$

Using the power of the proposed method, we apply it for hexaquarks with other quantum numbers.

For spins smaller than 3, the spin WF is no longer trivial. For $S=1, S_z=0$, the general symmetry structure can be obtained from Young tableaux. It is antisymmetry in two pairs of indices and symmetry in the other two. The tensor structure to be permuted is therefore $[\uparrow\downarrow][\uparrow\downarrow]\{\uparrow\downarrow\}$, where symmetric and antisymmetric combinations are

$$[\uparrow\downarrow]=(\uparrow\downarrow-\downarrow\uparrow)/\sqrt{2}$$

$$\{\uparrow\downarrow\}=(\uparrow\downarrow+\downarrow\uparrow)/\sqrt{2},$$

with large number of permutations of indices. Yet, after *orthogonalization* the basis of independent states possesses only nine linearly independent vectors. Therefore, two S_6 generators $\sigma^1_{\rm spin}$ and $\sigma^2_{\rm spin}$ in this basis are 9×9 matrices, given as examples in Eq. (8).

$$P^1_{\rm spin}=\begin{pmatrix}
-1&0&0&0&0&0&0&0&0\\
0&-1&0&0&0&0&0&0&0\\
0&0&-1&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1
\end{pmatrix},$$

$$P^2_{\rm spin}=\begin{pmatrix}
\frac{1}{4}&\frac{1}{4\sqrt{3}}&\frac{1}{\sqrt{6}}&-\frac{\sqrt{3}}{4}&-\frac{1}{4}&-\frac{1}{\sqrt{2}}&0&0&0\\
-\frac{\sqrt{3}}{4}&\frac{1}{12}&\frac{1}{3\sqrt{2}}&-\frac{1}{4}&\frac{1}{12\sqrt{3}}&\frac{1}{3\sqrt{6}}&-\frac{\sqrt{\frac{2}{3}}}{3}&-\frac{4}{3\sqrt{3}}&0\\
0&-\frac{\sqrt{2}}{3}&\frac{1}{6}&0&-\frac{\sqrt{\frac{2}{3}}}{3}&\frac{1}{6\sqrt{3}}&-\frac{1}{3\sqrt{3}}&\frac{1}{6\sqrt{6}}&-\frac{\sqrt{\frac{5}{2}}}{2}\\
-\frac{\sqrt{3}}{4}&-\frac{1}{4}&-\frac{1}{\sqrt{2}}&-\frac{1}{4}&-\frac{1}{4\sqrt{3}}&-\frac{1}{\sqrt{6}}&0&0&0\\
\frac{3}{4}&-\frac{1}{4\sqrt{3}}&-\frac{1}{\sqrt{6}}&-\frac{1}{4\sqrt{3}}&\frac{1}{36}&\frac{1}{9\sqrt{2}}&-\frac{\sqrt{2}}{9}&-\frac{4}{9}&0\\
0&\sqrt{\frac{2}{3}}&-\frac{1}{2\sqrt{3}}&0&-\frac{\sqrt{2}}{9}&\frac{1}{18}&-\frac{1}{9}&\frac{1}{18\sqrt{2}}&-\frac{\sqrt{\frac{5}{6}}}{2}\\
0&0&0&\sqrt{\frac{2}{3}}&-\frac{\sqrt{2}}{9}&-\frac{4}{9}&-\frac{1}{9}&-\frac{2\sqrt{2}}{9}&0\\
0&0&0&0&\frac{8}{9}&-\frac{2\sqrt{2}}{9}&-\frac{1}{9\sqrt{2}}&\frac{1}{36}&-\frac{\sqrt{\frac{5}{3}}}{4}\\
0&0&0&0&0&0&\frac{\sqrt{\frac{3}{10}}}{2}+\frac{7}{2\sqrt{30}}&\frac{\sqrt{\frac{3}{5}}}{4}-\frac{2}{\sqrt{15}}&-\frac{1}{4}
\end{pmatrix} \tag{8}$$

TABLE I. Hexaquarks with *uuuddd* flavor content, with different values of spin S (rows) and its projections S_z (columns). When the number of states is 0, as it is for $S = 0$ and 2, the row is all zeroes. When the number of antiperiodic wave functions found is one, we give the dimension of the orthogonalized basis for tensor product of color-flavor-spin states.

$S\backslash S_z$	0	1	2	3
0	0			
1	$\{1, 5\times5\times9\}$	$\{1, 5\times5\times9\}$		
2	0	0		
3	$\{1, 5\times5\times1\}$	$\{1, 5\times5\times1\}$	$\{1, 5\times5\times1\}$	$\{1, 5\times5\times1\}$

The next step is to define these two generators ($k = 1, 2$ for P_{12} and $P_{\rm cycle}$) written as tensor product incorporating every sector of the WF, e.g.,

$$P^k_{\rm total} = P^k_{\rm color} \otimes P^k_{\rm flavor} \otimes P^k_{\rm spin}. \tag{9}$$

In the previous subsection—hexaquarks with maximal spin $S = 3$—these were $5 \cdot 5 \cdot 1 = 25$-dimensional matrices of permutations. For other spin values and *uuuddd* quarks, those we found to be matrices in the following minimal dimensions: for $S = 2$, they are in 125-d space, for $S = 1$ in 225-d, and for spin 0, they are matrices in 125-dimensions again.

While all of them are too large to be given here, we still emphasize that these dimensions are many times smaller than that of the full space of monoms, 12^6. Important that in practice there is absolutely no problem to operate with them inside *Mathematica*. In particularly, all are generated in a second, and diagonalized as quickly. For the details of simultaneous diagonlization and procedure to find common antisymmetric eigenstates, see Appendix D.

The particular number of solutions for each spin depends only on the spin value S, and of course, not on its projection S_z, as follows from rotational symmetry. Yet the calculation themselves are not technically identical, so we did it for all values of S_z to check for their mutual consistency. Some of the results are shown in Tables I and II.

At $S = 0$ and $S = 2$, we found no solutions were possible with the permutation antisymmetry desired. At $S = 1$ and $S = 3$, however, we found a single antiperiodic wave function for each value of S_z.

Let us now change the flavor content, adding two strange quarks to *uuddss* hexaquark. The flavor becomes $SU(3)$ and its treatment is similar to that of color, if the total adds to zero. The resulting antisymmetric states are reported in Table III.

Completing the hexaquark discussion, let us consider another simplified case, of same-flavor quarks (e.g., *cccccc*). The number of good states is in the Table IV.

Let us explain some cases without solutions first. If both flavor and spin is flat-symmetric, then Fermi statistics requirement falls on color WF, which cannot be fulfilled for six quarks.

TABLE II. Number of antisymmetric $6q$ states (per choice of m_s and m_l) at each combination of total orbital and spin angular momentum for the light quark hexaquark. For example, last in row 1 of the $I = 0$ plot is the KKO state with $S = 3$, $L = 0$. At $L = 0$, there are three possible combinations of S and I, with the ability to swap them identically as they are both $SU(2)$: (1,0), (3,0), and (2,1). It is worth noting that these exactly match the allowed quantum number combinations for dibaryons made from nucleons and Δ's first calculated in 1964 by Dyson and Xuong [17].

$I = 0$				
L \ S	0	1	2	3
0	0	1	0	1
1	1	1	2	0
2	4	9	5	2
$I = 1$				
L \ S	0	1	2	3
0	1	0	1	0
1	1	4	2	1
2	9	15	10	2
$I = 2$				
L \ S	0	1	2	3
0	0	1	0	0
1	2	2	1	0
2	5	10	5	1

TABLE III. Number of *udsuds* antisymmetric states (per choice of m_s and m_l) at each combination of total orbital and spin angular momentum for *udsuds* hexaquark.

L \ S	0	1	2	3
0	0	1	0	0
1	0	2	1	0
2	5	7	4	1

TABLE IV. Number of antisymmetric states (per choice of m_s and m_l) at each combination of total orbital and spin angular momentum for *cccccc* hexaquark.

L \ S	0	1	2	3
0	1	0	0	0
1	0	1	0	0
2	2	2	1	0

TABLE V. Number of antisymmetric $udud\bar{q}$ states (per choice of m_s and m_l) at each combination of total orbital and spin angular momentum before Clebsch-Gordoning on the last antiquark.

L \ S	0	1	2
0	0	1	0
1	2	3	1
2	8	12	4

TABLE VI. Number of antisymmetric states (per choice of m_s and m_l) at each combination of total orbital and spin angular momentum for $udud\bar{q}$ pentaquark after accounting for the antiquark.

L \ S	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
0	1	1	0
1	5	4	1
2	20	16	4

V. PENTAQUARKS

A. Pentaquark components of baryons

In atomic and nuclear physics, it is well known that any state can be viewed as the lowest state in some mean field potential, plus infinite (but convergent) sum over particle-hole pairs. The same is true for hadrons, in particular the baryon wave function includes the basis 3-q sector, plus 5-q sector with an extra quark-antiquark pair, etc.

Such description is especially natural in the light cone formulation, where one of the central physics issues is calculation of the "antiquark sea" and its observed flavor asymmetry, large difference between the $\bar{u}$ and $\bar{d}$ PDFs of the proton.

Most popular description of the "antiquark sea" is done via a combination of approaches, such as most traditional DGLAP (gluon-based $\bar{q}q$ production), or pion-based $\bar{q}q$ production. In our own paper [18], quark pair production is attributed to a four-fermion t'Hooft instanton-based Lagrangian. Common to all of them is *incoherent* (or probabilistic kinetic) approximation, *assuming* that no interference between the produced and original quarks occurs, so that one can calculate *probability* of quark pair production as if it happens in empty space.

Strictly speaking, all of these mechanisms should be treated coherently, by adding pertinent operators to the Hamiltonian, connecting 5-q and 3-q sectors dynamically, and only then calculating additions to the wave function. Yes, there will be "mostly 3-q" and "mostly 5-q" states, after Hamiltonian gets diagonalized.

While we leave this ambitions program for future, in this paper, we focus on Center of Mass (CM) frame and 5-q component alone. In this sector, we set antiquark aside (e.g., think of it as heavy c, b) and focus on WFs of the four quarks, getting the wave function as required by Fermi statistics. So, the symmetry group considered in this section is S_4.

B. $udud\bar{Q}$ pentaquarks

The simplest way to approach this is to find the wave function of the four quarks first, and then Clebsch-Gordon the relevant antiquark wave function on to the result.

The pentaquark color symmetry will be antisymmetric in three indices, and the last index will be controlled by the color of the antiquark. The starting tensor with this symmetry we chose was $\epsilon_{c_1c_2c_3}\delta_{c_4 1}$, where without loss of generality (because color is never directly observed), we have chosen the color of the antiquark to correspond to the first color basis vector.

After writing and then orthogonalizing every flattened rearrangement of this tensor, we found three linearly independent elements.

For the $udud\bar{q}$ pentaquark, a tensor of the flavor of the quarks can be written $\epsilon_{f_1f_2}\epsilon_{f_3f_4}$. There are two linearly independent combinations of its rearrangements.

For spin, we must consider the possible representations of the four quarks first. They will fall into classifications with integer total spins 0, 1, or 2. Just as with the hexaquark, the starting tensor we choose for a given combination of total and projected spin will be the product of some symmetrized and some antisymmetrized basis spinors, i.e., for $S = 1$, $S_z = 1$ $[\uparrow\downarrow]\uparrow\uparrow$.

For coordinate angular momentum, we use the same Jacobi coordinates, but only look at the representations of the generators of S_4 with them. The first generator (1 2) is the same, but the second one is (1 2 3 4) instead of (1 2 3 4 5).

Once we have the permutation matrices for each sector, we can take their Kronecker product and then simultaneously diagonalize them to find the allowed states for the first four quarks. The multiplicity of states for each combination of total spin and orbital angular momentum is given in Table V.

After these states are found, the full system's wave functions can be found multiplying by the relevant multiple by the $S = \frac{1}{2}$ doublet. For instance, a single $S = 1$ state of four quarks becomes an $S = \frac{1}{2}$ state and an $S = \frac{3}{2}$ state of the pentaquark, because $\mathbf{3} \otimes \mathbf{2} = \mathbf{4} \oplus \mathbf{2}$. Because the multiplicity across S_z is the same in the initial representations, it remains constant across the final representations too. The counts of each quark's final wave function are shown in Table VI.

VI. INCLUSION OF ORBITAL ANGULAR MOMENTUM

If one wishes to generalize the method to excited states beyond the S shell, it is done by an addition of a new sector: angular coordinates. Here it helps to think of the system in

terms of the modified Jacobi coordinates. For instance, in six dimensions, the transformation to those is

$$\begin{pmatrix}\rho_1\\ \rho_2\\ \rho_3\\ \rho_4\\ \rho_5\\ \rho_6\end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0\\ 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{2}{\sqrt{5}} & 0 & 0\\ 0 & 0 & 0 & 0 & \sqrt{\frac{5}{6}} & 0\\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{6}{7}} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0\\ \frac{1}{2} & \frac{1}{2} & -1 & 0 & 0 & 0\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -1 & 0 & 0\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -1 & 0\\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & -1\\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6 \end{pmatrix}. \tag{10}$$

If we assume the wave function depends radially only on the hyperdistance, sum of these coordinates squared, the angular dependence of the wave function can be written as either tensors or spherical harmonics in terms of angles.

At $L = 1$, the spherical harmonics are linear in the components of the ρ_i so the transformation is simple. If one writes the n-dimensional permutation σ as a $n \times n$ matrix S_σ in the traditional way, the effect of the two permutation generators will be

$$S_{Jacobi} = U S_\sigma U^T. \tag{11}$$

Matrices for two generators of S_6, as 5*5 matrices for hexaquarks, have been calculated. The tensor product to those should be included together with other sectors during step 6, with color, flavor, and spin ones. (The resultant projection back to monoms has to be taken carefully, with the understanding that $n-1$ of the overall dimensions represent Jacobi coordinates.)

At $L = 2$, instead of being linear in the cartesian coordinates of the ρ_i, the spherical harmonics are quadratic. Therefore, the natural approach is to include *two* permutations of the Jacobi coordinates, $S_{Jacobi} \otimes S_{Jacobi}$; one corresponding to the first factor of the factorized quadratic and the other to the second factor. There is still no way for a permutation to risk rotating the actual coordinates, only mixing them, so our approach of separating different values of L_z is still valid.

TABLE VII. Number of antisymmetric states (per choice of m_s and m_l) at each combination of total orbital and spin angular momentum for $udsc\bar{Q}$ pentaquark.

L \ S	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
0	0	0	0
1	2	1	0
2	9	6	1

To generalize to higher values of angular momentum, it is natural to conclude that all one must do is append more tensor products of S_{Jacobi}, because the Lth order spherical harmonic is a degree L polynomial in its Cartesian coordinates. Of course, as more matrices are added the algorithm for finding the eigenvalues increases cubicly in runtime, but this procedure is in principle applicable for any combination of angular momenta one could want. In experiments pentaquarks with charm quarks has been discovered, so we also did calculations for 4 quarks with 4 (u,d,s,c) flavors, see Table VII.

Completing this section, we remind that experimental findings in which negative parity ($L = 1$) hexaquark part of dibaryon WFs play some role were discussed in review [3].

VII. MATRIX ELEMENTS OF BASIC OPERATORS AND HEXAQUARK MASSES

With the color-flavor-spin wave functions available, one can attempt to calculate the average values of pertinent operators. The obvious step one is to do that perturbatively, for a gluon exchanges. The lowest order gluon exchange generates potentials proportional to "relative color" operators made out of color generators $\langle \lambda_i^A \lambda_j^A/4\rangle$, where $A = 1..8$ and $i, j = 1..n$. Relativistic corrections lead to spin-spin, spin-orbit, and tensor forces, as usual. Perturbative one-gluon exchange require that those also are proportional both to products of color and spin generators

$$\left\langle \sum_{i>j} \left(\frac{1}{4}\lambda_i^A \lambda_j^A\right)(\vec{S}_i \vec{S}_j) \right\rangle.$$

For $S-shell, L = 0$ hadrons the spin-spin forces are the only relativistic corrections.

The resulting masses are shown for light hexaquarks in Table VIII and for all other $L = 0$ states in Table IX. Note that the ordering of the compact hexaquark states in Table VIII (fourth column) is rather different from the ordering of the dibaryon molecules with the same quantum

TABLE VIII. $6q$ matrix elements of color and color-spin operators for all of the $L = 0$ light hexaquark states. The resulting masses (in MeV) obtained from simple additive model [16] are in the fourth column (using $m_q = 330$ MeV, $m_s = 500$ MeV, and $m_c = 1270$ MeV). The fifth column show prediction of molecular dibaryons model [17]. The sixth column show experimentally measured values of the resonances with corresponding quantum numbers (taken from compilation [3]).

(I,S)	$\langle\lambda\lambda\rangle$	$\langle\lambda\lambda SS\rangle$	$6q$ mass	Dibaryon model	Experiment
(0,1)	−16	−2/3	2098	1876	1876
(1,0)	−16	−2	2196	1876	1878
(1,2)	−16	−4	2342	2160	2160
(2,1)	−16	−20/3	2536	2160	2160
(0,3)	−16	−4	2342	2350	2380
(3,0)	−16	−12	2926	2350	2464

numbers (fifth column). The (0,3) state uniquely breaks the pattern: not only it is degenerate with the (1,2) state and lighter than the (2,1) state, it is lighter than (or at least close to) the dibaron molecule state with spin 3 and isospin 0, which was predicted to have a mass of 2350 MeV. Every other channel show that a molecule is lighter than compact hexaquark. Perhaps this is the only hexaquark state that has been experimentally observed.

Higher order gluon exchanges lead to operators with higher orders of Gell-Mann matrices. Those diagrams can best be obtained from an expansion of the set of Wilson lines convoluted with color wave functions, see, e.g., Fig. 3 for hexaquarks. For example, for hexaquarks, one can either put two color epsilons with all 6! permutations or simplify it to just five "good basis" color convolutions. As far as we know, next order gluon exchanges were not yet used in spectroscopy.

The nonperturbative confining potentials are defined via correlators of n Wilson lines,

$$\langle\Psi|(\delta^{c1'}_{c1}\ldots\delta^{cn'}_{cn} - W^{c1'}_{c1}\ldots W^{c1n'}_{cn})|\Psi'\rangle, \tag{12}$$

with path-ordered exponential of color generators,

TABLE IX. Matrix elements and masses for all other $L = 0$ states found in this paper, using the same fit from [16].

State	$\langle\lambda\lambda\rangle$	$\langle\lambda\lambda SS\rangle$	Mass (MeV)
$udsuds$, $S = 0$	−16	6	1611
$cccccc$, $S = 0$	−16	−12	7819
$ududq$, $S = \frac{3}{2}$	$-\frac{20}{3}$	7/4	1623
$ududq$, $S = \frac{1}{2}$	$-\frac{20}{3}$	1/2	1714
$ududc$, $S = \frac{3}{2}$	−7.654	1.117	2607
$ududc$, $S = \frac{1}{2}$	−7.654	1.442	2582
$ududQ$, $m_Q \to \infty$, $S = \frac{3}{2}, \frac{1}{2}$	−8	4/3	$1319 + m_Q$

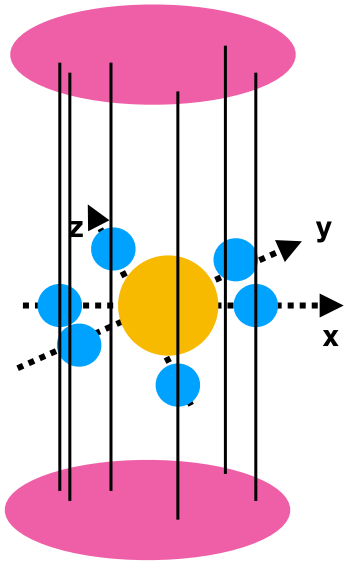

FIG. 3. The setting for calculation of the effective potential for $n = 6$ quarks, located at (blue) points in three dimensions. The vertical direction is Euclidean time, six vertical lines are Wilson lines. Red ovals above and below indicate color wave functions to which Wilson lines are traced. Yellow circle indicate gauge fields of an instanton.

$$W = P\exp[i(\lambda^a/2)\int gA^a_0 dt]^{c1'}_{c1}.$$

The setting is shown schematically in Fig. 3.

Note that while total wave functions Ψ is a spin-tensor with many different indices (color, flavor, spin etc.) the main perturbative $\lambda\lambda$ operator has only color indices. So, if Ψ consists of several factorizable parts,

$$W = \sum_A C^A \psi^A_{\rm color}\psi^A_{\rm flavor}\psi^A_{\rm spin},$$

one can sum over noncolor indices using normalization of those wave functions and put this operator as "sums of squares" of the color wave functions,

$$\sum_A C^2_A \psi^A_{\rm color}(c)(\delta^{c1'}_{c1}\ldots\delta^{cn'}_{cn} - W^{c1'}_{c1}\ldots W^{c1n'}_{cn})\psi^A_{\rm color}(c').$$

We have calculated it explicitly for various states. For hexaquarks, their five color wave functions are explicitly given in Appendix C, and all coefficients of $A = 1..5$ are $|C^A|^2 = 1/5$. We also of course convoluted it with Wilson lines, but the expression is too long to be given here (can be obtained from the authors upon request). This expression can be directly evaluated on the lattice, or by any vacuum model (e.g., in the instanton model), producing forces among quarks in all hexaquarks.

VIII. NINE AND TWELVE-QUARK S-SHELL STATES

Simple observation tells us that u, d quarks with three colors states and two spins have $2 \cdot 3 \cdot 2 = 12$ states, and thus, it should be the "magic number" completing the $1S, L = 0$ shell.

The corresponding quantum numbers are those of alpha particle $^4\mathrm{He} = ppnn$ nucleus, which is a "double magic" one by itself, being very well bound and compact. Large literature exist discussing whether nuclei like $^{12}\mathrm{C}$,^{16} O do or do not include some "alpha clusters" in their WF. Similarly, one can ask whether WF of alpha particles themselves should be calculated including compact 12-quark 1S state.

Let us start from the theory of the ^{4}He WF, which of course has a very long history originating in 1960's. Let us just mention application of hyperdistance approximation in 9-d setting, recently borrowed for discussion of (fully charmed) tetraquarks [12,13].

Avoiding any approximation, one can perform numerically Path Integral Monte Carlo [19]. Recently, this approach has been revived in [20] looking for "alpha preclustering" at temperatures $T \sim 100$ MeV corresponding to freezeouts in heavy ion collisions. Without going into detail, let us comment that the problem remains quite challenging. At one hand, six nuclear potentials combined have deep minimum $\approx 6 \cdot 50$–300 MeV, yet corresponding to small fraction of configurations: but this technical problem can still be solved by persistent use of Monte Carlo algorithm. The "precluster" component is seen in the WF, being robust enough to survive even at $T \sim 100$ MeV.

A nontechnical issue here is the following. While nucleon scattering data, with application of the renormalization group allowed to fix nuclear forces at low energy *uniquely*, this is not yet the case for repulsive "nuclear core". (Indeed, various sets fitted to spectra and scattering phases predict short-distance nuclear potentials with large spread.) Six of such potentials added naively lead to multi-GeV repulsion, which is too large and too uncertain to believe. As many people noted before, this issue better be addressed at the quark level, as we are going to discuss.

Yet before we do so, let us make two comments. The first, as proper, is on experimental observations. It was noted in [21] that the standard theory of p-nuclei scattering based on many successful applications does not work as well for scattering on ^{4}He. Quantities considered include total cross section, magnitude of the diffraction slope, and the location of the diffractive minimum. Inclusion of 12-q component in the WF with certain parameters can remedy all three; see details in [21]. Later in [22], similar treatment was generalized to scattering data of the pion-^4He scattering. The proposed parameters of the cluster in both papers indicate that the 12-q "core object" is not so small in size.

The second comment is theoretical. Let us start, for a moment, with a "good diquark model". We already argued in the Introduction that its reasoning is wrong logically and leads to phenomena never observed. There is no reason to single out some (six in the 12q case) pairwise forces out many pairs (66): all should be included. It can be shown why it is so on the perturbative level (see below).

At the nonpertubative level, one has to explain first where strong diquark binding comes from. As argued in [9,10], the main part of it for light u, d, s quarks comes from the instanton-induced 't Hooft Lagrangian. It, in turn, is the consequence of fermion zero modes. To produce $6 \cdot B_{qq}$ binding of six diquarks, it would take six instantons inside the 12-q cluster. This is quite problematic to accomplish, as the instantons in the QCD vacuum are dilute. In summary, for 12-q objects made of light u, d, s quarks the issue naturally is elevated to multiinstanton correlations in the nonperturbative vacuum; the problem we are not ready to attack at this time.

Let us however approach the problem assuming that quarks are heavy enough, so that it is treatable via nonrelativistic Schrodinger equation, with perturbative Coulomb plus (perhaps instanton-induced) potentials. We will start with the most symmetric case, with two quark flavors (c, b) possessing the same heavy mass M. We will call those 12-Q systems. We restrict our discussion here to the basic issue addressed in this work, namely construction of the S-shell WFs satisfying quark Fermi statistics.

As we have shown above, for 4,5,6 quark systems, one can construct WFs using the "brute force"—building explicitly the Kroneker products of the color-flavor-spin components of the WF and diagoninalizing generators. For 9-Q and 12-Q cases, it is not so easy to do. The total space of monoms is in the latter case $12^{12} \approx 8.9 \times 10^{12}$-dimensional. The 3^{12} color states is not that large, but direct reduction to "good basis" needs to start with all (9! or 12!) permutations of the open-index tensors made of (3) 4 Levi-Civita symbols, which is not practically possible to perform as such.

Still, reducing to smaller set of permutations we were able to find the "good basis" sets. For $n = 9$, the color basis is 33 dimensional, and for flavor 42 dimensional. If spin is maximal $S = 9/2$ and those variables are trivialized, the Kroneker color-spin good basis space is thus $33 \times 42 = 1386$ dimensional. Two generators of S_9 group were calculated as matrices in such dimension, with the unfortunate conclusion that common eigenvectors with correct eigenvalue -1 does *not* exist.

Going for nonmaximal spin adds tensor product by another 42 dimensions, and operating in $33 \times 42 \times 42$ dimensions, we were not able to do. Perhaps we would be able to move further in the subsequent publications.

While actually computing these WFs becomes increasingly difficult as n increases, by looking at tables of Kronecker coefficients such as [2], it is possible to predict whether these large multiquark states do or do not exist. The results for lowest spin and isospin (1/2 or 0) up to 24

TABLE X. Computed from a table of Kronecker coefficients, the multiplicities of spin statistics-obeying u and d states at higher quark numbers, with $I = S = 0$ for even n and $I = S = 1/2$ for odd (in the odd case, this is per choice of S_z as before).

n	States
9	1
12	1
15	0
18	0
21	0
24	1

quarks are given in the Table X. In principle, this technique can be used to predict the existence of any quantum number combination, up to as high a value of n as mathematicians provide.

From this table, one can conclude that, for instance, some nuclei have multiquarks with quantum numbers that allow for mixing, and thus modification of their masses and cross sections, while others do not. Two cases discussed, $n = 9$ and $n = 12$, both have 1, corresponding to clusters which can mix with, e.g., ^{3}He and ^{4}He. That phenomenology was already discussed. The rows for $n = 15$–21 give zero, and the next quark cluster existing with such quantum numbers appear only at $n = 24$, or at eight nucleons.

Exploration of the existence of quark clusters mixing with "nuclear" states has been discussed for decades and is still an interesting direction for future research to take. Needless to say, possible existence of standalone "exotic" quark states (with "hidden color") are even more fascinating: those with heavy quarks are now appearing in experiment. We however have not discuss those yet in this paper, limiting flavor content to only light or light + strange quarks.

IX. SUMMARY

This paper shows how one can construct the WFs of multiquark hadrons using the representations of the S_n symmetry group. The novel method we developed had started from excited (3-q) nucleons and 4-q tetraquarks, and extended here to the 5-q pentaquarks and 6-q hexaquarks. Naturally, in doing, so we focus on the most symmetric cases first, extended it further whenever possible. We also discussed physics of some 9-q and 12-q states made of u, d quarks, although getting explicit WFs of those turn out to be numerically challenging even for our method.

This paper follows an approach of our previous work on excited baryons [1]. The main idea is that instead of building the wave function of multiquark hadrons starting from quark pairs (mesons or diquarks), then combining their color, flavor, spin etc. into corresponding tensor products, as done traditionally, one can work directly with the tensor products of generators of the symmetry groups S_n. In many cases considered, it leads to unique (or very few) wave functions, possessing the required Fermi statistics.

In [1], we worked directly with spin-tensor form of the wave functions, e.g., for $P - shell, L = 1$ nucleons. The number of components in this case were all monoms in $SU(2)$ flavor and spin, in total $2^3 \times 2^3 = 64$, with an explicit dependence of orbital part on angle of two Jacobi coordinate, ρ, λ. In this case, we constructed pertinent wave functions symmetric under S_3 permutation, as Fermi statistics requires.

In this work, we generalized this method to multiquark hadrons, focusing mostly on 6-q (hexaquark) and 5-q (pentaquark) states. It turned out to be possible *not* to calculate all $n!$ permutations, but basically construct Kronecker products of all sectors for only *two* generators of the S_n symmetry group, P_{12} and $P_{\rm cycle}$.

A good representative is the spin $S = 3$ hexaquark state, already discussed in literature, with explicit construction of the wave function by [16]. As we have shown, our method provides much more direct way toward it. Furthermore, we were able to "unfreeze" spin value and derive WFs of ud hexaquark with other spins, notably $S = 1$. We also show why $S = 2$ and $S = 0$ do not have states with required symmetry. We also constructed WFs for pentaquarks, considering symmetry group S_4. The method generalizes to states with nonzero orbital momentum, deriving permutation matrices for the set of modified Jacobi coordinates.

The WFs obtained should be used to calculate matrix elements of various operators in the Hamiltonian. We did so only for operators $\vec{\lambda}_i\vec{\lambda}_j$ and $(\vec{\lambda}_i\vec{\lambda}_j)(\vec{S}_i\vec{S}_j)$. The resulting masses are for hexaquarks in Tables VIII and for all other $L = 0$ states in Table IX.

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APPENDIX A: SYMMETRY GROUPS S_n

Let us start reminding the strategy in our previous work on excited baryons [1] based on representations of the S_3 group. This symmetry group consists of six elements, unity, and various permutations,

$$P_{i=1,\ldots,6} = I, (12), (13), (23), (123), (132) \quad \text{(A1)}$$

(hope notations are self-evident). In mathematics, symmetric groups usually are defined via some geometric maps, e.g., S_3 as self-maps of the equilateral triangle, S_4 by that of tetrahedron, and so on.

The (12) and (23) permutations are then improper (out of plane) $O(3)$ rotations with determinant equal to -1. Other three permutations are then in-plane rotations by $\frac{\pi}{3}$ of which

P_4 is an example. The same definition of self-maps generalizes to any S_n group.

After the action of any permutation, the wave function can be expressed as a matrix in the original monom basis. In our previous work, we were located states with the required permutation symmetry by finding common eigenstates of the two of them, P_{12} and P_{23}. By "common," we mean both having eigenvalues either 1 or -1, as needed. One can also calculate the commutator of those two matrices and locate eigenstate with eigenvalue 0. After these were found, one can observe that they in fact are common to the whole group.

In this work, we generalize the approach to higher permutation groups $S_n, n = 4, 5, 6$. Those have $n!$ elements, e.g., for hexaquarks, we will have a group of 720 elements. It would be hard to explicitly form their matrices and check their common eigenstates. Fortunately, all elements of the permutation group of order n can be generated by only two group generators, the full n-cycle P_{cycle},

$$\{1, 2, 3, \ldots n\} \to \{2, 3, 4, \ldots 1\},$$

and the original transposition (12) [23]. Therefore, our proposed strategy would be to look for common eigenstate for just two matrices. If found, their common eigenstate is guaranteed to be invariant under *all possible* permutations of S_n.

The dimensions for arbitrary representations of S_n are given exactly by the hook length formula [24]. Because the representations of $SU(2)$ and $SU(3)$ have predictable shapes as Young tableaux, it is possible to use this formula to determine the dimensions of their corresponding permutation group representations. For $SU(2)$ with total spin (or isospin), s and n quarks the dimension is given by

$$\begin{aligned} n'_{SU(2)} &= \frac{2s+1}{\frac{n}{2}+s+1} \frac{n!}{(\frac{n}{2}+s)!(\frac{n}{2}-s)!} \\ &= \frac{2s+1}{\frac{n}{2}+s+1} \binom{n}{\frac{n}{2}+s}. \end{aligned} \tag{A2}$$

For $SU(3)$ color/isospin singlets with n quarks, the dimension is

$$n'_{SU(3)} = \frac{2}{(\frac{n}{3}+2)(\frac{n}{3}+1)^2} \frac{n!}{(\frac{n}{3}!)^3}. \tag{A3}$$

A key part of our method involves the finding of a "good basis" on the space of monoms (step 3). The dimension of this basis is given by these formulas, and though they may appear to have large factorial components, plotting them reveals the dimensions are all under 100 for $n < 10$ and are roughly exponential for larger n—a significant improvement on the $n!$ basis of monoms we would otherwise have to work with.

APPENDIX B: PERMUTATIONS OF MONOMS

The wave function-finding procedure requires a way to represent the two generators as matrices acting on the space of all monoms, so we present one possible way of finding these matrices below.

The general superposition of the product of n $\mathbf{C}^N$ basis vectors $\sum_{i_1 i_2 \ldots i_n} \psi^{i_1 i_2 \ldots i_n} \hat{e}_{i_1} \otimes \hat{e}_{i_2} \otimes \ldots \hat{e}_{i_n}$ maps from n N-dimensional indices to 1 N^n-dimensional index via the prescription $k = i_1 + i_2 N + i_3 N^2 + \ldots i_n N^{n-1}$. Essentially, converting the i's into the digits of an n-digit number in base N. If one then wants to know what the effect of swapping two quarks or any other permutation σ is on the $(\hat{e} \otimes \hat{e} \otimes \ldots \hat{e})_k$ basis vector is, one just needs to write k as a number in base N, swap the corresponding digits according to σ, and then convert the new number k_σ back to base 10. The total $N^n \times N^n$ dimensional permutation matrix can be written as the matrix where position $(k, k_\sigma) = 1$ for all integers k from 1 to N^n, and all other entries are 0. For instance, the generator P_{12} acting on $\mathbf{C}^{2^3}$ takes the form,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{B1}$$

where, for example, position (3,5) has a 1 because in binary, $011 \to 101$ under $P = (1\,2)$.

APPENDIX C: GOOD BASIS FOR COLOR WFS FOR HEXAQUARKS

Example of drastic simplification by the orthogonalization procedure used. For hexaquarks, there are 3^6 values of color indices (monoms) and 6! permutations of indices in $LeviChivita[3][[c1, c2, c3]] * LeviChivita[3][[c4, c5, c6]]$. The orthogonalization procedure leads to only *five* "GoodBasis" states, which in the monom basis are the following five orthonormal vectors

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{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
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4 Sqrt[6]), 0, 1/(4 Sqrt[6]), 0, 0, 0, 0, 0, 0, 0, 1/(4 Sqrt[6]), 0,
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0, -(1/(6 Sqrt[2])), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
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0, -(1/(6 Sqrt[2])), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, -(1/(6 Sqrt[2])), 0, -(1/(6 Sqrt[2])), 0, 0, 0, 0, 0, 1/(
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0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0}}
```

APPENDIX D: SIMULTANEOUS DIAGONALIZATION

Our procedure requires the diagonalization of the two C_n group generators, P_{12} and P_{cycle}, with then identification of their common eigenvectors with appropriate eigenvalues, -1 or 1. There are many techniques to compute the simultaneous eigenvectors of two matrices, but the one we used was simply to choose two random real numbers a and b, and then diagonalize $aP_{12} + bP_{\text{cycle}}$, looking for eigenvectors with an eigenvalue exactly equal to $a + b$ in the symmetric case or $-a + (-1)^{n-1}b$ in the antisymmetric case. [The $(-1)^{n-1}$ is necessary because the second generator, the n-cycle, is equal to the product of $n-1$ transpositions, each of which should flip the sign of the antisymmetric state once.] Since the odds of hitting exactly these arbitrary real numbers any way other than by the vector being an eigenvector of both generators with the correct eigenvalues are extremely low, we can conclude with good confidence that these spinors satisfy the desired relation. And of course, it is very easy to check by hand afterwards that they are the correct eigenvectors.

APPENDIX E: USING KRONECKER COEFFICIENT TABLES

To determine the existence of a wave function with large quark numbers, one may use the resource: https://homepages.physik.uni-muenchen.de/~vondelft/Papers/ClebschGordan/; https://www.jgibson.id.au/articles/characters/. To use this website to find if the needed Kronecker coefficient is zero or not, one first needs to write the representations of all its sectors as Young tableaux. The notation used index each tableau (which correspond to what are called Specht modules) as a list of the lengths of each row of boxes in order. For instance, with $S = 3$ hexaquark $n = 6$, the Young tableaux are shown in Fig. 2. The color tensor form $\epsilon_{c_1c_2c_3}\epsilon_{c_4c_4c_6}$ in this web resource is written as $s[2,2,2]$ or s[2^3]. The flavor one $\epsilon_{f_1f_2}\epsilon_{f_3f_4}\epsilon_{f_5f_6}$ is denoted by s[3,3] or s[3^2].

Set the value of S_n to the correct value of n—in this case 6. This will bring up a list of irreducible representations for the group, which will include s[2^3] and s[3^2]. To view their tensor product, simply select *both* rows from the table. This will bring up another table containing Kroneker decompositions of higher powers of it, in terms of exterior, symmetric, and tensor products. In most cases, the only column needed in this table is the first one, corresponding to the decomposition of just the state. The row we are looking for is the one for the totally antisymmetric representation. This is always the final row, s[1^n], because that it is the Young tableau of n blocks stacked vertically (known to mathematicians as the "sign" representation). In the case considered, there is a "1" in the first column of the last row, which tells us there does exist *one* antisymmetric state that can be built from the tensor product of this particular color and flavor combination. This is the KKO state we also explicitly found by our procedure.

Note that there are "0" in many other cases of that table, meaning no such state with those quantum numbers exist. Numbers larger than one are very rare but present: it would imply higher multiplicities of states.

The program handles the combination of many different representations easily, though dealing with the case where flavor and spin are identical with nontrivial color is somewhat difficult because there is no way to select one rep twice and another once. One solution is to look at the product of the color sector with s[1^n] and then try to find the resultant rep in the $\otimes^2 \chi$ column of the flavor/spin rep. This works because the Kronecker coefficients are symmetric under interchanges of the representation indices.

APPENDIX F: *MATHEMATICA* CODE

This section will provide a complete set of the code necessary to find these wave functions, and an example in the context of the *ududud* hexaquark.

First, it is useful to have a way to write the two permutation generators $P_{(1\,2)}$ and P_{cycle} as $n \times n$ matrices (where n is still the number of identical quarks).

```
Permutes[n_] :=
  PermutationMatrix[#, n] & /@ GroupGenerators[SymmetricGroup[n]]
```

The transformation to modified Jacobi coordinates in n we used can implemented as a matrix [as in Eq. (10)] with Jacobi, and the generators can be projected into that basis with JacobiPermute.

```
Jacobi[n_] :=
  Inverse[DiagonalMatrix[
    Table[Sqrt[i +1]/Sqrt[i], {i, 1, n}]]]. (Table[
      Join[Table[1/i, i], Table[0, n - i]], {i, 1, n}] +
    DiagonalMatrix[Table[-1, n - 1], 1])
```

```
JacobiPermute[n_, i_] :=
  SparseArray[(Jacobi[n]. Permutes[n][[i]]. Inverse[Jacobi[n]])[[
    1;; n - 1, 1;; n - 1]]]
```

BigPermute returns the full monom basis permutation matrices, obtained using the technique described in Appendix B.

```
ColumnSwap[n_, dim_, i_, x_] :=
FromDigits[Permutes[n][[i]]. IntegerDigits[x, dim, n], dim]
BigPermute[n_, dim_, i_] :=
SparseArray[
     Table[{x +1, ColumnSwap[n, dim, i, x] +1} -> 1, {x, 0,
     dim^n - 1}]]
```

Now, we have the necessary tools to create PermuteInBasis, which accomplishes steps 2 through 4 of the method described in Sec. II A, given any starting tensor T with rank equal to the number of quarks.

```
PermuteInBasis[T_] := Module[{n, dim, tensorList, basis},
  n = TensorRank[T];
  dim = Length[T];
  tensorList=
   Flatten /@ (Transpose[T, PermutationList[#, n]] & /@
   GroupElements[SymmetricGroup[n]]);
   basis = DeleteCases[Orthogonalize[tensorList], {0..}];
   {SparseArray[basis. BigPermute[n, dim, 1]. Transpose[basis]],
   SparseArray[basis. BigPermute[n, dim, 2]. Transpose[basis]]}
   ]
```

If one just wants the basis of minimal linearly independent combination of permutations of tensor indices, GoodBasis can be used.

```
GoodBasis[T_] := Module[{n, dim, tensorList, basis},
  n = TensorRank[T];
  tensorList=
    Flatten /@ (Transpose[T, PermutationList[#, n]] & /@
      GroupElements[SymmetricGroup[n]]);
    DeleteCases[Orthogonalize[tensorList], {0..}]
    ]
```

For the example of nucleon flavor, it returns the Jacobi coordinates ρ and λ as expected.

```
GoodBasis[TensorProduct[LeviCivitaTensor[2],u]]
{{0,0,1/Sqrt[2],0,-(1/Sqrt[2]),0,0,0},{0,Sqrt[2/3],-(1/Sqrt[6]),0,-(1/
Sqrt[6]),0,0,0}}
```

There is one more piece missing. Given two matrices, M[[1]] and M[[2]], SymmetricStates and AntiSymmetricStates apply the simultaneous diagonalization technique described in Appendix D, which is step 6 in our procedure.

```
SymmetricStates[M_] :=
  Module[
    {a, b, test},
    a = RandomReal[];
    b = RandomReal[];
    test=a M[[1]] + b M[[2]];
    Chop[Select[Transpose[Eigensystem[test]],
     Round[#[[1]], 10^-3] == Round[a + b, 10^-3] &][[;;, 2]]]
    ]
AntiSymmetricStates[M_, n_] := Module[
  {a, b, test},
  a = RandomReal[];
  b = RandomReal[];
  test=a M[[1]] + b M[[2]];
  Chop[Select[Transpose[Eigensystem[test]],
    Round[#[[1]], 10^-3] ==
      Round[-a + (-1)^(n - 1) b, 10^-3] &][[;;, 2]]]
  ]
```

With PermuteInBasis, AntiSymmetricStates, and JacobiPermute, we now have all that is needed to find the wave functions for any particle. Consider the *ududud* hexaquark. For step 1, we need to write the tensors with the correct index symmetries. For spin, Young tableaux give us the following:

```
u = {1, 0}; d = {0, 1};
hexspin0 = {TensorProduct[LeviCivitaTensor[2], LeviCivitaTensor[2],
    LeviCivitaTensor[2]]};
hexspin1 = {TensorProduct[LeviCivitaTensor[2], LeviCivitaTensor[2],
    Symmetrize[TensorProduct[u, d]]],
  TensorProduct[LeviCivitaTensor[2], LeviCivitaTensor[2], u, u]};
hexspin2 = {TensorProduct[LeviCivitaTensor[2],
    Symmetrize[TensorProduct[u, u, d, d]]],
  TensorProduct[LeviCivitaTensor[2],
    Symmetrize[TensorProduct[u, u, u, d]]],
  TensorProduct[LeviCivitaTensor[2],
    Symmetrize[TensorProduct[u, u, u, u]]]};
  hexspin3 = {Symmetrize[TensorProduct[u, u, u, d, d, d]],
    Symmetrize[TensorProduct[u, u, u, u, d, d]],
    Symmetrize[TensorProduct[u, u, u, u, u, d]],
    Symmetrize[TensorProduct[u, u, u, u, u, u]]};
```

Color is more straightforward, always being just two Levi-Cevitas as discussed previously, and orbital angular momentum is a number of JacobiPermute's equal to L. Once we have all of these tensors, we can use PermuteInBasis to make tables of the permutation generators in each basis. The color, spin, and orbital angular momentum projections are the same for every hexaquark.

```
colorPermute6 =
  PermuteInBasis[
    TensorProduct[LeviCivitaTensor[3], LeviCivitaTensor[3]]];
spinPermute6 = {PermuteInBasis /@ hexspin0,
  PermuteInBasis /@ hexspin1, PermuteInBasis /@ hexspin2,
  PermuteInBasis /@ hexspin3}
coordPermute6 ={{{{1}}, {{1}}}, {JacobiPermute[6, 1],
  JacobiPermute[6, 2]},
{KroneckerProduct[JacobiPermute[6, 1], JacobiPermute[6, 1]],
KroneckerProduct[JacobiPermute[6, 2], JacobiPermute[6, 2]]}}
```

For *ududud*, flavor is an $SU(2)$ singlet.

```
flavorPermute6ududud =
  PermuteInBasis[
    TensorProduct[LeviCivitaTensor[2], LeviCivitaTensor[2],
      LeviCivitaTensor[2]]]
```

The generators on the full space can now be constructed (step 5). They may look different for different values of S, S_z, and L, so it is worth making a large table over these angular momenta and getting what we need from it later.

```
fullPermute6ududud =
  Table[{KroneckerProduct[colorPermute6[[1]],
    flavorPermute6ududud[[1]], #[[1]], coordPermute6[[l +1, 1]]],
  KroneckerProduct[colorPermute6[[2]],
    flavorPermute6ududud[[2]], #[[2]],
    coordPermute6[[l +1, 2]]]} & /@ spinPermute6[[sp +1]], {sp,
  0, 3}, l, 0, 2}];
```

We are now able to apply the simultaneous diagonalization technique to the two generators. UdududStates gives a list of every possible antisymmetric *ududud* state for a given angular momentum combination.

```
  UdududStates[sp_, sz_, l_] :=
  AntiSymmetricStates[fullPermute6ududud[[sp +1, l +1, sz +1]], 6]
```

For $S = 3$, $S_z = 3$, $L = 0$, there is only the KKO state.

```
  UdududStates[3, 3, 0]
```

```
{{0, 0, 0, 0, -0.447214, 0, 0, 0, 0.447214, 0, 0, 0, -0.447214, 0, 0,
0, -0.447214, 0, 0, 0, 0.447214, 0, 0, 0, 0}}
```

To obtain tables like Table I, one can run the following block:

```
TableForm[
Table[Table[Dimensions[UdududStates[sp, sz, 0]], {sz, 0, sp}], {sp, 0,
  3}], TableHeadings -> {Range[0, 3], Range[0, 3]}]
```


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