

Comparison of the mixed-fermion-action effects using different fermion and gauge actions with 2 + 1 and 2 + 1 + 1 flavors

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The leading-order low-energy constant Δ_{mix} in mixed-action chiral perturbation theory is calculated using 2 + 1 + 1-flavor gauge ensembles with HISQ fermions and a tadpole-improved Symanzik gauge action at four lattice spacings $a \in [0.048, 0.111]$ fm. By comparing our results to those from different actions and a 2 + 1-flavor case, we find that the fermion action has the dominant impact, the gauge action has a secondary but measurable effect, and the contribution from charm quark loops is negligible within our current uncertainties.

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I. INTRODUCTION

Lattice QCD provides the primary nonperturbative framework for *ab initio* calculations of strong interaction physics. A central challenge in the field is the choice of fermion discretization, which must balance computational cost against the preservation of fundamental symmetries, most notably chiral symmetry. In this context, the *mixed-action* approach has emerged as a powerful pragmatic strategy. By generating gauge ensembles with a computationally efficient “sea” fermion action and calculating observables using a more theoretically clean but expensive “valence” fermion action, one can leverage the statistical advantages of large ensembles while maintaining good control over systematic

errors. Mixed-action lattice QCD has been employed in calculations of glue helicity [1], nucleon charges (e.g., [2–4]), hadron vacuum polarization [5], and other observables.

The success of this approach hinges on a quantitative understanding of the additional discretization artifacts introduced by the action mismatch, known as *mixed-action effects*. A key calculable quantity for quantifying these effects is the mass $m_{\pi,vs}$ of the mixing pion with one valence quark and one sea antiquark, which leads to the leading-order low-energy constant in the mixed-action partially quenched chiral perturbation theory (MAPQ χ PT) [6–8]

$$\Delta_{\text{mix}} \equiv m_{\pi,vs}^2 - \frac{m_{\pi,vv}^2 + m_{\pi,ss}^2}{2} \Big|_{m_{\pi,vv}=m_{\pi,ss}}, \quad (1)$$

where $m_{\pi,vv}$ ($m_{\pi,ss}$) is the pion mass with the valence (sea) quark and antiquark.

In the past two decades, numerical lattice QCD studies of mixed-action effects have remained relatively sparse and fragmented [9–18]. This is largely because such effects are typically treated as a component of the systematic

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uncertainty for specific valence-sea fermion combinations rather than as a primary subject of study. A systematic numerical investigation into the impact of specific valence and sea actions was absent until Ref. [19].

Based on high-precision calculations of Δ_{mix} , Ref. [19] established that when the sea fermion action preserves chiral symmetry, Δ_{mix} scales as $\mathcal{O}(a^4)$ —a significant improvement over the expected $\mathcal{O}(a^2)$ behavior. This favorable scaling has been observed over a wide range of lattice spacings, $a \in [0.04, 0.19]$ fm, for chirally symmetric actions such as DW [20–22] and HISQ fermions [23]. In contrast, when a clover sea fermion action with explicit chiral symmetry breaking is used, Δ_{mix} remains substantially larger, independent of the valence action.

However, the comparison in Ref. [19] presents potential loopholes. The clover fermion ensembles used the tadpole-improved tree-level Symanzik (TITLS) gauge action, while the HISQ and DW ensembles used different gauge actions (one-loop improved Symanzik and Iwasaki, respectively). Consequently, the possibility that the gauge action itself significantly impacts Δ_{mix} cannot be excluded. Furthermore, the number of dynamical fermion flavors differs between the clover ($N_f = 2 + 1$) and HISQ ($N_f = 2 + 1 + 1$) ensembles, introducing another confounding factor.

In this work, we perform a controlled study to isolate these effects. We generate new HISQ fermion ensembles using the same tadpole-improved Symanzik gauge action as the clover studies, with $N_f = 2 + 1 + 1$ flavors at four lattice spacings, and three $N_f = 2 + 1$ flavor ensembles at $a \sim 0.11$ fm with different gauge actions. By calculating Δ_{mix} on these new ensembles, we enable a fair comparison to precisely determine the impact of the gauge action and the dependence on the number of dynamical fermion flavors.

II. SETUP AND BASIC TESTS OF THE GENERATED GAUGE ENSEMBLES

The fermion action used for the gauge ensembles generated in this work is the HISQ action [23], which effectively mitigates taste symmetry breaking and discretization errors while retaining acceptable chiral symmetry. It is formulated in the following form:

$$S_{\text{HISQ}} = \sum_x \bar{\psi}(x) (\gamma \cdot D^{\text{HISQ}} + m) \psi(x), \quad (2)$$

with the $\mathcal{O}(a^2)$ improved covariant derivative operator

$$D_\mu^{\text{HISQ}} \equiv \Delta_\mu(W) - \frac{a^2}{6} (1 + \epsilon) \Delta_\mu^3(X), \quad (3)$$

where the smeared link field $W_\mu(x)$ is constructed via a two-stage Fat7 smearing process with intermediate SU(3) projection, and $X_\mu(x)$ denotes links smeared with a single application of the ASQTAD smearing operator. The

parameter ϵ provides improved dispersion relation for heavy quarks with bare mass m by canceling tree-level $\mathcal{O}((am)^4)$ errors.

We consider a general gauge action of the form

$$S_g = \hat{\beta} \left[\sum_P \left(1 - \frac{1}{3} \text{ReTr}(P) \right) + C_R \sum_R \left(1 - \frac{1}{3} \text{ReTr}(R) \right) \right], \quad (4)$$

where the sums run over all 1×1 plaquettes (P) and 2×1 rectangles (R). Specific gauge actions are characterized by the value of the parameter C_R

- (i) $C_R = 0$ for the Wilson action (W);
- (ii) $C_R = -1/20$ for the tree-level Symanzik action ($S^{(0)}$);
- (iii) $C_R = -1/(20u_0^2)$ for the tadpole-improved Symanzik action, with $u_0 = (\langle \text{ReTr}(P) \rangle / 3)^{1/4}$. This was employed in previous CLQCD ensembles with clover fermions [24,25];
- (iv) $C_R = -0.331/(1 + 8 \cdot 0.331) = -0.0907346$ for the Iwasaki gauge action.

In this work, we generated a set of $2 + 1 + 1$ -flavor gauge ensembles using the aforementioned fermion and gauge actions. The ensemble parameters are detailed in Table I (with the symbol $\tilde{X} \equiv Xa^n$ for any quantity X with mass dimension n), including the gauge coupling $\hat{\beta}$, the tadpole factor u_0 , lattice volume $\tilde{L}^3 \times \tilde{T}$, and the bare dimensionless quark masses $\tilde{m}_{l,s,c}^b$. For all ensembles at $\hat{\beta} = 7.29$ and 7.54 (except c24P31s to be discussed later), we fixed the charm-to-strange mass ratio to $\tilde{m}_c^b / \tilde{m}_s^b = 11.766$, corresponding to the $2 + 1 + 1$ flavor FLAG average [26–30]. For the ensembles at $\beta = 7.75$ and 8.20 , we used $\tilde{m}_c^b / \tilde{m}_s^b = 11.74$, a value consistent with the FLAG average of $11.766(30)$ within its uncertainty. The strange-to-light mass ratio, $\tilde{m}_s^b / \tilde{m}_l^b$, was set to 27.18 [31] for the physical point ensemble (c48P13), to 10 for c24P22 and c32P22, and to 5 for the remaining ensembles (excluding c24P31s). The bare strange quark mass \tilde{m}_s^b was then tuned to ensure the resulting m_{n_s} mass was close to the value of $689.63(18)$ MeV [32] which corresponds to the physical strange quark mass, and the tadpole factor u_0 was tuned to ensure consistency within 0.001% between its input value and the value from thermalized configurations. Finally, to investigate the impact of the strange sea quark mass, we generated the c24P31s ensemble using the same $\hat{\beta}$ and $\tilde{m}_{l,c}^b$ as the c24P31 ensemble but with a \tilde{m}_s^b enlarged by 17% .

The values of u_0 in Table I indicate a clear hierarchy in the parameter dependencies. The gauge coupling $\hat{\beta}$ is the dominant factor. Contributions from the light and strange quark masses, though much weaker [$\mathcal{O}(10^{-5})$], are still relevant at our level of statistical uncertainty. Finally, any

TABLE I. Input parameters $\hat{\beta}$, u_0 , $\tilde{L}^3 \times \tilde{T}$, and $\tilde{m}_{l,s,c}^b$ of the $2 + 1 + 1$ flavor gauge ensembles with HISQ fermion and S^{tad} gauge actions.

Ensemble	$\hat{\beta}$	$\tilde{L}^3 \times \tilde{T}$	u_0	\tilde{m}_l^b	\tilde{m}_s^b	\tilde{m}_c^b
c24P31s	7.29	$24^3 \times 48$	0.879440(3)	0.00944	0.055	0.5555
c24P31	7.29	$24^3 \times 48$	0.879452(3)	0.00944	0.04721	0.5555
c32P31	7.29	$32^3 \times 48$	0.879451(2)	0.00944	0.04721	0.5555
c24P22	7.29	$24^3 \times 48$	0.879469(3)	0.00472	0.04721	0.5555
c32P22	7.29	$32^3 \times 48$	0.879468(2)	0.00472	0.04721	0.5555
c48P13	7.29	$48^3 \times 48$	0.879472(1)	0.00174	0.04721	0.5555
e32P31	7.54	$32^3 \times 64$	0.886360(2)	0.007434	0.03715	0.4371
g32P31	7.75	$32^3 \times 64$	0.891434(1)	0.00579	0.02895	0.34
g48P31	7.75	$48^3 \times 64$	0.891432(1)	0.00579	0.02895	0.34
h48P31	8.20	$48^3 \times 96$	0.900600(1)	0.003526	0.01763	0.207

finite volume effect is negligible, remaining consistent with zero within a 2σ confidence interval.

To mitigate autocorrelation between configurations, we set the molecular dynamics time between trajectories to

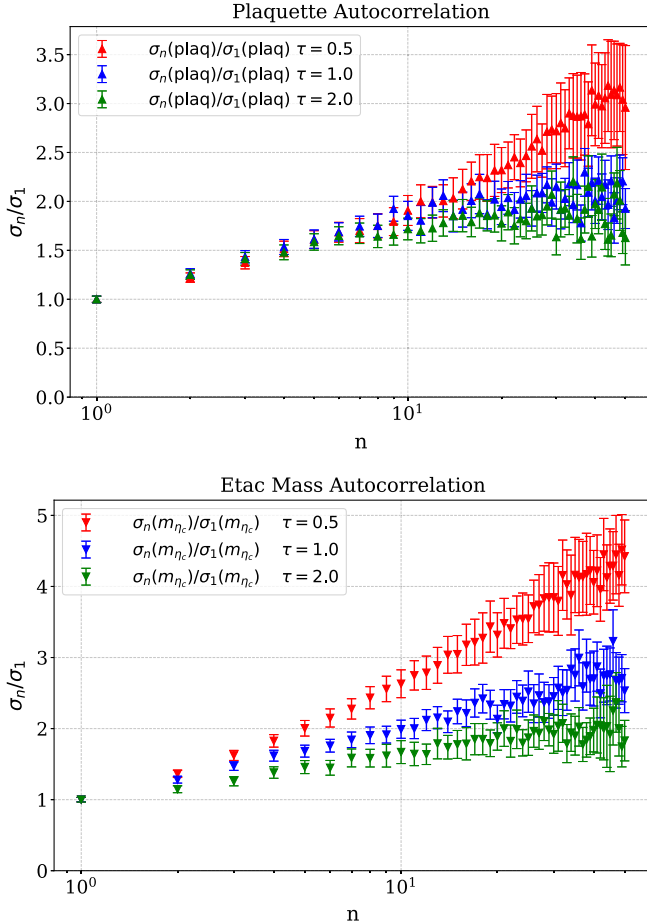


FIG. 1. The normalized variance ratio σ_n/σ_1 of the averaged plaquette $\text{ReTr}(P)$ (upper panel) and m_{η_c} (lower panel), using the molecular-dynamics time $\tau = 0.5$ (red), 1.0 (blue) and 2.0 (green) per trajectory.

$\tau = 2$ and employed multiple integration steps with $\delta\tau = 0.02$ to maintain a high acceptance rate. We quantified autocorrelations by computing the normalized variance ratio σ_n/σ_1 for various bin sizes n , using the averaged plaquette $\text{ReTr}(P)$, the topological charge Q , the pion mass m_π , and the η_c mass m_{η_c} , as detailed in the Appendix.

The results, shown in Fig. 1, reveal that $\text{ReTr}(P)$ (upper panel) and m_{η_c} (lower panel) exhibit stronger autocorrelations, requiring larger bin sizes n for the variance ratio to saturate. Notably, the autocorrelation of m_{η_c} is significantly suppressed with increasing τ , showing a marked improvement from $\tau = 0.5$ to $\tau = 1.0$. In contrast, a larger τ provides only a minor reduction in the autocorrelation of $\text{ReTr}(P)$. Both Q and m_π demonstrate much weaker autocorrelations, remaining negligible even at the finest lattice spacing of $a = 0.047$ fm.

We determine the lattice spacing a_{w_0} for each ensemble using the gradient flow scale with $w_0 = 0.1725(7)$ fm, based on the FLAG average [26,32–36]. The results are summarized in Table II. To perform a global fit, we adopt the empirical formula from Ref. [24], originally used for CLQCD clover fermion ensembles. The fit function is

TABLE II. a_{w_0} on different ensembles and the fitted values through the functional form in Eq. (5). The values in the column labeled by a_{w_0} only show the statistical errors. Other fitted results include the uncertainty from $w_0 = 0.17250(70)$ fm.

	a_{w_0} (fm)	$a_{w_0}^{\text{fit}}$ (fm)	Fit parameters	
c24P31s	0.11121(14)	0.11098(46)	$a(7.29)$	0.10840(44)
c24P31	0.11089(14)	0.11089(47)	$a(7.54)$	0.08668(37)
c24P22	0.10938(15)	0.10944(45)	$a(7.75)$	0.07103(30)
c32P31	0.11094(08)	0.11100(46)	$a(8.20)$	0.04730(30)
c32P22	0.10943(09)	0.10940(45)	c_l	0.311(17)
c48P13	0.10843(05)	0.10843(44)	c_s	-0.022(18)
e32P31	0.08880(11)	0.08879(38)	c_{u_0}	-5(12)
g32P31	0.07276(14)	0.07283(30)		
g48P31	0.07278(07)	0.07276(30)		
h48P31	0.04845(22)	0.04845(30)		

$$\begin{aligned}
a_{w_0}(\hat{\beta}, \tilde{m}_\pi, \tilde{m}_{\eta_s}, \delta u_0) = & a(\hat{\beta}) \left[1 + c_l \left(\frac{\tilde{m}_\pi^2}{a(\hat{\beta})^2} - m_{\pi, \text{phys}}^2 \right) \right. \\
& + c_s \left(\frac{\tilde{m}_{\eta_s}^2}{a(\hat{\beta})^2} - m_{\eta_s, \text{phys}}^2 \right) \\
& \left. + c_{u_0} (u_0 - u_0^I) \right], \quad (5)
\end{aligned}$$

Here, $m_{\pi, \text{phys}} = 134.98$ MeV [37] is the physical pion mass with QED corrections highly suppressed, $m_{\eta_s, \text{phys}} = 689.63(18)$ MeV [32] corresponds to the physical strange quark mass, and u_0^I denotes the input value of u_0 . The fit yields a $\chi^2/\text{d.o.f.}$ of 1.3, and the resulting parameters are also provided in Table II.

To isolate the effect of the charm sea quark and gauge actions, we also generated three 2 + 1-flavor ensembles with a light-to-strange mass ratio of $\tilde{m}_s^b/\tilde{m}_l^b = 5$: x24P31 using the $S^{(0)}$ gauge action, y24P31 using the S^{tad} action gauge, and z24P31 using the Iwasaki gauge actions. The parameters \tilde{m}_s^b and $\hat{\beta}$ were tuned to match both the m_{η_s} and the lattice spacing of the 2 + 1 + 1-flavor ensemble c24P30, enabling a direct comparison of quantities with and without charm sea quarks.

The input parameters of the 2 + 1 flavor ensembles are collected in Table III. Based on fitted value of the

TABLE III. Input parameters $\hat{\beta}$, u_0 , and $\tilde{m}_{l,s}^b$ of the 2 + 1 flavor gauge ensembles with size $\tilde{L} \times \tilde{T} = 24^3 \times 48$ HISQ fermion and different gauge actions.

Ensemble	C_R	$\hat{\beta}$	u_0	\tilde{m}_l^b	\tilde{m}_s^b
x24P31	-1/20	6.743	0.872958(4)	0.01027	0.05135
y24P31	-1/(20 u_0^2)	7.2133	0.877265(3)	0.009796	0.04898
z24P31	-0.0907346	8.322	0.886683(3)	0.010582	0.05291

coefficient c_l , the lattice spacing a_{w_0} exhibits a -2% shift when moving from $m_\pi \sim 300$ MeV to the physical point. We therefore apply a 0.98 rescaling factor to a_{w_0} for the x(y/z)24P31 ensembles. Given that Δ_{mix} has mass dimension 2, we assign a 4% systematic uncertainty to its value for the x(y/z)24P31 ensembles to account for this rescaling.

Furthermore, we observe that u_0 is comparable for all three ensembles, despite considerable variation in $\hat{\beta}$. This feature is expected, as the effective gauge coupling defined through tadpole improvement,

$$\alpha_s^{\text{tad}} \equiv \frac{6(1 - 40/3C_R u_0^2)}{4\pi\hat{\beta}u_0^4}, \quad (6)$$

is found to be similar at the 2%–3% level for all the three C_R values used in this work.

Using the rescaled lattice spacing, the η_s meson mass m_{η_s} for both the x24P31 and z24P31 ensembles is found to be 2%–3% larger than the value corresponding to the physical strange quark mass. The resulting rescaled strange quark masses are 0.0483(3) and 0.0508(3), respectively. These values show a clear increase with the gauge coupling $\hat{\beta}$ across all three ensembles studied.

Table IV lists the pseudoscalar meson masses m_π , m_{η_s} , and m_{η_c} , determined using the jointly fitted lattice spacing. The m_{η_s} values on all the 2 + 1 + 1 flavor ensembles are consistent with the our target value of 689.63(18) MeV [32] within 2σ , except for the c24P31s ensemble used to test strange mass mistuning. The m_{η_c} results also reproduce the physical value to within approximately 1%.

Using Table IV, we perform a second-order polynomial interpolation (or extrapolation) to determine the values of $\hat{\beta}$, u_0 , and \tilde{m}_s^b at the lattice spacings of the $N_f = 2 + 1$ CLQCD ensembles. These interpolated values are collected in the Appendix.

TABLE IV. Summary table on the lattice spacing, $\tilde{L} \times \tilde{T}$, $m_\pi L$ and m_{π, η_s, η_c} (In unit of MeV) of the CLQCD ensembles with HISQ fermion.

Ensemble	a (fm)	$\tilde{L}^3 \times \tilde{T}$	$m_\pi L$	m_π	m_{η_s}	m_{η_c}
c24P31s	0.1084(4)	$24^3 \times 48$	4.13	313(2)	745(3)	2.973(12)
c24P31		$24^3 \times 48$	4.07	309(2)	687(3)	2.972(12)
c32P31		$32^3 \times 48$	5.44	310(1)	686(3)	2.972(12)
c24P22		$24^3 \times 48$	2.94	223(2)	685(3)	2.970(12)
c32P22		$32^3 \times 48$	3.87	220(1)	684(3)	2.970(12)
c48P13		$48^3 \times 48$	3.53	134(1)	683(3)	2.970(12)
e32P31	0.0867(4)	$32^3 \times 64$	4.41	313(2)	694(3)	3.015(13)
g32P32	0.0710(3)	$32^3 \times 64$	3.65	317(3)	692(3)	2.981(13)
g48P31		$48^3 \times 64$	5.38	311(2)	691(3)	2.983(13)
h48P31	0.0473(3)	$48^3 \times 96$	3.60	313(3)	692(5)	2.947(19)
x24P31	0.1114(6)	$24^3 \times 48$	4.35	321(2)	711(4)	—
y24P31	0.1116(6)	$24^3 \times 48$	4.22	311(2)	688(4)	—
z24P31	0.1128(7)	$24^3 \times 48$	4.35	317(2)	704(4)	—

TABLE V. Comparison of the lattice spacing a , tadpole factors u_0 and $v_0^{(\text{HYP})}$, and the bare light quark mass m_l^b (for $m_\pi \simeq 300$ MeV) using the HYP-smearred clover action. Symbols: HI (HISQ), SC (stout-smearred clover), $S^{(0)}$ (tree-level Symanzik), S^{tad} (tadpole-improved Symanzik), $S^{(1)}$ (one-loop-improved Symanzik), I (Iwasaki).

Action	N_f	$\hat{\beta}$	a (fm)	u_0	$v_0^{(\text{HYP})}$	$\tilde{m}_l^{b(\text{HYP})}$
HI + S^{tad}	2 + 1 + 1	7.29	0.1084(04)	0.87944	0.9862	-0.0529
		7.54	0.0867(03)	0.88636	0.9878	-0.0438
		7.75	0.0710(03)	0.89143	0.9889	-0.0397
		8.20	0.0473(03)	0.90060	0.9905	-0.0329
HI + $S^{(0)}$	2 + 1	6.74	0.1114(06)	0.87296	0.9852	-0.0578
HI + S^{tad}	2 + 1	7.21	0.1116(06)	0.87727	0.9858	-0.0558
HI + I	2 + 1	8.32	0.1128(07)	0.88668	0.9870	-0.0485
SC + S^{tad}	2 + 1	6.20	0.1052(06)	0.85545	0.9830	-0.0328
		6.41	0.0775(05)	0.86346	0.9851	-0.0208
		6.72	0.0520(03)	0.87338	0.9871	-0.0135
HI+S $^{(1)}$	2 + 1 + 1	6.00	0.1222(03)	0.86373	0.9836	-0.0708
		6.30	0.0879(02)	0.87417	0.9863	-0.0514
		6.72	0.0566(01)	0.88578	0.9887	-0.0398
		7.00	0.0426(01)	0.89218	0.9897	-0.0365

Table V summarizes the parameters $\hat{\beta}$, a , and tadpole factors u_0 and $v_0^{(\text{HYP})}$ for the following ensembles:

- (i) HI + S^{tad} : The 2 + 1 + 1 and 2 + 1 flavor HISQ fermion ensembles with the tadpole-improved Symanzik gauge action generated in this work;
- (ii) HI + $S^{(0)}$ or HI + I: The 2 + 1 flavor HISQ fermion ensembles with the tree-level Symanzik (or Iwasaki) gauge action generated in this work;
- (iii) SC + S^{tad} : The 2 + 1 flavor stout-smearred clover (SC) fermion ensembles with the same S^{tad} gauge action from the CLQCD collaboration [24];
- (iv) HI + $S^{(1)}$: The 2 + 1 + 1 flavor HISQ fermion ensembles with a one-loop-improved Symanzik ($S^{(1)}$) gauge action from the MILC collaboration [38,39] (data taken from Ref. [19]).

A comparison reveals that at a similar lattice spacing of $a = 0.1116$ fm, the value of $\hat{\beta}$ is 0.6% larger for the 2 + 1 + 1 flavor ensembles than for the 2 + 1 flavor case. This indicates that the tadpole improved bare strong coupling α_s^{tad} decreases by approximately 1.1% when charm sea quark effects are included. Using the same formulation, we find that α_s^{tad} obtained with the 2 + 1 flavor HISQ action is 1.2% and 2.8% smaller when paired with the tree-level Symanzik or Iwasaki gauge action, respectively, compared to the result with the tadpole-improved Symanzik gauge action. In contrast, changing either the fermion action from SC to HI or the gauge action from S^{tad} to $S^{(1)}$ leads to a much more significant shift in $\hat{\beta}$, corresponding to an $\sim 18\%$ change in α_s^{tad} .

For the Symanzik gauge action, the strong coupling α_s can be estimated at leading-order perturbation theory through the tadpole factor u_0 [40], defined as

$$\alpha_s^{u_0} \equiv -\frac{4}{3.06839} \log u_0. \quad (7)$$

As illustrated in Fig. 2, the ratio $\alpha_s^{\text{tad}}/\alpha_s^{u_0}$ depends on the specific combination of fermion and gauge actions. However, this ratio consistently approaches unity with deviations of only a few percent in the weak coupling limit $\alpha_s \rightarrow 0$, using extrapolation with a polynomial form $\sum_{i=0}^2 c_i \alpha_s^i$.

As shown in Table V, we observe that the HYP-smearred tadpole factor $v_0^{(\text{HYP})}$ deviates from unity by less than 2% in

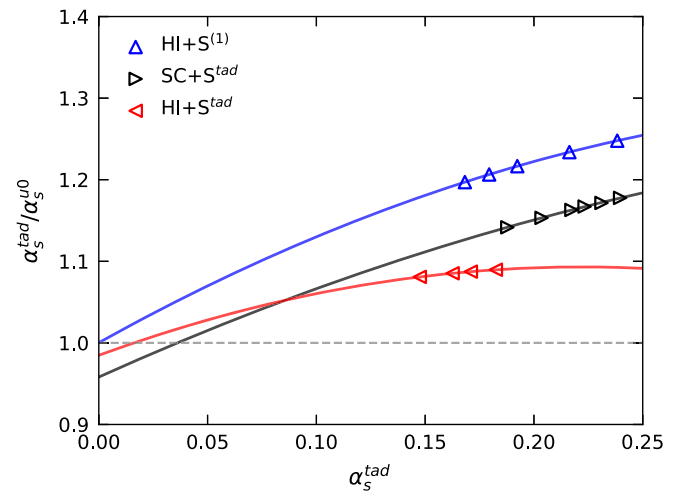


FIG. 2. The ratio $\alpha_s^{\text{tad}}/\alpha_s^{u_0}$ of two definitions of effective α_s using different fermion and gauge action combinations. They approach to 1 within a few percent deviations in the weak coupling limit $\alpha_s \rightarrow 0$.

all ensembles, with a scaling that is faster than $\alpha_s(a)$. This implies that $\mathcal{O}(a^n)$ discretization effects can be as important as $\alpha_s(a)$ corrections. The same logic applies to the dimensionless bare quark mass $\tilde{m}_l^b \equiv m_l^b a$ which corresponds to $m_\pi \simeq 300$ MeV, and its scaling provides clear evidence for both the α_s^n/a divergence and $\mathcal{O}(a^n)$ discretization effects in the clover fermion's bare mass parameter.

III. RESULT OF Δ_{mix}

We calculate the leading mixed-action effects Δ_{mix} on $2 + 1 + 1$ flavor gauge ensembles c24P31, e32P31, g32P31, and h48P31 at four lattice spacings with $m_\pi \sim 310$ MeV, and also $2 + 1$ flavor gauge ensembles x24P31, y24P31, and z24P31 with different gauge actions. For the valence fermion actions, we use:

- (i) SC: Clover fermion with 1-step STOUT smearing and tree level tadpole improved clover coefficient c_{sw} ;
- (ii) HC: Clover fermion with 1-step HYP smearing and tree level tadpole improved clover coefficient c_{sw} ;
- (iii) OV: Overlap fermion with 1-step HYP smearing and $\rho = 1.5$.

Even-grid source (including $\tilde{L}^3/8$ points at all the spatial positions where the x/y/z indices are even) are used generate propagator for all the valence fermion actions to improve the statistics. In practice, on each ensemble we compute quark propagators for the sea (HISQ) action and for the valence action at several valence quark masses m_v . We construct pseudoscalar two-point functions for the sea-sea (ss), valence-valence (vv), and valence-sea (vs) channels and extract $m_{\pi,ss}$, $m_{\pi,vv}(m_v)$, and $m_{\pi,vs}(m_v)$ from standard fits. We then interpolate in m_v to the matching point where $m_{\pi,vv} = m_{\pi,ss}$, and evaluate Δ_{mix} using Eq. (1) at this point. Since our sea action employs HISQ fermions, calculating Δ_{mix} necessitates an expansion of the Dirac indices for the corresponding HISQ quark propagator S^{HISQ} . For this purpose, we extend S^{HISQ} into unitary matrices in the Dirac space, and multiply products of gamma matrices at

each sink point $(x_{\text{snk}}, y_{\text{snk}}, z_{\text{snk}}, t_{\text{snk}})$ of the even-grid source propagator, namely $S^{\text{Dirac}} = \gamma_t^t \gamma_z^z \gamma_y^y \gamma_x^x S^{\text{HISQ}}$.

All our $2 + 1 + 1$ flavor results of Δ_{mix} are collected in Table VI, together with a generalized mixed action effect between valence fermion actions [19],

$$\bar{\Delta}_{\text{mix}}^{\text{B+C/A}} \equiv m_{\pi,BC}^2 - \frac{m_{\pi,BB}^2 + m_{\pi,CC}^2}{2} \Big|_{m_{\pi,AA}=m_{\pi,BB}=m_{\pi,CC}}, \quad (8)$$

which quantifies the effect of using two different valence actions B and C on the same sea action A. For computational efficiency, we calculate Δ_{mix} using the overlap valence fermion action only at two representative lattice spacings, $a \sim 0.11$ and 0.07 fm.

The lattice spacing dependence of Δ_{mix} on the $2 + 1 + 1$ flavor HISQ ensembles is shown in Fig. 3. The solid (dashed) line passes through the HC/HI Δ_{mix} point at the coarsest spacing and indicates a^4 (a^2) scaling. The SC and HC results exhibit a clear $\mathcal{O}(a^4)$ scaling across the four lattice spacings, despite a large relative uncertainty at the finest lattice spacing ($a = 0.048$ fm). The overlap results, though limited to two data points, are also consistent with this $\mathcal{O}(a^4)$ scaling behavior. We further parametrize Δ_{mix} with the following form,

$$\Delta_{\text{mix}}(a) = c_2 a^2 + c_4 a^4, \quad (9)$$

and list the fitted (or solved in the overlap case) coefficients $c_{2,4}$ in Table VI. It is evident that the values of c_2 are significantly nonzero in cases involving the stout-link smeared clover action (SC), although they remain much smaller than the corresponding c_4 values. Since a^4 scaling is observed for both the exactly chiral overlap fermion and the HYP-smeared clover fermion (with its suppressed chiral symmetry breaking), the scaling can be strongly related to chiral symmetry.

TABLE VI. Δ_{mix} and $\bar{\Delta}_{\text{mix}}$ on different ensembles with different valence quark actions. The symbols OV, SC, and HC correspond to the overlap, stout-smeared clover, and HYP smeared clover fermion actions.

Action	Symbol	a(fm)	$m_{\pi,ss}$ (MeV)	Δ_{mix} (GeV ²)			$\bar{\Delta}_{\text{mix}}$ (GeV ²)			
				OV	SC	HC	OV + SC	OV + HC	SC + HC	
$2 + 1 + 1$	c24P30	0.108	309	0.01724(76)	0.01545(45)	0.01138(35)	0.01344(54)	0.00700(40)	0.00292(12)	
	e32P31	0.087	313	—	0.00668(20)	0.00447(15)	—	—	0.00156(06)	
	g32P31	0.071	317	0.00313(31)	0.00370(17)	0.00220(13)	0.00321(28)	0.00126(23)	0.00100(06)	
	h48P31	0.047	313	—	0.00104(18)	0.00063(18)	—	—	0.00027(03)	
	c_2 (GeV ⁴)				-0.001(05)	0.010(02)	0.001(02)	0.010(04)	-0.001(03)	0.005(01)
	c_4 (GeV ⁶)				0.196(21)	0.135(10)	0.120(08)	0.118(17)	0.080(13)	0.181(03)
$2 + 1$	x24P31	0.111	321	0.02366(80)	0.01965(50)	0.01449(41)	0.01787(65)	0.00952(43)	0.00383(15)	
	y24P31	0.112	311	0.01997(82)	0.01725(49)	0.01259(36)	0.01563(93)	0.00902(76)	0.00325(15)	
	z24P31	0.113	317	0.01378(59)	0.01272(35)	0.00937(26)	0.01113(41)	0.00554(30)	0.00245(08)	

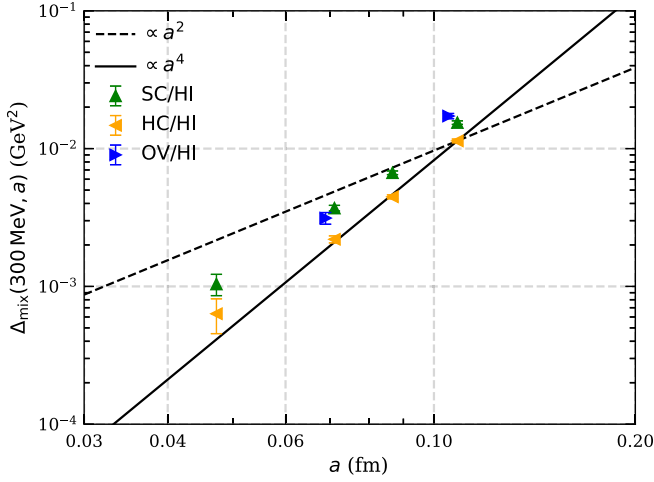


FIG. 3. Mixed-action effects of the stout link smeared clover fermion (SC/HI, green triangles) and HYP smeared clover fermion (HC/HI, yellow left triangles), overlap fermion (OV, blue right triangles, at $a = 0.108$ and 0.073 fm only) on the $2 + 1 + 1$ flavor HISQ ensembles at different lattice spacings.

Given that the lattice spacings of the $x/z24P31$ ensembles ($a = 0.111/0.113$ fm) differ from that of the $y24P31$ ensemble ($a = 0.112$ fm) by only 1%, we interpolated all the $2 + 1 + 1$ flavor results to a common reference point of $a = 0.112$ fm using the parametrization in Eq. (9). This enables a direct comparison of Δ_{mix} across different fermion-gauge action combinations and charm quark content. The interpolated results are presented in Fig. 4.

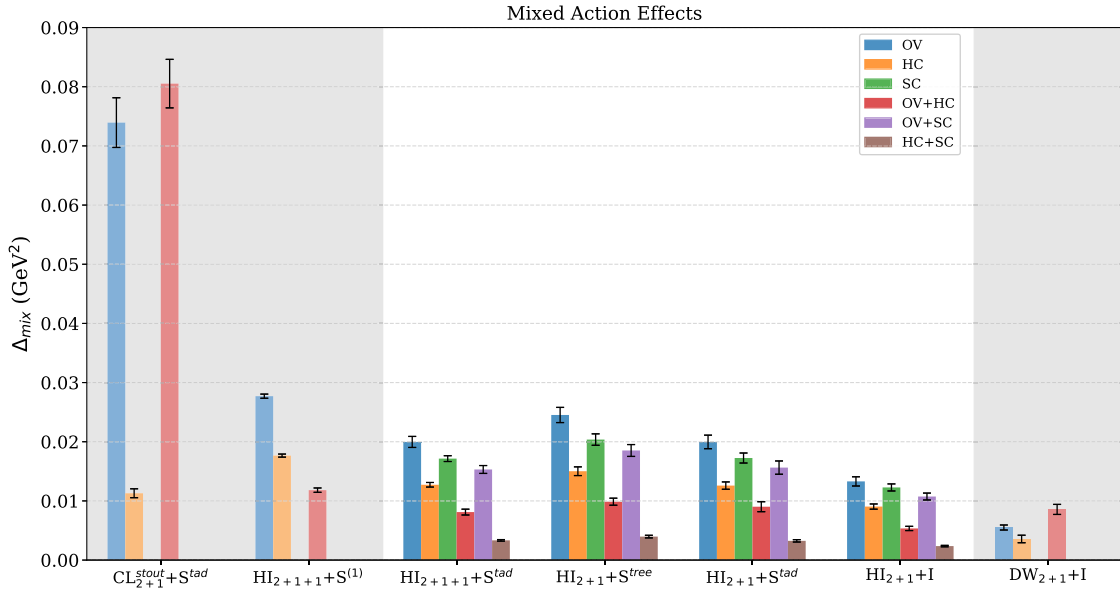


FIG. 4. The mixed action effects Δ_{mix} at $m_\pi \sim 0.30$ GeV for various valence fermion actions (OV, SC and HC) and $\bar{\Delta}_{\text{mix,uni}}$ for different valence fermion actions (OV + HC, OV + SC and HC + HC) on the $2 + 1$ flavor HI + I, HI + S^{lad} , HI + $S^{(0)}$ ensembles, $2 + 1 + 1$ flavor HI + S^{lad} ensembles, and also includes the values from Ref. [19] (shown as histograms in the shaded region) for comparison. All the quantities are in unit of GeV^2 , and are interpolated to $a = 0.112$ fm and $m_\pi \sim 300$ MeV to make a fair comparison.

A comparison between the $2 + 1$ and $2 + 1 + 1$ flavor HISQ ensembles with the same S^{lad} gauge action reveals that Δ_{mix} is consistent within statistical uncertainties, regardless of the inclusion of charm sea quarks. In contrast, both of these Δ_{mix} values are lower than the values [19] on the $2 + 1 + 1$ flavor HISQ ensemble with the $S^{(1)}$ gauge action.

For the $2 + 1$ flavor cases, Δ_{mix} exhibits an obvious decrease with increasing absolute value of $|C_P|$, rendering its value with the Iwasaki action being $0.61(4)$ of that with the tree level Symanzik gauge actions. Nevertheless, Δ_{mix} with DW fermions is still smaller by a factor of $2.5(3)$ compared with HISQ fermions under the same Iwasaki gauge action. When using the same S^{lad} gauge action, Δ_{mix} with HISQ fermions is significantly smaller than with stout clover (SC) fermions, confirming the importance of using a chirally symmetric fermion action for generating gauge configurations on suppression Δ_{mix} .

The preceding calculations enable us to isolate the specific origins of variations in Δ_{mix} by comparing our results with existing literature:

- (1) OV@TM₂ + S^{tree} [15]: Using ensembles with 2-flavor twisted mass sea quarks and a tree-level Symanzik gauge action at $m_\pi \in [376, 484]$ MeV and $a \in [0.05, 0.08]$ fm, the reported Δ_{mix} for a 1-step HYP-smeared overlap valence fermion ($\rho = 1$) is $0.34(4)$ GeV^2 when extrapolated to $a = 0.112$ fm using an $a^2 + a^4$ ansatz Eq. (9). The dominant scaling was a^2 , with an a^4 term coefficient of $c_4 = 0.4(6)$ GeV^6 . This value is substantially larger than the $0.071(4)$ GeV^2 found for OV@CL₂₊₁ + S^{lad} [19].

This significant difference is likely attributable to the different sea fermion actions, as the effects from the number of sea flavors and the gauge action are probably insufficient to explain it.

- (2) OV@CL₂₊₁ + S^{tree} [9]: Using a 2 + 1 flavor ensemble with 6-step stout-smearing clover sea quarks and a tree-level Symanzik gauge action at $m_\pi \in [190, 300]$ MeV and $a = 0.088$ fm, Δ_{mix} for a 3-step HYP-smearing overlap valence fermion ($\rho = 1$) was $0.12(5)$ GeV². Scaling to $a = 0.112$ fm assuming a^2 dependence gives $0.19(8)$ GeV². This is consistent with the $0.071(4)$ GeV² from OV@CL₂₊₁ + S^{tad} [19] within the large uncertainty, especially considering the expected increase when switching from S^{tad} to S^{tree}.
- (3) OV@HI₂₊₁₊₁ + S⁽¹⁾ [16]: Using a 2 + 1 + 1 flavor HISQ ensemble with a 1-loop Symanzik gauge action at $m_\pi = 310$ MeV and $a \simeq 0.12$ fm, Δ_{mix} for a 1-step HYP-smearing overlap valence fermion is $0.031(3)$ GeV² at $a = 0.112$ fm, based on the a^4 scaling verified for this setup in Ref. [19] (note: taste-breaking effects were unsubtracted [18]). This value is in excellent agreement with the $0.027(3)$ GeV² from Ref. [19].
- (4) DW@HI₂₊₁₊₁ + S⁽¹⁾ [17]: Using 2 + 1 + 1 flavor HISQ ensembles with a 1-loop Symanzik gauge action at $m_\pi \in [220, 310]$ MeV and $a \in [0.09, 0.15]$ fm, Δ_{mix} for a Mobius domain wall valence fermion is $0.024(1)$ GeV² at $a = 0.112$ fm, based on an $a^2 + a^4$ fit Eq. (9) with coefficients $c_2 = 0.04(1)$ GeV⁴ and $c_4 = 0.09(3)$ GeV⁶. This is slightly smaller than the OV@HI₂₊₁₊₁ + S⁽¹⁾ value, suggesting valence Mobius domain wall fermions may be a better choice on HISQ ensembles at coarser lattice spacings. However, the significantly nonzero c_1 term warrants further investigation.
- (5) DW@AS₂₊₁ + S⁽¹⁾ [10–12]: Using 2 + 1 flavor Asqtad-improved staggered (AS) sea quarks with a 1-loop Symanzik gauge action at $m_\pi \in [260, 510]$ MeV and $a \in [0.09, 0.13]$ fm, Δ_{mix} for a domain wall valence fermion is $0.065(5)$ GeV² at $a = 0.112$ fm. The data were fit with an $a^2 + a^4$ ansatz, but the uncertainty was too large to distinguish between the scaling terms. This value is roughly twice that of the most similar case (OV@HI₂₊₁₊₁ + S⁽¹⁾), a difference we attribute to the AS sea action, given that domain wall and overlap fermions yield similar Δ_{mix} on HISQ ensembles.
- (6) OV@DW₂₊₁ + I [13,14]: Using 2 + 1 flavor domain wall sea quarks with an Iwasaki gauge action at $m_\pi \in [300, 400]$ MeV and $a \in [0.082, 0.114]$ fm, Δ_{mix} for a 1-step HYP-smearing overlap valence fermion is $0.010(2)$ GeV² at $a = 0.112$ fm. This is larger than the $0.0053(4)$ GeV² reported in Ref. [19],

as the partially quenched chiral extrapolation used in Ref. [14] can about twice inflate the value.

- (7) BC@AS₂₊₁ + S⁽¹⁾ [18]: Using 2 + 1 flavor AS sea quarks with a 1-loop Symanzik gauge action at $m_\pi \sim 300$ MeV, Δ_{mix} for a Borici-Creutz valence fermion [41,42] was $0.017(6)$ GeV² at $a = 0.15$ fm and consistent with zero ($0.02(5)$ GeV²) at $a = 0.13$ fm. Considering the tendency of the AS sea action and S⁽¹⁾ gauge action to enlarge Δ_{mix} , the Borici-Creutz fermion appears to be a promising candidate for mixed-action setups, though higher-precision calculations at finer lattice spacings are required for confirmation.

IV. SUMMARY

In this work, we have performed a systematic lattice QCD calculation of the leading-order low-energy constant Δ_{mix} in mixed-action chiral perturbation theory. Our study utilized 2 + 1 + 1 flavor HISQ sea fermions with a tadpole-improved Symanzik gauge action across four lattice spacings, and also 2 + 1 flavor HISQ sea fermion with 3 kinds of gauge action at a fixed lattice spacing, enabling a controlled investigation into the sources of mixed-action artifacts.

Our central finding is the observation of a favorable $\mathcal{O}(a^4)$ scaling for Δ_{mix} when valence overlap or clover fermions are used on gauge ensembles generated with 2 + 1 + 1 flavor HISQ sea fermions. This result is particularly significant as it was achieved using the same tadpole-improved Symanzik gauge action previously associated with a sizable Δ_{mix} in CLQCD clover fermion ensembles [19]. This comparison directly verifies that the use of a dynamical fermion action with chiral symmetry is the dominant factor in suppressing mixed-action artifacts, irrespective of the gauge action. Furthermore, by comparing different action combinations at a fixed lattice spacing, we have quantified additional, secondary influences on Δ_{mix} . Our results show that the larger correction from rectangle loops in the gauge action can provide an additional suppression of Δ_{mix} , while the effect of dynamical charm quarks is statistically insignificant.

A further comparison with values from the literature [10–12,15] indicates that using either twisted mass or AS fermions as the sea action does not reduce mixed-action effects compared to the HISQ fermion. For the valence fermion, the domain wall fermion yields a Δ_{mix} [17] comparable to that of either the clover or overlap fermion. If clover valence fermions are used, HYP smearing is preferable. The Borici-Creutz fermion appears to be a promising candidate for the valence fermion action [18], although higher-precision calculations at finer lattice spacings are required to confirm this.

In summary, our studies here that the HISQ sea fermion with the Iwasaki gauge action is the so far most cost-effective combination for suppressing Δ_{mix} with the flexibility in

selecting valence fermion actions. If clover valence fermions are used, HYP smearing is preferable. The tadpole-improved Symanzik gauge action, while somewhat less effective at suppressing Δ_{mix} , benefits from marginally weaker autocorrelations (see Appendix). Further investigation on the Borici-Creutz fermion and different choice of the staggered fermion (likes the stout smearing with different steps and smearing sizes) would be helpful to further suppress Δ_{mix} and worth studies in the future.

Furthermore, a preliminary study [43] of physical observable using clover valence fermions on our HISQ ensembles indicates that the discretization errors for quantities such as f_K and m_Ω are suppressed compared to those from unitary clover fermion calculations [24,44]. This suggests that the mixed-action setup can offer superior control over systematic uncertainty from continuum extrapolation, compared to the one with unitary valence-sea fermion action. Consequently, further investigations into a wider range of observables using this setup are warranted to identify optimal, cost-efficient strategies for lattice QCD calculations that maintain rigorous control over systematic errors.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: AUTOCORRELATION ANALYSIS

To assess the statistical independence of gauge configurations generated by the hybrid Monte Carlo algorithm, we

perform an autocorrelation analysis using the binning method following the approach in Ref. [50]. The analysis focuses on four key observables: the average plaquette $\langle \text{Tr}U_p \rangle / 3$, the topological charge $Q = \int d^4x q(x)$ with $q(x) = \frac{g^2}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{Tr} [F^{\mu\nu}(x) F^{\rho\sigma}(x)]$, the pion mass m_π , and the η_c mass m_{η_c} .

For a given observable O , the normalized variance ratio σ_n/σ_1 is computed as a function of bin size n . First, the overall mean is calculated as $\bar{O} = \frac{1}{N} \sum_{i=1}^N O_i$, where O_i denotes an individual measurement and N is the total number of configurations. For a bin size n , the data is partitioned into $M = \lfloor \frac{N}{n} \rfloor$ bins, and the mean for the k th bin is given by $\bar{O}_{n,k} = \frac{1}{n} \sum_{i=(k-1)n+1}^{kn} O_i$. The variance of these binned means is then computed as $\sigma_n^2(O) = \frac{1}{M} \sum_{k=1}^M \sigma_{n,k}^2(O)$ with $\sigma_{n,k}^2(O) = \frac{1}{M-1} (\bar{O}_{n,k} - \bar{O})^2$. To estimate the uncertainty of this variance, we calculate its standard error, $\text{SE}(\sigma_n^2) = \sqrt{\frac{1}{M(M-1)} \sum_{k=1}^M (\sigma_{n,k}^2 - \sigma_n^2)^2}$.

Finally, the normalized ratio σ_n/σ_1 and its propagated error are evaluated. The error is given by $\delta(\frac{\sigma_n}{\sigma_1}) = \sqrt{(\frac{\delta\sigma_n}{\sigma_1})^2 + (\frac{\sigma_n\delta\sigma_1}{\sigma_1^2})^2}$, where $\delta\sigma_n = \text{SE}(\sigma_n^2)/(2\sigma_n)$ and $\delta\sigma_1 = \text{SE}(\sigma_1^2)/(2\sigma_1)$. This ratio quantifies the reduction in statistical error with increasing bin size, serving as our measure of autocorrelation.

The upper three panels of Fig. 5 illustrate the ratio σ_n/σ_1 as a function of bin size n for all four observables, using data from the ensemble with $m_\pi \sim 0.3$ GeV at $a = 0.108$ fm. These panels correspond to different molecular dynamics time steps: $\tau = 0.5$ (upper left), 1.0 (central left), and 2.0 (lower left). As τ increases, the dependence of the statistical error on bin size weakens for all quantities.

The three panels in the middle present the cases for ensembles at finer lattice spacings ($a = 0.09, 0.07,$ and 0.05 fm from top to bottom), all with $\tau = 2$. These plots suggest that the sensitivity to binning does not become significantly stronger at smaller lattice spacings.

The lower three panels show the cases with different gauge actions with increasing $|C_P|$, tree level Symanzik, tadpole improved Symanzik, and Iwasaki. We can see that the autocorrelation effect is somehow stronger for the pion mass with the Iwasaki gauge action.

In summary, we observe weaker autocorrelation on the HISQ ensembles compared to the CLQCD ensembles, which we attribute to the larger Monte Carlo time ($\tau = 2$) per trajectory in our simulations. These results ensure that the statistical uncertainties quoted throughout this work are not substantially affected by autocorrelation effects, thereby providing confidence in the reliability of our physical predictions.

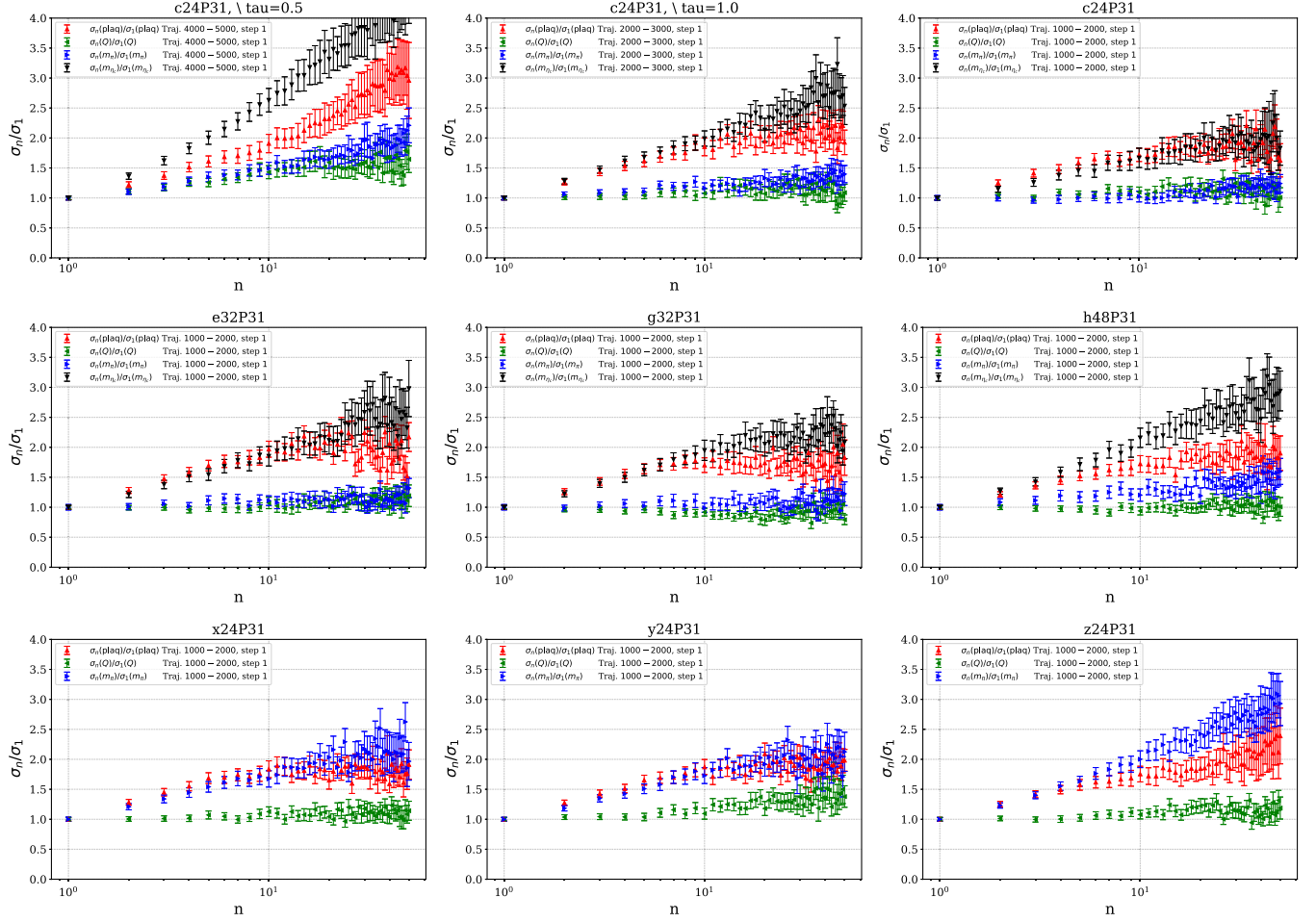


FIG. 5. Autocorrelation analysis across multiple lattice ensembles, depicting the bin-size dependence of variance for four fundamental quantities: the plaquette operator, topological charge, pion mass, and η_c meson mass. The notation $\{\text{Traj. } n_{\min}, -n_{\max}, \text{step } \Delta n\}$ accompanying each observable specifies the minimal and maximal trajectory indices, along with the stride parameter employed for configuration subsampling to ensure statistical independence in our analysis.

APPENDIX B: INTERPOLATED PARAMETERS ACCORDING TO THE LATTICE SPACINGS OF CLQCD ENSEMBLES

Initial estimates for the parameters $\hat{\beta}$, u_0 , and \tilde{m}_s^b at the lattice spacings of the $N_f = 2 + 1$ CLQCD ensembles are obtained via second-order polynomial interpolation or extrapolation of our data; these values are collected in Table VII. The bare strange quark mass \tilde{m}_s^b is rescaled to the value corresponding to the physical strange quark mass using ensembles with $m_s/m_l = 5$. It is then increased by

0.5% to account for light quark mass dependence, based on the $m\eta_s$ values from the c24P31 and c48P13 ensembles.

These parameters can be further refined to obtain the same L with specific spatial lengths $\tilde{L} = \{24, 28, 32, 36, 40, 48, 64\}$ by fine-tuning the lattice spacing; the resulting adjusted values are compiled in Table VIII. It should be noted that these parameters serve as initial estimates and will require further adjustment during actual production to achieve self-consistent u_0 , the target lattice spacing, and m_{η_s} which corresponds to physical strange quark mass.

TABLE VII. Initial guesses of $\hat{\beta}$, u_0 , and \tilde{m}_s^b corresponding to the lattice spacings of the $N_f = 2 + 1$ CLQCD ensembles.

$a(\text{CLQCD})$	0.10542(64)	0.09015(58)	0.07761(46)	0.06896(44)	0.05239(32)	0.03761(23)
β	7.3216(82)	7.4950(83)	7.6564(78)	7.7821(81)	8.0790(88)	8.531(32)
u_0	0.88031(23)	0.88505(22)	0.88919(19)	0.89219(19)	0.89837(16)	0.90447(20)
\tilde{m}_s^b	0.04590(38)	0.03838(34)	0.03222(28)	0.02798(26)	0.01988(20)	0.01268(21)

TABLE VIII. Initial guesses of $\hat{\beta}$, u_0 , and \tilde{m}_s^b corresponding to the same $L \equiv \tilde{L}a$.

a [fm]	0.10417	0.08929	0.07812	0.06944	0.05208	0.03906
β	7.3351(46)	7.5055(45)	7.6493(45)	7.7746(45)	8.0856(57)	8.462(21)
u_0	0.88068(13)	0.88533(12)	0.88901(11)	0.89202(10)	0.89849(10)	0.90384(16)
\tilde{m}_s^b	0.04528(21)	0.03796(18)	0.03248(16)	0.02822(14)	0.01973(13)	0.01339(17)
$\tilde{L}^3 \times \tilde{T}$	$24^3 \times 72$	$28^3 \times 84$	$32^3 \times 96$	$36^3 \times 108$	$48^3 \times 144$	$64^3 \times 192$

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