

Fixed points of the renormalization group running of quark and fermion mixing matrices in the standard model and beyond

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The renormalization group running of fermion mixing matrices in the Standard Model and beyond is studied. For the massless one-loop running with three generations, six fixed points are found. Their associated anomalous dimension matrices are calculated and the nature of each fixed point, whether attractive, repulsive, or mixed, is determined. An argument is given that the fixed points found at one loop must remain fixed points to all orders in perturbation theory and even nonperturbatively, as they are associated with certain differential geometric properties of vector fields on the space of mixing matrices. With N_g dark or sterile neutrinos there are at least $N_g!$ fixed points of the fermion mixing matrix.

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I. INTRODUCTION

A central concept in understanding the Standard Model of particle physics, and relating experimental results to the 19 fundamental parameters in the Standard Model Lagrangian, is the running of the parameters as a function of energy: the β functions. Understanding the behavior of β functions is also essential in any attempt to go beyond the Standard Model. The running of the gauge couplings and the discovery of asymptotic freedom [1,2] was a crucial step in the development of the Standard Model and the running of the Higgs quartic coupling, and the top quark Yukawa coupling is important for the stability of the Higgs sector at high energies [3,4].

The running of the parameters in the Cabibbo-Kobayashi-Maskawa (CKM) matrix [5] is not so well known perhaps for two reasons: the measured magnitude of the quark Yukawa couplings is too small for the running of the CKM parameters to have any physical relevance, and, even at one-loop, the β functions are not simple. The running of the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [6,7] is more relevant beyond the Standard Model, and the mixing matrices in both the quark and the lepton sectors can be treated in parallel with the same techniques: for the Standard Model with the addition of three gauge singlet, Weyl, right-handed neutrinos the analysis is identical to that of the quark sector.

The β functions for the CKM parameters were investigated in [8,9], in the limit where the top Yukawa coupling y_t dominates, and it was observed there is an infrared fixed point. The full one-loop β functions for the CKM matrix were calculated in [10]. The one-loop β functions for mixing matrices were further analyzed in [11–17] but the full expressions are somewhat complicated, and they are significantly simpler in the small angle approximation: approximate one-loop β functions, with one dominant Yukawa coupling and/or at least one small mixing angle, were analyzed in [18,19] (for higher loops, see Refs. [20–25]). However these approximations obscure an analytic pattern in the β functions which shall be explored in the present paper.

We shall therefore study the analytic form of the full one-loop massless renormalization group (RG) equations for the mixing matrix: fixed points are determined and the matrix of anomalous dimensions is calculated at each of the fixed points (full two-loop β functions are given in [26,27], but we leave this for later analysis). The β functions are of course scheme dependent, but the existence of fixed points, and the eigenvalues of the matrix of anomalous dimensions at the fixed points, are scheme independent. There are six fixed points¹ and the associated mixing matrices form a unitary representation of a group of order 6, the group of permutations of three objects, S_3 . The Jarlskog invariant [28] vanishes at all of the fixed points.

The parameters in the mixing matrices consist of three angles and one phase, parametrizing a space that is topologically the double coset

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¹After this work was complete we became aware of Ref. [25] where these six fixed points were also found.

$$U(1) \times U(1) \backslash SU(3)/U(1) \times U(1),$$

where $U(1) \times U(1) \approx T$ is the Cartan torus of $SU(3)$ (generated by λ_3 and λ_8 in the Gell-Mann representation). The right-coset $SU(3)/U(1) \times U(1)$ is the flag manifold F_3 , a compact complex manifold that admits a metric with isometry group $SU(3)$. The left action of T on F_3 has six fixed points, and these are precisely the six fixed points of the one-loop RG running.

If the RG running is lifted to F_3 , which can be done in a well-defined manner, and it is assumed that the left action of T on F_3 commutes with this running, then a proof is given² that fixed points of the left action of T on F_3 must necessarily be fixed points of the RG. With this assumption the fixed points that are found at one-loop must remain fixed points at all orders in perturbation theory, and even nonperturbatively,

The one-loop running is reviewed in Sec. II and a technique for extracting the running of the mixing parameters from the running of the Yukawa matrices is presented in Sec. III. As a simple example the special case of two generations, the Cabibbo angle, is presented in detail in Sec. IV before the three generation one-loop β functions for the mixing matrices are presented in Sec. V. The eigenvalues of the matrix of anomalous dimensions at each of the fixed points are determined in Sec. VI, and presented in detail in Appendix C. A proof that fixed points of the left action of T on F_3 must be fixed points of the RG is given in Sec. VII. The significance of the results and possible future developments are discussed in Sec. VIII. Full expression for the one-loop β functions are given in Eqs. (21), (22), and Appendix B.

II. YUKAWA COUPLINGS

The Yukawa couplings in the Standard Model are

$$\mathcal{L}_{\text{Yukawa}} = - \sum_{\bar{a}, b=1}^3 (Y_{\bar{a}b} \bar{\Psi}_{\bar{a}}^a \Phi_c \Psi_R^b + Y'_{\bar{a}b} \bar{\Psi}_{\bar{a}}^a \Phi \Psi_R^b) - \text{H.c.}, \quad (1)$$

where $\Phi = (\phi_{\bar{\phi}}^+)$ is the Higgs doublet; $\Phi_c = -i\sigma_2 \Phi^*$ the conjugate Higgs; Ψ_L^a are left-handed $SU(2)$ doublets; Ψ_R^a and Ψ_R^a right-handed singlets. The Yukawa couplings $Y_{\bar{a}b}$ and $Y'_{\bar{a}b}$ are 3×3 complex matrices, with $a, b = 1, 2, 3$ labeling generations.

In the quark sector

$$\Psi_L = \begin{pmatrix} u_L & c_L & t_L \\ d_L & s_L & b_L \end{pmatrix}, \quad \Psi_R = (u_R, c_R, t_R)$$

and $\Psi'_R = (d_R, s_R, b_R)$,

²I thank Charles Nash for pointing out the significance of the fact that these actions commute for the fixed points.

but the ensuing analysis is equally applicable to the Standard Model with the leptonic sector extended by three right-handed gauge-singlet Weyl neutrinos, in which case³

$$\Psi_L = \begin{pmatrix} \nu_{e,L} & \nu_{\mu,L} & \nu_{\tau,L} \\ e_L & \mu_L & \tau_L \end{pmatrix}, \quad \Psi'_R = (e_R, \mu_R, \tau_R),$$

and $\Psi'_R = (\nu_{1,R}, \nu_{2,R}, \nu_{3,R})$.

There are 18 parameters in $Y_{\bar{a}b}$ and $Y'_{\bar{a}b}$ but, as is well known, these are not all physical. The Weyl fermions Ψ_L and Ψ_R can be rotated by two different $U(3)$ transformations in generation space, to bring $Y_{\bar{a}b}$ to real diagonal form, with just three Yukawa couplings; a further eight parameters can be removed from $Y'_{\bar{a}b}$ with a further $SU(3)$ transformation on Ψ'_R , leaving 10 parameters in $Y'_{\bar{a}b}$. But we are still free to perform individual $U(1)$ phases transformations on the three generations in Ψ_L (compensating with phases transformation on Ψ_R to keep $Y_{\bar{a}b}$ real) to remove another three parameters from $Y'_{\bar{a}b}$, leaving seven parameters in $Y'_{\bar{a}b}$: three real Yukawa couplings and the four mixing parameters of the CKM (or PMNS) matrix.

The one-loop running of the Yukawa matrices Y and Y' can be determined by standard techniques; the Feynman diagrams that contribute are shown in Appendix A. Using dimensional regularization, with $t = \ln \mu$, the massless running is [10],

$$\frac{dY}{dt} = \frac{1}{16\pi^2} \left[\frac{3}{2} (YY^\dagger - Y'Y'^\dagger) + 4\pi(\Pi - \alpha)\mathbf{1} \right] Y, \quad (2)$$

$$\frac{dY'}{dt} = \frac{1}{16\pi^2} \left[\frac{3}{2} (Y'Y'^\dagger - YY^\dagger) + 4\pi(\Pi - \alpha')\mathbf{1} \right] Y', \quad (3)$$

where Π arises from summing over fermion loops in the Higgs leg of the Yukawa vertex and α and α' are quadratic combinations of the gauge couplings α_3 , α_2 , and α_1 (their explicit form will not be relevant for the following analysis).⁴

It is convenient to express the running in terms of Hermitian matrices

$$Z = \frac{1}{4\pi} YY^\dagger \quad \text{and} \quad Z' = \frac{1}{4\pi} Y'Y'^\dagger,$$

³Majorana neutrinos are not considered here: if the neutrinos are Majorana the analysis would be more complicated as there are more mixing parameters and extra terms involving Majorana masses [29].

⁴Explicitly, $\Pi = \frac{1}{4\pi} (3\text{Tr}(Y_q Y_q^\dagger + Y'_q Y_q'^\dagger) + \text{Tr}(Y'_l Y_l'^\dagger))$, where q stands for quarks and l for leptons (for ν SM there is a fourth term: $\frac{1}{4\pi} \text{Tr}(Y_l Y_l'^\dagger)$). For quarks $\alpha = 8\alpha_3 + \frac{9}{4}\alpha_2 + \frac{17}{12}\alpha_1$ and $\alpha' = 8\alpha_3 + \frac{9}{4}\alpha_2 + \frac{5}{12}\alpha_1$, while for leptons $\alpha = \frac{9}{4}\alpha_2 + \frac{3}{4}\alpha_1$ and $\alpha' = \frac{9}{4}\alpha_2 + \frac{15}{4}\alpha_1$, $\alpha_i = \frac{g_i^2}{4\pi}$ being the gauge couplings (α_1 is hypercharge).

in terms of which

$$\Lambda_a = \frac{y_a^2}{4\pi}, \quad \Lambda'_a = \frac{y'_a{}^2}{4\pi}.$$

$$\frac{dZ}{dt} = \frac{1}{4\pi} \left\{ 3Z^2 - \frac{3}{2}(ZZ' + Z'Z) + 2(\Pi - \alpha)Z \right\}, \quad (4)$$

If Z and Z' do not commute, then $U \neq U'$ and

$$\frac{dZ'}{dt} = \frac{1}{4\pi} \left\{ 3Z'^2 - \frac{3}{2}(ZZ' + Z'Z) + 2(\Pi - \alpha')Z' \right\}. \quad (5)$$

$$V = U^\dagger U'$$

The Hermitian matrices Z and Z' can be diagonalized with U and $U' \in SU(3)$:

$$\Lambda = U^\dagger Z U, \quad \Lambda' = U'^\dagger Z' U', \quad (6)$$

where the diagonal components of Λ and Λ' are related to the usual Yukawa couplings by

is the mixing matrix, the CKM matrix for quarks and the PMNS matrix for leptons.

If the right-handed neutrinos are Weyl, and not Majorana, the $U(1)$ phases of the leptons, as well as the quarks, are unobservable and, for both quarks and leptons, the space of physical parameters in the mixing matrix V is the double coset $U(1) \times U(1) \backslash F_3$. This is a four-dimensional space so there are four physical parameters

in V .⁵ A general $SU(3)$ matrix can be parametrized in terms of the Gell-Mann matrices as

$$V = e^{i(\psi_6\lambda_6 + \psi_7\lambda_7)} e^{i(\psi_4\lambda_4 + \psi_5\lambda_5)} e^{i(\psi_1\lambda_1 + \psi_2\lambda_2)} e^{i(\psi_3\lambda_3 + \psi_8\lambda_8)}.$$

The phases ψ_3 and ψ_8 can then be eliminated by the right action of $U(1) \times U(1)$ to render $V \in SU(3)/U(1) \times U(1)$. Changing variables to

$$\begin{aligned} \psi_1 &= -\theta_3 \sin \phi_3, & \psi_2 &= \theta_3 \cos \phi_3, \\ \psi_4 &= \theta_2 \sin \phi_2, & \psi_5 &= \theta_2 \cos \phi_2, \\ \psi_6 &= -\theta_1 \sin \phi_1, & \psi_7 &= \theta_1 \cos \phi_1 \end{aligned}$$

this is⁶

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & s_1 e^{i\phi_1} \\ 0 & -s_1 e^{-i\phi_1} & c_1 \end{pmatrix} \begin{pmatrix} c_2 & 0 & s_2 e^{-i\phi_2} \\ 0 & 1 & 0 \\ -s_2 e^{i\phi_2} & 0 & c_2 \end{pmatrix} \begin{pmatrix} c_3 & s_3 e^{i\phi_3} & 0 \\ -s_3 e^{-i\phi_3} & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (7)$$

where $s_i = \sin \theta_i$ and $c_i = \cos \theta_i$. The space of all such V is the complex manifold $SU(3)/U(1) \times U(1) \approx F_3$. This form of V is preserved by the adjoint action of $h \in U(1) \times U(1)$,

$$hVh^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & s_1 e^{i\tilde{\phi}_1} \\ 0 & -s_1 e^{-i\tilde{\phi}_1} & c_1 \end{pmatrix} \begin{pmatrix} c_2 & 0 & s_2 e^{-i\tilde{\phi}_2} \\ 0 & 1 & 0 \\ -s_2 e^{i\tilde{\phi}_2} & 0 & c_2 \end{pmatrix} \begin{pmatrix} c_3 & s_3 e^{i\tilde{\phi}_3} & 0 \\ -s_3 e^{-i\tilde{\phi}_3} & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $\tilde{\phi}_1 + \tilde{\phi}_2 + \tilde{\phi}_3 = \phi_1 + \phi_2 + \phi_3$. The two angles in h can be chosen so that $\tilde{\phi}_1 = \tilde{\phi}_3 = 0$ putting V into the standard form [31],

⁵In [30] a proof is given that $T \backslash F_3$ is topologically S^4 , though it is not everywhere differentiable—in the same way as the surface of a cube, while not a differentiable manifold, is topologically S^2 . I thank Charles Nash for bringing my attention to this reference.

⁶The usual physics notation is $\theta_1 = \theta_{23}$, $\theta_2 = \theta_{31}$, $\theta_3 = \theta_{12}$, which emphasizes the generations being mixed: the notation adopted here is chosen to reduce the number of indices.

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & s_1 \\ 0 & -s_1 & c_1 \end{pmatrix} \begin{pmatrix} c_2 & 0 & s_2 e^{-i\delta} \\ 0 & 1 & 0 \\ -s_2 e^{i\delta} & 0 & c_2 \end{pmatrix} \begin{pmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ = \begin{pmatrix} c_3 c_2 & s_3 c_2 & s_2 e^{-i\delta} \\ -s_3 c_1 - c_3 s_1 s_2 e^{i\delta} & c_3 c_1 - s_3 s_1 s_2 e^{i\delta} & s_1 c_2 \\ s_3 s_1 - c_3 c_1 s_2 e^{i\delta} & -c_3 s_1 - s_3 c_1 s_2 e^{i\delta} & c_1 c_2 \end{pmatrix}, \quad (8)$$

with $\tilde{\phi}_2 = \delta$. This parametrization can be used for both the CKM matrix and for the PMNS matrix with Weyl neutrinos (strictly speaking it would be V^\dagger that is the standard convention for the PMNS matrix), but, more generally, we can use (7) with CP -violating phase $\delta = \phi_1 + \phi_2 + \phi_3$.

The physics is invariant under independent left and right actions of $U(1) \times U(1)$,

$$V \rightarrow h V h'^\dagger,$$

with h acting on Ψ_R and h' acting on Ψ'_R . Such transformations will be referred to as phase transformations, it being understood that h and h' are global, but they may depend on the RG scale μ . We shall also refer to y_a and y'_a as Yukawa couplings and $(\theta_1, \theta_2, \theta_3, \delta)$ as mixing angles and a phase.

III. RUNNING COUPLINGS

Allowing for possible scale dependence $t = \ln \mu$ in $U(\mu)$, $U'(\mu)$, the RG evolution of Eq. (6) gives

$$\frac{dZ}{dt} = U \left(\frac{d\Lambda}{dt} - i[A, \Lambda] \right) U^\dagger, \quad (9)$$

$$\frac{dZ'}{dt} = U' \left(\frac{d\Lambda'}{dt} - i[A', \Lambda'] \right) U'^\dagger, \quad (10)$$

where

$$A = iU^\dagger \frac{dU}{dt}, \quad A' = iU'^\dagger \frac{dU'}{dt}. \quad (11)$$

In terms of A and A'

$$\frac{dV}{dt} = i(AV - VA'), \quad (12)$$

or

$$\frac{dV}{dt} = iBV$$

with

$$B = -i \frac{dV}{dt} V^\dagger = A - VA'V^\dagger. \quad (13)$$

The off-diagonal components of the two matrices A and A' can be obtained from Eqs. (4) and (5), expressed in terms of the diagonal matrices Λ and Λ' :

$$\frac{d\Lambda}{dt} = \frac{1}{4\pi} \left\{ 3\Lambda^2 - \frac{3}{2}(\Lambda V \Lambda' V^\dagger + V \Lambda' V^\dagger \Lambda) \right. \\ \left. + 2(\Pi - \alpha)\Lambda \right\} + i[A, \Lambda], \quad (14)$$

$$\frac{d\Lambda'}{dt} = \frac{1}{4\pi} \left\{ 3\Lambda'^2 - \frac{3}{2}(\Lambda' V^\dagger \Lambda V + V^\dagger \Lambda V \Lambda') \right. \\ \left. + 2(\Pi - \alpha')\Lambda' \right\} + i[A', \Lambda'], \quad (15)$$

by demanding that the right-hand side of each these equations is diagonal. Since A and A' are Hermitian matrices in the Lie algebra of $SU(3)$ they can both be expanded in terms of Gell-Mann matrices, $A = A_I \lambda_I$, $A' = A'_I \lambda_I$. However Eqs. (14) and (15) do not involve the diagonal components A_3, A_8, A'_3 , or A'_8 ; (14) and (15) are sufficient to fix $A_1, A_2, A_4, A_5, A_6, A_7$ and $A'_1, A'_2, A'_4, A'_5, A'_6, A'_7$ in terms of the off-diagonal components of the anticommutators,

$$\{\Lambda, V \Lambda' V^\dagger\} \quad \text{and} \quad \{\Lambda', V^\dagger \Lambda V\},$$

but A_3, A_8, A'_3 , and A'_8 remain undetermined. A natural choice is to choose fermion phases which preserve the parametrization (8) of V at all energies. This requires that the four components $\frac{dV_{11}}{dt}, \frac{dV_{12}}{dt}, \frac{dV_{23}}{dt}$, and $\frac{dV_{33}}{dt}$ of $\frac{dV}{dt}$ must be real and Eq. (12) then gives linear equations for A_3, A_8, A'_3 , and A'_8 , which can be solved to determine them uniquely as functions of the parameters in V and the off-diagonal components of A and A' , which have already been calculated.

An important consequence of this is that A and A' do not depend on Π, α , or α' ; they are completely determined by demanding that the off-diagonal components of

$$-\frac{3}{8\pi} \{\Lambda, V \Lambda' V^\dagger\} + i[A, \Lambda] \quad \text{and} \quad -\frac{3}{8\pi} \{V^\dagger \Lambda V, \Lambda'\} + i[A', \Lambda']$$

vanish, together with the phase convention that $\dot{V}_{11}, \dot{V}_{12}, \dot{V}_{23}$, and \dot{V}_{33} be real. In this way A and A' can be expressed uniquely as functions of the four parameters in V and the six Yukawa couplings in Λ and Λ' . Once A and A' are known the β functions for the parameters in V are obtained from (12).

More generally one can use (7), with the weaker condition that only \dot{V}_{11} and \dot{V}_{33} are real. This is sufficient to fix the differences $A_3 - A'_3$ and $A_8 - A'_8$, but leaves the sums $A_3^+ = A_3 + A'_3$ and $A_8^+ = A_8 + A'_8$ arbitrary. Only

$\delta = \phi_1 + \phi_2 + \phi_3$ is physical; one is free to choose arbitrary energy-dependent phases $\phi_1(t)$ and $\phi_3(t)$ in order to determine A_3^+ and A_8^+ , but all physical quantities are independent of A_3^+ and A_8^+ and only the combination δ remains. It is useful to perform all calculations in a general such “gauge,” as a check on our results: ϕ_1 , ϕ_3 , A_3^+ , and A_8^+ will be left arbitrary, but they must drop out in the calculation of any physically quantity, only $\delta = \phi_1 + \phi_2 + \phi_3$ should remain. This is the strategy that will be adopted in Sec. V, but we first examine the case of two generations to warm up.

IV. TWO GENERATIONS: THE CABIBBO ANGLE

Consider first the quark sector in the case of two generations, $\begin{pmatrix} u \\ d \end{pmatrix}$ and $\begin{pmatrix} c \\ s \end{pmatrix}$, when the parameters of the CKM matrix reduce to the Cabibbo angle θ_C , with $\theta_C \in [0, \frac{\pi}{2}]$, and V reduces to the 2×2 matrix

$$V = \begin{pmatrix} \cos \theta_C & \sin \theta_C \\ -\sin \theta_C & \cos \theta_C \end{pmatrix}.$$

Equations (14) and (15) give the off-diagonal components of $i[A, \Lambda]$ and $i[A, \Lambda']$ from the off-diagonal components of the anticommutators

$$\frac{3}{2} \{V\Lambda'V^T, \Lambda\} \quad \text{and} \quad \frac{3}{2} \{V^T\Lambda V, \Lambda'\} \quad (16)$$

respectively, with

$$\Lambda = \frac{1}{16\pi^2} \begin{pmatrix} y_u^2 & 0 \\ 0 & y_c^2 \end{pmatrix} \quad \text{and} \quad \Lambda' = \frac{1}{16\pi^2} \begin{pmatrix} y_d^2 & 0 \\ 0 & y_s^2 \end{pmatrix}.$$

The anticommutators are purely real and

$$A = \begin{pmatrix} A_3 & -\frac{3i}{64\pi^2} \left(\frac{(y_c^2 + y_u^2)(y_s^2 - y_d^2)}{y_c^2 - y_u^2} \right) \sin 2\theta_C \\ \frac{3i}{64\pi^2} \left(\frac{(y_c^2 + y_u^2)(y_s^2 - y_d^2)}{y_c^2 - y_u^2} \right) \sin 2\theta_C & -A_3 \end{pmatrix},$$

$$A' = \begin{pmatrix} A'_3 & -\frac{3i}{64\pi^2} \left(\frac{(y_d^2 + y_s^2)(y_c^2 - y_u^2)}{y_d^2 - y_s^2} \right) \sin 2\theta_C \\ \frac{3i}{64\pi^2} \left(\frac{(y_d^2 + y_s^2)(y_c^2 - y_u^2)}{y_d^2 - y_s^2} \right) \sin 2\theta_C & -A'_3 \end{pmatrix},$$

with A_3 and A'_3 yet to be determined. Demanding that $\frac{dV}{dt}$ is real in the two-generations version of (12),

$$\frac{dV}{dt} = i(AV - VA'),$$

then requires $A_3 = A'_3 = 0$, so

$$A = \frac{3}{64\pi^2} \left(\frac{(y_c^2 + y_u^2)(y_s^2 - y_d^2) \sin 2\theta_C}{y_c^2 - y_u^2} \right) \sigma_2,$$

$$A' = \frac{3}{64\pi^2} \left(\frac{(y_d^2 + y_s^2)(y_c^2 - y_u^2) \sin 2\theta_C}{y_d^2 - y_s^2} \right) \sigma_2.$$

These expressions for A and A' then give the Cabibbo angle β function for the two-generation version of (13) as

$$B = A - VA'V^T - A = A - A' = \beta_C \sigma_2$$

with [8]

$$\beta_C = \frac{3 \sin 2\theta_C}{64\pi^2} \left[\left(\frac{y_c^2 + y_u^2}{y_c^2 - y_u^2} \right) (y_s^2 - y_d^2) + \left(\frac{y_s^2 + y_d^2}{y_s^2 - y_d^2} \right) (y_c^2 - y_u^2) \right].$$

It proves convenient to define the variables (similar variables will be used extensively in the next section)

$$z = \frac{y_c^2 - y_u^2}{y_c^2 + y_u^2}, \quad z' = \frac{y_s^2 - y_d^2}{y_s^2 + y_d^2},$$

which lie in the range $[-1, 1]$: observationally these are both close to 1. In terms of z and z'

$$\frac{d\theta_C}{dt} = \beta_C = \frac{3 \sin 2\theta_C}{32\pi^2} \left[\frac{y_c^2 z}{z'(1+z)} + \frac{y_s^2 z'}{z(1+z')} \right]. \quad (17)$$

There are fixed points of θ_C at $\theta_C = 0$ and $\frac{\pi}{2}$.

Equation (17) is easily solved, with $\theta_C = \theta_0$ at $t = \ln \frac{\mu}{\mu_0} = 0$, to give

$$\tan \theta_C = \tan \theta_0 e^{\int b dt} \approx \tan \theta_0 \left(\frac{\mu}{\mu_0} \right)^b, \quad (18)$$

where

$$b = \frac{3}{16\pi^2} \left[\frac{y_c^2 z}{z'(1+z)} + \frac{y_s^2 z'}{z(1+z')} \right].$$

Thus $\theta_C \rightarrow 0$ or $\frac{\pi}{2}$ for $\mu \rightarrow \infty$ (depending on the sign of b).⁷ With the hierarchy $y_u \approx y_d \ll y_s^2 \ll y_c^2$,

$$\frac{d\theta_C}{dt} \approx \frac{3y_c^2 \sin 2\theta_C}{64\pi^2}.$$

Note that the poles in β_C at $z = 0$ and $z' = 0$ are not a pathology [14]. For example, if the RG trajectories of y_s and y_d cross at some energy t , so $y_s^2 = y_d^2$ and $z' = 0$ there, then Λ' is proportional to the identity matrix at that energy. With $y_c^2 > y_u^2$ and $y_s^2 - y_d^2 = \epsilon > 0$, $\frac{d\theta_C}{dt} \rightarrow \infty$ as $\epsilon \rightarrow 0$ and the RG evolution drives θ_C infinitely quickly to $\frac{\pi}{2}$ as $\epsilon \rightarrow 0$. This is physically perfectly reasonable.

Geometrically the Cabibbo angle parametrizes the double coset $U(1) \backslash SU(2)/U(1)$. Performing the right action first $SU(2)/U(1) \approx S^2$, however, the left action of $U(1)$ on S^2 has fixed points at the N and S poles. Using (ϑ, φ) as standard polar coordinates on S^2 , $\vartheta = 2\theta_C$ and the left action of $U(1)$ is the Killing vector $\frac{\partial}{\partial \varphi}$. In terms of $\sin \theta_C$, $U(1) \backslash SU(2)/U(1)$ is just the unit line interval $[0, 1]$ (see Fig. 1).

V. THE FERMION MIXING MATRIX FOR THREE GENERATIONS

With three generations, including three right-handed Weyl, gauge singlet, neutrinos, the CKM matrix, and the PMNS matrix, can be treated with the same formalism. Let V denote either the CKM matrix in the quark sector or the PMNS matrix in the leptonic sector, so $(y_1, y_2, y_3) = (y_u, y_c, y_t)$, $(y'_1, y'_2, y'_3) = (y_d, y_s, y_b)$ in the quark sector and $(y'_1, y'_2, y'_3) = (y_e, y_\mu, y_\tau)$, $(y_1, y_2, y_3) = (y_{\nu_1}, y_{\nu_2}, y_{\nu_3})$ in the leptonic sector. To exhibit the explicit form of the β functions it is convenient to define the ratios

$$z_1 = \frac{y_3^2 - y_2^2}{y_3^2 + y_2^2}, \quad z_2 = \frac{y_3^2 - y_1^2}{y_3^2 + y_1^2}, \quad z_3 = \frac{y_2^2 - y_1^2}{y_2^2 + y_1^2}, \quad (19)$$

$$z'_1 = \frac{y_3'^2 - y_2'^2}{y_3'^2 + y_2'^2}, \quad z'_2 = \frac{y_3'^2 - y_1'^2}{y_3'^2 + y_1'^2}, \quad z'_3 = \frac{y_2'^2 - y_1'^2}{y_2'^2 + y_1'^2}. \quad (20)$$

⁷This analysis is only for illustrative purposes: Eq. (18) is not valid for energies of the order of, or less than, the Higgs mass, since only massless RG evolution is considered here: the running will freeze for energies close to, and below, the Higgs mass.

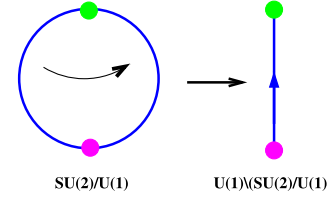


FIG. 1. The left action of $U(1)$ on $SU(2)/U(1)$, with the N and S poles fixed points, reduces the sphere to a line of constant longitude.

These all lie in the range $[-1, 1]$. They are not all independent, of course: for example

$$z_3 = \frac{z_2 - z_1}{1 - z_1 z_2}.$$

In these variables the β functions are

$$\beta_i = \dot{\theta}_i = \frac{3}{32\pi^2} \sum_{j,k=1}^3 \left[y_3^2 P_i^{jk} \frac{z_k}{z'_j(1+z_k)} + y_3'^2 Q_i^{jk} \frac{z'_k}{z_j(1+z'_k)} \right], \quad (21)$$

$$\beta_\delta = \dot{\delta} = \frac{3 \sin \delta}{32\pi^2} \sum_{j,k=1}^3 \left[y_3^2 P_\delta^{jk} \frac{z_k}{z'_j(1+z_k)} + y_3'^2 Q_\delta^{jk} \frac{z'_k}{z_j(1+z'_k)} \right], \quad (22)$$

with the P s and Q s polynomials in the trigonometric functions in V . Explicit expressions for all the P s and Q s are given in Appendix B.

There are poles when any of the z_j or z'_j are zero: as for the Cabibbo angle above, near a pole the RG flow will push V very quickly to a fixed point of the left action of $U(1) \times U(1)$ on the flag manifold $SU(3)/U(1) \times U(1)$.

Using the P s and Q s in Appendix B, the β functions (21) and (22) vanish when

$$(\theta_1^*, \theta_2^*, \theta_3^*) = (0, 0, 0), \left(0, 0, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, 0, 0\right), \left(\frac{\pi}{2}, 0, \frac{\pi}{2}\right), \\ \times \left(0, \frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right), \left(0, \frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right),$$

with $\delta = 0$ or π . For $\theta_2 = 0$ there are four such points

$$V^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

choosing $\delta = 0$ or π does not give different points, δ only appears in the mixing matrix V in the combination $\sin \theta_2 e^{\pm i\delta}$.

For $\theta_2 = \frac{\pi}{2}$ the situation is more subtle. For $\delta = 0$ the four points give

$$V^* = \left(\begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} \right), \left(\begin{matrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{matrix} \right), \left(\begin{matrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{matrix} \right), \left(\begin{matrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{matrix} \right),$$

for $\delta = \pi$ they give

$$V^* = \left(\begin{matrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} \right), \left(\begin{matrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} \right), \left(\begin{matrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{matrix} \right), \left(\begin{matrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \right).$$

Up to phase transformations only two of these eight are distinct physical points. Furthermore note that, when $\theta_2 = \frac{\pi}{2}$, the mixing matrix V in (8) takes the form

$$V = \begin{pmatrix} 0 & 0 & 1 \\ -\sin \theta_+ & \cos \theta_+ & 0 \\ -\cos \theta_+ & -\sin \theta_+ & 0 \end{pmatrix} \text{ for } \delta = 0, \quad (23)$$

$$V = \begin{pmatrix} 0 & 0 & -1 \\ \sin \theta_- & \cos \theta_- & 0 \\ \cos \theta_- & -\sin \theta_- & 0 \end{pmatrix} \text{ for } \delta = \pi, \quad (24)$$

with $\theta_- = \theta_3 - \theta_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, but Eqs. (23) and (24) are physically the same up to phase transformation. For $\delta = 0$ the range

$$0 \leq \theta_+ \leq \pi$$

with $\theta_+ = \theta_3 + \theta_1 \in [0, \pi]$ and

double counts physical values, as the sign of $\cos \theta_+$ in (23) can be changed by a phase transformation. All distinct

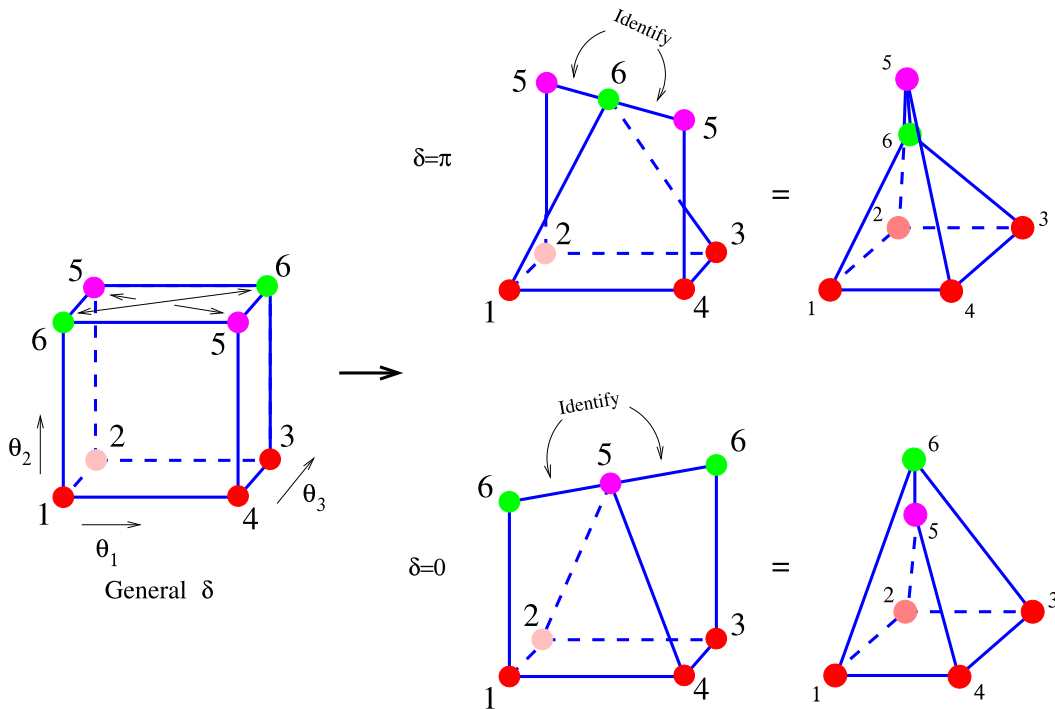


FIG. 2. When $\delta = 0$ or π the $\theta_2 = \frac{\pi}{2}$ face of the cube degenerates to a line segment which is topologically equivalent to the line on the right of Fig. 1.

physical values of V with $\theta_2 = \frac{\pi}{2}$ are achieved with the smaller range $0 \leq \theta_+ \leq \frac{\pi}{2}$.

The upshot of this is that $\theta_2 = \frac{\pi}{2}$, with δ either 0 or π , is a one-dimension line in physical V space, given by (23) with $0 \leq \theta_+ \leq \frac{\pi}{2}$, and there are only two physically distinct fixed points where the β functions vanish, (21) and (22), $\theta_+ = 0$ and $\theta_+ = \frac{\pi}{2}$.⁸ This gives the two distinct fixed points when $\theta_2 = \frac{\pi}{2}$ as

$$V^* = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

where all entries have been rendered positive by phase transformations of the fermions.

Although there are a total of ten parameters in the Yukawa sector, it makes physical sense to focus on the

fixed points of the mixing parameters alone. Label the six Yukawa couplings by $y_\alpha = (y_\alpha, y'_\alpha)$, $\alpha = 1, \dots, 6$, and the four mixing parameters by $\vartheta_\mu = (\theta_1, \theta_2, \theta_3, \delta)$, $\mu = 1, \dots, 4$. A fixed point of the full Yukawa sector requires $\beta_\alpha = \frac{dy_\alpha}{dt} = 0$ and $\beta_\mu = \frac{d\vartheta_\mu}{dt} = 0$. But the six fixed points of β_μ , determined by (21) and (22), require all the P 's and Q 's in Appendix B to vanish, regardless of the prefactors, so if ϑ_μ are chosen so that $\beta_\mu(\vartheta) = 0$ for some y_α , then $\beta_\mu(\vartheta) = 0$ for all y_α , independently of the values of the β_α . It is shown in Appendix C that the 10×10 matrix of anomalous dimensions is block diagonal when (21) and (22) vanish, so it makes physical sense to view the above six points where $\beta_\mu = 0$ to be fixed points of the mixing matrix flow, independently of the Yukawa coupling β -functions β_α .

In summary, there are six fixed points of the one-loop RG flow of the mixing matrix, and V at these points is (up to phase transformations)

$$\begin{aligned} V_1^* &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, (\theta_1^*, \theta_2^*, \theta_3^*) = (0, 0, 0); & V_2^* &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, (\theta_1^*, \theta_2^*, \theta_3^*) = \left(0, 0, \frac{\pi}{2}\right); \\ V_3^* &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, (\theta_1^*, \theta_2^*, \theta_3^*) = \left(\frac{\pi}{2}, 0, \frac{\pi}{2}\right); & V_4^* &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, (\theta_1^*, \theta_2^*, \theta_3^*) = \left(\frac{\pi}{2}, 0, 0\right); \\ V_5^* &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, (\theta_1^*, \theta_2^*, \theta_3^*) = \begin{cases} \left(0, \frac{\pi}{2}, \frac{\pi}{2}\right), \\ \left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right); \end{cases} & V_6^* &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, (\theta_1^*, \theta_2^*, \theta_3^*) = \begin{cases} \left(0, \frac{\pi}{2}, 0\right), \\ \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right). \end{cases} \end{aligned} \quad (25)$$

These are labeled 1 to 6 in the above order: with this labeling of the fixed points the situation can be visualised as shown in Fig. 2.

The six matrices $V_1^* - V_6^*$ actually form a group; they furnish a unitary representation of the symmetric group acting on three objects. The appearance of this group is related to the fact that S_3 is the Weyl group of $SU(3)$ and these six points are also fixed points of the left action of the Cartan torus of $SU(3)$, $T \approx U(1) \times U(1)$, on the flag manifold F_3 .⁹

Note that the Jarlskog invariant [28],

$$\begin{aligned} \mathcal{J} &= \frac{i}{4} \det[Z', Z] = (y_3^2 - y_2^2)(y_2^2 - y_1^2)(y_1^2 - y_3^2)(y_3'^2 - y_2'^2)(y_2'^2 - y_1'^2)(y_1'^2 - y_3'^2)J, \\ \text{with } J &= \frac{1}{8} \sin 2\theta_1 \sin 2\theta_2 \sin 2\theta_3 \cos \theta_2 \sin(\phi_1 + \phi_2 + \phi_3) \\ &= \frac{1}{8} \sin 2\theta_1 \sin 2\theta_2 \sin 2\theta_3 \cos \theta_2 \sin \delta, \end{aligned}$$

vanishes at all of the fixed points.

⁸This line is a RG invariant space in $(U(1) \times U(1)) \backslash SU(3) / (U(1) \times U(1))$; it is a double coset $U(1) \backslash SU(2) / U(1)$.

⁹The fact that there are six fixed points is related to fixed point theorems: the Euler characteristic of F_3 is 6. I am grateful to Charles Nash for bringing my attention to fixed point theorems and the significance of the Weyl group of $SU(3)$ in this context.

VI. OPERATOR MIXING

In any quantum field theory, with couplings x_A and β -functions $\beta_A = \frac{dx_A}{dt}$ with fixed points x_A^* , operator mixing under the RG flow is controlled by the matrix Γ with entries

$$\Gamma_A^B = \left(\frac{\partial \beta_A}{\partial x_B} \right).$$

The nature of each fixed point, whether attractive, repulsive, or unstable, is determined by the signs of the eigenvalues of Γ at the fixed point. Although the components of Γ_* depend on the parametrization x_A its eigenvalues do not; they are invariant under a reparametrization, $x_A \rightarrow x'_A(x)$, and hence are not scheme dependent.

It is shown in Appendix C that, at the six fixed points, ϑ^* , of the fermion mixing matrix, the 10×10 matrix Γ is block diagonal at one loop,

$$\Gamma_A^B = \begin{pmatrix} \frac{\partial \beta_\alpha}{\partial y_\beta} & 0 \\ 0 & \frac{\partial \beta_\mu}{\partial \vartheta_\nu} \Big|_* \end{pmatrix} \quad (26)$$

for any values of y_α , not just at y_α^* . It therefore makes sense to focus on¹⁰

$$\gamma_* = \begin{pmatrix} \frac{\partial \beta_1}{\partial \theta_1} & \frac{\partial \beta_1}{\partial \theta_2} & \frac{\partial \beta_1}{\partial \theta_3} & \frac{\partial \beta_1}{\partial \delta} \\ \frac{\partial \beta_2}{\partial \theta_1} & \frac{\partial \beta_2}{\partial \theta_2} & \frac{\partial \beta_2}{\partial \theta_3} & \frac{\partial \beta_2}{\partial \delta} \\ \frac{\partial \beta_3}{\partial \theta_1} & \frac{\partial \beta_3}{\partial \theta_2} & \frac{\partial \beta_3}{\partial \theta_3} & \frac{\partial \beta_3}{\partial \delta} \\ \frac{\partial \beta_\delta}{\partial \theta_1} & \frac{\partial \beta_\delta}{\partial \theta_2} & \frac{\partial \beta_\delta}{\partial \theta_3} & \frac{\partial \beta_\delta}{\partial \delta} \end{pmatrix} \Big|_*$$

separately.

A subtlety is that for $\theta_2 = \frac{\pi}{2}$, and $\delta = 0$ or π , $\frac{\partial \beta_\delta}{\partial \delta}$ is indeterminate, because the $(\theta_1, \theta_2, \theta_3, \delta)$ coordinates are singular there: better is to change coordinates to $(\theta_1, \theta_2, \theta_3, J)$ to use

$$\gamma_* = \begin{pmatrix} \frac{\partial \beta_1}{\partial \theta_1} & \frac{\partial \beta_1}{\partial \theta_2} & \frac{\partial \beta_1}{\partial \theta_3} & \frac{\partial \beta_1}{\partial J} \\ \frac{\partial \beta_2}{\partial \theta_1} & \frac{\partial \beta_2}{\partial \theta_2} & \frac{\partial \beta_2}{\partial \theta_3} & \frac{\partial \beta_2}{\partial J} \\ \frac{\partial \beta_3}{\partial \theta_1} & \frac{\partial \beta_3}{\partial \theta_2} & \frac{\partial \beta_3}{\partial \theta_3} & \frac{\partial \beta_3}{\partial J} \\ \frac{\partial \beta_J}{\partial \theta_1} & \frac{\partial \beta_J}{\partial \theta_2} & \frac{\partial \beta_J}{\partial \theta_3} & \frac{\partial \beta_J}{\partial J} \end{pmatrix} \Big|_*, \quad (27)$$

where $\beta_J = \frac{dJ}{dt}$, and then $\frac{\partial \beta_J}{\partial J} \Big|_*$ is well defined for $\theta_2 = \frac{\pi}{2}$ and $\delta = 0$ or π . The eigenvalues of γ_* at the six fixed points (at one loop) are given in Appendix C.

¹⁰ γ_* is the matrix of anomalous dimensions, but since angles are necessarily dimensionless, this may not be an appropriate name in this context.

In the following subsection we shall examine γ_* in more detail for the quark sector of the Standard Model.

A. The CKM matrix

In the quark sector the Standard Model hierarchy, with $y_u^2 \ll y_c^2 \ll y_t^2$ and $y_d^2 \ll y_s^2 \ll y_b^2$, it is a good approximation to set $z_i = z'_i = 1$. In this limit (21) and (22) reduce to

$$\beta_1 = \frac{3 \sin 2\theta_1}{64\pi^2} (y_t^2 + y_b^2 \cos^2 \theta_2), \quad (28)$$

$$\beta_2 = \frac{3 \sin 2\theta_2}{64\pi^2} (y_t^2 \cos^2 \theta_1 + y_b^2), \quad (29)$$

$$\beta_3 = \frac{3y_t^2}{64\pi^2} [\sin 2\theta_3 (\sin^2 \theta_1 - \cos^2 \theta_1 \sin^2 \theta_2) - 2 \cos \delta \cos^2 \theta_3 \sin 2\theta_1 \sin \theta_2], \quad (30)$$

$$\beta_\delta = \frac{3y_t^2}{64\pi^2} \sin \delta \cot \theta_3 \sin 2\theta_1 \sin \theta_2. \quad (31)$$

Assuming the above hierarchy persists under RG flow of the Yukawa couplings, the signs of the eigenvalues of the operator mixing matrix at the six fixed points can be found from the expressions in Appendix C with $y_t^2 \gg y_c^2 \gg y_u^2$, $y_b^2 \gg y_s^2 \gg y_d^2$, and $z_i = z'_i = 1$ (so $\zeta_{ij} = \zeta'_{ij} = \frac{1}{2}$ in Appendix C). Defining $\hat{\lambda}_i = \frac{32\pi^2}{3} \lambda_i$ the eigenvalues in the UV direction, to leading order, are

Fixed point	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{\lambda}_J$
1	$y_t^2 + y_b^2$	$y_t^2 + y_b^2$	0	$2(y_t^2 + y_b^2)$
2	$y_t^2 + y_b^2$	$y_t^2 + y_b^2$	0	$2(y_t^2 + y_b^2)$
3	$-(y_t^2 + y_b^2)$	y_b^2	$-y_t^2$	$-2y_t^2$
4	$-(y_t^2 + y_b^2)$	y_b^2	y_t^2	0

for the fixed points 1–4. For the fixed points 5 and 6 it is better to rotate the coordinates in the $\theta_1 - \theta_3$ plane to $\theta_+ = \theta_3 + \theta_1$ and $\theta_- = \theta_3 - \theta_1$ (see Appendix C) and $\lambda_\pm = \frac{\partial \beta_\pm}{\partial \theta_\pm} \Big|_*$ are eigenvalues. For the fixed point 5, the directions along the edges 25 and 35 in Fig. 2 are eigendirections near the point 5; and for the fixed point 6, the directions along the edges 16 and 46 eigendirections near the point 6: the corresponding eigenvalues are labeled λ_{25} , λ_{45} , λ_{16} , and λ_{36} in the tables below:

Fixed point	$\hat{\lambda}_+$	$\hat{\lambda}_{25}$	$\hat{\lambda}_{45}$	$\hat{\lambda}_J$
5	y_t^2	$-y_b^2$	$-(y_t^2 + y_b^2)$	$-2y_b^2$

Fixed point	$\hat{\lambda}_-$	$\hat{\lambda}_{16}$	$\hat{\lambda}_{36}$	$\hat{\lambda}_J$
6	$-y_t^2$	$-(y_t^2 + y_b^2)$	$-y_b^2$	$-2(y_t^2 + y_b^2)$

for the fixed points 5 and 6.

One could fill in the zeros by relaxing the conditions $z_i = z'_i = 1$. For example letting z_1 and z'_1 deviate slightly from unity, keeping $z_2 = z_3 = z'_2 = z'_3 = 1$, gives $\lambda_3 = \frac{3}{32\pi^2}(y_c^2 + y_s^2)$ at the fixed point 1 and $\lambda_3 = -\frac{3}{32\pi^2}(y_c^2 + y_s^2)$ at the fixed point 2: in particular this choice makes fixed point 1 fully repulsive in the UV direction, but higher loops could modify this conclusion. Thus, in this limit, fixed point 6 is fully attractive in the UV while fixed point 5 is fully attractive in the IR.

VII. FIXED POINTS AT ALL ORDERS

It is not a coincidence that the six fixed points of the one-loop RG equations for the mixing angles and phase coincide with the six elements of the Weyl group of $SU(3)$ that are fixed points of the left action of the Cartan torus T on the flag manifold F_3 ; indeed this must be true to all orders in perturbation theory, and even nonperturbatively, if it is assumed that the action of T on F_3 commutes with the RG flow on F_3 .

Let $\Theta = (\theta_1, \theta_2, \theta_3, \phi_1, \phi_2, \phi_3)$ be a point in F_3 in an arbitrary gauge (in the sense of the energy dependence of the fermion phases), and let $\varphi(\Theta)$ be an infinitesimal RG transformation of Θ . Let Θ^* be a fixed point of the action of T on F_3 , so $\mathbf{t}(\Theta^*) = \Theta^*$ for some $\mathbf{t} \in T$. If Θ^* were not a RG fixed point, it could be moved to an infinitesimally close point $\varphi(\Theta^*) \neq \Theta^*$ with a RG transformation, φ . Demanding that $\mathbf{t} \circ \varphi(\Theta^*) = \varphi \circ \mathbf{t}(\Theta^*) = \varphi(\Theta^*)$ then implies that $\varphi(\Theta^*)$ is also left invariant by \mathbf{t} . But the six fixed points of T are isolated, so there are no other fixed points of T infinitesimally close to Θ^* . Hence $\varphi(\Theta^*) = \Theta^*$ and Θ^* must be a RG fixed point.¹¹ This must even be true nonperturbatively; all that has been assumed is that T commutes with the RG flow on F_3 .

This argument does not preclude the possibility that there are fixed points of the RG that are not fixed points of the left action of T . But, if the Weyl group also commutes with the RG flow, then any such fixed points must come in sets of six. The argument is also only valid for Dirac fermions: for Majorana fermions there are more parameters and bare masses that are not considered in the present work (see Refs. [33,34] for partial two-loop corrections with Majorana neutrinos).¹²

VIII. DISCUSSION

It has been shown that one-loop running of the mixing angles and the CP -violating phase for the Standard Model with three generations gives rise to six RG fixed points.

¹¹Again I thank Charles Nash for pointing out the significance of the fact that these actions commute for the fixed points. The version of the argument presented here was used in the context of modular symmetry and the quantum Hall effect in [32].

¹²I thank the referee for pointing out these references.

These six points are related to the Weyl group of $SU(3)$, whose elements correspond to fixed points of the left action of the Cartan torus on F_3 . The same analysis applies to both the quark sector and the leptonic sector with three right-handed, gauge singlet, Weyl (but not Majorana) neutrinos.

Only massless RG running has been considered here, so the present analysis only applies to energies significantly above the Higgs and t -quark masses, a few hundred GeV and higher. In the IR direction one would need to include the effect of masses and take into account the fact that particles drop out of the running as the energy falls below their mass threshold. It is not the case that θ_1, θ_2 , and θ_3 are all driven to zero in the IR in the Standard Model hierarchy, rather they will freeze at their observed values for energies below a few hundred GeV. In any case the analysis presented here is not directly relevant to the physics of the Standard Model. The observed magnitudes of the Yukawa couplings are so small that the one-loop running of the CKM parameters in the UV direction is so slow that there are no significant physical effects even up to the largest conceivable energies.

Nevertheless the analysis could be useful for more formal aspects of the theory. For example the idea of gradient flow [35] requires introducing a metric on the space of couplings and a natural metric on the double $SU(3)$ coset $T \backslash F_3$ could be obtained from the restriction of the $SU(3)$ -invariant metric on the flag manifold; this will be direction of future investigations.

The ideas presented here could also be relevant to models beyond the Standard Model: dark matter could arise from a gauge theory, similar to the Standard Model, with a portal to the Standard Model Higgs but with no coupling to the photon. If there were N_g dark generations of chiral fermions with Yukawa coupling matrices, the mixing angles and CP -violating phases would be parametrized by $U(1)^{N_g-1} \backslash SU(N_g) / U(1)^{N_g-1}$ which has dimension $(N_g - 1)^2$, consisting of $\frac{1}{2}N_g(N_g - 1)$ angles and $\frac{1}{2}(N_g - 1)(N_g - 2)$ CP -violating phases. The left action of $U(1)^{N_g-1}$ has $N_g!$ fixed points, corresponding to the Weyl group of $SU(N_g)$ (which is the symmetric group acting on N_g objects, S_{N_g}) and these will be fixed points of the RG. If the relevant Yukawa couplings were large enough the system could be driven in the infrared toward significant CP violation before the running cuts off due to masses, satisfying one of the conditions for the observed CP violation in our Universe.

The present work has not considered Majorana fermions, which would introduce more parameters and masses. Such an analysis would be more involved but, at least in principle, could be tackled using similar techniques: this is left for future work.

The manipulations necessary to derive the polynomials P and Q and the matrix of anomalous dimensions were all performed using the symbolic manipulation software *Mathematica*.

ACKNOWLEDGMENTS

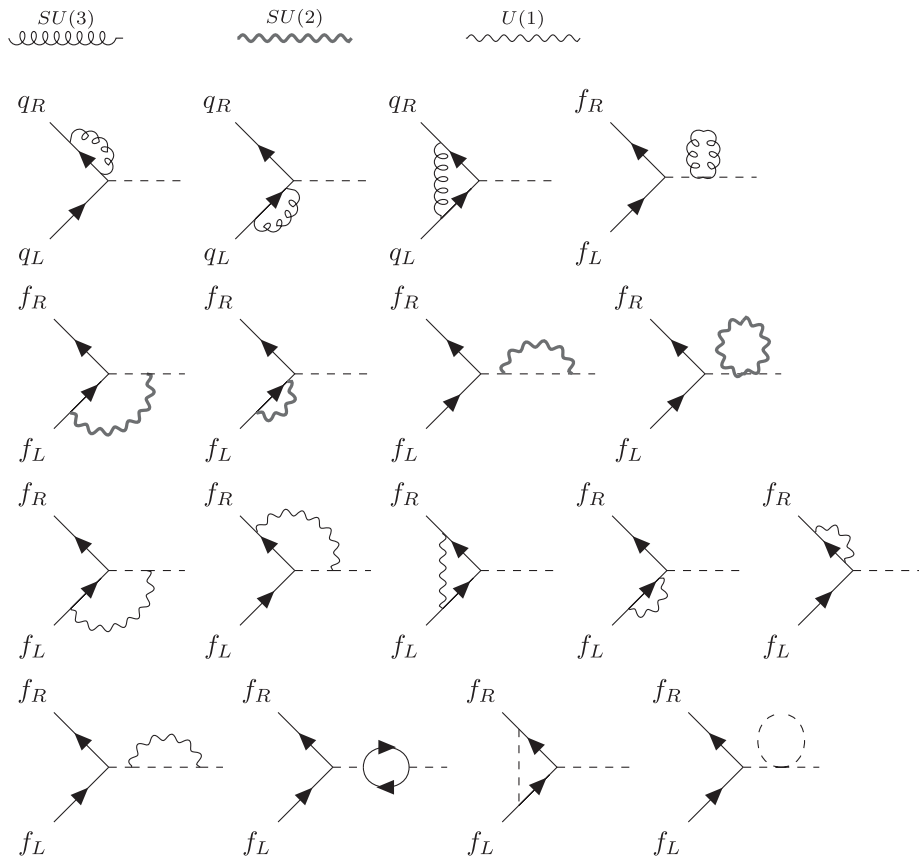
It is a pleasure to thank Charles Nash and Denjoe O’Connor for useful conversations on invariants of multimatrix systems and fixed points.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: FEYNMAN DIAGRAMS

The one-loop Feynman diagrams that contribute to the running of the Yukawa couplings are shown below:



APPENDIX B: THE POLYNOMIALS P AND Q

The polynomials P and Q in Eqs. (21) and (22) are

$$\begin{aligned}
P_1^{11} &= \sin 2\theta_1 \cos^2 \theta_3 - \cos \delta \sin^2 \theta_1 \sin \theta_2 \sin 2\theta_3, \\
P_1^{21} &= \sin 2\theta_1 \sin^2 \theta_3 + \cos \delta \sin^2 \theta_1 \sin \theta_2 \sin 2\theta_3, \\
P_1^{12} &= \cos \delta \sin 2\theta_3 \sin \theta_2, \\
P_1^{22} &= -\cos \delta \sin 2\theta_3 \sin \theta_2, \\
Q_1^{11} &= \sin 2\theta_1 (\cos^2 \theta_3 - \sin^2 \theta_2 \sin^2 \theta_3) + \cos \delta \cos 2\theta_1 \sin \theta_2 \sin 2\theta_3, \\
Q_1^{21} &= \sin 2\theta_1 \sin^2 \theta_2 \sin^2 \theta_3 + \cos \delta \sin^2 \theta_1 \sin \theta_2 \sin 2\theta_3, \\
Q_1^{31} &= -\sin 2\theta_1 \sin^2 \theta_2 \sin^2 \theta_3 + \cos \delta \cos^2 \theta_1 \sin \theta_2 \sin 2\theta_3, \\
Q_1^{12} &= \sin 2\theta_1 (\sin^2 \theta_3 - \sin^2 \theta_2 \cos^2 \theta_3) - \cos \delta \cos 2\theta_1 \sin \theta_2 \sin 2\theta_3, \\
Q_1^{22} &= \sin 2\theta_1 \sin^2 \theta_2 \cos^2 \theta_3 - \cos \delta \sin^2 \theta_1 \sin \theta_2 \sin 2\theta_3, \\
Q_1^{32} &= -\sin 2\theta_1 \sin^2 \theta_2 \cos^2 \theta_3 - \cos \delta \cos^2 \theta_1 \sin \theta_2 \sin 2\theta_3.
\end{aligned}$$

$$\begin{aligned}
P_2^{11} &= -\sin 2\theta_2 \sin^2 \theta_3 \sin^2 \theta_1 + \frac{1}{2} \cos \delta \cos \theta_2 \sin 2\theta_3 \sin 2\theta_1, \\
P_2^{21} &= -\sin 2\theta_2 \cos^2 \theta_3 \sin^2 \theta_1 - \frac{1}{2} \cos \delta \cos \theta_2 \sin 2\theta_3 \sin 2\theta_1, \\
P_2^{12} &= \sin 2\theta_2 \sin^2 \theta_3, \\
P_2^{22} &= \sin 2\theta_2 \cos^2 \theta_3, \\
Q_2^{21} &= \sin 2\theta_2 \sin^2 \theta_3 \cos^2 \theta_1 + \frac{1}{2} \cos \delta \cos \theta_2 \sin 2\theta_3 \sin 2\theta_1, \\
Q_2^{31} &= \sin 2\theta_2 \sin^2 \theta_3 \sin^2 \theta_1 - \frac{1}{2} \cos \delta \cos \theta_2 \sin 2\theta_3 \sin 2\theta_1, \\
Q_2^{22} &= \sin 2\theta_2 \cos^2 \theta_3 \cos^2 \theta_1 - \frac{1}{2} \cos \delta \cos \theta_2 \sin 2\theta_3 \sin 2\theta_1, \\
Q_2^{32} &= \sin 2\theta_2 \cos^2 \theta_3 \sin^2 \theta_1 + \frac{1}{2} \cos \delta \cos \theta_2 \sin 2\theta_3 \sin 2\theta_1.
\end{aligned}$$

$$\begin{aligned}
P_3^{11} &= \sin 2\theta_3 \sin^2 \theta_1 \sin^2 \theta_2 - \cos \delta \cos^2 \theta_3 \sin 2\theta_1 \sin \theta_2, \\
P_3^{21} &= -\sin 2\theta_3 \sin^2 \theta_1 \sin^2 \theta_2 - \cos \delta \sin^2 \theta_3 \sin 2\theta_1 \sin \theta_2, \\
P_3^{31} &= \sin 2\theta_3 (\sin^2 \theta_1 \sin^2 \theta_2 - \cos^2 \theta_1) - \cos \delta \cos 2\theta_3 \sin 2\theta_1 \sin \theta_2, \\
P_3^{12} &= -\sin 2\theta_3 \sin^2 \theta_2, \\
P_3^{22} &= \sin 2\theta_3 \sin^2 \theta_2, \\
P_3^{32} &= \sin 2\theta_3 \cos^2 \theta_2, \\
Q_3^{21} &= -\sin 2\theta_3 \sin^2 \theta_1 - \cos \delta \sin^2 \theta_3 \sin 2\theta_1 \sin \theta_2, \\
Q_3^{31} &= -\sin 2\theta_3 \cos^2 \theta_1 + \cos \delta \sin^2 \theta_3 \sin 2\theta_1 \sin \theta_2, \\
Q_3^{22} &= \sin 2\theta_3 \sin^2 \theta_1 - \cos \delta \cos^2 \theta_3 \sin 2\theta_1 \sin \theta_2, \\
Q_3^{32} &= \sin 2\theta_3 \cos^2 \theta_1 + \cos \delta \cos^2 \theta_3 \sin 2\theta_1 \sin \theta_2.
\end{aligned}$$

$$\begin{aligned}
P_\delta^{11} &= \frac{\cot \theta_3 \tan \theta_1}{\sin \theta_2} [\cos 2\theta_3 (1 - \sin^2 \theta_1 \cos^2 \theta_2) + (3\cos^2 \theta_1 \sin^2 \theta_2 - \cos^2 \theta_1 - \sin^2 \theta_2)], \\
P_\delta^{21} &= \frac{\tan \theta_3 \tan \theta_1}{\sin \theta_2} [\cos 2\theta_3 (1 - \sin^2 \theta_1 \cos^2 \theta_2) - (3\cos^2 \theta_1 \sin^2 \theta_2 - \cos^2 \theta_1 - \sin^2 \theta_2)], \\
P_\delta^{12} &= -2 \sin 2\theta_3 \cot 2\theta_1 \sin \theta_2, \\
P_\delta^{22} &= 2 \sin 2\theta_3 \cot 2\theta_1 \sin \theta_2, \\
P_\delta^{31} &= \frac{2 \sin 2\theta_1 \sin \theta_2}{\sin 2\theta_3}, \\
Q_\delta^{11} &= -\frac{2 \sin 2\theta_3 \sin \theta_2}{\sin 2\theta_1}, \\
Q_\delta^{21} &= -\frac{\tan \theta_3 \tan \theta_1}{\sin \theta_2} [\cos 2\theta_3 (\cos^2 \theta_1 \cos^2 \theta_2 - \sin^2 \theta_2) - \sin^2 \theta_1 \sin^2 \theta_2 + \cos^2 \theta_1], \\
Q_\delta^{31} &= \frac{\tan \theta_3 \cot \theta_1}{\sin \theta_2} [\cos 2\theta_3 (\sin^2 \theta_1 \cos^2 \theta_2 - \sin^2 \theta_2) - \cos^2 \theta_1 \sin^2 \theta_2 + \sin^2 \theta_1], \\
Q_\delta^{12} &= \frac{2 \sin 2\theta_3 \sin \theta_2}{\sin 2\theta_1}, \\
Q_\delta^{22} &= -\frac{\cot \theta_3 \tan \theta_1}{\sin \theta_2} [\cos 2\theta_3 (\cos^2 \theta_1 \cos^2 \theta_2 - \sin^2 \theta_2) + \sin^2 \theta_1 \sin^2 \theta_2 - \cos^2 \theta_1], \\
Q_\delta^{32} &= \frac{\cot \theta_3 \cot \theta_1}{\sin \theta_2} [\cos 2\theta_3 (\sin^2 \theta_1 \cos^2 \theta_2 - \sin^2 \theta_2) + \cos^2 \theta_1 \sin^2 \theta_2 - \sin^2 \theta_1]
\end{aligned}$$

(all other P s and Q s vanish).

Observe that

$$\begin{aligned}
\sum_{k=1}^3 \sum_{h=1}^2 P_1^{kh} &= \sin 2\theta_1, & \sum_{k=1}^3 \sum_{h=1}^2 Q_1^{kh} &= \sin 2\theta_1 \cos^2 \theta_2, \\
\sum_{k=1}^3 \sum_{h=1}^2 P_2^{kh} &= \sin 2\theta_2 \cos^2 \theta_1, & \sum_{k=1}^3 \sum_{h=1}^2 Q_2^{kh} &= \sin 2\theta_2, \\
\sum_{k=1}^3 \sum_{h=1}^2 P_3^{kh} &= \sin 2\theta_3 (\sin^2 \theta_1 - \cos^2 \theta_1 \sin^2 \theta_2) - 2 \cos \delta \cos^2 \theta_3 \sin 2\theta_1 \sin \theta_2, \\
\sum_{k=1}^3 \sum_{h=1}^2 Q_3^{kh} &= 0, \\
\sum_{k=1}^3 \sum_{h=1}^2 P_\delta^{kh} &= 2 \sin 2\theta_1 \sin \theta_2 \cot \theta_3, & \sum_{k=1}^3 \sum_{h=1}^2 Q_\delta^{kh} &= 0.
\end{aligned}$$

We also note that $\dot{\theta}_2 = 0$ when $\theta_2 = \frac{\pi}{2}$, so $\theta_2 = \frac{\pi}{2}$ is an invariant hypersurface of the RG flow in $(\theta_1, \theta_2, \theta_3, \delta)$ space.

APPENDIX C: OPERATOR MIXING

At a global fixed point $y_\alpha^*, \vartheta_\mu^*$, where $\beta_\alpha^* = \beta_\mu^* = 0$, the operator mixing matrix is

$$\Gamma_* = \left(\begin{array}{cc} \frac{\partial \beta_\alpha}{\partial y_\beta} & \frac{\partial \beta_\alpha}{\partial \vartheta_\nu} \\ \frac{\partial \beta_\mu}{\partial y_\beta} & \frac{\partial \beta_\mu}{\partial \vartheta_\nu} \end{array} \right) \Big|_*. \quad (\text{C1})$$

This appendix gives the eigenvalues of the operator mixing matrix γ_* in (27) at each of the six fixed points at one loop. We first prove that, at one loop, Γ_* is block diagonal,

$$\Gamma_* = \left(\begin{array}{cc} \frac{\partial \beta_\alpha}{\partial y_\beta} & 0 \\ 0 & \frac{\partial \beta_\mu}{\partial \vartheta_\nu} \end{array} \right) \Big|_{\vartheta = \vartheta_*} \quad (\text{C2})$$

for ϑ_μ^* and any y_α .

- (i) As noted at the end of the Sec. V, at the fixed points ϑ_μ^* , all the P 's and Q 's must vanish, regardless of the prefactors involving z_j and z'_j in (21) and (22). This implies that $\frac{\partial \beta_\mu}{\partial y_\beta} \Big|_* = 0$.
- (ii) Next note that, in Eq. (14), $A(\vartheta_\mu)$ is chosen to cancel the off-diagonal components of $\frac{3}{8\pi}(\Lambda V \Lambda' V^\dagger + \Lambda' V^\dagger \Lambda) + i[A, \Lambda]$ for all θ_μ , so the off-diagonal components of

$$\frac{\partial \left\{ \frac{3}{8\pi}(\Lambda V \Lambda' V^\dagger + \Lambda' V^\dagger \Lambda) + i[A, \Lambda] \right\}}{\partial \vartheta_\mu}$$

vanish for all values of ϑ_μ by construction—we only need to prove that the diagonal entries of the six matrices

$$\frac{\partial(\Lambda V \Lambda' V^\dagger + \Lambda' V^\dagger \Lambda)}{\partial \vartheta_\mu}$$

vanish at the six points ϑ_μ^* , and this is easily checked directly. Exactly the same arguments applies to (15).

On the $\theta_2 = 0$ hypersurface it is straightforward to calculate the eigenvalues of γ at the fixed points: γ_* in (27) is diagonal in $(\theta_1, \theta_2, \theta_3, J)$ coordinates at the four fixed points 1, 2, 3, and 4 and the eigenvalues are just the four diagonal elements.

For the two fixed points on the $\theta_2 = \frac{\pi}{2}$ hypersurfaces the situation is rather more subtle. First rotate the $\theta_1 - \theta_3$ plane through 45° , to $\theta_\pm = \theta_3 \pm \theta_1$, then at the fixed points 5 and 6, γ_* diagonal. However, expressing $\frac{\partial J}{\partial J}$ in terms of δ , the four choices $(\theta_1^*, \theta_2^*, \theta_3^*, \delta^*) = (0, \frac{\pi}{2}, \frac{\pi}{2}, 0)$, $(0, \frac{\pi}{2}, \frac{\pi}{2}, \pi)$, $(\frac{\pi}{2}, \frac{\pi}{2}, 0, 0)$, and $(\frac{\pi}{2}, \frac{\pi}{2}, 0, \pi)$ give the same physical mixing matrix $V_{*,5}$, but with different γ_* for the fixed point 5; similarly the four choices $(\theta_1^*, \theta_2^*, \theta_3^*, \delta^*) = (0, \frac{\pi}{2}, 0, 0)$, $(0, \frac{\pi}{2}, 0, \pi)$, $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0)$, and $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \pi)$ give the same physical mixing matrix $V_{*,6}$ but with different γ_* for the fixed point 6: the differences lie in the fact that the eigenvalues are permuted, and some signs are changed. This is due to the singularity when $\theta_2 = \frac{\pi}{2}$ and $\delta = 0$ or π described in Sec. V: the space of physical mixing matrices when $\theta_2 = \frac{\pi}{2}$, $\delta = 0, \pi$ is only one-dimensional, not two-dimensional.

To unravel this consider γ_* in $(\theta_+, \theta_-, \theta_2)$ coordinates on the hypersurface $\theta_2 = \frac{\pi}{2}$. In the left-hand diagram in Fig. 2, $\theta_+ = \frac{\pi}{2}$ at the two vertices corresponding to fixed point 5 (purple), but θ_- is different at these two vertices, it is not a good coordinate there: similarly $\theta_- = 0$ at the two vertices corresponding to fixed point 6 (green), but θ_+ is different at these two vertices. We therefore use θ_+ for fixed point 5 and θ_- for fixed point 6.

For convenience we define

$$\zeta_{jk} = \frac{z_k}{z'_j(1+z_k)} \quad \text{and} \quad \zeta'_{jk} = \frac{z'_k}{z_j(1+z'_k)}, \quad (\text{C3})$$

as these combinations appear frequently and this notation tidies up the following formulas.

From (21) and Appendix B:

$$\begin{aligned} \beta_+ &= \frac{d\theta_+}{dt} = \frac{3}{32\pi^2} [y_3'^2(\zeta'_{11} - \zeta'_{12}) - y_3^2 \zeta_{31}] \sin 2\theta_+ \quad \text{for } \theta_2 = \frac{\pi}{2}, \quad \delta = 0; \\ \beta_- &= \frac{d\theta_-}{dt} = \frac{3}{32\pi^2} [y_3'^2(\zeta'_{11} - \zeta'_{12}) - y_3^2 \zeta_{31}] \sin 2\theta_- \quad \text{for } \theta_2 = \frac{\pi}{2}, \quad \delta = \pi, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \beta_+}{\partial \theta_+} &= \frac{3}{16\pi^2} [y_3'^2(\zeta'_{11} - \zeta'_{12}) - y_3^2 \zeta_{31}] \cos 2\theta_+ \quad \text{for } \theta_2 = \frac{\pi}{2}, \quad \delta = 0; \\ \frac{\partial \beta_-}{\partial \theta_-} &= \frac{3}{16\pi^2} [y_3'^2(\zeta'_{11} - \zeta'_{12}) - y_3^2 \zeta_{31}] \cos 2\theta_- \quad \text{for } \theta_2 = \frac{\pi}{2}, \quad \delta = \pi. \end{aligned}$$

Using $\theta_+ = \frac{\pi}{2}$, $\delta = 0$ for fixed point 5, and $\theta_- = 0$, $\delta = \pi$ for fixed point 6 yields

$$\lambda_+ = \left. \frac{\partial \beta_+}{\partial \theta_+} \right|_{\theta_+ = \frac{\pi}{2}, \delta = 0} = \frac{3}{16\pi^2} [y_3'^2 (\zeta'_{12} - \zeta'_{11}) + y_3^2 \zeta_{31}] \quad \text{at fixed point 5;}$$

$$\lambda_- = \left. \frac{\partial \beta_-}{\partial \theta_-} \right|_{\theta_- = 0, \delta = \pi} = -\frac{3}{16\pi^2} [y_3'^2 (\zeta'_{12} - \zeta'_{11}) + y_3^2 \zeta_{31}] \quad \text{at fixed point 6.}$$

For $\lambda_2 = \left. \frac{\partial \beta_2}{\partial \theta_2} \right|_*$ on the $\theta_2 = \frac{\pi}{2}$ hypersurface $\theta_* = (0, \frac{\pi}{2}, \frac{\pi}{2})$ and $\theta_* = (\frac{\pi}{2}, \frac{\pi}{2}, 0)$ give different eigenvalues for λ_2 at fixed point 5, and $\theta_* = (0, \frac{\pi}{2}, 0)$ and $\theta_* = (\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ give different eigenvalues for λ_2 at fixed point 6. The reason for this can be again understood from Fig. 2: for fixed point 5 one eigenvalue is from the direction of the 2–5 edge and the other from the direction of the 4–5 edge; for fixed point 6 one eigenvalue is from the direction of the 1–6 edge and the other from the direction of the 3–6 edge: these are labeled accordingly below.

The four eigenvalues of γ_* at each of the fixed points 1–6 obtained from this strategy are

$$\begin{aligned} \mathbf{1)} \quad & (\theta_1^*, \theta_2^*, \theta_3^*, J^*) = (0, 0, 0, 0) & \mathbf{2)} \quad & (\theta_1^*, \theta_2^*, \theta_3^*, J^*) = \left(0, 0, \frac{\pi}{2}, 0\right) \\ \lambda_{1,1} &= \frac{3}{16\pi^2} (y_3^2 \zeta_{11} + y_3^2 \zeta'_{11}), & \lambda_{2,1} &= \frac{3}{16\pi^2} (y_3^2 \zeta_{21} + y_3^2 \zeta'_{12}), \\ \lambda_{1,2} &= \frac{3}{16\pi^2} (y_3^2 \zeta_{22} + y_3^2 \zeta'_{22}), & \lambda_{2,2} &= \frac{3}{16\pi^2} (y_3^2 \zeta_{12} + y_3^2 \zeta'_{21}), \\ \lambda_{1,3} &= -\frac{3}{16\pi^2} [y_3^2 (\zeta_{31} - \zeta_{32}) + y_3^2 (\zeta'_{31} - \zeta'_{32})], & \lambda_{2,3} &= \frac{3}{16\pi^2} [y_3^2 (\zeta_{31} - \zeta_{32}) + y_3^2 (\zeta'_{31} - \zeta'_{32})] = -\lambda_{1,3}, \\ \lambda_{1,J} &= \lambda_{1,1} + \lambda_{1,2} + \lambda_{1,3}; & \lambda_{2,J} &= \lambda_{2,1} + \lambda_{2,2} + \lambda_{2,3}; \\ \mathbf{3)} \quad & (\theta_1^*, \theta_2^*, \theta_3^*, J^*) = \left(\frac{\pi}{2}, 0, \frac{\pi}{2}, 0\right) & \mathbf{4)} \quad & (\theta_1^*, \theta_2^*, \theta_3^*, J^*) = \left(\frac{\pi}{2}, 0, 0, 0\right) \\ \lambda_{3,1} &= -\frac{3}{16\pi^2} (y_3^2 \zeta_{21} + y_3^2 \zeta'_{12}) = -\lambda_{2,1}, & \lambda_{4,1} &= -\frac{3}{16\pi^2} (y_3^2 \zeta_{11} + y_3^2 \zeta'_{11}) = -\lambda_{1,1}, \\ \lambda_{3,2} &= -\frac{3}{16\pi^2} [y_3^2 (\zeta_{11} - \zeta_{12}) - y_3^2 \zeta'_{31}], & \lambda_{4,2} &= \frac{3}{16\pi^2} [y_3^2 (\zeta_{22} - \zeta_{21}) + y_3^2 \zeta'_{32}], \\ \lambda_{3,3} &= -\frac{3}{16\pi^2} [y_3^2 \zeta_{32} + y_3^2 (\zeta'_{22} - \zeta'_{21})] = -\lambda_{4,3}, & \lambda_{4,3} &= \frac{3}{16\pi^2} [y_3^2 \zeta_{32} + y_3^2 (\zeta'_{22} - \zeta'_{21})], \\ \lambda_{3,J} &= \lambda_{3,1} + \lambda_{3,2} + \lambda_{3,3}; & \lambda_{4,J} &= \lambda_{4,1} + \lambda_{4,2} + \lambda_{4,3}; \\ \mathbf{5)} \quad & (\theta_1^*, \theta_2^*, \theta_3^*, J^*) = \left(0, \frac{\pi}{2}, \frac{\pi}{2}, 0\right) \quad \text{or} \quad \left(\frac{\pi}{2}, \frac{\pi}{2}, 0, 0\right) & \mathbf{6)} \quad & (\theta_1^*, \theta_2^*, \theta_3^*, J^*) = \left(0, \frac{\pi}{2}, 0, 0\right) \quad \text{or} \quad \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0\right) \\ \lambda_{5,+} &= \frac{3}{16\pi^2} [y_3^2 \zeta_{31} - y_3^2 (\zeta'_{11} - \zeta'_{12})], & \lambda_{6,-} &= -\frac{3}{16\pi^2} [y_3^2 \zeta_{31} - y_3^2 (\zeta'_{11} - \zeta'_{12})] = -\lambda_{5,+}, \\ \lambda_{5,25} &= \frac{3}{16\pi^2} [y_3^2 (\zeta_{21} - \zeta_{22}) - y_3^2 \zeta'_{32}] = -\lambda_{4,2}, & \lambda_{6,16} &= -\frac{3}{16\pi^2} [y_3^2 \zeta_{22} + y_3^2 \zeta'_{22}] = -\lambda_{1,2}, \\ \lambda_{5,45} &= -\frac{3}{16\pi^2} (y_3^2 \zeta_{12} + y_3^2 \zeta'_{21}) = -\lambda_{2,2}, & \lambda_{6,36} &= \frac{3}{16\pi^2} [y_3^2 (\zeta_{11} - \zeta_{12}) - y_3^2 \zeta'_{31}] = -\lambda_{3,2}, \\ \lambda_{5,J} &= \lambda_{5,+} + \lambda_{5,25} + \lambda_{5,45}; & \lambda_{6,J} &= \lambda_{6,-} + \lambda_{6,16} + \lambda_{6,36}. \end{aligned}$$

Note that, for all six fixed points $A = 1, \dots, 6$, $\lambda_{A,J}$ is the sum of the other three eigenvalues, so

$$\text{Tr}(\gamma_{*,A}) = 2\lambda_{A,J};$$

and the six traces can be put into two subsets $\{1, 3, 5\}$ and $\{2, 4, 6\}$, satisfying

$$\text{Tr}(\gamma_{*,1}) + \text{Tr}(\gamma_{*,3}) + \text{Tr}(\gamma_{*,5}) = \text{Tr}(\gamma_{*,2}) + \text{Tr}(\gamma_{*,4}) + \text{Tr}(\gamma_{*,6}) = 0,$$

corresponding to even and odd permutations of the permutation group S_3 , $\{V_1, V_3, V_5\}$ and $\{V_2, V_4, V_6\}$ in Eq. (25).

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