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Generalized Extended Uncertainty Principles, Liouville theorem and density of states: Snyder-de Sitter and Yang models

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ABSTRACT

Modifications in quantum mechanical phase space lead to changes in the Heisenberg uncertainty principle, which can result in the Generalized Uncertainty Principle (GUP) or the Extended Uncertainty Principle (EUP), introducing quantum gravitational effects at small and large distances, respectively. A combination of GUP and EUP, the Generalized Extended Uncertainty Principle (GEUP or EGUP), further generalizes these modifications by incorporating noncommutativity in both coordinates and momenta. This paper examines the impact of GEUP on the analogue of the Liouville theorem in statistical physics and density of states within the classical limit of non-relativistic quantum mechanics framework. We find a weighted phase space volume element, invariant under the infinitesimal time evolution, in the cases of Snyder-de Sitter and Yang models, presenting how GEUP alters the density of states, potentially affecting physical (thermodynamical) properties. Special cases, obtained in certain limits from the above models are discussed. New higher order types of GEUP and EUP are also proposed.

1. Introduction

The search for the fundamental theory of Quantum Gravity has been supported by the development of phenomenological models that explore possible modifications to the known quantum mechanical or gravitational phenomena. In this context the various purely phenomenological proposals have appeared aiming to capture potential signatures of quantum effects of gravity, hinting towards possible experimental set-ups which would help guide the way in the full formulation of the theory. What is known is that one needs to challenge the concepts of classical space-time. The idea that the structure of space-time should be modified is a common feature of various approaches to Quantum Gravity, such as String Theory, Loop quantum gravity, Causal Dynamical Triangulations, Asymptotically Safe Quantum Gravity, Horava-Lifshitz Gravity and Noncommutative Geometry, just to name a few [1–6]. Many of these share also the concept of minimum length [7]. Such minimal length can be implemented in quantum mechanics through introducing the modifications in Uncertainty Principles (UP)[1]:

$$\Delta x\Delta p \geq \frac{1}{2}\,|\langle[x,p]\rangle| \tag{1}$$

where Δ denotes standard deviation and $\langle\ \ \rangle$ the quantum expectation value on a given state. From the above relation, it is clear that any modifications in the canonical Heisenberg commutation relations between coordinates and momenta will imply changes in the UP. Various generalizations of UP have been considered in the literature. The most common type of such modifications is the

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[1] For simplicity, to illustrate the main points, we use 1-dimensional case throughout the introduction.

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Generalized Uncertainty Principle (GUP) with the momenta-dependent right hand side of (1). It was firstly proposed in [8] and then related to the specific algebraic structure of quantum phase space [9–12]. The right hand side of (1) may include quadratic terms in momenta (resulting in QGUP), see e.g. [11–13] or have linear and quadratic terms (LQGUP), see e.g. [14–18]. In momenta-dependent GUP relations one introduces the parameter, usually denoted by β, which is related to the Planck (length) scale l_P, hence linking GUPs with Planck scale physics and quantum gravitational effects. Such phenomenological models have attracted a lot of attention [9–35], see also [36] for a recent review and more references on the topic.

On the other hand, the symmetry between the position and momenta in the canonical (quantum mechanical) commutation relations, as well as Born reciprocity [37], suggests the possibility of introducing corrections to the Heisenberg uncertainty principle by including the modifications proportional to coordinates instead, i.e. so that the RHS of (1) is quadratic or linear in coordinates, instead of momenta. This complimentary type of uncertainty relation has been called an Extended Uncertainty Principle (EUP) [38,39]. Here, the parameter of the model, usually denoted by α, is related with the non-vanishing cosmological constant Λ and this way it can be embedded in the non-relativistic quantum mechanics. It has been shown [40] that this type of modification to the UP is related with the (Anti-)de Sitter geometric background, and the parameter α is then naturally linked with the (Anti-)de Sitter radius. While GUP, with $\beta \sim l_P$, exposes the gravitational modifications in quantum mechanics at the small distances; EUP, with $\alpha \sim \Lambda$ introduces the idea of modifications at large distances. Relying on the analogy with GUPs, various types of such coordinate-dependent models have been introduced leading to different interesting physical effects, see e.g. [38,39,41,42].

Combination of GUP and EUP (i.e. considering both coordinate and momenta dependent terms on the RHS of (1)) appeared firstly in the construction of noncommutative quantum mechanics with quantum groups as a symmetry [43], where both nonzero minimal uncertainties have appeared for position and momentum observables, respectively. Later on the relation of the type:

$$\Delta x \Delta p \geq \frac{\hbar}{2}(1 + \alpha^2(\Delta x)^2 + \beta^2(\Delta p)^2) \tag{2}$$

under the name of Generalized Extended Uncertainty Principle (GEUP, or alternatively EGUP), has been studied in various contexts, see e.g. [38–40,44], for more recent work see e.g. [45] where Born reciprocity is also promoted or see e.g. [46] for applications in statistical physics.

Even though the origins of GUPs, EUPs and GEUPs can be tracked to the noncommutative geometry and quantum groups as the underlying mathematical frameworks [10,43], these models are mainly treated as purely phenomenological in the majority of the literature related to this subject. They have been used (and have been quite fruitful) in providing predictions for various phenomenological effects. However, such departure from the underlying mathematical framework resulting in the specific form of GUP, EUP or GEUPs together with the non-uniqueness of the defining commutation relations between coordinates and momenta has lead to many conceptual shortcomings and ambiguities (see e.g. the recent review [36] or [47] for discussion of some of the arising issues). Moreover, it is worth to point out that the minimal length may not necessarily appear in such models if one bases only on the modifications of the quantum mechanical phase space, see e.g. [48–51].

In this paper, following the point of view that the noncommutative geometry is the underlying framework (mathematical language) of the possible fundamental theory, we assume that the quantization process of general relativity, includes the quantization of space-time [52,53], i.e. requiring the space-time coordinates to become noncommutative $[\hat{x}_\mu, \hat{x}_\nu] \neq 0$. The noncommutativity of space-time introduces corrections to the canonical quantum-mechanical phase space relations. Hence modifications to the UP appear as a natural consequence. In this view, quadratic GUPs have been mainly linked with the Snyder model [10,34,35], with noncommuting coordinates and commuting momenta. EUPs, on the other hand can be linked with the (anti-)de Sitter geometric background and algebraic structure with commuting coordinates but noncommuting momenta [40]. The main aim of this paper is to show how the noncommutativity of both coordinates $[\hat{x}_i, \hat{x}_j] \neq 0$ and momenta $[\hat{p}_i, \hat{p}_j] \neq 0$ leading to GEUPs (and in some cases, the appearance of the nonzero minimal uncertainties in positions and momenta separately) affects the density of states and the analogue of the Liouville theorem in statistical mechanics. For this reason we limit ourselves to the case of non-relativistic quantum mechanics. We find the new form of weighted phase space volume element in the presence of GEUPs in the specific cases of Snyder-de Sitter (SdS) and Yang models. We show that noncommuting coordinates and noncommuting momenta with the corresponding GEUP require introducing the modification in the density states and this may impact various physical effects and thermodynamical properties (as it has been shown in the case of various GUP models).

In the next section, we summarise the framework generalized to the case where both coordinates and momenta do not commute and present the general formulae for the Jacobian arising from the variable change under the infinitesimal time evolution. In Sec. 3, we specify the model to the Snyder-de Sitter (SdS) algebra, as only then we can identify the required weight factor (which depends on both coordinates and momenta) for the phase space volume element. The factor we obtain in this case does not depend on the dimension D. We also discuss the special cases obtained in the certain limits of the parameters, giving GUP or EUP relations (by reducing the starting SdS algebra to Snyder or dS algebras, respectively) and we give the expressions for the phase space volume element in these cases. In Sec. 4, we discuss the Yang model and consider specific realizations of its generators on the canonical phase space leading to another type for (higher order) GEUP relation. The weighted phase space volume element is obtained for this case in 1 dimension by adapting the result from SdS model. Special cases, obtained in certain limits are also discussed, one provides the known "square-root" (or Maggiore) GUP and the other leads to the new higher order ("square-root") type of EUP. In Sec. 5, the Lie algebraic case with commuting momenta is briefly considered with the fuzzy sphere as an example. It is shown that in this case the phase space volume element stays invariant under the time evolution and there is no change in the density of states.

2. Preliminaries

In this section we set up the most general framework for investigating the effects of noncommuting coordinates and noncommuting momenta (and in principle the appearance of the nonzero length and momentum uncertainties) on the density of states in the phase space so that we can adapt the Liouville theorem in statistical physics to this new scenario. For this reason we do not fix the specific choice for the noncommutativity of space-time or momenta and the resulting deformation of the quantum phase space commutation relations at this point yet.

We start with the following (most general) set of commutation relations describing the noncommutative quantum mechanical phase space algebra:

$$[\hat{x}_i, \hat{x}_j] = i\hbar a_{ij}(\hat{x}, \hat{p}), \qquad [\hat{p}_i, \hat{p}_j] = i\hbar b_{ij}(\hat{x}, \hat{p}) \tag{3}$$

$$[\hat{x}_i, \hat{p}_j] = i\hbar c_{ij}(\hat{x}, \hat{p}), \tag{4}$$

where $a_{ij}(\hat{x}, \hat{p}), b_{ij}(\hat{x}, \hat{p}), c_{ij}(\hat{x}, \hat{p})$ are functions which may include all kinds of terms (linear, quadratic, higher order etc.) in space-time coordinates $\hat{x}$ and momenta $\hat{p}$, such that the Jacobi identities are satisfied. We will focus on the non-relativistic general case in any dimension D with $i, j = 1, 2, \dots, D$. When none of the above commutators are zero (see e.g. [45,54–57]), this will lead to the interesting types of the GEUPs: $\Delta\hat{x}_i \Delta\hat{p}_j \geq \frac{\hbar}{2}\left|\langle c_{ij}(\hat{x}, \hat{p})\rangle\right|$ with the specific form of RHS depending on the concrete choice of the algebra (3), (4).

In the phenomenological approaches, one considers the right hand side of (4) as the definition of a new effective value of $\hbar$ which (in the most general case) may depend on both coordinates and momenta. This means that the size of the unit cell that each quantum state occupies in the phase space can be thought of as being also coordinates and momenta dependent. This will have an effect on the density of states and as a consequence affect physical, for example thermodynamical, properties. For this interpretation to be valid, the volume of phase space must evolve in such a way that the number of states does not change with time, in other words we are looking for the analogue of the Liouville theorem in statistical physics. Similar investigation has been done in the case of the quadratic GUP [12,13] or GUPs with higher order terms [58]. Note that here we assume that both coordinates and momenta do not commute. Nevertheless, later on we shall see how this more general case can be reduced to the special cases like the GUP or EUP (which include some commutative generators).

Starting with the quantum mechanical commutation relations (3), (4), these will correspond to the Poisson brackets in classical mechanics:

$$\frac{1}{i\hbar}[\hat{A}, \hat{B}] \xrightarrow{classical\ limit} \{A, B\}. \tag{5}$$

Therefore, in the classical limit[2] we obtain (in our shortcut notation):

$$\{x_i, x_j\} = a_{ij}(x, p), \qquad \{p_i, p_j\} = b_{ij}(x, p), \tag{6}$$

$$\{x_i, p_j\} = c_{ij}(x, p). \tag{7}$$

The time evolution of the coordinates and momenta is governed by the equations (where the summation convention is assumed)

$$\dot{x}_i = \{x_i, H\} = \{x_i, p_j\}\frac{\partial H}{\partial p_j} + \{x_i, x_j\}\frac{\partial H}{\partial x_j} = c_{ij}\frac{\partial H}{\partial p_j} + a_{ij}\frac{\partial H}{\partial x_j}, \tag{8}$$

$$\dot{p}_i = \{p_i, H\} = -\{x_j, p_i\}\frac{\partial H}{\partial x_j} + \{p_i, p_j\}\frac{\partial H}{\partial p_j}\delta t = -c_{ji}\frac{\partial H}{\partial x_j} + b_{ij}\frac{\partial H}{\partial p_j}. \tag{9}$$

The Liouville theorem requires that the phase space volume is invariant under time evolution. Hence we consider an infinitesimal time interval δt and the evolution of the coordinates and momenta during δt is:

$$x'_i = x_i + \delta x_i, \quad p'_i = p_i + \delta p_i$$

where

$$\delta x_i = \dot{x}_i \delta t = \left(c_{ij}\frac{\partial H}{\partial p_j} + a_{ij}\frac{\partial H}{\partial x_j}\right)\delta t, \qquad \delta p_i = \dot{p}_i \delta t = \left(-c_{ji}\frac{\partial H}{\partial x_j} + b_{ij}\frac{\partial H}{\partial p_j}\right)\delta t. \tag{10}$$

The infinitesimal phase space volume after this infinitesimal time evolution will be:

$$d^D x' d^D p' = J d^D x d^D p$$

with the Jacobian

[2] It is worth to note that the classical limit in GUP models may be more involved than just applying the above transformation, see e.g. [59] where a possible way out is suggested to derive the GUP relations from the (explicitly state dependent) deformed commutators between coordinates and momenta. Here, on the contrary to the phenomenological GUP approaches, we are starting from the noncommutative model as a possible description of the quantization of space-time, hence the starting algebra is fixed from the beginning and modified uncertainty relations and all effects are a consequence of the starting choice of quantum-deformed phase space.

$$J = \left| \frac{\partial \left(x'_1, \dots, x'_D, p'_1, \dots, p'_D\right)}{\partial \left(x_1, \dots, x_D, p_1, \dots, p_D\right)} \right| = 1 + \left(\frac{\partial \delta x_i}{\partial x_i} + \frac{\partial \delta p_i}{\partial p_i} \right) + \dots \tag{11}$$

where we used:

$$\frac{\partial x'_i}{\partial x_j} = \delta_{ij} + \frac{\partial \delta x_i}{\partial x_j}, \quad \frac{\partial x'_i}{\partial p_j} = \frac{\partial \delta x_i}{\partial p_j}, \tag{12}$$

$$\frac{\partial p'_i}{\partial x_j} = \frac{\partial \delta p_i}{\partial x_j}, \quad \frac{\partial p'_i}{\partial p_j} = \delta_{ij} + \frac{\partial \delta p_i}{\partial p_j}. \tag{13}$$

Up to the first order in δt (based on (10)) we get:

$$\begin{aligned}\left(\frac{\partial \delta x_i}{\partial x_i} + \frac{\partial \delta p_i}{\partial p_i} \right) &= \frac{\partial}{\partial x_i} \left(c_{ij} \frac{\partial H}{\partial p_j} + a_{ij} \frac{\partial H}{\partial x_j} \right) \delta t + \frac{\partial}{\partial p_i} \left(-c_{ji} \frac{\partial H}{\partial x_j} + b_{ij} \frac{\partial H}{\partial p_j} \right) \delta t \\ &= \left[\left(\frac{\partial}{\partial x_i} a_{ij} - \frac{\partial}{\partial p_i} c_{ji} \right) \frac{\partial H}{\partial x_j} + \left(\frac{\partial}{\partial x_i} c_{ij} + \frac{\partial}{\partial p_i} b_{ij} \right) \frac{\partial H}{\partial p_j} \right] \delta t \end{aligned} \tag{14}$$

where the terms with mixed derivatives have cancelled each other and the antisymetricity of the Poisson brackets was used to cancel the remaining terms. This way we find the expression for the time evolved infinitesimal phase space volume (in the first order of δt) as:

$$d^D x' d^D p' = d^D x d^D p \left(1 + \left[\left(\frac{\partial}{\partial x_i} a_{ij} - \frac{\partial}{\partial p_i} c_{ji} \right) \frac{\partial H}{\partial x_j} + \left(\frac{\partial}{\partial x_i} c_{ij} + \frac{\partial}{\partial p_i} b_{ij} \right) \frac{\partial H}{\partial p_j} \right] \delta t + O\left(\delta t^2\right) \right) \tag{15}$$

valid for any Poisson algebra (6), (7) obtained as the classical limit of any noncommutative model (3), (4). Since in general the terms in the brackets will not cancel out, already here we see the need to introduce the weight to the phase space volume element so that the analogue of the Liouville theorem is satisfied. The specific factor has to be chosen in such a way that the weighted phase space volume is invariant under the time evolution, i.e. so that:

$$\frac{d^D x' d^D p'}{F(x', p')} \sim \frac{d^D x d^D p}{F(x, p)}. \tag{16}$$

It is straightforward to notice from (15) that the noncanonical Poisson brackets for coordinates and for momenta (6), as well as the mixed relation between coordinates and momenta (7) are crucial in the choice of the weighted phase space volume. Hence the underlying noncommutative geometry (and the choice of (3), (4)) is the intrinsic feature of phenomenological models investigating the possible effects arising from such modified phase space volume element (and consequently affecting the thermodynamical properties of physical systems). To be able to investigate the time evolution of the weight factor $F(x, p)$, we need to consider the concrete noncommutative model, which we shall do in the next section.

3. Snyder-de Sitter model and the density of states

Snyder-de Sitter (SdS) model, which includes the noncommutative space-time coordinates and the noncommutative momenta, was proposed [54] as a generalization of the Snyder model [60] to a space-time background of constant curvature and it was investigated in many contexts, see e.g. [61], [62]. In SdS model, the noncommutativity among space-time coordinates corresponds to the curved momentum space, and vice-versa noncommutative momenta lead to the curved space-time. Since we are interested in investigating how the noncommutativity of both coordinates and momenta affects the density of states we choose SdS set of commutation relations which exhibit these features:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\hbar\beta^2 \hat{M}_{\mu\nu}, \qquad [\hat{p}_\mu, \hat{p}_\nu] = i\hbar\alpha^2 \hat{M}_{\mu\nu}, \tag{17}$$

$$[\hat{M}_{\mu\nu}, \hat{x}_\rho] = i\hbar(\eta_{\mu\rho}\hat{x}_\nu - \eta_{\nu\rho}\hat{x}_\mu), \tag{18}$$

$$[\hat{M}_{\mu\nu}, \hat{p}_\rho] = i\hbar(\eta_{\mu\rho}\hat{p}_\nu - \eta_{\nu\rho}\hat{p}_\mu), \tag{19}$$

$$[\hat{M}_{\mu\nu}, \hat{M}_{\rho\tau}] = i\hbar(\eta_{\mu\rho}\hat{M}_{\nu\tau} - \eta_{\mu\tau}\hat{M}_{\nu\rho} + \eta_{\nu\tau}\hat{M}_{\mu\rho} - \eta_{\nu\rho}\hat{M}_{\mu\tau}), \tag{20}$$

with the modified quantum mechanical phase space relation:

$$[\hat{x}_\mu, \hat{p}_\nu] = i\hbar \left(\eta_{\mu\nu} + \alpha^2 \hat{x}_\mu \hat{x}_\nu + \beta^2 \hat{p}_\mu \hat{p}_\nu + \alpha\beta(\hat{x}_\mu \hat{p}_\nu + \hat{p}_\mu \hat{x}_\nu - \hat{M}_{\mu\nu}) \right). \tag{21}$$

Here $\eta_{\mu\nu}$ is the flat metric with Lorentzian signature and $\mu, \nu = 0, 1, \dots, D$. Depending on the sign[3] of α^2 the Lorentz generators $\hat{M}_{\mu\nu}$ and momenta $\hat{p}_\mu$ generate de Sitter (dS) or Anti-de Sitter (AdS) subalgebras. This model involves, besides the speed of light,[4] two

[3] For the Jacobi identities to be satisfied both coupling constants α^2 and β^2 must have the same sign. Following the convention of [61] when α^2 and β^2 are negative then $\alpha\beta$ is negative.

[4] We set $c = 1$ throughout the paper.

other observer-independent constants,[5] the Planck length as well as the de Sitter radius which is related to the cosmological constant. More precisely, the parameter $|\alpha^2|$ has the dimension of the inverse of the square of length and it can be identified with the (Anti-)de Sitter radius and the cosmological constant as its inverse, while the parameter $|\beta^2|$ has the dimension of the inverse square of mass and it can be is identified with $1/M_P^2 = l_P^2$, where l_P and M_P are the Planck length and mass, respectively. In general, α^2 and β^2 can take positive or negative value, which generates models with very different properties [61].

In the following we focus on the non-relativistic quantum mechanical counterpart (which in $D = 3$ would give the Snyder model restricted to a three-dimensional sphere) however we keep unspecified dimension D, $i, j = 1, 2, \dots, D$. The GEUP corresponding to (21) has the form similar to (2) [40],[6] but additional (mixed) terms may appear on the RHS depending on the choice of the representation of the angular momentum used, see e.g. [61]. Without any additional assumptions we can, at most, write:

$$\left(\Delta\hat{x}_i\right)\left(\Delta\hat{p}_i\right) \geq \frac{\hbar}{2}\left|1 + \alpha^2 (\Delta\hat{x})^2 + \beta^2 (\Delta\hat{p})^2 + \gamma\right| \tag{22}$$

where we used $(\Delta\hat{x})^2 = \left\langle \hat{x}^2 \right\rangle - \langle \hat{x}\rangle^2$, $(\Delta\hat{p})^2 = \left\langle \hat{p}^2 \right\rangle - \langle \hat{p}\rangle^2$ and $\gamma = \alpha^2 \langle \hat{x}\rangle^2 + \beta^2 \langle \hat{p}\rangle^2 + \alpha\beta(\langle \hat{x}\hat{p}\rangle + \langle \hat{p}\hat{x}\rangle)$ as well as the standard notation $\hat{x}^2 = \hat{x}_i\hat{x}_i$, $\hat{p}^2 = \hat{p}_i\hat{p}_i$, $\hat{x}\hat{p} = \hat{x}_i\hat{p}_i$ was assumed, while on the LHS there is no summation. In 1-dimensional case, one can show [61] that, for both $\alpha^2 > 0$ and $\beta^2 > 0$ the bounds on the localization in position and momentum space arise, while when $\alpha^2 < 0$ and $\beta^2 < 0$ a combination of spatial and momentum coordinates becomes bounded instead.

The classical (non-relativistic) Snyder-de Sitter Poisson algebra is given by[7]:

$$\begin{aligned} &\{x_i, x_j\} = \beta^2(x_i p_j - x_j p_i) \equiv a_{ij}, \qquad \{p_i, p_j\} = \alpha^2(x_i p_j - x_j p_i) \equiv b_{ij}, \\ &\{x_i, p_j\} = \delta_{ij} + \alpha^2 x_i x_j + \beta^2 p_i p_j + 2\alpha\beta p_i x_j \equiv c_{ij}. \end{aligned} \tag{23}$$

and supplemented by the relations with Lorentz generators. By using the general set up from the previous section and now the explicit form of SdS model (23) we obtain that the infinitesimal phase space volume element after the infinitesimal time evolution as:

$$d^D x' d^D p' = d^D x d^D p \left(1 + 2\left[\left(\alpha^2 x_j + \alpha\beta p_j\right)\frac{\partial H}{\partial p_j} - \left(\beta^2 p_j + \alpha\beta x_j\right)\frac{\partial H}{\partial x_j}\right]\delta t + O\left(\delta t^2\right)\right) \tag{24}$$

where we explicitly used (23) in (15) with the following:

$$\frac{\partial}{\partial x_i} a_{ij} = \beta^2 (D-1) p_j, \qquad \frac{\partial}{\partial x_i} c_{ij} = \alpha^2 (D+1) x_j + 2\alpha\beta p_j, \tag{25}$$

$$\frac{\partial}{\partial p_i} b_{ij} = \alpha^2 (1-D) x_j = -\alpha^2 (D-1) x_j, \qquad \frac{\partial}{\partial p_i} c_{ji} = \beta^2 (D+1) p_j + 2\alpha\beta x_j. \tag{26}$$

In the remaining part of this section we will show that for the SdS Poisson algebra (23) the analogue of the Liouville theorem is satisfied when we consider the following weighted phase space volume

$$\frac{d^D x d^D p}{\left(1 + \alpha^2 x^2 + \beta^2 p^2 + 2\alpha\beta x \cdot p\right)} \tag{27}$$

which is invariant under the infinitesimal time evolution.

To show this, we consider the time evolution of each of the terms in the proposed factor $F(x, p) = 1 + \alpha^2 x^2 + \beta^2 p^2 + 2\alpha\beta x \cdot p$ during the infinitesimal time interval δt. At first we keep all the formulae as general as possible and express them in terms of a_{ij}, b_{ij} and c_{ij} and only at the end we will specify to SdS case (23). We get the following expressions:

$$\begin{aligned} x'^2 &= \left(x_i + \delta x_i\right)^2 \\ &= x^2 + 2x_i\delta x_i + O\left(\delta t^2\right) \\ &= x^2 + 2x_i\left(c_{ij}\frac{\partial H}{\partial p_j} + a_{ij}\frac{\partial H}{\partial x_j}\right)\delta t + O\left(\delta t^2\right), \end{aligned} \tag{28}$$

$$\begin{aligned} p'^2 &= \left(p_i + \delta p_i\right)^2 \\ &= p^2 + 2p_i\delta p_i + O\left(\delta t^2\right) \\ &= p^2 + 2p_i\left(-c_{ji}\frac{\partial H}{\partial x_j} + b_{ij}\frac{\partial H}{\partial p_j}\right)\delta t + O\left(\delta t^2\right), \end{aligned} \tag{29}$$

and for the last term

[5] SdS is also known as doubly special relativity (DSR) in de Sitter space [56] or triply special relativity (TSR) model [54].

[6] Note that the same GEUP can be realized by a different deformed Heisenberg algebras, see e.g. [45].

[7] Where we used the fact that the above algebra (and relation (21)) can be implemented on the canonical phase space with Lorentz generators defined as $M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$.

$$x' \cdot p' = x'p' = \left(x_i + \delta x_i\right)\left(p_i + \delta p_i\right)$$
$$= xp + p_i \delta x_i + x_i \delta p_i + O\left(\delta t^2\right)$$
$$= xp + p_i \left(c_{ij}\frac{\partial H}{\partial p_j} + a_{ij}\frac{\partial H}{\partial x_j}\right)\delta t + x_i\left(-c_{ji}\frac{\partial H}{\partial x_j} + b_{ij}\frac{\partial H}{\partial p_j}\right)\delta t + O\left(\delta t^2\right). \tag{30}$$

Therefore the time evolution of the whole factor $F(x,p)$ can be first written generally as (where for simplicity we use $2\alpha\beta = \gamma$):

$$1 + \alpha^2 x'^2 + \beta^2 p'^2 + \gamma x' p' = 1 + \alpha^2 x^2 + \beta^2 p^2 + \gamma xp +$$
$$+ \left[2\alpha^2 x_i c_{ij} + 2\beta^2 p_i b_{ij} + \gamma\left(p_i c_{ij} + x_i b_{ij}\right)\right]\frac{\partial H}{\partial p_j}\delta t +$$
$$+ \left[2\alpha^2 x_i a_{ij} - 2\beta^2 p_i c_{ji} + \gamma\left(p_i a_{ij} - x_i c_{ji}\right)\right]\frac{\partial H}{\partial x_j}\delta t + O\left(\delta t^2\right). \tag{31}$$

Now specialising this to the case of SdS (23), after plugging in the expressions for a_{ij}, b_{ij} and c_{ij}, we obtain (in the first order of δt):

$$1 + \alpha^2 x'^2 + \beta^2 p'^2 + \gamma x' p' = 1 + \alpha^2 x^2 + \beta^2 p^2 + \gamma xp$$
$$+ \left[2\alpha^2\left(x_j + \alpha^2 x^2 x_j + \beta^2 (xp) p_j + \gamma (xp) x_j\right) + 2\beta^2\alpha^2 (xp) p_j - 2\beta^2\alpha^2 x_j p^2 + \gamma\left(p_j + \alpha^2 x^2 p_j + \beta^2 p^2 p_j + \gamma p^2 x_j\right)\right]\frac{\partial H}{\partial p_j}\delta t +$$
$$+ \left[-2\beta^2\left(p_j + \alpha^2 x_j (xp) + \beta^2 p_j p^2 + \gamma (xp) p_j\right) + 2\alpha^2\beta^2 x^2 p_j - 2\alpha^2\beta^2 x_j (xp) - \gamma\left(x_j + \alpha^2 x^2 x_j + \beta^2 p^2 x_j + \gamma x^2 p_j\right)\right]\frac{\partial H}{\partial x_j}\delta t. \tag{32}$$

Since the aim is to factorise the whole expression $(1 + \alpha^2 x^2 + \beta^2 p^2 + \gamma xp)$ on the right hand side of this equality, we need to rearrange all the terms in the square brackets in such a way so that we can recognize the whole factor $F(x,p)$. In this way, we obtain (once we returned to the notation $\gamma = 2\alpha\beta$):

$$1 + \alpha^2 x'^2 + \beta^2 p'^2 + 2\alpha\beta x' p' =$$
$$= 1 + \alpha^2 x^2 + \beta^2 p^2 + 2\alpha\beta(xp)$$
$$+ \left[2\alpha^2\left(1 + \alpha^2 x^2 + \beta^2 p^2 + 2\alpha\beta (xp)\right) x_j + 2\alpha\beta\left(1 + \alpha^2 x^2 + \beta^2 p^2 + 2\alpha\beta xp\right) p_j\right]\frac{\partial H}{\partial p_j}\delta t +$$
$$+ \left[-2\beta^2\left(1 + \alpha^2 x^2 + \beta^2 p^2 + 2\alpha\beta (xp)\right) p_j - 2\alpha\beta\left(1 + \alpha^2 x^2 + \beta^2 p^2 + 2\alpha\beta x \cdot p\right) x_j\right]\frac{\partial H}{\partial x_j}\delta t + O\left(\delta t^2\right)$$
$$= \left(1 + \alpha^2 x^2 + \beta^2 p^2 + 2\alpha\beta(xp)\right)[1 + \left(2\alpha^2 x_j + 2\alpha\beta p_j\right)\frac{\partial H}{\partial p_j}\delta t - \left(2\beta^2 p_j + 2\alpha\beta x_j\right)\frac{\partial H}{\partial x_j}\delta t] + O\left(\delta t^2\right). \tag{33}$$

We can see that the weighted volume element will stay invariant since the weight factor we introduced produces the same terms as the Jacobian in (24) under the infinitesimal time evolution (up to the first order in δt):

$$\frac{d^D x' d^D p'}{1 + \alpha^2 x'^2 + \beta^2 p'^2 + 2\alpha\beta x' p'} =$$
$$= \frac{d^D x d^D p\left(1 + 2\left[\left(\alpha^2 x_j + \alpha\beta p_j\right)\frac{\partial H}{\partial p_j} - \left(\beta^2 p_j + \alpha\beta x_j\right)\frac{\partial H}{\partial x_j}\right]\delta t + O\left(\delta t^2\right)\right)}{\left(1 + \alpha^2 x^2 + \beta^2 p^2 + 2\alpha\beta x \cdot p\right)\left(1 + 2\left[\left(\alpha^2 x_j + \alpha\beta p_j\right)\frac{\partial H}{\partial p_j} - \left(\beta^2 p_j + \alpha\beta x_j\right)\frac{\partial H}{\partial x_j}\right]\delta t + O\left(\delta^2\right)\right)}$$
$$= \frac{d^D x d^D p}{\left(1 + \alpha^2 x^2 + \beta^2 p^2 + 2\alpha\beta xp\right)}. \tag{34}$$

We point out this result holds to any order of parameters α and β (as we have not done any approximations in the noncommutative parameters).

We note that from the SdS model considered above we can obtain two well known special cases. Namely:

- Snyder model [60][8] is obtained when we take $\alpha \to 0$ in SdS algebra (17)-(21), i.e. we obtain the model with noncommutative coordinates and commutative momenta (curved momentum space):

$$[\hat{x}_\mu, \hat{x}_\nu] = i\hbar\beta^2 \hat{M}_{\mu\nu}, \qquad [\hat{p}_\mu, \hat{p}_\nu] = 0, \qquad [\hat{x}_\mu, \hat{p}_\nu] = i\hbar(\eta_{\mu\nu} + \beta^2 \hat{p}_\mu \hat{p}_\nu), \tag{35}$$

[8] Snyder model was the first proposed noncommutative space-time model preserving Lorentz symmetry.

supplemented by the Lorentz covariance conditions (18)-(20). Such modification of quantum mechanical phase space relations leads, in the non-relativistic case, to the quadratic GUP (QGUP)[9]:

$$\Delta\hat{x}_i\Delta\hat{p}_i \geq \frac{\hbar}{2}(1+\beta^2(\Delta\hat{p})^2). \tag{36}$$

We note that the algebraic set of commutation relations (35) is only one of the many possible realizations of the Snyder model and more general realizations may be considered, see e.g. [34,35]. The non-relativistic classical Poisson algebra:

$$\{x_i, x_j\} = \beta^2(x_i p_j - x_j p_i), \qquad \{p_i, p_j\} = 0, \qquad \{x_i, p_j\} = \delta_{ij} + \beta^2 p_i p_j \tag{37}$$

will result in the following weighted phase space volume element:

$$\frac{d^D x d^D p}{1+\beta^2 p^2} \tag{38}$$

(in any dimension D), obtained as the limit $\alpha \to 0$ in (27).

- (Anti-)de Sitter model (dual Snyder model) is obtained when we take $\beta \to 0$ in SdS algebra (17)-(21), i.e. we obtain the model with commutative coordinates but noncommutative momenta (curved space-time[10]):

$$[\hat{x}_\mu, \hat{x}_\nu] = 0, \qquad [\hat{p}_\mu, \hat{p}_\nu] = i\hbar\alpha^2 \hat{M}_{\mu\nu}, \qquad [\hat{x}_\mu, \hat{p}_\nu] = i\hbar\left(\eta_{\mu\nu} + \alpha^2 \hat{x}_\mu \hat{x}_\nu\right), \tag{39}$$

with the Lorentz covariance given by (18)-(20). The non-relativistic case results in the quadratic EUP (QEUP)[11]:

$$\Delta\hat{x}_i\Delta\hat{p}_i \geq \frac{\hbar}{2}(1+\alpha^2(\Delta\hat{x})^2). \tag{40}$$

The non-relativistic classical Poisson algebra is:

$$\{x_i, x_j\} = 0, \qquad \{p_i, p_j\} = \alpha^2(x_i p_j - x_j p_i), \qquad \{x_i, p_j\} = \delta_{ij} + \alpha^2 x_i x_j. \tag{41}$$

And the weighted phase space volume element (in any dimension D), obtained as the limit $\beta \to 0$ in (27), is:

$$\frac{d^D x d^D p}{1+\alpha^2 x^2}. \tag{42}$$

It is worth to point out that there exists a way to transform SdS algebra (17)-(21) generated by $(\hat{x}, \hat{p}, \hat{M})$ into the Snyder algebra (35) generated by $(\hat{x}^S, \hat{p}^S, \hat{M})$ by the following linear maps:

$$\hat{x}_i = \hat{x}_i^S + \frac{\beta}{\alpha}\lambda \hat{p}_i^S, \quad \hat{p}_i = (1-\lambda)\hat{p}_i^S - \frac{\alpha}{\beta}\hat{x}_i^S \tag{43}$$

where λ is a free parameter and we have temporarily denoted the Snyder algebra (35) generators by the upper index S. This relation of the SdS algebra with the Snyder algebra was first presented in [61]. Through such noncanonical change of basis one can use the already known machinery of realizations in Snyder spaces and consider various applications of SdS algebra, for example to find harmonic oscillator solutions [63] or in applications to quantum field theory [64].

One can also consider various generalizations of the SdS model. For example, in [65] the following generalization of the last relation in SdS algebra (21) was proposed:

$$[\hat{x}_\mu, \hat{p}_\nu] = i\hbar\left(\eta_{\mu\nu}\varphi_1 + \left(\alpha^2\hat{x}_\mu\hat{x}_\nu + \beta^2\hat{p}_\mu\hat{p}_\nu + \alpha\beta\hat{x}_\mu\hat{p}_\nu + \alpha\beta\hat{p}_\mu\hat{x}_\nu\right)\varphi_2 - \alpha\beta\hat{M}_{\mu\nu}\right) \tag{44}$$

where the functions φ_1 and φ_2 need to satisfy specific conditions (due to Jacobi identities). When $\varphi_1 = \varphi_2 = 1$ we get back the original SdS model (21). Various choices of φ_1, φ_2 are discussed in [65] and for example for one specific choice of φ_1 and φ_2 we can obtain the following relation:

$$[\hat{x}_\mu, \hat{p}_\nu] = i\hbar\eta_{\mu\nu}\sqrt{1-\left(\alpha^2\hat{x}^2 + \beta^2\hat{p}^2 + \alpha\beta\hat{x}\hat{p} + \alpha\beta\hat{p}\hat{x}\right)} - \alpha\beta\hat{M}_{\mu\nu}. \tag{45}$$

Such generalizations would be interesting to investigate further in the context of GEUPs and their influence on the density of states.

Before completing this section, since the effects of modifications of UPs on the density of states have been investigated previously few comments are in order. In [12,13], in the case of GUP (36)[12] the invariant weighted phase space volume el-

[9] For the simple case in which $\langle\hat{p}_i\rangle = 0$.

[10] We recall, that the parameter α still will play the role of the inverse of the de Sitter radius R i.e. space-time curvature, which is linked to the cosmological constant.

[11] For the simple case in which $\langle\hat{x}_i\rangle = 0$.

[12] In [12,13] more general case, than (36), is considered with two parameters β, β'. In the comparison we take $\beta' = 0$ since this is the most similar option to the Snyder model discussed here. Nevertheless, the case with $\beta' \neq 0$ can also be associated with the Snyder model, but requires different (more general) realization see e.g. [34,35].

ement is obtained as: $\frac{d^D x d^D p}{\left(1+\beta^2 p^2\right)^D}$, where the power D (dimension) is necessary since the Jacobian obtained is: $d^D x' d^D p' = d^D x d^D p \left(1 - 2\beta^2 D p_k \frac{\partial H}{\partial x_k}\delta t + O\left(\delta t^2\right)\right)$. This is different than the case considered here, where in the limit $\alpha \to 0$ we obtain $\frac{d^D x d^D p}{1+\beta^2 p^2}$ (38). The difference arises from the fact that commutation relations used in [12] for $[x_i, p_j]$ have the term proportional to p^2 while here we have the terms with $p_i p_j$ instead, cf. (37). In the case of the Snyder model with (37) the terms with D in the Jacobian (24) cancel out, hence the power D is not appearing in the weight factor of the volume form (38).

Other important point is that often in the context of modified UP (1) also the inner product on the Hilbert space[13] becomes modified with appropriately chosen measure so that the observables (satisfying the modified commutation relations) stay symmetric on the dense domain of functions decaying faster than any power [11]. For the inner product modifications in the case of GUP see, e.g. [12,13] and for the relativistic case, see e.g. [66]. The inner product modifications in the case of EUP, i.e. in AdS and dS spaces were investigated e.g. in [41]. Subsequently the modified inner product can be used to study effects on solutions of Schrödinger equation, see e.g. [12,13,41]. Such modifications in the measure in the inner product have not been under investigation of the present paper.

4. Yang model

The Yang model introduced in [55] is a Lorentz invariant model incorporating noncommutative space-time coordinates as well as noncommutative momenta, depending on the pair of dimensionful parameters α and β related with the curvatures of quantum space-time and momentum spaces (in similarity to SdS model). The defining relations are as follows:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\hbar\beta^2 \hat{M}_{\mu\nu}, \qquad [\hat{p}_\mu, \hat{p}_\nu] = i\hbar\alpha^2 \hat{M}_{\mu\nu} \tag{46}$$

$$[\hat{M}_{\mu\nu}, \hat{x}_\rho] = i\hbar(\eta_{\mu\rho}\hat{x}_\nu - \eta_{\nu\rho}\hat{x}_\mu), \tag{47}$$

$$[\hat{M}_{\mu\nu}, \hat{p}_\rho] = i\hbar(\eta_{\mu\rho}\hat{p}_\nu - \eta_{\nu\rho}\hat{p}_\mu), \tag{48}$$

$$[\hat{M}_{\mu\nu}, \hat{M}_{\rho\tau}] = i\hbar(\eta_{\mu\rho}\hat{M}_{\nu\tau} - \eta_{\mu\tau}\hat{M}_{\nu\rho} + \eta_{\nu\tau}\hat{M}_{\mu\rho} - \eta_{\nu\rho}\hat{M}_{\mu\tau}). \tag{49}$$

However, the quantum phase space relation is described by an additional generator $\hat{r}$ (central charge):

$$[\hat{x}_\mu, \hat{p}_\nu] = i\hbar\eta_{\mu\nu}\hat{r}, \tag{50}$$

hence to obtain the full Yang algebra we need the additional relations:

$$[\hat{r}, \hat{x}_\mu] = i\hbar\beta^2 \hat{p}_\mu, \qquad [\hat{r}, \hat{p}_\mu] = -i\hbar\alpha^2 \hat{x}_\mu, \qquad [\hat{M}_{\mu\nu}, \hat{r}] = 0. \tag{51}$$

The uncertainty principle corresponding to the Yang model, in the non-relativistic case, can be written in general as:

$$\Delta\hat{x}_i \Delta\hat{p}_j \geq \frac{\hbar\delta_{ij}}{2} \left|\langle\hat{r}\rangle\right|, \tag{52}$$

where the generator $\hat{r}$ can be realized in terms of the phase space variables $\hat{r} = \hat{r}(\hat{x}, \hat{p})$. It is also worth to mention that the Yang model is covariant (self-dual) under the Born reciprocity:

$$B: \quad \hat{x}_\mu \to \hat{p}_\mu, \quad \hat{p}_\mu \to -\hat{x}_\mu, \quad \hat{M}_{\mu\nu} \leftrightarrow \hat{M}_{\mu\nu}, \quad \hat{r} \leftrightarrow \hat{r}, \quad \alpha \leftrightarrow \beta. \tag{53}$$

The similarity between the Yang model and the SdS model considered in the previous section is not coincidental. It has been shown [67] that the SdS algebra (17)-(21) can be viewed as a nonlinear realization of the Yang model (46)-(51).

The classical limit of the Yang model (cf. [68], see also [69]) is:

$$\{x_i, x_j\} = \beta^2(x_i p_j - x_j p_i), \qquad \{p_i, p_j\} = \alpha^2(x_i p_j - x_j p_i) \tag{54}$$

and one of the possible realizations for the $\hat{r}$ generator on the canonical phase space [68], for example, gives:

$$\{x_i, p_j\} = \delta_{ij}\sqrt{1 - \alpha^2 x^2 - \beta^2 p^2 - \alpha^2\beta^2(x^2 p^2 - (xp)^2)}. \tag{55}$$

The remaining relations are obtained in a straightforward way. In D = 1 this would simplify to:

$$\{x, p\} = \sqrt{1 - \alpha^2 x^2 - \beta^2 p^2}, \tag{56}$$

with the corresponding GEUP[14]:

$$\Delta x \Delta p \geq \frac{\hbar}{2}\sqrt{1 - \alpha^2(\Delta x)^2 - \beta^2(\Delta p)^2}. \tag{57}$$

[13] The Heisenberg algebra (3), (4) is represented on the space of states in which one usually chooses a basis of position or momentum eigenvectors. In the case of GUP one usually chooses the momentum space, while in the case of GEUP Bergmann-Fock construction can be used [43].

[14] In higher dimensions additional mixed terms would be present in the GEUP relation.

We see that in such 1-dimensional case higher order terms would appear, which is not uncommon for GUPs see e.g. [70–72] but this would be the first such example considering higher order GEUPs, up to our knowledge.

Expanding the RHS of (56) in deformation parameters, up to α^2 and β^2, we can apply the results of the previous section (27) and obtain the invariant weighted phase space volume element as:

$$\frac{dxdp}{1-\frac{1}{2}\left(\alpha^2x^2+\beta^2p^2\right)}. \tag{58}$$

From the Yang model (46)-(51), in the realization (55) for $\hat{r}$ generator we can obtain two special cases. Namely:

- In the limit when $\alpha \to 0$ we obtain the so-called "square-root modified" or "Maggiore algebra" (see e.g. [73]):

  $$[\hat{x}_\mu,\hat{x}_\nu]=i\hbar\beta^2\hat{M}_{\mu\nu}, \qquad [\hat{p}_\mu,\hat{p}_\nu]=0, \qquad [\hat{x}_\mu,\hat{p}_\nu]=i\hbar\eta_{\mu\nu}\sqrt{1-\beta^2\hat{p}^2}, \tag{59}$$

  supplemented by the Lorentz covariance conditions (47)-(49). Such modification of quantum mechanical phase space relations leads, in the non-relativistic case, to the higher order type of GUP[15] of the form:

  $$\Delta\hat{x}_i\Delta\hat{p}_i \geq \frac{\hbar}{2}\sqrt{1-\beta^2\left(\Delta\hat{p}\right)^2}. \tag{60}$$

  This version of GUP no longer produces a minimum observable length [48,72]. But it would still result in the weighted phase space volume element:

  $$\frac{dxdp}{1-\frac{1}{2}\beta^2p^2} \tag{61}$$

  affecting the density of states.
- By taking $\beta \to 0$ in (46)-(51), we obtain

  $$[\hat{x}_\mu,\hat{x}_\nu]=0, \qquad [\hat{p}_\mu,\hat{p}_\nu]=i\hbar\alpha^2\hat{M}_{\mu\nu}, \qquad [\hat{x}_\mu,\hat{p}_\nu]=i\hbar\eta_{\mu\nu}\sqrt{1-\alpha^2\hat{x}^2}, \tag{62}$$

  leading to the higher order type of EUP[16]:

  $$\Delta\hat{x}_i\Delta\hat{p}_i \geq \frac{\hbar}{2}\sqrt{1-\alpha^2\left(\Delta\hat{x}\right)^2} \tag{63}$$

  with the weighted phase space volume element as:

  $$\frac{dxdp}{1-\frac{1}{2}\alpha^2x^2} \tag{64}$$

  affecting the density of states.

We postpone the investigation of the full D-dimensional case of Yang model (46)-(51) in the context of density of states to the future work. It is also worth to mention that in [74] generalizations of the Snyder algebra to a curved space-time background with de Sitter symmetry were considered where the SdS model and Yang model were obtained as special cases. The realizations of these algebras were considered in terms of canonical phase space coordinates, up to the fourth order in the deformation parameters. Therefore the results of the present paper could be generalized to these type of models and realizations as well.

5. Lie-algebraic case with commuting momenta: fuzzy sphere

Many noncommutative (quantum) space-times proposed in the quantum gravity motivated literature have the Lie-algebraic form for the noncommutativity of coordinates and include commuting momenta. Hence, for the sake of completeness we discuss here one example of such model defined by the following commutation relations:

$$\left[\hat{x}_i,\hat{x}_j\right]=i\hbar\epsilon_{ijk}\hat{x}_k, \qquad \left[\hat{x}_i,\hat{p}_j\right]=i\hbar\epsilon_{ijk}\hat{p}_k, \qquad \left[\hat{p}_i,\hat{p}_j\right]=0 \tag{65}$$

where ϵ_{ijk} is totally skew-symmetric tensor, $\epsilon_{123}=1$. The subalgebra $[\hat{x}_i,\hat{x}_j]=i\hbar\epsilon_{ijk}\hat{x}_k$ supplemented by the relation $\hat{x}^i\hat{x}_i=r$ corresponds to the fuzzy sphere, with r the constant radius of the sphere.

Since we are interested in the effects on the density of states in this Lie algebraic case we follow the steps outlined in Sec. 2. In the classical limit, we obtain:

$$\{x_i,x_j\}=\epsilon_{ijk}x_k \equiv a_{ij}, \qquad \{x_i,p_j\}=\epsilon_{ijk}p_k \equiv c_{ij}, \qquad \{p_i,p_j\}=0 \equiv b_{ij}, \tag{66}$$

[15] We call this type of GUP "higher order" due to the expansion of the square root function in powers of p thanks to which, through series of inequalities, one obtains the formula (60), see e.g. [72].

[16] By similar expansion of the square-root function in powers of x.

where we identified RHSs with the notation used in Sec. 2. Directly plugging these relations into (15) we obtain:

$$
\begin{aligned}
d^D x' d^D p' &= d^D x d^D p \left(1 + \left[\left(\frac{\partial}{\partial x_i}\left(\epsilon_{ijk} x_k\right) - \frac{\partial}{\partial p_i}\left(\epsilon_{jik} p_k\right)\right)\frac{\partial H}{\partial x_j} + \left(\frac{\partial}{\partial x_i}\left(\epsilon_{ijk} p_k\right)\right)\frac{\partial H}{\partial p_j}\right]\delta t + O\left(\delta t^2\right)\right) \\
&= d^D x d^D p \left(1 + \left[\left(\epsilon_{ijk}\delta_{ik} - \epsilon_{jik}\delta_{ik}\right)\frac{\partial H}{\partial x_j} + 0\right]\delta t + O\left(\delta t^2\right)\right) = d^D x d^D p.
\end{aligned}
\tag{67}
$$

Hence we see that the infinitesimal phase space volume element stays invariant under the time evolution without the need to introduce any additional factor and there will be no change in the density of states for the case of fuzzy sphere.

6. Final remarks

In this paper we have focused on models exhibiting noncommutativity in both space-time coordinates and in momenta (i.e. models with curved space-time and curved momentum space), such that in the quantum phase space relations both the Planck length l_P and the cosmological constant Λ appear as fundamental parameters on equal footing. The modified Heisenberg commutation relations lead to GEUPs where the symmetry between position and momentum is preserved (which is not the case in the usual GUPs or EUPs). Such symmetric GEUPs may be seen as an indication of quantum gravitational corrections to the classical space-time and standard quantum mechanics, at both very small and very large scales. We point out that, on the contrary to many works which have proposed multiple variants of the GUPs, EUPs and GEUPs by arbitrarily choosing specific forms of the commutation relations between space-time coordinates and momenta $[x, p]$ with supposedly desirable properties, we have focused here on studying the consequences of known models which arise in the noncommutative geometry approach to quantum gravity.

In general since the canonical commutation relations are modified, one expects that thermodynamics and statistical mechanics will be affected by the introduced modifications, possibly leading to some new effects. As a consequence of the GEUP arising from the cases of Snyder-de Sitter and Yang models we have shown that the analogue of the Liouville theorem in statistical physics requires considering the weighted phase space volume and introduces modification in the density of states, with the weight factor depending on both coordinates and momenta. Since such modification of the density states is required this will influence the statistical and thermodynamical properties of physical systems. Various applications can now be studied and the effects of both noncommutativity in coordinates and momenta (or the presence of the Planck length and the cosmological constant in modified UPs) on atomic physics, condensed matter physics, preheating phase of the universe and black holes etc. can now be investigated.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Anna Pachol reports financial support was provided by NCN grant number 2022/45/B/ST2/01067. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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