Spin polarization for massive fermion in a shear flow: Complete results at $O(\partial)$

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(Received 15 December 2024; accepted 24 February 2025; published 18 March 2025)

Motivated by the key role of shear induced polarization in understanding the local spin polarization puzzle of Λ hyperons in heavy ion collisions, we perform a complete analysis of spin polarization of massive fermion in a quantum electrodynamic plasma with shear flow. Apart from the well-known spin-shear coupling in free theory, we include two more collision dependent contributions: one is a nondynamical contribution fixed by shifted spin-averaged distribution in steady state; the other is a dynamical contribution following from spin evolution. Despite of the dependencies on collision, we find the dependencies on coupling drop out in the final results. These contributions can lead to significant enhancement of the spin-shear coupling in phenomenologically interesting regime.

DOI: 10.1103/PhysRevD.111.056023

I. INTRODUCTION

The measurement of spin is one of the most challenging and important experimental tasks. The observation of a global spin polarization of Λ hyperons [1] confirms the interplay between spin and rotation in the relativistic heavy-ion physics [2,3]. Such global spin polarization of final particle can be well described based on a spin-orbit coupling picture [4–6] together with a vorticity in quark-gluon plasma predicted through model calculations [7–9]. However, measurement of Λ hyperon local polarization [10] shows an overall sign difference from theoretical predictions [11–13].

Recently it has been realized that shear tensor can also contribute to spin polarization [14–16]. Phenomenological implementations have shown the right trend toward the measured local polarization results [17–21]. The situation is complicated by the fact that shear cannot be consistently described by collisionless theories. It is known that a shear flow necessarily drives the system into a steady state, in which the distribution function has parametrically large deviation from local equilibrium distribution in weakly coupled medium [22,23]. We illustrated that the large deviation can compensate the smallness in collision term [24,25], giving rise to a collisional contribution to spin polarization, which is parametrically the same as the kinematic contribution obtained in free theory, see also [26].

In the scenario proposed in [2], the spin polarization of Λ hyperons inherits the spin of strange quark. The finite mass of strange quark introduces a further twist to the story. Unlike the massless case where spin is locked to the momentum, spin is an independent degree of freedom for massive quark, and is described by an independent kinetic equation. This leads to an extra dynamical contribution to the spin polarization.

All three contributions can be systematically treated in the framework of collisional quantum kinetic theory (QKT). The spin polarization is described by axial component of Wigner function given by [27–29]

$$\mathcal{A}^\mu = 2\pi\delta(P^2 - m^2)(a^\mu f_A + S^{\mu\nu}_{n,m}\mathcal{D}_\nu f), \tag{1}$$

$a^\mu f_A$ is a dynamical contribution [29–32], which needs to be determined by spin kinetic equation. It can be identified as the Pauli-Lubanski vector, which describes the spin in a given frame specified by the frame vector n_μ. $S^{\mu\nu}_{n,m}\mathcal{D}_\nu f$ is the nondynamical contribution, where $S^{\mu\nu}_{n,m} = \frac{\epsilon^{\mu\nu\alpha\beta}P_\alpha n_\beta}{2(P\cdot n+m)}$ and $\mathcal{D}_\nu f = \partial_\nu f + \Sigma^>_{V\nu} f - \Sigma^<_{V\nu}\bar{f}$. f and $\bar{f} = 1 - f$ are the spin-averaged distribution function and the Pauli blocking factor. $\Sigma^{\gtrless}_{V\nu} = \frac{1}{4}\mathrm{tr}[\gamma_\nu \Sigma^{\gtrless}]$ corresponds to vector component of greater/lesser self-energy accounting for the collisional contribution. The ∂f and Σf terms are recognized as the magnetization current and the displacement current, corresponding to kinematic and collisional contributions to spin polarization respectively [25]. In [24] we determined the collisional contribution for massive fermion in a shear flow, leaving the dynamical contribution undetermined.

In this paper, we fill the gap by determining the dynamical contribution, thus providing complete results for spin polarization at $O(\partial)$. We shall treat the less abundant strange quark as probe to the medium consisting of light quarks and gluons. Since the QKT framework for QCD is not available for now, we study a simplified version of this problem: probe massive fermion in massless QED plasma with a shear flow. In the limit of large fermion mass, Compton scattering and pair annihilation are suppressed. The collisional and dynamical contributions to probe fermion come entirely from Coulomb scattering with medium fermions. The collisional contribution has been studied in our previous work [24], where we assumed the probe fermion in local equilibrium while the medium fermions reach steady state. We found the collisional contribution suppresses the kinematic contribution. In this work, we will consider both probe and medium fermions in steady state. The dynamical contribution will be determined for the first time. We shall show the spin kinetic equation leads to a detailed balance condition with vanishing collision term in the long time limit, which allows us to fix the dynamical contribution.

This paper is organized as follows, in Sec. II we collect and calculate the nondynamical contribution to the massive probe fermion. In Sec. III we review the frame independence of the axial-vector component of Wigner function $\mathcal{A}_\mu$ and use it to determine $\mathcal{A}_\mu$ for massless fermion. We further generalize the procedure to determine the leading mass correction to $\mathcal{A}_\mu$ for massive fermions. In Sec. IV, we derive the collision term of spin kinetic equation. The equation is analyzed in the long time limit, in which we obtain the static solution of $\mathcal{A}_\mu$ and the dynamical term $a_\mu f_A$ from the vanishing of the collision term. In Sec. V we provide conclusion and outlook. Calculation details are presented in the Appendix.

In this paper, we take the mostly negative convention of metrix $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. The Levi-Civita symbol is chosen as $\epsilon^{0123} = -\epsilon_{0123} = +1$. We use majuscule letter for four-dimension covariant momentum such as P^μ and use minuscule latter for its component such as p^0 and its module such as $p = |\vec{p}|$. We use the projector $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ to project a vector onto direction perpendicular to the fluid velocity u^μ, such as $P_\perp^\mu = \Delta^{\mu\nu} P_\nu$ and define $\hat{P}_\perp^\mu = P_\perp^\mu / p$ with $p = (-P_\perp^\mu P_{\perp\mu})^{1/2}$. The symmetrization and antisymmetrization of two symbols are defined through $X_{(\alpha} Y_{\beta)} = X_\alpha Y_\beta + X_\beta Y_\alpha$ and $X_{[\alpha} Y_{\beta]} = X_\alpha Y_\beta - X_\beta Y_\alpha$.

II. SELF-ENERGY CORRECTION

We start with some general analysis based on symmetries. For the spin polarization induced by hydrodynamic gradients, it is naturally to count it as $\mathcal{A}_\mu \sim \mathcal{O}(\partial)$. Consider a fluid where shear tensor is the only flow, $\mathcal{A}_\mu$ can in general be decomposed into

$$\mathcal{A}_\mu = 2\pi\delta(P^2 - m^2)(u_\mu I^P_{\nu\xi} N_{1P} + \hat{P}_{\perp\mu} I^P_{\nu\xi} N_{2P} + J^P_{\mu,\nu\xi} N_{3P} + L^P_{\mu,\nu\xi} N_{4P}) f'_P \sigma^{\nu\xi}. \tag{2}$$

With the shear tensor defined as $\sigma_{\mu\nu} = \frac{1}{2}(\partial^\perp_\mu u_\nu + \partial^\perp_\nu u_\mu) - \frac{1}{3}\Delta_{\mu\nu}\partial\cdot u$, with u^μ being fluid velocity. $f_P \equiv f_V(\beta u\cdot P)$ is the distribution of vector charge, and β is inverse temperature. We have used the abbreviation $f'_P = \partial_{u\cdot P} f_P^{eq}$ and $f_P^{eq} = (\exp(\beta u\cdot P) + 1)^{-1}$. N_{iP} are arbitrary scalar functions of momentum. The projectors $I^P_{\nu\xi}$, $J^P_{\mu,\nu\xi}$, and $L^P_{\mu,\nu\xi}$ are defined as

$$\begin{aligned} I^P_{\xi\nu} &= \hat{P}_{\perp\xi}\hat{P}_{\perp\nu} + \frac{1}{3}\Delta_{\xi\nu}, \\ J^P_{\mu,\xi\nu} &= -\hat{P}_{\perp\xi}\Delta_{\mu\nu} - \hat{P}_{\perp\nu}\Delta_{\mu\xi} + \frac{2}{3}\hat{P}_{\perp\mu}\Delta_{\xi\nu}, \\ L^P_{\mu,\xi\nu} &= \frac{1}{2}(\epsilon_{\mu\xi\lambda\sigma} u^\lambda \hat{P}^\sigma_\perp \hat{P}_{\perp\nu} + \epsilon_{\mu\nu\lambda\sigma} u^\lambda \hat{P}^\sigma_\perp \hat{P}_{\perp\xi}). \end{aligned} \tag{3}$$

As has been discussed in [14], $\mathcal{A}^\mu$ transforms under time reversal as $\mathcal{A}^\mu \to (\mathcal{A}^0, -\mathcal{A}^i)$ and transforms under parity reversal as $\mathcal{A}^\mu \to (-\mathcal{A}^0, \mathcal{A}^i)$. In the absence of parity odd parameter such as μ_5, N_{1P} through N_{4P} can only be parity even functions. In follows that only term with $L_{\mu,\nu\xi}\sigma^{\nu\xi}$ in (2) has the same behavior under time and parity reversal. Thus $\mathcal{A}_\mu$ can be written as

$$\mathcal{A}_\mu = 2\pi\delta(P^2 - m^2)\frac{\epsilon_{\mu\nu\alpha\beta} P^\alpha_\perp u^\beta}{2P\cdot u} P_{\perp\xi}\sigma^{\nu\xi} f'_P N_P. \tag{4}$$

On the other hand, in a collisional QKT [27–29], the axial component of Wigner function for fermion in the local rest frame of the fluid $n_\mu = u_\mu$ is given by (1). $\mathcal{A}_\mu$ contains the dynamical part $a_\mu f_A$ as well as nondynamical part $S^{\mu\nu}_{u,m}\mathcal{D}_\nu f$, with $\mathcal{D}_\nu f = \partial_\nu f + \Sigma^>_{V\nu} f - \Sigma^<_{V\nu}\bar{f}$. With the same logic, we can parametrize the dynamical, kinematic and collisional contributions separately as

$$\mathcal{A}_\mu^a = 2\pi\delta(P^2-m^2)a_\mu f_A = 2\pi\delta(P^2-m^2)\frac{\epsilon_{\mu\nu\alpha\beta}P_\perp^\alpha u^\beta}{2P\cdot u}P_{\perp\xi}\sigma^{\nu\xi}f'_P N_a,$$
$$\mathcal{A}_\mu^\partial = 2\pi\delta(P^2-m^2)S_{\mu\nu}^{m,u}\partial^\nu f_p = 2\pi\delta(P^2-m^2)\frac{\epsilon_{\mu\nu\alpha\beta}P_\perp^\alpha u^\beta}{2P\cdot u}P_{\perp\xi}\sigma^{\nu\xi}f'_P N_\partial,$$
$$\mathcal{A}_\mu^\Sigma = 2\pi\delta(P^2-m^2)S_{\mu\nu}^{m,u}(\Sigma_{V\nu}^> f - \Sigma_{V\nu}^< \bar{f}) = 2\pi\delta(P^2-m^2)\frac{\epsilon_{\mu\nu\alpha\beta}P_\perp^\alpha u^\beta}{2P\cdot u}P_{\perp\xi}\sigma^{\nu\xi}f'_P N_\Sigma, \tag{5}$$

The quantities N_∂ and N_Σ can be further related by scalar kinetic equation. Using the rotational symmetry, $\Sigma_{V\nu}^> f - \Sigma_{V\nu}^< \bar{f}$ in the shear flow can be decomposed as

$$\Sigma_{V\nu}^> f - \Sigma_{V\nu}^< \bar{f} = (u_\nu I_{\alpha\beta}T_{1p} + \hat{P}_{\perp\nu}I_{\alpha\beta}T_{2p} + J_{\nu,\alpha\beta}T_{3p})f'_P\sigma_{\alpha\beta}. \tag{6}$$

Thus $\mathcal{D}_\nu f$ becomes

$$\mathcal{D}_\nu f = P^\alpha\sigma_{\alpha\nu}f'_P + (u_\nu I^{\alpha\beta}T_{1p} + \hat{P}_{\perp\nu}I^{\alpha\beta}T_{2p} + J_\nu^{\alpha\beta}T_{3p})f'_P\sigma_{\alpha\beta}. \tag{7}$$

The scalar kinetic equation $P^\nu\mathcal{D}_\nu f = 0$ [27,28] gives a constraint between the above parameters,

$$T_{1p} = \frac{p^2}{P\cdot u}\left(\frac{T_{2p}}{p} + \frac{2T_{3p}}{p} - 1\right), \tag{8}$$

where $p = (-P_\perp^\mu P_{\perp\mu})^{1/2}$ is the module of the three-momentum. With the decomposition above, the dynamical part, the kinematic part and the collisional part have the following decomposition, and the axial-vector component as the sum of the above three parts $\mathcal{A}_\mu = \mathcal{A}_\mu^a + \mathcal{A}_\mu^\partial + \mathcal{A}_\mu^\Sigma$. Comparing (4) with (5) one can directly find N_P is the sum of three parts

$$N_P = N_a + N_\partial + N_\Sigma = N_a + \frac{P\cdot u}{P\cdot u + m}\left(1 - \frac{2T_{3p}}{p}\right), \tag{9}$$

where in the last equality we have used $N_\partial = \frac{P\cdot u}{P\cdot u+m}$ and $N_\Sigma = -\frac{P\cdot u}{P\cdot u+m}\frac{T_3}{p}$ following from (5) and (6) respectively.

Before moving on to the detailed calculation, we first recall the results in related studies. Results obtained in linear response theory [14] and statistical field theory [15] correspond $N_P = 1$. Chiral kinetic theory also gives $N_P = 1$ in the massless limit [16]. Its extrapolation to massive case corresponds to N_∂ [19]. All three results do not contain collision dependent contributions, which are generically present in N_a and N_Σ. As we are going to see, $\mathcal{A}_\mu$ for massive spin carriers with collisional effect determined in a static state is much more complicated.

We first work out the collisional contribution for probe fermion in the massless QED plasma with a steady shear flow with both probe and medium fermions (and also photons) in steady state. We start by collecting the result in our previous work [25]. Consider a neutral QED plasma with N_f flavor of massless fermions in a shear flow. To the leading logarithmic order, only elastic scatterings among fermions and photons are relevant. The presence of shear flow leads to redistribution of fermions and photons, which has been solved in Refs. [22,23,33]. The deviation from local equilibrium distribution can be parametrized as $\delta f_P = \beta f_P^{eq}(1-f_P^{eq})I_{\mu\nu}^P\sigma^{\mu\nu}\chi_p$ with $I_{\mu\nu}^P$ being a symmetric traceless projector defined in (3). Approximate analytic expressions of χ_p have been obtained in [24], the redistribution for massless medium fermion is characterized by δf_p with χ_p given by

$$\chi_p^{\rm med} = \frac{(2\pi)^3 C_f}{e^4\ln e^{-1}}\beta^2 p^2, \tag{10}$$

with $C_f = \frac{3(1+2N_f)}{4\pi^2 N_f^2}$ and e being the QED coupling constant. This redistribution leads to collisional contribution through the self-energy correction. The collisional contribution of the massless medium fermion includes Coulomb, Compton and annihilation processes [25]. We quote the result in [25] for the spin polarization of massless fermion in a shear flow with vanishing dynamical contribution

$$\begin{aligned}\mathcal{A}_\mu &= 2\pi\delta(P^2)S_{\mu\nu}\mathcal{D}^\nu f_P \\ &= -2\pi\delta(P^2)\frac{\epsilon_{\mu\nu\alpha\beta}P_\perp^\alpha u^\beta}{2P\cdot u}P_{\perp\xi}\beta\sigma^{\nu\xi} \\ &\quad\times\left(1 - \frac{2T_{3p}^{\rm med}}{p}\right)f_P^{\rm eq}(1-f_P^{\rm eq}).\end{aligned} \tag{11}$$

The ratio $\frac{2T_{3p}^{\rm med}}{p}$ delineates the importance of collisional contribution in comparison to the well investigated kinematic contribution. $T_{3p}^{\rm med}$ is the coefficient contributed from the self-energy correction [25].

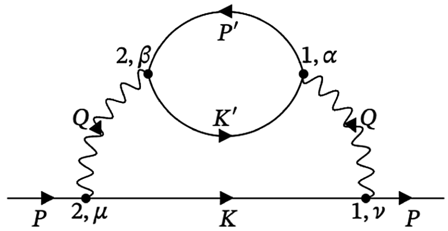

FIG. 1. Two-loop diagram for fermion self-energy containing propagator corrections [28,34]. The unprimed momenta and primed momenta correspond to probe and medium fermions respectively.

$$T_{3p}^{\mathrm{med}} = \frac{C_f N_f (9T\zeta(3)(5Fp - 4T) + \pi^2 p(4T - Fp))}{6p} + \frac{(C_f - C_b)\pi^4 T^2}{48p}\frac{1 + f_{\gamma,P}^{\mathrm{eq}}}{1 - f_P^{\mathrm{eq}}}. \tag{12}$$

T_3 splits into contribution from Coulomb scattering (first line) and from Compton and annihilation (second line), with $F = 1–2f_P^{\mathrm{eq}}$, f_P^{eq}, and $f_{\gamma,P}^{\mathrm{eq}}$ are the distribution functions for fermions and photons at local equilibrium respectively, and $(C_b - C_f) = \frac{2}{\pi^2 N_f}$.

Next, we work out the self-energy part $\mathcal{A}_\mu^\Sigma$ in the spin polarization of the probe fermion. We assume the probe fermion also reaches a steady state. At the leading logarithmic order we consider, the Compton scattering and pair annihilation are suppressed as long as $m \gg eT$. This is because the Compton scattering and pair annihilation involving the probe fermion are screened by the its mass, giving rise to an infrared enhancement factor $\ln\frac{T}{m}$, which is much less than the counterpart $\ln\frac{T}{eT}$ from Coulomb scattering. Therefore, we can obtain the redistribution of probe fermion by consider Coulomb scattering with medium fermions only. The self-energy diagram corresponding to Coulomb scattering is depicted in Fig. 1.

The relevant scalar kinetic equation for the massive probe fermion reads

$$P\cdot\partial f_P = -4e^4(2\pi)^3 \int_{K,P',K'} (2\pi)^4\delta^4(P - K - K' + P')\delta(K^2 - m^2)\delta(P'^2)\delta(K'^2)|\mathcal{M}|^2_{\mathrm{Coul}}(f_P f_{P'}\bar{f}_K\bar{f}_{K'} - \bar{f}_P\bar{f}_{P'}f_K f_{K'}), \tag{13}$$

where $\int_{K,P',K'} = (2\pi)^{-12}\int d^4K d^4P' d^4K'$ and $|\mathcal{M}|^2_{\mathrm{Coul}}$ is the amplitude square for Coulomb scattering with overall coupling constants factored out

$$|\mathcal{M}|^2_{\mathrm{Coul}} = \frac{2N_f \cdot 2(K\cdot P' P\cdot K' + P\cdot P' K\cdot K' - m^2 P'\cdot K')}{(Q^2)^2}. \tag{14}$$

The unprimed momenta and primed momenta correspond to probe and medium fermions respectively. $Q = P - K = K' - P'$ is the small momentum transfer. $2N_f$ in the numerator comes from the N_f flavor fermion and antifermion in the loop. The redistribution of medium fermions is already known. The counterpart of probe fermion can be parametrized as $\delta f_P = \beta f_P^{eq}(1 - f_P^{eq})I_{\mu\nu}^P\sigma^{\mu\nu}\chi_p^{\mathrm{prob}}$. After linearizing the Boltzmann equation and performing the phase space integral, (13) is then turned into a differential equation of χ_p

$$\begin{aligned}\frac{2}{3}p^2 f_P^{eq}(1 - f_P^{eq}) = \frac{e^4\ln e^{-1}}{(2\pi)^4}\frac{4\pi^3T^3N_f}{3p\tilde{v}^4}\Big\{&(\tilde{v}(3\tilde{v}^2 - 1) + (\tilde{v}^2 - 1)^2\eta_p)\chi_p^{\mathrm{prob}} \\ &+ \frac{p\tilde{v}^2}{3T}(Fp(\tilde{v}^2 - 1)\eta_p + Fp\tilde{v} - 2T\tilde{v}^4)\chi_p^{\mathrm{prob}'} - \frac{1}{3}p^2\tilde{v}^2((\tilde{v}^2 - 1)\eta_p + \tilde{v})\chi_p^{\mathrm{prob}''} \\ &+ \frac{72\pi p\zeta(3)C_f}{T^2}(\tilde{v}^2 - 1)(Fp(\tilde{v}^2 - 3)\eta_p + 3Fp\tilde{v} + 4T\tilde{v}^4)\Big\}f_P^{eq}(1 - f_P^{eq}),\end{aligned} \tag{15}$$

where $F = 1–2f_P^{eq}$ and $\tilde{v} = p/p_0$, the rapidity $\eta_p \equiv \mathrm{arctanh}\,\tilde{v} \equiv 2^{-1}\ln[(p_0 + p)/(p_0 - p)]$, with p the module of three-momentum, and $p_0 = (p^2 + m^2)^{1/2}$. The redistribution of the probe fermion in the shear flow can be characterized with χ_p^{prob}. We solve (15) in the phenomenological interesting regime $p \gg T$. An approximate analytical solution can be found by keeping only the leading powers of p in (15), which singles out $\chi_p^{\mathrm{prob}'}$ terms on the right-hand side (rhs). It follows then

$$\chi_p^{\mathrm{prob}} = \frac{(2\pi)^3 C_f^{\mathrm{prob}}}{e^4\ln e^{-1}}\beta^2 p^2. \tag{16}$$

In comparison to (10), $C_f^{\rm prob}$ of the probe fermion is a function of momentum and mass

$$C_f^{\rm prob} = \frac{3}{2\pi^2 N_f}\cdot\frac{\tilde{v}^2}{\tilde{v} + (\tilde{v}^2-1)\operatorname{arctanh}\tilde{v}}. \tag{17}$$

In the massless limit, namely as $\tilde{v}\to 1$, the above coefficient has the limit $C_f^{\rm prob}\to\frac{3}{2\pi^2 N_f}$. This is different from the corresponding coefficient $C_f = \frac{3(1+2N_f)}{4\pi^2 N_f^2}$ of the massless medium fermion since only Coulomb scattering is considered for massive fermion.

With the redistribution of the medium and probe fermion determined above, we can work out $\Sigma^>_{V\nu}f - \Sigma^<_{V\nu}\bar{f}$ for the probe fermion. By using the complete basis for the Clifford algebra, the Wigner function and the self-energy are decomposed into

$$S^< = \mathcal{S}^< + i\mathcal{P}^<\gamma^5 + \mathcal{V}^<_\mu\gamma^\mu + \mathcal{A}^<_\mu\gamma^5\gamma^\mu + \frac{1}{2}\mathcal{S}^<_{\mu\nu}\sigma^{\mu\nu},$$
$$\Sigma^< = \Sigma^<_S + i\Sigma^<_P\gamma^5 + \Sigma^<_{V\mu}\gamma^\mu + \Sigma^<_{A\mu}\gamma^5\gamma^\mu + \frac{1}{2}\Sigma^<_{T\mu\nu}\sigma^{\mu\nu}, \tag{18}$$

and likewise for the greater components, we have used the shorthand notation $\mathcal{A}_\mu \equiv \mathcal{A}^<_\mu$. The calculation is quite similar to those in [24,25], here we just collect the key steps. For the massive probe fermion, only Coulomb scattering contributes at leading logarithmic order, the self-energy is depicted by Fig. 1. From Fig. 1 the greater self-energy is

$$\Sigma^>(P) = 2N_f e^4\int_{K',P',K}(2\pi)^4\delta^4(P-K+P'-K')\delta(K^2-m^2)\delta(P'^2)\delta(K'^2)\epsilon(K\cdot u)\epsilon(P'\cdot u)\epsilon(K'\cdot u)$$
$$\times\gamma^\mu S^>(K)\gamma^\nu D^{22}_{\mu\beta}(P-K)D^{11}_{\alpha\nu}(P-K)\mathrm{Tr}[\gamma^\alpha S^{(0)<}(P')\gamma^\beta S^{(0)>}(K')], \tag{19}$$

with the unprimed momenta and primed momenta correspond to probe and medium fermions respectively. $2N_f$ comes from N_f flavors of fermion and antifermion in the loop of Fig. 1. For simplicity, we will choose Feynman gauge for photon propagator $D^{22}_{\mu\beta}$ and $D^{11}_{\alpha\nu}$, namely $D^{22}_{\mu\beta}(Q)=\frac{ig_{\mu\beta}}{Q^2}$ and $D^{11}_{\alpha\nu}(Q)=\frac{-ig_{\alpha\nu}}{Q^2}$. The vector component of self-energy is obtained through projecting out the vector component of (19). Combining both the lessor and greater components of the self-energy, one then has $\Sigma^>_{V\mu}f_P - \Sigma^<_{V\mu}\bar{f}_P$ as

$$\begin{aligned}\Sigma^>_{V\mu}f_P - \Sigma^<_{V\mu}\bar{f}_P &= 4e^4(2\pi)^3\int_{K',P',K}(2\pi)^4\delta^4(P-K+P'-K')\delta(K^2-m^2)\delta(P'^2)\delta(K'^2)\epsilon(k_0)\epsilon(k_0')\epsilon(p_0')\\
&\quad\times M^{\rm Coul}_\mu(f_{P'}f_P\bar{f}_K\bar{f}_{K'} - \bar{f}_P\bar{f}_{P'}f_Kf_{K'})\\
&= 4e^4(2\pi)^3\int_{K',P',K}(2\pi)^4\delta^4(P-K+P'-K')\delta(K^2-m^2)\delta(P'^2)\delta(K'^2)\epsilon(k_0)\epsilon(k_0')\epsilon(p_0')\\
&\quad\times M^{\rm Coul}_\mu[I^{\hat{P}'}_{\alpha\beta}\chi^{\rm med}_{p'} - I^{\hat{K}'}_{\alpha\beta}\chi^{\rm med}_{k'} + I^{\hat{P}}_{\alpha\beta}\chi^{\rm prob}_{p} - I^{\hat{K}}_{\alpha\beta}\chi^{\rm prob}_{k}]f^{eq}_Pf^{eq}_{P'}\bar{f}^{eq}_K\bar{f}^{eq}_{K'}\beta\sigma^{\alpha\beta}\\
&= T^{\rm prob}_{\mu,\alpha\beta}(P)\beta\sigma^{\alpha\beta},\end{aligned} \tag{20}$$

with

$$M^{\rm Coul}_\mu = \frac{2N_f\cdot 2(P'_\mu K\cdot K' + K'_\mu K\cdot P')}{(Q^2)^2}. \tag{21}$$

where in the second equal sign, we have linearized in the shear tensor, and in the third equal sign we have considered that general analysis from (6) through (8) applies to $\Sigma^>_{V\mu}f_P - \Sigma^<_{V\mu}\bar{f}_P$, so that we can write

$$\Sigma^>_{V\mu}f_P - \Sigma^<_{V\mu}\bar{f}_P = (u_\nu I_{\alpha\beta}T^{\rm prob}_{1p} + \hat{P}_{\perp\nu}I_{\alpha\beta}T^{\rm prob}_{2p} + J_{\nu,\alpha\beta}T^{\rm prob}_{3p})f'_P\sigma_{\alpha\beta}, \tag{22}$$

with

$$\begin{aligned}T^{\rm prob}_{2p} &= \frac{6C_fN_fT\zeta(3)}{p\tilde{v}^4}[(9\tilde{v}^4-7\tilde{v}^2+15)\tilde{v}^2T + (17\tilde{v}^2-30)p\tilde{v}F + 3((\tilde{v}^4-9\tilde{v}^2+10)pF + (3\tilde{v}^2-5)\tilde{v}T)\eta_p]\\
&\quad-\frac{C_f^{\rm prob}N_f\pi^2}{3\tilde{v}^6}[(4\tilde{v}^6+2\tilde{v}^4+9\tilde{v}^2+3)\tilde{v}T - (4\tilde{v}^2+3)p\tilde{v}^2F + ((-2\tilde{v}^4+3\tilde{v}^2+3)p\tilde{v}F + (2\tilde{v}^6+\tilde{v}^4-8\tilde{v}^2-3)T)\eta_p],\end{aligned} \tag{23}$$

$$T_{3p}^{\mathrm{prob}} = \frac{3TC_f N_f \zeta(3)}{2p^3\tilde{v}^3}[2(-3\tilde{v}^4+7\tilde{v}^2-6)p^2\tilde{v}T+(24-19\tilde{v}^2)p^3F+3m^2\tilde{v}((\tilde{v}^2-8)pF+4T\tilde{v})\eta_p] + \frac{C_f^{\mathrm{prob}}N_f\pi^2}{6p^2\tilde{v}^5}[2(-4\tilde{v}^4+9\tilde{v}^2-3)p^2T+(2\tilde{v}^2-3)p^3\tilde{v}F+m^2\tilde{v}(3p\tilde{v}F+(6-14\tilde{v}^2)T)\eta_p]. \tag{24}$$

In massless limit, T_{3p}^{prob} becomes

$$T_{3p}^{\mathrm{prob},m=0} = \frac{3C_f N_f T\zeta(3)(5Fp-4T)}{2p} + \frac{C_f^{\mathrm{prob}}N_f\pi^2(4T-Fp)}{6}. \tag{25}$$

The ratio $R^{\Sigma/\partial} \equiv N_\Sigma/N_\partial = -2T_{3p}^{\mathrm{prob}}/p$ delineates the importance of collisional contribution A_μ^Σ in comparison to the kinematic contribution A_μ^∂. As has been discussed in [25], the sign of this ratio depends on p. The ratio $R^{\Sigma/\partial}$ versus p/T for different masses is presented in Fig. 2. The ratio for $p \lesssim T$ is unreliable because of the use of approximate solution (16).

Note that in the limit $p \gg T$, the ratio $R^{\Sigma/\partial}$ still depends the mass of the probe fermion. In the relativistic limit $p \gg m$,

$$R^{\Sigma/\partial}|_{p\gg m, p\gg T} \to \frac{1}{2}. \tag{26}$$

In the nonrelativistic limit $m \gg p$,

$$R^{\Sigma/\partial}|_{m\gg p} \to -\frac{18C_f N_f \zeta(3)T^2}{p^2} - \frac{2mT}{5p^2}. \tag{27}$$

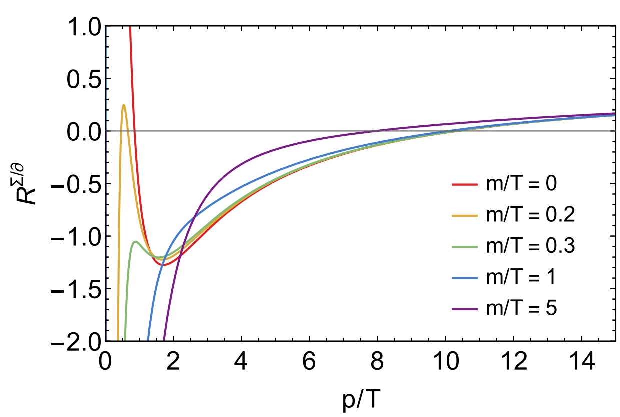

FIG. 2. Ratio $R^{\Sigma/\partial} \equiv N_\Sigma/N_\partial = -2T_{3p}^{\mathrm{prob}}/p$ versus p/T for different masses for $N_f = 2$.

III. FRAME DEPENDENCE

In this section, we constrain the dynamical part from the frame independence of $\mathcal{A}_\mu$. We start our analysis by collecting the known result in [35], see also [36]. It has long been realized that Lorentz covariance and the requirement of conservation of angular momentum together generate a frame-dependent particle position, which is a tunnelinglike motion of the particle known as the side-jump. For a massless fermion with helicity λ, the Lorentz covariant current in presence of a 2-by-2 collision $PK \to P'K'$ in a frame specified by vector n^μ is given by

$$J_\lambda^\mu = P^\mu f_\lambda + \lambda S_n^{\mu\nu}\partial_\nu f_\lambda + \lambda \int_{KK'P'} C^\lambda \bar{\Delta}^\mu, \tag{28}$$

with $\lambda = \pm$ corresponding to right/left-handed particles. $\lambda S_n^{\mu\nu} = \lambda \frac{\epsilon^{\mu\nu\alpha\beta}P_\alpha n_\beta}{2P\cdot n}$ is the spin tensor for fermion with helicity λ in frame n, $\lambda\bar{\Delta}^\mu \equiv \lambda\Delta_{\bar{n}n}^\mu = \lambda\frac{\epsilon^{\mu\alpha\beta\gamma}P_\alpha\bar{n}_\beta n_\gamma}{2(P\cdot n)(P\cdot\bar{n})}$ is the side-jump of the particle between the "no-jump frame" $\bar{n} = (P+K)/\sqrt{s}$ and the arbitrary frame n^μ. $\int_{KK'P'}$ is the phase space integral and C^λ is the collision kernel with the following explicit form [37].

$$C^\lambda = |\mathcal{M}|^2(2\pi)^4\delta(P+K-P'-K')[f_{\bar{n}}^\lambda(P')f_{\bar{n}}^\lambda(K') \times(1-f_{\bar{n}}^\lambda(P))(1-f_{\bar{n}}^\lambda(K)) - (P,K\leftrightarrow P',K')], \tag{29}$$

with $f_{\bar{n}}$ being the distribution function in the no-jump frame. The axial-vector current can be constructed from J_λ^μ as

$$\mathcal{A}^\mu = J_+^\mu - J_-^\mu = P^\mu f_A + S_n^{\mu\nu}\partial_\nu f_P + \int_{KK'P'} C[f_P]\bar{\Delta}^\mu, \tag{30}$$

With a little bit of algebra, one can translate (30) into the corresponding expression in [38]

$$\mathcal{A}^\mu = 2\pi\delta(P^2)(P^\mu f_A + S_n^{\mu\nu}\mathcal{D}_\nu f_P), \tag{31}$$

The generalization from massless to massive case (1) is minimal [29]. By construction, $a^\mu f_A$ is identified as the Pauli-Lubanski vector which describes the spin in a given frame. It is a dynamical variable which needs to be determined by the kinetic theory. On the other hand, $S_n^{\mu\nu}\partial_\nu f$ can be recognized as the magnetization current and $S_n^{\mu\nu}(\Sigma_{V\nu}^> f - \Sigma_{V\nu}^< \bar{f})$ as the displacement current.

Since $\mathcal{A}^\mu$ is frame independent, it requires that the frame dependences in dynamical and nondynamical part to cancel. As we shall show below this requirement can fully fix f_A in the massless case and can fix the leading mass correction to $a^\mu f_A$ in the massive case. We begin with the massless case, in which the requirement leads to

$$\begin{aligned} P^\mu(f'_A - f_A) &= (S_n^{\mu\nu} - S_{n'}^{\mu\nu})\mathcal{D}_\nu f_P \\ &= (P^\nu \Delta^\mu_{nn'} - P^\mu \Delta^\nu_{nn'})\mathcal{D}_\nu f_P, \end{aligned} \tag{32}$$

where f'_A and f_A are the axial-charge in frame n' and n respectively. The frame dependence in f_P occurs only at higher order in gradient, so we may take f_P in any frame. Using the kinetic equation $P\cdot\mathcal{D}f_P = 0$, we easily find

$$f'_A - f_A = -\Delta^\nu_{nn'}\mathcal{D}_\nu f_P. \tag{33}$$

We proceed by writing down the general frame independent decomposition of $\mathcal{D}_\nu f_P$ with the shear tensor $\sigma_{\alpha\beta}$ as

$$-\mathcal{D}_\nu f_P = P^\alpha P^\beta \sigma_{\alpha\beta}(A_1 P_\nu + A_2 u_\nu) + A_3 P^\mu \sigma_{\nu\mu}. \tag{34}$$

The term proportional to P_ν vanishes upon contraction with $\Delta^\nu_{nn'}$. The coefficients A_2 and A_3 are related by the kinetic equation $P\cdot\mathcal{D}f_P = 0$ as $A_2 = -\frac{1}{P\cdot u}A_3$. Thus we have

$$f'_A - f_A = \frac{\epsilon^{\mu\nu\alpha\beta}P_\nu n_\alpha n'_\beta}{2(P\cdot n)(P\cdot n')}\left(-\frac{P^\xi P^\rho \sigma_{\xi\rho}}{P\cdot u}u_\mu + p^\xi\sigma_{\mu\xi}\right)A_3. \tag{35}$$

Using the following contracted Schouten identity

$$\begin{aligned} &(\epsilon^{\mu\nu\alpha\beta}P^\xi + \epsilon^{\nu\alpha\beta\xi}P^\mu + \epsilon^{\alpha\beta\xi\mu}P^\nu + \epsilon^{\beta\xi\mu\nu}P^\alpha + \epsilon^{\xi\mu\nu\alpha}P^\beta) \\ &\quad\times P_\nu n_\alpha n'_\beta \frac{P^\rho\sigma_{\xi\rho}u^\mu}{(P\cdot n)(P\cdot n')(P\cdot u)} = 0, \end{aligned} \tag{36}$$

and noting the middle term vanishes by the on-shell condition $P^2 = 0$, we obtain

$$f'_A - f_A = \left(\frac{\epsilon^{\mu\nu\alpha\beta}u_\nu P_\alpha n'_\beta P^\xi\sigma_{\mu\xi}}{2P\cdot n'} - (n'\to n)\right)\frac{A_3}{P\cdot u}. \tag{37}$$

It is solved by $f_A = \frac{\epsilon^{\mu\nu\alpha\beta}u_\nu P_\alpha n_\beta P^\xi\sigma_{\mu\xi}}{2P\cdot n}\frac{A_3}{P\cdot u}$. We cannot add another frame-independent function to f_A, which is a pseudoscalar. The only possible structure $\sim P^\mu P^\xi \sigma_{\mu\xi}$ is excluded by parity. This is in contrast to the case with vorticity, where one can have $\sim P^\mu\omega_\mu$.

For massive fermion, we use a frame independent vector N^μ to parametrize $a^\mu f_A$ as

$$a^\mu f_A + \frac{\epsilon^{\mu\nu\alpha\beta}P_\alpha n_\beta \mathcal{D}_\nu f_P}{2(P\cdot n + m)} \equiv N^\mu. \tag{38}$$

N^μ should be chosen such that f_A has the correct massless limit and a^μ satisfies the constraint $P\cdot a = P^2 - m^2 = 0$ [29]. Recall that for $n^\mu = u^\mu$, with $f_A = 0$, we have the following explicit expression for N^μ in the massless limit.

$$\begin{aligned} N^\mu(m=0) &= \frac{\epsilon^{\mu\nu\alpha\beta}P_\alpha u_\beta \mathcal{D}_\nu f_P}{2P\cdot u} \\ &= \frac{\epsilon^{\mu\nu\alpha\beta}P_\alpha u_\beta}{2P\cdot u}P_{\perp\xi}\sigma^{\nu\xi}f'_P\left(1 - \frac{2T_{3p}(m=0)}{p}\right). \end{aligned} \tag{39}$$

In small mass limit, we choose the corresponding N^μ to have the same expression as above except with $T_{3p}(m=0)$ replaced by the massive counterpart T_{3p}.

$$N^\mu = \frac{\epsilon^{\mu\nu\alpha\beta}P_\alpha u_\beta}{2P\cdot u}P_{\perp\xi}\sigma^{\nu\xi}f'_P\left(1 - \frac{2T_{3p}}{p}\right). \tag{40}$$

In writing down this ansatz, we have assumed that $\mathcal{A}^\mu$ is an even function of m. This is true for $\mathcal{A}^\mu$ in free theory in background vorticity and magnetic field after a cancellation of odd terms in m [29]. We assume this also holds for the shear case. This assumption excludes first order term of m in the small mass limit. Inserting the decomposition (7) and the ansatz (40) back to (38), and using the Schouten identity, the dynamical term $a^\mu f_A$ can be rewritten as

$$\begin{aligned} a^\mu f_A = &\left(P^\mu \frac{\epsilon^{\nu\alpha\beta\rho}P_\alpha n_\beta u_\nu}{2(P\cdot n + m)P\cdot u} + m^2\frac{\epsilon^{\mu\nu\alpha\rho}n_\alpha u_\nu}{2(P\cdot n + m)P\cdot u}\right. \\ &\left. + m\frac{\epsilon^{\mu\nu\alpha\rho}P_\alpha u_\nu}{2(P\cdot n + m)P\cdot u}\right)\left(\frac{2T_{3p}}{p} - 1\right)f'_P P^\xi \sigma_{\xi\rho} \\ &- m^2\frac{\epsilon^{\mu\nu\alpha\beta}P_\alpha n_\beta u_\nu}{2(P\cdot n + m)P\cdot u}\frac{T_{2p}}{p^3}f'_P P^\xi P^\sigma \sigma_{\xi\sigma}. \end{aligned} \tag{41}$$

It is easy to find from the above expression $P^\mu a_\mu = 0$ is satisfied. In small mass limit, taking $n^\mu = u^\mu$, we have

$$a^\mu f_A = m\left(1 - \frac{2T_{3p}(m=0)}{p}\right)\frac{\epsilon^{\mu\nu\alpha\beta}P_\alpha u_\beta}{2(P\cdot u)^2}f'_P P^\xi\sigma_{\xi\nu} + \mathcal{O}(m^2). \tag{42}$$

Now we comment on the uniqueness of the ansatz: the structure of N^μ is fixed by Lorentz covariance and parity. The normalization is subject to possible mass correction as $1 - \frac{2T_{3p}(m=0)}{p} \to 1 - \frac{2T_{3p}(m=0)}{p} + O(m^2)$. However, the $O(m^2)$ correction will not affect our result (42). The dynamical part beyond small mass limit can only be obtained through detailed balance condition. As we are going to see, (42) agrees with the result based on detailed balance.

As a by product, we find the dynamical contribution nicely explains the discrepancy between field theory approaches [14,15] and kinetic approach [29] on shear induced spin polarization for massive fermions. In the kinetic approach [29], if one consider only thermal shear, the polarization should be

$$N_{\mu}^{ki}=a_{\mu}f_A+\frac{\epsilon_{\mu\nu\alpha\beta}P_{\perp}^{\alpha}u^{\beta}}{2(P\cdot u+m)}P_{\perp\xi}\sigma^{\nu\xi}f_P', \tag{43}$$

which contains both kinematic and dynamical contributions. While in field theory approaches such as linear response theory [14], both kinematic and dynamical contributions are included implicitly, giving the shear induced polarization

$$N_{\mu}^{th}(P)=\frac{\epsilon_{\mu\nu\alpha\beta}P_{\perp}^{\alpha}u^{\beta}}{2P\cdot u}P_{\perp\xi}\sigma^{\nu\xi}f_P'. \tag{44}$$

Obviously, in the massless case, both expressions above agree with each other. Turning off the collisional contribution by setting $T_{3p}^{prob}=0$, and inserting $O(m)$ term of $a_{\mu}f_A$ (42) into (43), then the difference between (43) and (44) will be of $O(m^2)$ order, which means the $O(m)$ term in (42) precisely accounts for the difference between the two approaches in the small mass limit. This is also proved by our numerical solution in the next section.

IV. DETAILED BALANCE

A. Collision term

Now we specify what we mean by detailed balance. For a plasma with steady shear flow, we expect f_P and $\mathcal{A}^{\mu}$ to reach steady functions [39]. f_P satisfies the Boltzmann equation. It is known long ago that the presence of a steady shear will drive f_P away from the local equilibrium distribution such that the collision term balance the gradient term on the left-hand side (lhs) of the kinetic equation. The situation of $\mathcal{A}^{\mu}$ is different. $\mathcal{A}^{\mu}$ satisfies a separate axial kinetic equation, which schematically reads

$$P\cdot\partial\mathcal{A}^{\mu}=C_{\mathcal{A}}^{\mu}. \tag{45}$$

With our counting $f_A\sim O(\partial)$ and $f_P\sim O(\partial^0)$, $\mathcal{A}^{\mu}\sim O(\partial)$, thus the lhs is counted as $\sim O(\partial^2)$. It follows that the collision term should vanish at $O(\partial)$. This is to be referred as detailed balance condition.

The evolution of spin involves diffusion of the initial spin of the probe fermion as well as polarization induced by the shear flow through collision with medium fermions. Since we are going to use the detailed balance condition of the spin kinetic equation, it is required to evaluate the collision terms up to $\mathcal{O}(\partial)$. To $\mathcal{O}(\partial)$ order, the collision terms for axial-vector components follow from Kadanoff-Baym equation $C_{\mathcal{A}}^{\mu}=C_{\mathcal{A}}^{>\mu}-C_{\mathcal{A}}^{<\mu}$, with $C_{\mathcal{A}}^{>\mu}$ given by

$$\begin{aligned}C_{\mathcal{A}}^{>\mu}=&-p_{\mu}\Sigma_{A\nu}^{>}\mathcal{V}^{<\nu}+p_{\nu}\Sigma_{A\mu}^{>}\mathcal{V}^{<\nu}+m\Sigma_{S}^{>}\mathcal{A}_{\mu}^{<}+p_{\rho}\Sigma_{V}^{>\rho}\mathcal{A}_{\mu}^{<}\\&-\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}(\partial^{\sigma}\Sigma_{V}^{>\nu})\mathcal{V}^{<\rho}+\frac{m}{2}\epsilon_{\rho\sigma\lambda\mu}\Sigma_{T}^{>\rho\sigma}\mathcal{V}^{<\lambda}\\&+\frac{1}{2}\epsilon_{\rho\sigma\lambda\mu}(\Sigma_{V}^{>\rho}+\Sigma_{V}^{<\rho})\Sigma_{V}^{>\sigma}\mathcal{V}^{<\lambda},\end{aligned} \tag{46}$$

$C_{\mathcal{A}}^{<\mu}$ can be obtained by exchanging $>\leftrightarrow<$. The last term in (46) has been ignored in previous studies, since it contains an extra power of Σ_V thus is superficially higher order in coupling constant. However, in a steady state, distribution functions in $(\Sigma_V^{>\sigma}\mathcal{V}^{<\lambda}-\ >\leftrightarrow<)$ should be replaced by $f\to f_{eq}+\delta f$ with f_{eq} and δf corresponding to local equilibrium distribution and steady state deviation. The f_{eq} term vanishes by detailed balance, while terms linear in δf has additional dependence on coupling constant as (16), which precisely cancels the counterpart in $(\Sigma_V^{>\sigma}\mathcal{V}^{<\lambda}-\ >\leftrightarrow<)$ [24]. Besides, since δf is first order in gradient, the last term in (46) is of $\mathcal{O}(\partial)$ order. Hence the last term in (46) contributes at the same order of coupling constant and gradient as the other terms.

As reasoned earlier, for $m\gg eT$, only Coulomb scattering between probe fermion and medium fermions is relevant to leading logarithmic order. Figure 1 is the self-energy diagram corresponding to the Coulomb scattering of the massive probe fermion (P and K) with the massless medium fermion (P' and K'). Writing all the components of the self-energy accordingly, and substituting into (46), the collision terms for the Coulomb scattering is

$$\begin{aligned}C_{\mathcal{A}}^{\mu}=-\int_{K,Q,K',P'}\{&M_{\mu\nu}^{A1}(\bar{f}_Kf_{P'}\bar{f}_{K'}+f_K\bar{f}_{P'}f_{K'})N^{\nu}(P)+M_{\mu\nu}^{A2}(f_{P'}\bar{f}_{K'}f_P+\bar{f}_{P'}f_{K'}\bar{f}_P)N^{\nu}(K)\\&+M_{\mu\nu}^{A3}(\bar{f}_{K'}f_P\bar{f}_K+f_{K'}\bar{f}_Pf_K)S_u^{\nu\rho}\mathcal{D}_{\rho}f_{P'}+M_{\mu\nu}^{A4}(f_{P'}f_P\bar{f}_K+\bar{f}_{P'}\bar{f}_Pf_K)S_u^{\nu\rho}\mathcal{D}_{\rho}f_{K'}\\&+M_{\mu\nu}^{A5}(\bar{f}_Kf_{P'}\bar{f}_{K'}+f_K\bar{f}_{P'}f_{K'})\mathcal{D}^{\nu}f_P+M_{\mu\nu}^{A6}(f_{P'}\bar{f}_{K'}f_P+\bar{f}_{P'}f_{K'}\bar{f}_P)\mathcal{D}^{\nu}f_K\},\end{aligned} \tag{47}$$

with $\int_{K,Q,K',P'}=2\pi\delta(P^2-m^2)\int\frac{d^4Kd^4Qd^4K'd^4P'}{((2\pi)^4)^4}(2\pi)^8\delta(P-K-Q)\delta(Q+P'-K')\delta(K^2-m^2)\delta(P'^2)\delta(K'^2)$, and effective amplitudes $M_{\mu\nu}^{Ai}$ are given in Appendix (A1).

The collision term above is aimed to describe the evolution of massive probe spin in a steady shear flow. The first line of (47) describes the spin diffusion of the probe fermion, where $N^\nu(P)$ and $N^\nu(K)$ are the spin of the massive probe fermion defined in (38). The rest of terms in (47) are the polarization effect. The second line delineates the polarization contributed from spin of the medium fermion, where $S_u^{\nu\rho}\mathcal{D}_\rho f_{P'}$ and $S_u^{\nu\rho}\mathcal{D}_\rho f_{K'}$ are the spin polarization of the massless medium fermion at local equilibrium (31). The third line is the polarization effect induced by inhomogeneous distribution as well as the redistribution of the probe fermion. The collision terms delineate the competition between spin diffusion and polarization. When the evolution of the probe spin reaches a steady state, the diffusion and polarization will balance each other, resulting in a vanishing collision term. It is worth noticing that the self-energy correction enters the collision terms through both $S_u^{\nu\rho}\mathcal{D}_\rho f_{P'}(S_u^{\nu\rho}\mathcal{D}_\rho f_{K'})$ for medium fermions and $\mathcal{D}^\nu f_P(\mathcal{D}^\nu f_K)$ for probe fermion. With these quantities from (12) for medium fermions and from (23) and (24) for probe fermions, we are ready to complete the phase space integral and determine $N^\nu(P)$ through detailed balance condition.

Before carrying out the phase space integral, it is useful to decompose $C_{\mathcal{A}}^\mu$. Since $C_{\mathcal{A}}^\mu$ has the same parity and time-reversal properties as $\mathcal{A}^\mu$, we may as well decompose it as

$$C_{\mathcal{A}\mu} = C_A L^P_{\mu,\xi\nu}\beta\sigma^{\xi\nu}, \tag{48}$$

with $L_P^{\mu,\xi\lambda}$ defined in (3). C_A is a scalar function of momentum, it is obtained from the collision term (47) with straightforward but tedious calculation

$$\begin{aligned}C_A = 16N_f e^4(2\pi)^3\int_{K,Q,K',P'}\frac{1}{(Q^2)^2}\Bigg\{&\frac{J_1}{2P\cdot uP_\perp^2}\left(N_P-1+\frac{2T_{3p}^{\rm prob}}{p}\right)+\frac{J_2}{2K\cdot uP_\perp^2}\left(N_K-1+\frac{2T_{3k}^{\rm prob}}{k}\right)\\&+\frac{m^2}{2K\cdot uP_\perp^2}\left(J_3N_K-J_4\left(1-\frac{2T_{3k}^{\rm prob}}{k}\right)-J_5\frac{T_{2k}^{\rm prob}}{k}\right)+\frac{m^2J_6}{2P\cdot uP_\perp^2}\left(1-\frac{2T_{3p}^{\rm prob}}{p}\right)\\&+\frac{m^2J_7}{2K'\cdot uP_\perp^2}\left(1-\frac{2T_{3k'}^{\rm med}}{k'}\right)-\frac{m^2J_8}{2P'\cdot uP_\perp^2}\left(1-\frac{2T_{3p'}^{\rm med}}{p'}\right)\Bigg\}\bar{f}_K^{eq}\bar{f}_{K'}^{eq}f_{P'}^{eq}f_P^{eq},\end{aligned} \tag{49}$$

where the J_i are functions of momentum P, K, P', K' defined in (A2), $T_2^{\rm prob}$, $T_3^{\rm prob}$, and $T_3^{\rm med}$ comes from the self-energy correction with their explicit expressions given in (12), (23), and (24). The task of determining the local equilibrium $\mathcal{A}^\mu$ from the detailed balance is now turned into finding the coefficient N_P that eliminate C_A.

In the massless limit $C_A = 0$ has an exact solution, with terms proportional to m^2 vanishing in (49), one can directly read from (49) that

$$N_P = 1 - 2T_{3p}^{\rm prob}/p. \tag{50}$$

From (9), such solution of massless fermion indicates $N_a = 0$, in other word $P_\mu f_A = 0$ in the massless case. This agrees with our analysis in the Sec. III and [16,40]. For massive fermion, there is no such simple solution, however, the vanishing of collision term gives a constraint on N_P and N_K, which leads to a complicated integral equation. Fortunately, it can be simplified to a differential equation in the leading logarithmic order approximation: With the probe fermion and medium fermion being hard fermions, and the momentum transfer being soft $eT \ll q_0, q \ll T$, N_K can be expanded around N_P as

$$N_K = N_{P-Q} = N_P - Q^\nu\partial_{P^\nu}N_P + \frac{1}{2}Q^\rho\partial_{P^\rho}Q^\nu\partial_{P^\nu}N_P + \mathcal{O}(Q^3). \tag{51}$$

Since N_P is only a function of $|\vec{p}|$, the expansion above can be further simplified to be

$$N_K = N_P - q\cos\theta_q N'_P + \frac{1}{2p}(\sin^2\theta_q N'_P + \cos^2\theta_q N''_P)q^2 + \mathcal{O}(q^3), \tag{52}$$

with $\cos\theta_q \equiv \hat{q}\cdot\hat{p}$, $N'_P = \partial_p N_P$ and $q^2 = -Q_\perp^\mu Q_{\perp\mu}$. The calculation details of momentum integral are presented in the Appendix. After momentum integral, the detailed balance condition is converted into a differential equation of N_P

$$C_A = c_{\rm diff}^{(0)}N_P + c_{\rm diff}^{(1)}N'_P + c_{\rm diff}^{(2)}N''_P + c_{\rm pol} = 0, \tag{53}$$

where $c_{\rm diff}^{(0)}N_P + c_{\rm diff}^{(1)}N'_P + c_{\rm diff}^{(2)}N''_P$ is the diffusion of probe spin, leading to decrease of the probe spin with $c_{\rm diff}^{(0)} < 0$. $c_{\rm pol}$ is the polarization effect, which contains contribution from different parts $c_{\rm pol} = c_{\rm pol}^{\rm med\partial} + c_{\rm pol}^{\rm med\Sigma} + c_{\rm pol}^{\rm prob\partial} + c_{\rm pol}^{\rm prob\Sigma}$. Definitions and explicit expressions of all parts are given in the Appendix. $c_{\rm pol}^{\rm med\partial}$ is polarization effect contributed from the magnetization current of the medium fermion, $c_{\rm pol}^{\rm med\Sigma}$ from the displacement current of the medium fermion, $c_{\rm pol}^{\rm prob\partial}$ from the inhomogeneity of the probe distribution, and $c_{\rm pol}^{\rm prob\Sigma}$ from the redistribution of the probe fermion in the shear flow.

The competition between polarization and diffusion rate can be characterized through the ratio $-c_{\rm pol}/c_{\rm diff}^{(0)}$, which depends on temperature, momentum, and mass of the probe. The ratio between polarization and diffusion can be simplified in certain limit. In the relativistic limit $p \gg m$, the ratio between polarization and diffusion effect approaches

$$-\frac{c_{\rm pol}}{c_{\rm diff}^{(0)}}\bigg|_{p\gg m} = \frac{3}{2}, \tag{54}$$

wherein the polarization effect from the medium fermion does not contribute in such limit $c_{\rm pol}^{\rm med\partial}/c_{\rm diff}^{(0)} \to 0$, $c_{\rm pol}^{\rm med\Sigma}/c_{\rm diff}^{(0)} \to 0$, while that from the inhomogeneity of probe fermion $-c_{\rm pol}^{\rm prob\partial}/c_{\rm diff}^{(0)} \to 1$, $-c_{\rm pol}^{\rm prob\Sigma}/c_{\rm diff}^{(0)} \to 1/2$. In the nonrelativistic limit $m \gg p$, the ratio between polarization and diffusion effect is

$$-\frac{c_{\rm pol}}{c_{\rm diff}^{(0)}}\bigg|_{m\gg p} = \frac{1}{2} + \left(\frac{\pi^2(C_b - C_f)}{16} - C_f N_f\left(\frac{9\zeta(3)}{4\pi^2} + \ln 2\right)\right) + \frac{T(7m - 225 C_f N_f T\zeta(3))}{50p^2}$$
$$+ \tanh\left(\frac{m}{2T}\right)\left(\frac{173}{600} + \frac{21 C_f N_f T\zeta(3)}{5m}\right) - \frac{128T}{35m} + \frac{141 C_f N_f \zeta(3) T^2}{5m^2}. \tag{55}$$

The magnetization current of the medium fermion spin $-c_{\rm pol}^{\rm med\partial}/c_{\rm diff}^{(0)} \to 1/4$ and inhomogeneity of probe fermion $-c_{\rm pol}^{\rm prob\partial}/c_{\rm diff}^{(0)} \to 1/4$ make the first $1/2$ in the first line. The second and third terms in the first line come from the displacement current of the medium fermion $c_{\rm pol}^{\rm med\Sigma}$, which also approaches a constant in such limit. The redistribution of the probe fermion $c_{\rm pol}^{\rm prob\Sigma}$ contributes to the second line, such effect is linear in m in the large mass limit. Coefficients in (53) under both limits are presented in the Appendix.

B. Boundary condition and solution

In order to solve the differential equation (53), boundary conditions are required. We first work out the boundary condition. With the coefficients in the large momentum limit $p \gg m$ (A20), the corresponding differential equation can be solved analytically, keeping leading and subleading terms, the solution of the differential equation becomes

$$N_P|_{p\gg m} = \frac{3}{2} + \frac{21T}{2p}. \tag{56}$$

The homogeneous solution is discarded since without the inhomogeneous term sourced by shear, detailed balance for axial component would favor a vanishing axial component for the Wigner function. The solution above agrees with our analysis from (49) upon using $T_{3p}^{{\rm prob},m=0}$ (25) in the large momentum limit.

On the other hand, with the coefficients in the small momentum limit $p \ll m$ (A19), the analytical solution of the differential equation reads

$$N_P|_{p\ll m} = \frac{T(7m - 225T\zeta(3)C_f N_f)}{50p^2} + \frac{1}{2} + \left(\frac{\pi^2(C_b - C_f)}{16} - C_f N_f\left(\frac{9\zeta(3)}{4\pi^2} + \ln 2\right)\right)$$
$$+ \tanh\left(\frac{m}{2T}\right)\left(\frac{27 T C_f N_f \zeta(3)}{10m} + \frac{67}{200}\right) - \frac{38789T}{10500m} + \frac{3087 C_f N_f \zeta(3) T^2}{105m^2}, \tag{57}$$

where we keep the leading and subleading order. We see the enhancement of polarization in the infrared (IR) as $N_p \sim p^{-2}$. This can be traced back to the corresponding term in (55). As remarked before, this comes from collisional effect through redistribution of the probe fermion. An enhancement of the same origin has been found in the massless counterpart [25].

Between both limits, the differential equation can only be solved numerically. We take analytical solutions above as the boundary condition for numerically solving the

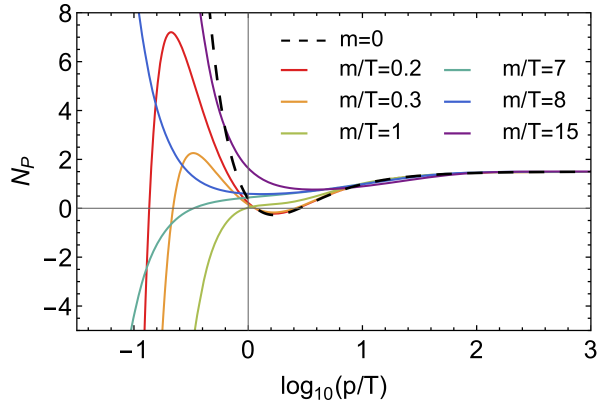

FIG. 3. N_P defined in (4) as a function of $\log_{10}(p/T)$ for different masses.

differential equation (53). The differential equation for two flavor medium fermion $N_f = 2$ is solved using shooting method. Numerical solution N_P (4) as a function of p/T for different masses is presented in Fig. 3. The black dashed line is the exact solution (50) in the massless limit, the colored lines are solutions with different masses. In the large momentum limit $p \gg m$ and also $p \gg T$, the solutions all approaches $3/2$ as fixed by our boundary condition (56). For moderate p, the curves deviate from each other. For $p \lesssim T$, the solution becomes unreliable. An input for the redistribution of probe fermions beyond the approximate solution (16) is needed.

Recall that from the analysis (9), N_P is the sum of dynamical contribution N_a, kinematic contribution N_∂ and collisional contribution N_Σ. We have calculated N_∂ and N_Σ in the Sec. II, subtracting these two terms from N_P will give the dynamical part N_a. We plot the dynamical part N_a as a function of momentum in Fig. 4. It shows that dynamical part $a_\mu f_A$ gets strongly suppressed at large momentum, and $a_\mu f_A \to 0$ when $p/T \to \infty$.

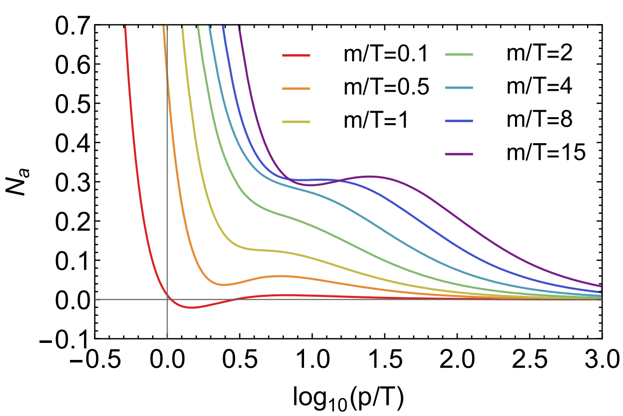

FIG. 4. The dynamical part N_a as a function of $\log_{10}(p/T)$ for different masses.

The dependence of the dynamical part N_a on mass is presented in Fig. 5. In the massless limit N_a always vanishes, this is in consistency with analysis (50). In the left panel, we present N_a at small mass for various momentum. The solid lines are the numerical solution, while the dashed lines give a comparison with N_a obtained through frame independence (42), namely

$$N_a|_{p\gg m} = \frac{m}{p}\left(1 - \frac{2T_{3p}(m=0)}{p}\right). \tag{58}$$

Clearly, in the limit $p \gg m$, (42) is a good approximation to N_a. In the right panel, numerical solutions to N_a in a wide range of mass are presented as the solid lines. For relatively large mass, $a_\mu f_A$ increases almost linearly with m/T. The dot-dashed lines represent the analytical approximation of N_a in the case $m \gg p$. This approximation is obtained through subtracting $N_\partial + N_\Sigma$ from (57) and taking the corresponding limit, giving

$$N_a|_{m\gg p} = \frac{17mT}{50p^2} + C_f N_f \left(\frac{9}{4}\zeta(3)\left(\frac{2T^2}{p^2} - \frac{1}{\pi^2}\right) - \ln 2\right) + \frac{241}{1200}. \tag{59}$$

As is shown in the figure, both lines agree when $m \gg p$.

Now we discuss implication for spin polarization in heavy ion collisions. Despite the complicated dependence of N_P on m and p, we see from Fig. 3 that the complete polarization is in general larger than $N_P = 1$ obtained ignoring collisions [14–16]. If we naively apply the QED results to QCD, taking p to be a few GeV, T to be the freeze-out temperature and fermion mass to be ~100 MeV for the strange quark, we seem to be in the limit $p \gg T$ and $p \gg m$, in which $N_P \to 3/2$. It indicates that collisional effect ignored in previous phenomenological studies actually enhances the spin-shear coupling by 50%! This is in contrast to the suppression found by us in [24]. The difference follows from different assumptions on distribution of the probe fermion: in the present work, the probe fermion is in steady state, while in [24], the probe fermion is assumed to be in local equilibrium. Phenomenologically, the QGP may be viewed as a steady shear flow, since the shear flow relaxes on the hydrodynamic scale, which is much slower than the relaxation of plasma constituents. Since the behavior of the kaon and pion elliptic flow as a function of p_T are nearly identical, one can expect equal elliptic flow of s and u, d quarks. With the s quarks be treated as the massive probe fermion while u and d quarks as the massless medium fermion, we expect both probe and medium to reach a steady state. The enhancement seems to be phenomenologically favored as existing studies suggest that the collisionless spin-shear coupling may not be sufficient to account for the local spin polarization [41].

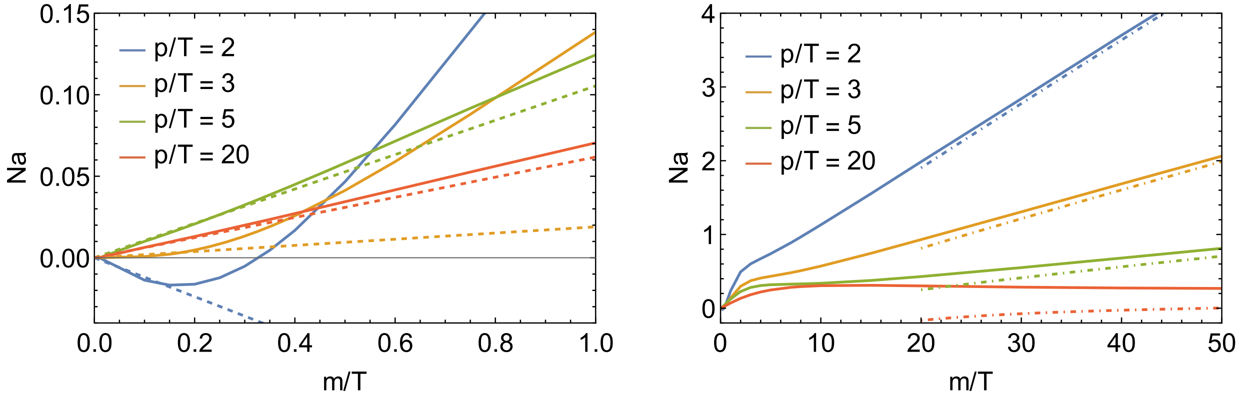

FIG. 5. The dynamical part N_a as a function of m/T for small masses (left) and large masses (right). Solid lines are from numerical solution to (53). Dashed lines and dot-dashed lines correspond to (42) and (57) respectively.

V. CONCLUSION AND OUTLOOK

In the QKT framework, the spin polarization $\mathcal{A}^\mu$ is composed of the dynamical term $a^\mu f_A$ and a nondynamical part $S^{\mu\nu}_{m,n}\mathcal{D}_\nu f$. The nondynamical part including collisional contribution in a shear flow has been obtained in [24,25]. The dynamical part is unique to massive theory and involves solving the axial kinetic equation governing the evolution of spin. Power counting indicates that the axial-vector component of Wigner function satisfies a detailed balance condition, in which the corresponding component of collision term vanishes. This allows us to fix $\mathcal{A}^\mu$ as a whole, which can then be split into three parts with existing knowledge about nondynamical part.

We perform analysis with the principle of frame independence of $\mathcal{A}^\mu$ in the small mass regime, which enables us to constrain the form of dynamical part of $\mathcal{A}^\mu$. The resulting dynamical part in the fluid frame explains the discrepancy in spin polarization between field theory approach and kinetic theory approach in the corresponding regime in the collisionless limit. We also perform explicit calculations for a massive probe fermion undergoing Coulomb scattering in a massless QED plasma with a steady shear flow, aiming at understanding strange quark polarization in QGP. We find the corresponding solution to $\mathcal{A}_\mu$ and $a^\mu f_A$. In the phenomenologically interesting region, we find the collisional effect enhances the spin-shear coupling by 50%, which may help better understand the measurement of local spin polarization of Λ hyperon in heavy ion collisions.

For phenomenological application, generalizations of the present work from QED to QCD case is necessary. In a QCD plasma, additional gluon loop presents in the collision terms, hence the redistribution of both quarks and gluons would contribute. We leave this for future studies.

ACKNOWLEDGMENTS

We are grateful to Shi Pu and Di-Lun Yang for fruitful discussions. The work is supported by NSFC Grants No. 12005112 (Z. W.) and No. 12075328, No. 11735007 (S. L.).

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: CALCULATION DETAILS

Effective amplitudes $M^{Ai}_{\mu\nu}$ in (47) are given by

$$
\begin{aligned}
M^{A1}_{\mu\nu} &= \frac{8N_f e^4(2\pi)^3}{(Q^2)^2} 2(-m^2 P'\cdot K' + P\cdot P' K\cdot K' + K\cdot P' P\cdot K') g_{\mu\nu},\\
M^{A2}_{\mu\nu} &= \frac{8N_f e^4(2\pi)^3}{(Q^2)^2} \{2(P'\cdot K' P\cdot K - P\cdot P' K'\cdot K - P\cdot K' P'\cdot K) g_{\mu\nu}\\
&\quad - 2P_\mu (P\cdot P' K'_\nu + P\cdot K' P'_\nu) + 2K_\mu (P\cdot P' K'_\nu + P\cdot K' P'_\nu - P'\cdot K' P_\nu)\\
&\quad + 2P'_\mu (K'\cdot K P_\nu - (P\cdot K - m^2) K'_\nu) + 2K'_\mu (P'\cdot K P_\nu - (P\cdot K - m^2) P'_\nu)\},
\end{aligned}
$$

$$M^{A3}_{\mu\nu} = \frac{8N_f e^4(2\pi)^3}{(Q^2)^2} 2(m^2 Q\cdot K' g_{\mu\nu} - m^2 K'_\mu Q_\nu + K\cdot K' P_\mu P_\nu - P\cdot K' P_\mu K_\nu),$$
$$M^{A4}_{\mu\nu} = \frac{8N_f e^4(2\pi)^3}{(Q^2)^2} 2(m^2 Q\cdot P' g_{\mu\nu} - m^2 P'_\mu Q_\nu + K\cdot P' P_\mu P_\nu - P\cdot P' P_\mu K_\nu),$$
$$M^{A5}_{\mu\nu} = -\frac{8N_f e^4(2\pi)^3}{(Q^2)^2} \epsilon_{\mu\nu\rho\sigma} P^\rho (K\cdot K' P'^\sigma + K\cdot P' K'^\sigma),$$
$$M^{A6}_{\mu\nu} = \frac{8N_f e^4(2\pi)^3}{(Q^2)^2} \{\epsilon_{\mu\nu\rho\sigma} K^\rho (P\cdot K' P'^\sigma + P\cdot P' K'^\sigma - P'\cdot K' P^\sigma) + \epsilon_{\nu\rho\sigma\lambda}(P'_\mu K'^\sigma + K'_\mu P'^\sigma) P^\lambda K^\rho\}. \tag{A1}$$

Additional factor $2N_f$ comes from scattering with N_f fermions and antifermions.

The J_i in (49) are functions of momentum P, K, P', K' defined below

$$
\begin{aligned}
J_1 &= (P\cdot P' K\cdot K' + K\cdot P' P\cdot K' - m^2 P'\cdot K') 2P_\perp^4,\\
J_2 &= -(P\cdot P' K'\cdot K + P\cdot K' P'\cdot K - P'\cdot K' P\cdot K)(3(P_\perp\cdot K_\perp)^2 - K_\perp^2 P_\perp^2)\\
&\quad - P'\cdot K' P_\perp\cdot K_\perp((P_\perp\cdot K_\perp)^2 - K_\perp^2 P_\perp^2)\\
&\quad + \{K\cdot K'(P_\perp\cdot K_\perp)((P'_\perp\cdot P_\perp)(P_\perp\cdot K_\perp) - (P'_\perp\cdot K_\perp)P_\perp^2)\} + \{P' \leftrightarrow K'\}\\
&\quad + \{P\cdot K'(P_\perp\cdot K_\perp)((P'_\perp\cdot K_\perp)(P_\perp\cdot K_\perp) - (P'_\perp\cdot P_\perp)K_\perp^2)\} + \{P' \leftrightarrow K'\}\\
&\quad - \{P\cdot K[(\vec{k}'\times\vec{p}\cdot\vec{q})^2 + (P'_\perp\cdot K'_\perp)(P_\perp\cdot K_\perp)^2 - (P'_\perp\cdot K_\perp)(K'_\perp\cdot P_\perp)(P_\perp\cdot K_\perp)]\} - \{P' \leftrightarrow K'\},\\
J_3 &= \{(\vec{k}'\times\vec{p}\cdot\vec{q})^2 + (P'_\perp\cdot K'_\perp)(P_\perp\cdot K_\perp)^2 - (P'_\perp\cdot K_\perp)(K'_\perp\cdot P_\perp)(P_\perp\cdot K_\perp)\} + \{P' \leftrightarrow K'\},\\
J_4 &= \{(P_\perp\cdot K_\perp)[(P'_\perp\cdot K_\perp)P_\perp^2 - (P'_\perp\cdot K_\perp)(K'_\perp\cdot P_\perp)]\} + \{P' \leftrightarrow K'\},\\
&\quad - \{P'\cdot K'(P_\perp\cdot K_\perp)P_\perp^2 + P\cdot K'(P'_\perp\cdot K_\perp)P_\perp^2 - 3P\cdot K'(P'_\perp\cdot P_\perp)(P_\perp\cdot K_\perp)\} - \{P' \leftrightarrow K'\},\\
J_5 &= 4(\vec{k}'\times\vec{p}\cdot\vec{q})^2(P_\perp\cdot K_\perp)/K_\perp^2,\\
&\quad - \{[2P\cdot K'(P_\perp\cdot P'_\perp K_\perp^2 - P_\perp\cdot K_\perp P'_\perp\cdot K_\perp) + P'\cdot K'((P_\perp\cdot K_\perp)^2 - P_\perp^2 K_\perp^2)](P_\perp\cdot K_\perp)/K_\perp^2\} - \{P' \leftrightarrow K'\},\\
J_6 &= 2P_\perp^2[K\cdot K'(P'_\perp\cdot P_\perp) + K\cdot P'(K'_\perp\cdot P_\perp) - P'\cdot K' P_\perp^2]\\
J_7 &= P'\cdot K'(3(K'_\perp\cdot P_\perp)^2 - K'^2_\perp P_\perp^2) + (\vec{k}'\times\vec{p}\cdot\vec{q})^2 + P'^2_\perp(K'_\perp\cdot P_\perp)^2 - (P'_\perp\cdot K'_\perp)(K'_\perp\cdot P_\perp)(P'_\perp\cdot P_\perp),\\
J_8 &= P'\cdot K'(3(P'_\perp\cdot P_\perp)^2 - P'^2_\perp P_\perp^2) + (\vec{k}'\times\vec{p}\cdot\vec{q})^2 + K'^2_\perp(P'_\perp\cdot P_\perp)^2 - (P'_\perp\cdot K'_\perp)(P'_\perp\cdot P_\perp)(K'_\perp\cdot P_\perp),
\end{aligned}
\tag{A2}
$$

where $\{P' \leftrightarrow K'\}$ means exchanging the two momentum in the bracket $\{\cdots\}$ ahead.

Assuming that the medium fermion and probe fermion are hard fermions with momentum comparable with temperature $p, k, p', k' \sim T$, while the momentum transfer is soft $q_0, q \ll T$, the phase space integral can be simplified using the small momentum transfer as well as momentum conservation and on-shell condition

$$
\begin{aligned}
&(2\pi)^3 \int \frac{d^4K d^4Q d^4P' d^4K'}{((2\pi)^4)^4} (2\pi)^8 \delta(P-K-Q)\delta(Q+P'-K')\delta(K^2-m^2)\delta(P'^2)\delta(K'^2)\\
&= \frac{1}{(2\pi)^5}\int dq_0 d^3q d^3k' \frac{1}{2p'_0 2k'_0 2k_0}\delta(p_0-k_0-q_0)\delta(p'_0-k'_0+q_0).
\end{aligned}
\tag{A3}
$$

The momentum integral is left with integral over Q and $\vec{k}'$. It is useful to decompose momentum $\vec{q}$ and $\vec{k}'$ into

$$
\begin{aligned}
\vec{k}' &= k'\cos\theta_k \hat{p} + k'\sin\theta_k\cos\varphi_k\hat{x} + k'\sin\theta_k\sin\varphi_k\hat{y},\\
\vec{q} &= q\cos\theta_q \hat{p} + q\sin\theta_q\cos\varphi_q\hat{x} + q\sin\theta_q\sin\varphi_q\hat{y},
\end{aligned}
\tag{A4}
$$

where we have denoted $\hat{p}$ as $\hat{z}$ for now. We introduce Ω as the angle between $\vec{k}'$ and $\vec{q}$, namely $\cos\Omega = \cos\theta_k\cos\theta_q - \sin\theta_k\sin\theta_q\cos\Delta\varphi$, with $\Delta\varphi = \varphi_q - \varphi_k$. The measure can be parametrized as

$$\int d^3qd^3k' = \int q^2dqd\cos\theta_q d\varphi_q k'^2dk'd\cos\theta_k d\Delta\varphi. \tag{A5}$$

Considering that loop fermions are light fermions, they can be treated as massless. Using $\vec{p}' = \vec{k}' - \vec{q}$ and $\vec{k} = \vec{p} - \vec{q}$, we can use the on-shell condition to cast the δ-function into

$$\begin{aligned}\delta(p_0 - k_0 - q_0) &\simeq \delta\left(q\frac{p}{p_0}\cos\theta_q - q^2\frac{(p^2\sin^2\theta_q + m^2)}{2p_0^3} - q_0\right),\\ \delta(p_0' - k_0' + q_0) &\simeq \delta\left(q\cos\Omega - q^2\frac{\sin^2\Omega}{2k'} - q_0\right),\end{aligned} \tag{A6}$$

with $p_0 = (p^2 + m^2)^{1/2}$. The angular integral over φ_q and φ_k can be performed to obtain,

$$\int d\varphi_q d\varphi_k \delta(p_0' - k_0' + q_0) \simeq 4\pi\frac{1}{q\left(1 + \frac{q_0}{k'}\right)}\frac{1}{[\sin^2\theta_q\sin^2\theta_k - (\cos\Omega - \cos\theta_q\cos\theta_k)^2]^{1/2}}. \tag{A7}$$

Note that the above δ–function constrain the unique solution of $\cos\Delta\varphi$, yet $\sin\Delta\varphi$ can take both solutions $\pm(1 - \cos^2\Delta\varphi)^{1/2}$. Thus integrals containing odd number of $\sin\Delta\varphi$ will be vanishing under the angular integral. The square root constrains the domain of $\cos\theta_k$ as $\cos(\theta_q - \Omega) < \cos\theta_k < \cos(\theta_q + \Omega)$. The other δ-function gives

$$\int d\cos\theta_q \delta(p_0 - k_0 - q_0) \simeq \frac{1}{\frac{pq}{p_0}\left(1 + \frac{q_0}{p_0}\right)}. \tag{A8}$$

From the δ-function, one can solve

$$\begin{aligned}\cos\Omega &\simeq \frac{q_0}{q} + \frac{q}{2k'}\left(1 - \frac{q_0^2}{q^2}\right) + \mathcal{O}(q^2), &\qquad \sin\Omega &\simeq \left(1 - \frac{q_0^2}{q^2}\right)^{1/2}\left(1 - \frac{q_0}{2k'}\right),\\ \cos\theta_q &\simeq \frac{p_0q_0}{pq} + \frac{q}{2p}\left(1 - \frac{q_0^2}{q^2}\right) + \mathcal{O}(q^2), &\qquad \sin\theta_q &\simeq \left(1 - \frac{p_0^2}{p^2}\frac{q_0^2}{q^2}\right)^{1/2}\left(1 - \frac{q^2 - q_0^2}{p^2 - p_0^2q_0^2}\frac{q_0p_0}{2}\right),\end{aligned} \tag{A9}$$

in obtaining the leading-log order result, it is enough to keep the above solution to the first order of q. Note that $-1 \le \cos\Omega, \cos\theta_q \le 1$ also set a limit to $x = q_0/q$ that $-\frac{p}{p_0} \le \frac{q_0}{q} \le \frac{p}{p_0}$. So that $\int dxdk'$ has the integration domain $\int dxdk' \to \int_0^\infty dk' \int_{-p/p_0}^{+p/p_0} dx$. With the above approximation of small momentum transfer, the collision term at leading logarithmic order can be explicitly calculated. The basic process is to collect all terms of integrand to Q^{-2}, after combining the measure, and integrating q ranges from $eT \ll q \ll T$, the log then arises from $\int_{eT}^T dq/q = \ln(1/e)$.

The various terms in the differential equation $c_{\rm diff}^{(0)}N_P + c_{\rm diff}^{(1)}N_P' + c_{\rm diff}^{(2)}N_P'' + c_{\rm pol} = 0$ are defined as follows. $c_{\rm diff}^{(0)}N_P + c_{\rm diff}^{(1)}N_P' + c_{\rm diff}^{(2)}N_P''$ is the diffusion of probe spin, which is defined as the corresponding parts in (49) by

$$c_{\rm diff}^{(0)}N_P + c_{\rm diff}^{(1)}N_P' + c_{\rm diff}^{(2)}N_P'' = 16N_f e^4(2\pi)^3\int_{K,Q,K',P'}\frac{1}{(Q^2)^2}\left\{\frac{J_1}{2P\cdot uP_\perp^2}N_P + \frac{(J_2 + m^2J_3)}{2K\cdot uP_\perp^2}N_K\right\}\bar{f}_k\bar{f}_{k'}f_{p'}f_p, \tag{A10}$$

due to the small momentum transfer, N_K is expanded around N_P by (52). After performing the phase space integral, the coefficients $c_{\rm diff}^{(i)}$ are given as the following,

$$c^{(0)}_{\rm diff} = c_A \frac{(p^2-2p_0^2)p^3p_0^3F + (-2p^6+6p_0^2p^4-11p_0^4p^2+3p_0^6)pT - m^2p_0\eta_p(Fp_0(p^2-2p_0^2)p^2+(2p^4-5p_0^2p^2+3p_0^4)T)}{p^3p_0^4},$$

$$c^{(1)}_{\rm diff} = c_A \frac{-Fp^3p_0^3 + pT(m^2p_0^2+2p^4+2p_0^4)+m^2p_0\eta_p(Fp_0p^2+(p^2-3p_0^2)T)}{p^2p_0^2},$$

$$c^{(2)}_{\rm diff} = c_A \frac{p_0T(pp_0-m^2\eta_p)}{p}, \tag{A11}$$

where $c_A = \frac{T^2N_f}{24\pi}$ is an overall constant. $c_{\rm pol}$ is the polarization effect, which contains contribution from different parts $c_{\rm pol} = c^{\rm med\partial}_{\rm pol} + c^{\rm med\Sigma}_{\rm pol} + c^{\rm prob\partial}_{\rm pol} + c^{\rm prob\Sigma}_{\rm pol}$. Where $c^{\rm med\partial}_{\rm pol}$ is the polarization effect contributed from the magnetization current of the medium fermion, the definition and expression are

$$\begin{aligned} c^{\rm med\partial}_{\rm pol} &= 16N_fe^4(2\pi)^3\int_{K,Q,K',P'}\frac{1}{(Q^2)^2}\left\{\frac{m^2J_7}{2K'\cdot uP_\perp^2} - \frac{m^2J_8}{2P'\cdot uP_\perp^2}\right\}\bar f^{eq}_K\bar f^{eq}_{K'}f^{eq}_{P'}f^{eq}_P \\ &= c_A\frac{m^2(Fp_0^2(m^2+2p_0^2)\eta_p - 3Fp_0^3p+2p^3T)}{2p^3p_0^2}. \end{aligned}\tag{A12}$$

$c^{\rm med\Sigma}_{\rm pol}$ from the displacement current of the medium fermion,

$$\begin{aligned} c^{\rm med\Sigma}_{\rm pol} &= 16N_fe^4(2\pi)^3\int_{K,Q,K',P'}\frac{1}{(Q^2)^2}\left\{-\frac{m^2J_7}{2K'\cdot uP_\perp^2}\frac{2T^{\rm med}_{3k'}}{k'} + \frac{m^2J_8}{2P'\cdot uP_\perp^2}\frac{2T^{\rm med}_{3p'}}{p'}\right\}\bar f^{eq}_K\bar f^{eq}_{K'}f^{eq}_{P'}f^{eq}_P \\ &= -\frac{9c_AC_fN_fm^2\zeta(3)[(F-2)p_0^2(3p_0^2-p^2)\eta_p - 3(F-2)p_0^3p+2p^3T]}{2\pi^2p^3p_0^2} \\ &\quad + \frac{2c_AC_fN_fm^2\ln 2[Fp_0^2(p^2-3p_0^2)\eta_p+3Fp_0^3p-2p^3T]}{p^3p_0^2} \\ &\quad + \frac{\pi^2c_A(C_b-C_f)m^2[p_0^2(3p_0^2-p^2)(2F+1+2\ln 2)\eta_p - 3p_0^3p(2F+1+2\ln 2)+4p^3T]}{16p^3p_0^2}. \end{aligned}\tag{A13}$$

$c^{\rm prob\partial}_{\rm pol}$ is the contribution from the inhomogeneity of the probe distribution,

$$\begin{aligned} c^{\rm prob\partial}_{\rm pol} &= 16N_fe^4(2\pi)^3\int_{K,Q,K',P'}\frac{1}{(Q^2)^2}\left\{\frac{m^2J_6-J_1}{2P\cdot uP_\perp^2} - \frac{m^2J_4+J_2}{2K\cdot uP_\perp^2}\right\}\bar f^{eq}_K\bar f^{eq}_{K'}f^{eq}_{P'}f^{eq}_P \\ &= c_A\frac{(9m^4+7m^2p^2+2p^4)p_0pF + 2(m^2+4p^2)p^3T - Fm^2p_0^2(9m^2+4p^2)\eta_p}{2p^3p_0^2}. \end{aligned}\tag{A14}$$

$c^{\rm prob\Sigma}_{\rm pol}$ is polarization effect contributed from the redistribution of the probe fermion in the shear flow. It is defined as the corresponding parts in (49) by

$$c^{\rm prob\Sigma}_{\rm pol} = 16N_fe^4(2\pi)^3\int_{K,Q,K',P'}\frac{1}{(Q^2)^2}\left\{\frac{(J_1-m^2J_6)}{2P\cdot uP_\perp^2}\frac{2T^{\rm prob}_{3p}}{p} + \frac{(J_2+m^2J_4)}{2K\cdot uP_\perp^2}\frac{2T^{\rm prob}_{3k}}{k} - \frac{m^2J_5}{2K\cdot uP_\perp^2}\frac{T^{\rm prob}_{2k}}{k}\right\}\bar f^{eq}_K\bar f^{eq}_{K'}f^{eq}_{P'}f^{eq}_P. \tag{A15}$$

After taking the phase space integral, it becomes

$$c^{\rm prob\Sigma}_{\rm pol} = c_fF_p + c_f^{(1)}F_p^{(1)} + c_f^{(2)}F_p^{(2)} + c_hH_p + c_h^{(1)}H_p^{(1)}, \tag{A16}$$

with the coefficients giving by

$$
\begin{aligned}
c_f &= c_A\left(\frac{Fm^2(5m^2+4p_0^2)\eta_p}{2p^3} - \frac{F(2m^4+7m^2p_0^2+2p^4)}{2p^2p_0} - \frac{2(3p^2+p_0^2)T}{2p_0^2}\right),\\
c_f^{(1)} &= c_A\left(m^2\eta_p\left(\frac{9m^2T+4p^2T}{p^3} - \frac{Fp}{p_0}\right) + Fp^2 - \frac{T(9m^6+16m^4p^2+9m^2p^4+4p^6)}{p^2p_0^3}\right),\\
c_f^{(2)} &= 2c_A pT\left(p - \frac{m^2\eta_p}{p_0}\right),\\
c_h &= \frac{c_A m^2(3m^2p_0^2\eta_p(Fp^2+2p_0T) - Fp_0p^3(3m^2+p^2) + 2pT(-3m^4-4m^2p^2+p^4))}{p^5p_0^2},\\
c_h^{(1)} &= -\frac{2c_A m^2 T(-3m^2p_0\eta_p + 3m^2p + p^3)}{p^3p_0}.
\end{aligned}
\tag{A17}
$$

And $F_p = 2T_{3p}^{\mathrm{prob}}/p$ and $H_p = T_{2p}^{\mathrm{prob}}/p$ with T_{3p}^{prob} and T_{2p}^{prob} given in (23) and (24), there expression are also presented here for completeness,

$$
\begin{aligned}
F_p =& -\frac{3C_fN_f\zeta(3)T}{p^5p_0^2}[Fp(-19m^2p_0^3-5p_0^5)+2p(3p^4-7p_0^2p^2+6p_0^4)T-3m^2p_0^2\eta_p(F(p^2-8p_0^2)+4p_0T)]\\
&-\frac{2N_fC_f^{\mathrm{prob}}\pi^2p_0}{p^7}[(3p_0^3p^3-2p_0p^5)F+(8p^5-18p_0^2p^3+6p_0^4p)T-m^2p_0\eta_p(3Fp_0p^2-14p^2T+6p_0^2T)],\\
H_p =& \frac{6C_fN_f\zeta(3)T}{p^5p_0^2}[(17p^2-30p_0^2)pp_0^3F+(9p^4-7p^2p_0^2+15p_0^4)pT+3p_0^2\eta_p((p^4-9p_0^2p^2+10p_0^4)F+(3p^2-5p_0^2)p_0T)]\\
&-\frac{2N_fC_f^{\mathrm{prob}}\pi^2}{6p^7p_0}[-(4p^2+3p_0^2)p^3p_0^3F+(4p^6+2p_0^2p^4+9p_0^4p^2+3p_0^6)pT\\
&+p_0\eta_p((-2p^4+3p_0^2p^2+3p_0^4)p^2p_0F+(2p^6+p_0^2p^4-8p_0^4p^2-3p_0^6)T)].
\end{aligned}
\tag{A18}
$$

$F_p^{(1)} = -\partial_{p_0}F_p(p_0,m)$, $F_p^{(2)} = \frac{1}{2}\partial_{p_0}^2F_p(p_0,m)$ and $H_p^{(1)} = -\partial_{p_0}H_p(p_0,m)$ can be obtained by taking p_0 and m as variable and taking derivatives with respective to p_0.

In the small momentum limit $p \ll m$, the coefficients in (53) are expanded as series of $v = p/m$ around $v = 0$. The coefficients becomes

$$
\begin{aligned}
c_{\mathrm{diff}}^{(0)}|_{p\ll m} &= -c_A 4T,\\
c_{\mathrm{diff}}^{(1)}|_{p\ll m} &= c_A 2Tp,\\
c_{\mathrm{diff}}^{(2)}|_{p\ll m} &= c_A \frac{2}{3}Tp^2,\\
c_{\mathrm{pol}}|_{p\ll m} &= c_A\left(\frac{T(7m-225T\zeta(3)C_fN_f)}{50p^2} + \frac{1}{2} + \left(\frac{\pi^2(C_b-C_f)}{16} - C_fN_f\left(\frac{9\zeta(3)}{4\pi^2}+\ln 2\right)\right)\right.\\
&\quad\left. + \tanh\left(\frac{m}{2T}\right)\left(\frac{27TC_fN_f\zeta(3)}{10m} + \frac{67}{200}\right) - \frac{38789T}{10500m} + \frac{3087C_fN_f\zeta(3)T^2}{105m^2}\right).
\end{aligned}
\tag{A19}
$$

In the large momentum limit $p \gg m$, the coefficients in (53) are expanded as series of $v = p/m$ around $v = \infty$. Keeping the leading and subleading orders, the coefficients becomes

$$\begin{aligned} c^{(0)}_{\rm diff}|_{p\gg m} &= -c_A(p+4), \\ c^{(1)}_{\rm diff}|_{p\gg m} &= -c_A(p^2-4p), \\ c^{(2)}_{\rm diff}|_{p\gg m} &= c_A p^2, \\ c_{\rm pol}|_{p\gg m} &= c_A\left(\frac{3}{2}p+6\right). \end{aligned} \tag{A20}$$

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