Phys. Lett. B 877 (2026) 140472

Contents lists available at ScienceDirect

Physics Letters B

journal homepage: www.elsevier.com/locate/physletb

Letter

Holographic entropy bound and a special class of spherical systems in cosmology

Hao Yu [a], Zichao Lin [a,b], Jin Li [a,*]

[a] Physics Department, Chongqing University, 401331, Chongqing, China
[b] School of Physics, Huazhong University of Science and Technology, 430074, Wuhan, Hubei, China

ARTICLE INFO

Editor: P Brax

Keywords:
Entropy bound
Holographic entropy bound
Spherical systems in cosmology

ABSTRACT

The holographic entropy bound is discussed in cosmology. Inspired by the work of Fischler and Susskind [hep-th/9806039], we aim to define a special class of spherical systems in cosmology, within which the entropy of matter remains compliant with the holographic entropy bound throughout the evolution of the universe, irrespective of the universe's components. It is found that if the entropy of matter per unit co-moving volume is bounded from above, such a special class of spherical systems indeed exists. Moreover, the matter contained within a unit co-moving volume can be replaced by a black hole of the same mass-energy. Provided that the entropy of the black hole consistently exceeds that of the matter it replaces, there is also a unified definition for these special spherical systems.

1. Introduction

The first entropy bound, i.e., the Bekenstein bound [1], was proposed by Jacob Bekenstein in 1981, which points out that the total entropy in a spherical system of radius R should follow an inequality $S \leq \frac{2\pi k_B E R}{\hbar c}$, where E is the total mass-energy contained within the system. More than a decade later, the holographic principle [2–7] indicates that the description of a volume of space is encoded on its lower-dimensional boundary. Thus, the degree of freedom of a spatial region resides on its boundary and the maximum entropy of the spatial region should be its area in Planck units, which is also known as the holographic (entropy) bound. Since the proposal of these two entropy bounds, physicists have suggested various entropy bounds [4,8–11]. For a review, refer to Refs. [7,12].

Most entropy bounds, particularly those arising from black hole physics, are inherently rooted in static space-times. Consequently, they are prone to violations in dynamical systems [7], such as collapsing stars [8,13], the universe [6], and weakly gravitating spherical thermodynamic systems in asymptotically flat space [7]. To address the problem and preserve the holographic principle, Bousso introduced a covariant version of the entropy bound, i.e., the Bousso bound [8,9,14], which states that the entropy should be evaluated on null hypersurfaces. The Bousso bound can be applicable to arbitrarily curved space-time [15–18], so it is considered a potential imprint of quantum gravity.

In 1998, Fischler and Susskind proposed to apply the holographic bound from nearly flat space to both flat and non-flat universes [19]. They found that, for the entropy contained within a volume of coordinate size R_H, the practicability of the holographic bound depends on multiple parameters in cosmology, so it does not universally hold true. In fact, as early as 1989, Bekenstein had already discovered that his entropy bound is violated in the early universe when applied to the entropy contained within the particle horizon of a given observer. Surprisingly, this phenomenon could potentially help solve the cosmological singularity problem [20,21]. In a similar work [22], it was also found that, as the universe evolves, the entropy contained within a unit co-moving volume can exceed both the Bekenstein bound and the holographic bound. These studies indicate that applying directly entropy bounds in cosmological contexts faces significant challenges.

Building on the failed attempts discussed above, we propose constructing alternative spherical systems in order to implement entropy bounds in cosmological settings. Our study does not introduce any fundamentally new entropy bound; rather, it generalizes the construction of the spherical system originally considered by Fischler and Susskind, namely the spherical system enclosed by the particle horizon [19]. In this sense, the present work is best viewed as a primarily mathematical device for enforcing entropy bounds. We refrain at this stage from investigating whether the resulting spherical systems carry any additional physical significance, and instead focus exclusively on whether these constructions are largely self-consistent from a mathematical perspective. This work can serve as a complement to the Bousso bound [8,9,14]: instead of switching to null hypersurfaces, we keep space-like regions while allowing their radii to acquire an explicit time dependence

* Corresponding author.
E-mail addresses: yuhaocd@cqu.edu.cn (H. Yu), cqujinli1983@cqu.edu.cn (J. Li).

https://doi.org/10.1016/j.physletb.2026.140472
Received 9 February 2026; Received in revised form 13 April 2026; Accepted 19 April 2026
Available online 21 April 2026

determined by the global dynamical evolution of the cosmological spacetime. We will attempt to generalize entropy bounds to cosmological settings in space-like regions.

2. A question on entropy bounds in cosmology

When entropy bounds are applied to cosmology, we can refer to them as cosmological entropy bounds. If entropy bounds seek to determine the maximum entropy contained within a closed spherical system, then cosmological entropy bounds can be seen as extending the concept for a closed system to an open system (a spherical system with an evolving boundary). However, the majority of existing studies [7,19–23] demonstrate that cosmological entropy bounds can be readily violated in a general spherical system. An analysis of previous studies reveals that the primary reason for the violation of entropy bounds in cosmology lies in the choice of the spherical system. In these studies, the radii of spherical systems are typically the characteristic length scales in cosmology, such as the particle horizon of a given observer [20,21], the cosmological apparent horizon [24], the Hubble horizon [13,25,26], the radius of $R \sim H^{-1} \sim t$ [27], and so forth. For these spherical systems, their internal entropy can readily violate entropy bounds at certain cosmological epochs. To elucidate the phenomenon, we perform the following analysis. Consider a spherical system, within which the maximum entropy permitted by a cosmological entropy bound takes the form: $S_{max} = f(t)\, R^n$ with n being a constant. Here, R is the radius of the spherical system and $f(t)$ is a variable independent of R. The dimension of $f(t)$ depends on the value of n; for example, when $n = 3$, $f(t)$ is the typical entropy density. This assumption is motivated by the fact that $S_{max} = f(t)\, R^n$ can effectively encompass both the holographic bound ($n = 2$) and the Bekenstein bound ($n = 4$). In our analysis, we restrict consideration to extensive entropy formulations, thereby excluding non-extensive entropy measures such as Tsallis entropy [28–31] and Barrow entropy [32–34]. Thus, the entropy of matter inside the spherical system can be expressed as $S(t) = s(t)R^3$ with $s(t)$ being a time-dependent entropy density. Note that $S(t)$ is the form that the matter entropy must take, as dictated by the properties of the extensive entropy, while S_{max} is the upper bound on the matter entropy imposed by the chosen cosmological entropy bound. We now analyze whether $S(t) = s(t)R^3$ can ever exceed $S_{max} = f(t)\, R^n$ as the universe evolves. In an expanding universe, since the radius R evolves with the scale factor, it must be vanishingly small in the early universe but grow to an extremely large scale at late times. As a result, entropy bounds with $n < 3$ are readily violated in the late universe due to the increase of R. Similarly, entropy bounds with $n > 3$ are violated in the early universe provided R is sufficiently small. For the case of $n = 3$, to guarantee the validity of entropy bounds during the entire evolution of the universe, the coefficient $f(t)$ must remain strictly greater than the entropy density $s(t)$ at all cosmological epochs. However, the implementation of this condition is challenging, since the entropy density $s(t)$ is highly sensitive to both the evolution and components of the universe.

Based on the analysis, an intriguing question arises: Does there exist a special class of spherical systems in cosmology, within which the entropy of matter remains compliant with a given entropy bound throughout the evolution of the universe, irrespective of the universe's components? Next, we employ the holographic bound as a case to answer this question. Remarkably, when the components of the universe are properly specified, such a special class of spherical systems can usually be found for the holographic bound. For example, in a universe full of radiation, we consider a spherical system of radius R. The holographic principle conjectures that the entropy of matter inside a given spatial region can not exceed the area of its boundary surface:

$$S \le \frac{k_B A}{4 l_p^2}, \tag{1}$$

where A is the boundary area. Therefore, the evolution of R need satisfy

$$\frac{\frac{4}{3}\pi R^3}{a(t)^3 V_0} S_0 \le \frac{4\pi R^2 k_B}{4\, l_p^2}, \tag{2}$$

where $S_0 = s(t)a(t)^3 V_0$ is a conserved quantity representing the radiation entropy per unit co-moving volume. The variable $s(t)$ denotes the entropy density of radiation. To facilitate dimensional analysis, the unit volume V_0 is not explicitly set to 1. Accordingly, the special class of spherical systems in the radiation universe is precisely the spheres of radii $R = c_0\, a(t)^3$ with $0 < c_0 \le \frac{3}{4} \frac{k_B V_0}{S_0 l_p^2}$.

3. Holographic bound and a universe composed of black holes and a cosmological constant

Considering that the critical condition for the holographic bound in static space-times corresponds exactly to black holes, we conjecture that black hole entropy plays a vital role in defining a class of spherical systems that satisfies the above requirement. In light of the conjecture, we study a universe composed of black holes and a cosmological constant. The role of the cosmological constant is solely to control the evolution of the universe. If we can find such a special class of spherical systems in this universe, the result may apply to all cosmological models.

As a heuristic example, we begin by considering a static universe that is assumed to be homogeneous and isotropic. Here, "static" refers to a constant scale factor, but black holes are still allowed to merge. For simplicity, the contribution of black hole motion to the mass-energy density of the universe is neglected. Furthermore, energy losses associated with black hole mergers and Hawking radiation are also disregarded. According to the holographic bound, for a spherical system of radius R in this universe, the boundary area should satisfy

$$\frac{\pi R^2 k_B}{l_p^2} \ge \frac{4\pi G k_B}{\hbar c}\left(\sum_i M_i\right)^2 \ge \frac{4\pi G k_B}{\hbar c}\sum_i M_i^2, \tag{3}$$

where M_i represents the mass of the i-th black hole in the spherical system. The first inequality indicates that even if all black holes in the spherical system collapse into a single black hole, the entropy contained within the spherical system still obeys the holographic bound. Since black holes are evenly distributed in the universe, black hole mergers everywhere in the universe should also be equiprobable. Since the universe is static, the average mass-energy density of the universe is a constant, which can be expressed as

$$\bar{\rho} = \left(\frac{4\pi}{3}R^3\right)^{-1}\sum_i M_i. \tag{4}$$

We use the average mass-energy density $\bar{\rho}$ of the universe to characterize the mass-energy density of the spherical system. Hence, the radius R that satisfies Eq. (3) is given by

$$R \le R_c = c\left(\frac{8\pi G}{3}\bar{\rho}\right)^{-\frac{1}{2}}. \tag{5}$$

Here, R_c can be interpreted as a critical radius, which, consistent with expectations, is a constant for a fixed $\bar{\rho}$. The critical radius R_c in relation to the spherical system in the static universe is similar to the Schwarzschild radius of a black hole in relation to a spherical system encompassing the black hole in static space-times. It should be noted that R_c is the upper bound on the radius of the spherical system, but the Schwarzschild radius of a black hole is the lower bound on the radius of a system encompassing the black hole.

If the universe evolves, the critical radius R_c should evolve as well. Here, "evolve" implies not only the merging of black holes but also accounts for the time-dependent nature of the scale factor. To simplify the calculation, we assume that black holes have an average mass of $\bar{M}$ and use this value to characterize the mass-energy density of the spherical system. Since black holes are continually merging, they can be modeled as a form of dust with a time-dependent average mass $\bar{M}(t)$ and a particle number density $n(t)$. Although black holes are constantly merging, we can still regard them as dust with variable mass. In this case, the scale factor can be given as

$$a(t) = b\, t^{\frac{2}{3}}, \tag{6}$$

where b is a constant depending on the energy density of the universe. Because of the merger of black holes, the number of black holes inside a co-moving volume will decrease with the expansion of the universe. We can define an annihilation rate to quantify the frequency of black hole mergers:

$$\Gamma = \frac{\mathrm{d}N}{\mathrm{d}t}\frac{1}{N} = \frac{\mathrm{d}n}{\mathrm{d}t}\frac{1}{n} + 3H, \tag{7}$$

where H is the Hubble parameter and $n = N/V = N/(\frac{4\pi}{3}a^3)$ is the number density of black holes. We set $n(t_0) = n_0$, which represents the number density of black holes at time t_0. Since the number of black holes inside the co-moving volume is decreasing, we have $\Gamma < 0$. Then, the number of black holes inside the co-moving volume at time t can be written as

$$N(t) = N_0 \exp\left[\int_{t_0}^{t} \Gamma(t)\,\mathrm{d}t\right] = N_0 \exp\left[\int_{t_0}^{t}\left(\frac{\mathrm{d}n}{\mathrm{d}t}\frac{1}{n} + 3H\right)\mathrm{d}t\right], \tag{8}$$

where $N(t_0) = N_0 = \frac{4\pi}{3}a_0^3 n_0$ is the number of black holes inside the co-moving volume at time t_0. According to the previous assumptions, the total mass-energy (i.e., the total mass of black holes) inside the co-moving volume is a constant unaffected by the merger of black holes, so the average mass of black holes at time t can be expressed as

$$\bar{M}(t) = \frac{\frac{4\pi}{3}a_0^3 n_0 \bar{M}_0}{N(t)} = \frac{\bar{M}_0 N_0}{N(t)}, \tag{9}$$

where $\bar{M}_0$ is the average mass of black holes at time t_0.

Here, we can take the co-moving volume as an example to illustrate that, for a general evolving radius R (such as the scale factor), the holographic bound does not always hold true. The total entropy of black holes inside the co-moving volume at time t is given by

$$S(t) = 4\pi \bar{M}(t)^2 N(t) = 4\pi \frac{\bar{M}_0^2 N_0^2}{N(t)}, \tag{10}$$

where, for convenience, we have set $c = k_B = G = \hbar = 1$. With the holographic bound, we need it to satisfy

$$S(t) = 4\pi \frac{\bar{M}_0^2 N_0^2}{N(t)} \le \pi a^2. \tag{11}$$

Substituting Eqs. (6) and (8) into Eq. (11) yields

$$4\frac{\bar{M}_0^2 N_0^2}{b^2} \le N(t)\cdot t^{\frac{4}{3}} = N_0 \exp\left[\int_{t_0}^{t}\Gamma\,\mathrm{d}t\right]\cdot t^{\frac{4}{3}}. \tag{12}$$

Apparently, if we need the above inequality to hold true at any time, Γ can not be arbitrary. Therefore, for the co-moving volume, it is difficult to guarantee that the holographic bound holds true throughout the evolution of the universe. Next, we try to define a typical R, for which the holographic bound is valid all the time regardless of the value of Γ.

For a general system of radius R_c, the number of black holes inside the system is

$$N_c(t) = \frac{R_c^3}{a^3}N(t), \tag{13}$$

where $N(t)$ is the number of black holes inside the co-moving volume. The holographic bound for the system is given as

$$S_c(t) = 4\pi \bar{M}(t)^2\, N_c(t) \le \pi R_c^2. \tag{14}$$

Therefore, R_c needs to satisfy

$$R_c \le \frac{a^3}{4\bar{M}_0^2 N_0^2} N_0 \exp\left[\int_{t_0}^{t}\Gamma\,\mathrm{d}t\right]. \tag{15}$$

Since these black holes are regarded as dust, the Friedmann equation can be given as $H^2 = \frac{8}{3}\pi\bar{\rho} = \frac{8}{3}\pi\bar{\rho}_0\, a^{-3}$, where $\bar{\rho}_0 = \bar{M}_0 N_0/(\frac{4}{3}\pi a_0^3)$ is the average mass-energy density of the universe at time t_0. Substituting it into the above inequality, we have

$$R_c \le \frac{\frac{8}{3}\pi\bar{\rho}_0\, H^{-2}}{4\bar{M}_0^2 N_0^2} N_0 \exp\left[\int_{t_0}^{t}\Gamma\,\mathrm{d}t\right] = \frac{1}{2\bar{M}_0 H^2 a_0^3 N_0} N(t). \tag{16}$$

If there exists a non-vanishing minimum function for the r.h.s. of the inequality and it is independent of Γ, then the non-vanishing minimum function can be regarded as a critical radius R_c of the spherical system. Thus, we need determine the lower bound on $N(t)$. If the minimum value of $N(t)$ vanishes, the existence of R_c becomes highly improbable. Otherwise, the spherical system must have a non-vanishing critical radius R_c.

In a universe with an initial singularity, as $a(t)$ approaches zero, the behavior of $N(t)$ is ambiguous. This suggests that the presence of the initial singularity may hinder our definition of the critical radius R_c. In this case, we require that the universe composed of black holes and a cosmological constant did not originate from an initial singularity. At the beginning of the universe, we have $N(t=0) > 0$. As the universe expands and black holes undergo mergers, $N(t)$ monotonically decreases. We postulate that there exists a critical moment during cosmic expansion at which black hole mergers effectively cease. The corresponding black hole number density is defined as the critical black hole number density. At this critical moment, the total entropy of the universe attains its maximum value and thereafter remains constant. A non-vanishing critical black hole number density guarantees a non-vanishing minimum value of $N(t)$. In general, the critical black hole number density depends on multiple factors, including the cosmological expansion rate, the relative velocity of black holes, the average black hole mass, and related quantities. Therefore, even within this simplified cosmological framework, deriving the critical black hole number density is a nontrivial task. Here, we present a heuristic scheme to demonstrate the existence of the non-vanishing critical black hole number density. Specifically, we can consider an extreme scenario and prove that the critical number density is nonzero under this scenario. On the other hand, we prove that, for all other cases, the non-vanishing critical black hole number density appears earlier than that obtained in the extreme scenario. Therefore, if the critical black hole number density is nonzero in the extreme scenario, then, the critical black hole number density should always be nonzero in any case.

Now consider an extreme scenario in which all black holes in the universe move at their maximum possible velocities. The constraint imposed by the speed of light on the relative velocity of black holes allows us to derive a "minimum" critical black hole number density that is independent of both the relative velocity and the average black hole mass. The rationale for using the term "minimum" is as follows. It is well known that the expansion of the universe can prevent two objects from ever encountering one another, even if they are moving toward each other at speeds close to that of light. Therefore, as the relative velocity of black holes approaches the speed of light, the corresponding critical black hole number density must be minimum value among all admissible relative velocities. In other words, for all other cases, the critical moment must appear earlier than the one corresponding to the scenario in which black holes propagate at the speed of light. In this extreme limit, the black hole mass would formally tend to infinity. However, because the relative velocity has already reached the speed of light, variations in the black hole mass and other related quantities do not further influence the critical black hole number density. At this point, whether black holes can effectively encounter each other is the dominant factor governing whether mergers occur. Therefore, the minimum critical black hole number density is determined solely by the expansion rate of the universe. If the minimum critical black hole number density is non-vanishing, it can be used naturally to give a non-vanishing minimum value of $N(t)$. We take an arbitrary black hole as an observer located at the origin. For the black hole closest to the observer, the co-moving coordinate distance between them can be approximated by the average co-moving coordinate distance $\bar{r}$ among black holes, which increases over time due to black hole mergers. The recessional velocity of the closest black hole is assumed to be approximately linear, i.e., $v_{\bar{r}} = H\,\bar{D} + v_p$, where $\bar{D} = a(t)\,\bar{r}$ is the average proper distance and v_p is the peculiar velocity of the closest black hole. If $v_{\bar{r}} \geq c$, then black holes can no longer encounter each other. In this case, we have $H\,\bar{D} \geq c - v_p$. In the limiting case where $v_p \to -c$, the lower bound on the average proper distance becomes $\bar{D} = \frac{2c}{H}$. In the following, we explore the evolution of $N(t)$ in relation to the expansion rate of the universe.

We begin by considering that the universe is decelerating, allowing us to set the cosmological constant to zero. In this case, the scale factor evolves as $a(t) = b\,t^{\frac{2}{3}}$ with b being a constant. The critical co-moving coordinate distance between the observer and the closest black hole is denoted by $\bar{r}_c = \frac{2c}{a(t)H} = \frac{3\,c}{b}t_c^{1/3}$, where t_c is a critical time to be determined. When $\bar{r} \geq \bar{r}_c$, black holes are no longer able to merge. The critical black hole number density at this moment is $n(t_c) = 1/(\frac{4\pi}{3}\bar{D}^3) = (36\pi c^3\,t_c^3)^{-1}$. Hence, the number of black holes per unit co-moving volume at the critical time is $N(t_c) = \frac{3\,a^3 V_0}{4\,\pi\bar{D}^3} = \frac{3\,V_0}{4\,\pi}\bar{r}_c^{-3} = \frac{b^3 V_0}{36\,\pi c^3 t_c}$, which is the lower bound on $N(t)$. Next, we analyze the value of $\bar{r}_c$. If $\bar{r}_c$ is a finite constant, then the lower bound on $N(t)$ is non-vanishing. Recalling $v_{\bar{r}} = H\,\bar{D} + v_p$, the recessional velocity of the closest black hole can be expressed as $v_{\bar{r}} = \frac{2}{3}\,b\,t^{-\frac{1}{3}}\,\bar{r}_c + v_p$. Assuming $\bar{r}_c$ is finite, it follows that $v_{\bar{r}}$ decreases over time. As $v_{\bar{r}}$ decreases, black holes can continue to merge, which contradicts the assumption that no mergers occur when $\bar{r} \geq \bar{r}_c$. Therefore, no finite $\bar{r}_c$ exists beyond which black holes are guaranteed to remain isolated and the lower bound on $N(t)$ must vanish as time progresses. This suggests that the spherical system may not exist in a decelerating universe composed of black holes.

Now, let us assume that the universe is undergoing accelerated expansion. When the cosmological constant is non-vanishing, the scale factor turns into

$$a(t) = e^{-\sqrt{\frac{\Lambda}{3}}c\,t}\left(\frac{1}{2}e^{\sqrt{3\Lambda}\,c\,t} - \frac{4\pi\,G\,\bar{\rho}_0}{c^2\Lambda}\right)^{\frac{2}{3}}. \tag{17}$$

To ensure the continued acceleration of the expansion, we require $\frac{8\pi\,G\,\bar{\rho}_0}{\Lambda\,c^2} \leq 2 - \sqrt{3}$. Our goal remains to determine a non-vanishing lower bound on $N(t)$. As in the previous case, the average proper distance $\bar{D}$ need satisfy the condition $\bar{D}\,H \geq c - v_p$. Using the scale factor (17), the critical average co-moving coordinate distance between black holes is given by

$$\bar{r}_c = \frac{2^{5/3}\sqrt{3}\,c^2\sqrt{\Lambda}\,e^{\sqrt{\frac{\Lambda}{3}}c\,t_c}\left(e^{\sqrt{3\Lambda}\,c\,t_c} - \frac{8\pi\,G\,\bar{\rho}_0}{\Lambda\,c^2}\right)^{\frac{1}{3}}}{c^2\Lambda\,e^{\sqrt{3\Lambda}\,c\,t_c} + 8\pi\,G\,\bar{\rho}_0}, \tag{18}$$

where t_c is still a critical time to be determined. Given the constraint $\frac{8\pi\,G\,\bar{\rho}_0}{\Lambda\,c^2} \leq 2 - \sqrt{3}$, it can be shown that $\frac{\mathrm{d}\bar{r}_c}{\mathrm{d}t_c} \leq 0$ for all $t_c \geq 0$. Then, the maximum value of $\bar{r}_c$ occurs at $\bar{r}_c(t_c = 0)$, and the lower bound on $N(t)$ is given by $N(t_c = 0) > 0$. Therefore, the radius R of the spherical system need satisfy

$$R \leq R_c = \frac{\sqrt{3}(c^2\Lambda + 8\pi\,G\,\bar{\rho}_0)^3 V_0^2}{2048\,G^2\sqrt{\Lambda}\,\pi^2\bar{M}_0^2 N_0^2(c^2\Lambda - 8\pi\,G\,\bar{\rho}_0)}a^3 = f_0 a^3, \tag{19}$$

where f_0 is a non-vanishing constant with the dimension of length. Hence, in an accelerating universe composed of black holes and a cosmological constant, there exists a special class of spherical systems (with radius $R \leq R_c = f_0 a^3$), within which the entropy of black holes consistently adheres to the holographic bound.

In fact, it can be demonstrated that the fundamental condition for the feasibility of the definition (19) is the existence of an upper bound on the entropy per unit co-moving volume, regardless of the expansion rate of the universe and the presence of an initial singularity. For a spherical system defined by (19), the entropy contained within the system satisfies

$$\frac{\frac{4\pi}{3}(f_0 a^3)^3}{a^3 V_0}S(t) \leq \frac{\frac{4\pi}{3}(f_0 a^3)^3}{a^3 V_0}S_{max} \leq \frac{k_B (f_0 a^3)^2}{l_p^2}, \tag{20}$$

where $S(t)$ is the entropy per unit co-moving volume at time t, and S_{mas} is the maximum value of $S(t)$ for the entire evolution of the universe. Then, we have

$$0 < f_0 \leq \frac{3\,k_B V_0}{16\,\pi\,l_p^2\,S_{max}}. \tag{21}$$

Based on the above calculations and analyses, we conclude that in any cosmological model in which the entropy per unit co-moving volume admits an upper bound, there exists a critical radius $R_c = f_0 a^3$ with f_0 being a non-vanishing constant. A spherical system of radius $R \leq R_c$ can accommodate entropy in a way that is consistent with the holographic bound. However, we have proved that the construction $R_c = f_0 a^3$ is not universally applicable in cosmological models. In a decelerating universe that originates from an initial singularity, the entropy per unit co-moving volume may diverge in both the early and late stages. As a result, the definition (19) does not hold in general, and it is typically restricted to accelerating cosmologies without an initial singularity. For realistic cosmological scenarios, such as the ΛCDM model, the definition (19) can be applicable provided that the model does not develop an initial singularity and that the matter entropy per unit co-moving volume remains bounded from above throughout the post-inflationary decelerating epoch.

4. Holographic bound and a general cosmological model

In the previous part, we have defined a special class of spherical systems in cosmology, within which the entropy of matter constantly adheres to the holographic bound, based on the assumption that the entropy per unit co-moving volume is bounded from above in the universe. In this part, we further explore the holographic bound in a general cosmological model and aim to establish a universal definition from the perspective of the entropy per unit co-moving volume.

We start with an arbitrary cosmological model, in which the entropy of matter per unit co-moving volume is described by a time-dependent function $\sigma(t)$. The holographic bound requires the entropy of matter contained within a spherical system of radius R to satisfy

$$S = \frac{\frac{4}{3}\pi R^3}{a(t)^3 V_0}\cdot\sigma(t) \leq \frac{k_B\,\pi R^2}{l_p^2}. \tag{22}$$

Given a mass-energy density $\rho(a)$, the corresponding entropy function $\sigma(t)$ may depend on multiple factors, and whether it possesses a universal upper bound is unknown. However, for any $\rho(a)$, we can define an auxiliary entropy function $\sigma_x(t)$ such that $\sigma_x(t) \geq \sigma(t)$ at all times, with an additional constraint that $\sigma_x(t)$ depends solely on the scale factor. If such a function $\sigma_x(t)$ exists, it can be used to derive a general form of the critical radius R_c applicable to any cosmological model. In the following, we present a method for constructing such a function $\sigma_x(t)$.

When all the matter contained within a unit co-moving volume is treated as a black hole of the same mass-energy $m(a) = a^3 V_0 \rho(a)$, the entropy of this black hole can be expressed as

$$\sigma_{BH}(t) = \frac{4\pi G k_B}{\hbar c}a^6 V_0^2 \rho(a)^2. \tag{23}$$

If $\sigma_{BH}(t)$ is the maximum entropy within unit co-moving volumes that share the same mass-energy, and if we can define a class of spherical systems based on $\sigma_{BH}(t)$ such that the entropy contained within these systems adheres to the holographic bound, then this definition should be applicable to other universes as well. According to the Bekenstein-Hawking entropy formula and the generalized second law of thermodynamics, the inequality $\sigma_{BH}(t) \geq \sigma(t)$ "appears to" hold for any form of matter. Here we temporarily assume that this condition holds for all matter, and later we will provide a counterexample.

To analyze the evolution of $\sigma_{BH}(t)$, we employ the Friedmann equation. Assuming the universe contains only a single component of matter, characterized by an energy density $\rho(a)$, one can use the Friedmann equation to derive that $\sigma_{BH}(t) \sim H^4 a^6$. Therefore, the radii of these special spherical systems need satisfy

$$R \le R_c = \frac{k_B \pi}{l_p^2} \frac{\hbar c}{4\pi G k_B} \frac{16\pi G^2}{3V_0} H^{-4} a^{-3} = \frac{4\pi}{3V_0} \frac{c^4}{H^4 a^3}. \tag{24}$$

The definition of R_c is independent of the specific components and expansion of the universe if $\sigma_{BH}(t) \ge \sigma(t)$ holds for all matter. As such, it provides a definition for these spherical systems ensuring that the entropy contained within them adheres to the holographic bound. Strictly speaking, this result relies only on two assumptions: 1) The evolution of the universe is governed by the Friedmann equation; 2) Among unit co-moving volumes with the same mass-energy, when the mass-energy is converted into a black hole, the entropy reaches its maximum. We can regard $\frac{4\pi}{3V_0}\frac{c^4}{H^4 a^3}$ as a characteristic length scale in the universe. It implies that as long as the radius of any spherical system in the universe does not exceed this scale, the entropy contained within it can remain compliant with the holographic bound. Defining $R = g_0 H^{-4} a^{-3}$ with $g_0 \le \frac{4\pi}{3V_0} c^4$, and differentiating it with respect to time, we have $\frac{\mathrm{d}R}{\mathrm{d}t} = g_0 \frac{\dot a^2 - 4a\ddot a}{\dot a^5}$. If the universe is undergoing decelerated expansion, R increases over time. But, in the case of accelerated expansion and $\ddot a > \frac{\dot a^2}{4a}$, R decreases over time.

It is important to emphasize that the condition $\sigma_{BH}(t) > \sigma(t)$ does not hold universally across cosmological models. For instance, consider a universe filled solely with photons. In this case, the entropy per unit co-moving volume, $\sigma_p(t)$, remains constant, whereas the internal energy per unit co-moving volume decreases with the scale factor. For sufficiently large scale factors, we necessarily obtain $\sigma_{BH}(t) < \sigma_p(t)$. As a consequence, the definition (24) is not universal, and it still requires the special condition $\sigma_{BH}(t) \ge \sigma(t)$ to hold for validity.

The photon-dominated universe can violate the condition $\sigma_{BH}(t) > \sigma(t)$ because, for low-energy photons, spontaneous collapse into black holes may correspond to a decrease in entropy [1] Therefore, from a more physical perspective, it would be appropriate for us to claim that the definition (24) applies only to particular cosmological models in which matter can evolve to form black holes spontaneously, without conflicting with the second law of thermodynamics. In realistic cosmological scenarios, such as the ΛCDM model, the definition (24) may be applicable because photons cease to be the dominant component at sufficiently large scale factors. However, given that the entropies of dark matter and dark energy remain unclear at present, a definitive conclusion is not yet within reach.

5. Discussion

We have identified two specific definitions of spherical systems in cosmology that guarantee the matter entropy enclosed within them satisfies the holographic bound throughout the evolution of the universe. However, both definitions suffer from limitations, being valid only under restricted cosmological conditions, as demonstrated by explicit counterexamples. Specifically, the definition (19) holds exclusively for cosmological models in which the entropy per unit co-moving volume is bounded from above, whereas the definition (24) is violated in a radiation-dominated universe. Thus, it appears impossible to construct a universal class of spherical systems in cosmology in which the matter entropy always satisfies the holographic bound. We stress once more that the spherical systems considered in this work do not correspond to any known cosmological horizon or physically meaningful scale; rather, they are constructed solely to enforce compliance with the holographic bound. Should a physical interpretation be sought, one may note that, within the specific cosmological models (an accelerating universe without an initial singularity or a universe not composed of photons), potential violations of the holographic bound are evaded only inside such spherical systems.

For the definition (19), the requirement that the entropy per unit co-moving volume has an upper bound, is almost certain to fail near the singularity of the universe. If a cosmological model satisfies this requirement, compliance with other entropy bounds can be achieved by appropriately adjusting the exponent of the scale factor in the definition (19). For instance, with regard to the Bekenstein bound, the radii of spherical systems are given by $R \le R_c = f_0 a$ with f_0 being a non-vanishing constant. In this case, counterexamples to the holographic bound may also arise for other entropy bounds, not due to a flaw in the entropy bounds themselves, but rather because of the similar definitions of spherical systems.

The fundamental reason why definition (24) fails in a radiation universe is that converting the mass-energy of photons within a system into a black hole may reduce the total entropy of the system, which does not occur in other known ordinary matter. This counterexample suggests that, in order to properly apply the holographic bound (as well as other entropy bounds) in cosmology, we must first clarify the maximal entropy state among all systems with the same mass-energy. If this problem can be resolved, it may pave the way toward identifying the most universal definition of such a special class of spherical systems. It now appears that the common assumption that black holes always represent the maximal entropy state is not entirely accurate. Finally, we emphasize that although we have only studied the methods and results related to the extension of the holographic bound in cosmology, the methods are applicable to most entropy bounds and the results [see Eqs. (19) and (24)] shed new light on the question addressed in Section 2. Therefore, we believe that our methods and results possess broad generality and conceptual novelty. In future work, we will further investigate realistic cosmological settings and generalized entropy in cosmology [35–38], which is motivated by the fact that entropy bounds can assume markedly different forms and exhibit distinct ranges of validity depending on the specific entropy. Such considerations will help place the present research within a broader and more modern theoretical context.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (Grants Nos. 11873001, 12047564, and 12247142), and the China Postdoctoral Science Foundation (Grant No. 2024M753825).

References

[1] J.D. Bekenstein, Phys. Rev. D 23 (1981) 287.
[2] G. 't Hooft, in: Conf. Proc. C 930308 (1993) 284–296.
[3] C.R. Stephens, G. 't Hooft, B.F. Whiting, Class. Quant. Grav. 11 (1994) 621–648.
[4] L. Susskind, J. Math. Phys. 36 (1995) 6377–6396.
[5] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253–291.
[6] L. Susskind, E. Witten, 1998. arXiv preprint arXiv:9805114 [hep-th].
[7] R. Bousso, Rev. Mod. Phys. 74 (2002) 825–874.
[8] R. Bousso, JHEP 07 (1999) 004.
[9] E.E. Flanagan, D. Marolf, R.M. Wald, Phys. Rev. D 62 (2000) 084035.
[10] R. Brustein, G. Veneziano, Phys. Rev. Lett. 84 (2000) 5695–5698.
[11] E.P. Verlinde, 2000, arXiv preprint arXiv:0008140 [hep-th].
[12] R.M. Wald, Living Rev. Relativ. 4 (2001).
[13] R. Easther, D.A. Lowe, Phys. Rev. Lett. 82 (1999) 4967–4970.
[14] A. Strominger, D.M. Thompson, Phys. Rev. D 70 (2004) 044007.

[1] Although we can not yet provide a first-principles explanation for why this phenomenon arises, it may motivate further study on the entropy properties of gravitational systems and the second law of thermodynamics in the future.

[15] R. Bousso, E.E. Flanagan, D. Marolf, Phys. Rev. D 68 (2003) 064001.
[16] S. Gao, J.P.S. Lemos, JHEP 04 (2004).
[17] A. Pesci, Class. Quant. Grav. 24 (2007) 6219–6226.
[18] A. Pesci, Class. Quant. Grav. 25 (2008) 125005.
[19] W. Fischler, L. Susskind, 1998. arXiv preprint arXiv:9806039 [hep-th].
[20] J.D. Bekenstein, Int. J. Theor. Phys. 28 (1989) 967–981.
[21] J.R. Powell, R. Lopez-Mobilia, R.A. Matzner, Entropy 22 (2020) 795.
[22] H. Yu, J. Li, Eur. Phys. J. C 83 (12) (2023) 1136.
[23] R. Brustein, Lect. Notes Phys. 737 (2008) 619–659.
[24] D. Bak, S.J. Rey, Class. Quant. Grav. 17 (2000) L83.
[25] G. Veneziano, Phys. Lett. B 454 (1999) 22–26.
[26] R. Brustein, Phys. Rev. Lett. 84 (2000) 2072.
[27] N. Kaloper, A.D. Linde, Phys. Rev. D 60 (1999) 103509.
[28] C. Tsallis, J. Statist. Phys. 52 (1988) 479–487.
[29] G. Wilk, Z. Wlodarczyk, Phys. Rev. Lett. 84 (2000) 2770.
[30] C. Tsallis, L.J.L. Cirto, Eur. Phys. J. C 73 (2013) 2487.
[31] E.N. Saridakis, K. Bamba, R. Myrzakulov, F.K. Anagnostopoulos, JCAP 12 (2018) 012.
[32] J.D. Barrow, Phys. Lett. B 808 (2020) 135643. arXiv preprint arXiv:2004.09444 [gr-qc], https://doi.org/10.1016/j.physletb.2020.135643
[33] E.N. Saridakis, Phys. Rev. D 102 (12) (2020) 123525.
[34] S. Nojiri, S.D. Odintsov, T. Paul, Phys. Lett. B 825 (2022) 136844. arXiv:2112.10159 [gr-qc], https://doi.org/10.1016/j.physletb.2021.136844
[35] S. Nojiri, S.D. Odintsov, T. Paul, Phys. Lett. B 831 (2022) 137189.
[36] I. Brevik, A.V. Timoshkin, Int. J. Geom. Meth. Mod. Phys. 21 (11) (2024) 2450181.
[37] M. Leizerovich, S.J. Landau, G.G. Luciano, A. Papatriantafyllou, E.N. Saridakis (2026). arXiv eprint arXiv:2603.03568 [astro-ph.CO].
[38] G.G. Luciano, E.N. Saridakis, 2026. arXiv preprint arXiv:2602.20004 [gr-qc].