Eur. Phys. J. C (2026) 86:567
https://doi.org/10.1140/epjc/s10052-026-15589-0

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

On dynamical stringy asymptotic flat Einstein–Cartan–Kalb–Ramond and teleparallel wormholes

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Received: 23 January 2026 / Accepted: 16 March 2026
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Abstract Inspired by the investigation of Kalb–Ramond strings and hairy black holes by Duncan et al., in this paper we investigate two approaches to determine the existence of asymptotic flatness and how the presence of cosmic string axionic torsion hair affects wormholes and black holes. In the case of Einstein–Cartan–Kalb–Ramond (ECKR) gravity, it is shown that by assuming asymptotic flatness in ECKR equations, the KR vanishes at both sides far away from the wormhole throat, indicating the absence of torsion hair. On the other hand, in the case of teleparallel gravity, we show that in a Schwarzschild black hole with a cosmic string, by considering the cosmic string tension as constant, the torsion invariant vanishes, showing no signatures of torsion hair. When a dynamical cosmic string with variable string tension is introduced in torsional wormholes, it is shown that no asymptotic flatness is obtained, and torsion does not vanish asymptotically. This indicates that dynamical cosmic strings must generate an imprint of torsion hair at wormholes away from the throat. We argue that in the future, the ideas here may be used to detect torsion from an astrophysical point of view.

1 Introduction

It is well known that one of the great advantages in the use of modified theories of gravity [1] for investigating wormholes [2] is that one does not need exotic matter to grant traversability in wormholes. Although Morris–Thorne (MT) wormholes [3,4], for example, are asymptotically flat in general relativity (GR), in modified theories of gravity, wormholes are non-asymptotically flat [5], as in some cosmological solutions where gravity links two Friedmann solution solutions, wormhole are non-asymptotically flat away from the wormhole throat. In this paper we analyze these and other topological properties of wormholes. For example, we show that for MT wormholes in teleparallel gravity with a torsion constant component, there is no asymptotic flatness, since not all torsion components vanish very far away from the throat zone of the wormhole. We show that asymptotically flat solutions of MT wormholes are possible in teleparallel gravity [6]. Asymptotic flatness is also tested for a wormhole with a cosmic string inside. Other theories of gravity such as Einstein–Cartan–Kalb–Ramond (ECKR) wormholes are also tested. The existence of ECKR wormhole manifolds are tested as well. On the other hand, embedding of topological defects [7] in a teleparallel [8] conformal spacetime called a teleparallel vacuum manifold were previously obtained by the author [9]. Topological defects of a planar domain wall were investigated as well [10], allowing the possibility of a parity-preserving torsion. In this case a $(2+1)$-dimensional domain wall is embedded in a $3+1$-dimensional teleparallel spacetime vacuum. In the case we shall briefly describe, a three-dimensional wormhole is embedded in a $(3+1)$-dimensional Friedman cosmological spacetime with a spatially flat section, where the embedding is simple and allows us to obtain the shape of the wormhole in terms of Kalb–Ramond (KR) vector field components. When the KR field vanishes, the throat shape is reduced to a general relativistic (GR) one. Recently, renewed interest in wormhole formation has arisen, due to the negative dark energy inside the throat [5]. Previously, the well-known Einstein–Rosen (ER) bridge [7] was commonly considered for wormholes as a topological structure solution of Einstein's GR theory [11] connecting two regions of spacetime. Since no wormholes have been observed so far, new attempts in modified gravity with torsion [6] have been considered to achieve this goal. Topologically, there is a connection between two spacetime

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Published online: 22 May 2026

regions, creating shortcuts between distant regions of spacetime. This model works as a theoretical tool for time travel, as discussed by Gott [12]. Teleparallel solutions describing wormholes have been discovered using torsion [1]. In this paper, we find two static solutions. The first, presented in Sect. 2, shows that the existence of radial KR fields is not possible. In Sect. 3, we show that there is a solution which represents a static black hole trapped inside the wormhole throat, bounded by the ER bridge. At a finite distance from the wormhole axis, the spacetime corresponds to a Kerr [13] ring-type singularity of a stationary black hole [14]. Moreover, it is important to realize that the Schwarzschild solution is a particular solution of ECKR field equations. In Sect. 4, one obtains traversable asymptotic conformal ECKR wormholes sourced by the product of KR vector fields. In Sect. 5, we show that teleparallel cosmic strings do not asymptotically represent a torsion hair in black holes, since the black holes would be asymptotically flat. This is shown by computing a vanishing torsion invariant. An important detail is that in this black hole, cosmic strings have constant string tension. This a key issue, since as we show in Sect. 6, in wormholes, the cosmic strings are dynamical or the string tension is not constant, and these cosmic strings violate asymptotic flatness. Also in Sect. 6 we discuss how the stability of MT spherical wormholes is affected by torsion in teleparallel gravity, following observations of Dzhunhushaliev et al. [15], and we discuss how a specific solution for a teleparallel wormhole for constant torsion cannot be asymptotically flat. Section 7 is left for discussion and outlook. Our attention was recently drawn to a paper by Dzhunhushaliev et al., which provided us with a solution of thin-shell torus T^2 wormholes, where some of the violations of energy conditions do not exist.

2 On the existence of asymptotic flatness in Einstein–Cartan–Kalb–Ramond wormholes

Hochberg and Visser [16] recently investigated in detail the null energy conditions for wormholes in the totally skew-symmetric torsional spacetime where torsion is proportional to KR fields of closed strings [17] of low-energy physics. They showed that the presence of KR fields, instead of allowing energy condition violations, could be dumped into torsion degrees of freedom; they weaken the weak energy condition (WEC), which does not support wormhole throat stability, and moreover, they help the wormhole to collapse. This simple proof is obtained by making use of recent results by SenGupta and Sur [18,19] which found static spherically symmetric KR geometry like the Schwarzschild metric. SenGupta and Sur showed that [18,19] there are no time-dependent wormhole solutions of ECKR gravity field equations. In this section, we prove that for ECKR wormhole solutions, the condition of asymptotic flatness implies that four components vanish. The proof is straightforward and comes from the ECKR system

$$\mathrm{e}^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)-\frac{1}{r^2}=k(h_1h^1+h_2h^2+h_3h^3-h_4h^4) \tag{1}$$

$$\mathrm{e}^{-\lambda}\left(\frac{1}{r^2}-\frac{\Phi'}{r}\right)-\frac{1}{r^2}=k(h_1h^1+h_2h^2-h_3h^3+h_4h^4) \tag{2}$$

$$\mathrm{e}^{-\lambda}\left(\Phi''+\frac{\Phi'^2}{2}-\frac{\Phi'\lambda'}{2}+\frac{\Phi'-\lambda'}{r}\right)-\mathrm{e}^{-\Phi}\left(\ddot{\lambda}+\frac{\dot{\lambda}^2}{2}-\frac{\dot{\lambda}\dot{\Phi}}{2}\right)$$
$$=2k(h_1h^1-h_2h^2+h_3h^3+h_4h^4) \tag{3}$$

$$\mathrm{e}^{-\lambda}\left(\Phi''+\frac{\Phi'^2}{2}-\frac{\Phi'\lambda'}{2}+\frac{\Phi'-\lambda'}{r}\right)-\mathrm{e}^{-\Phi}\left(\ddot{\lambda}+\frac{\dot{\lambda}^2}{2}-\frac{\dot{\lambda}\dot{\Phi}}{2}\right)$$
$$=2k(-h_1h^1+h_2h^2+h_3h^3+h_4h^4) \tag{4}$$

in the geometry

$$ds^2=\mathrm{e}^{\Phi(r,t)}dt^2-\mathrm{e}^{\lambda(r,t)}dr^2-r^2(d\theta^2+sin^2\theta d\phi^2), \tag{5}$$

where the remaining conditions on the ECKR gravity field are

$$\mathrm{e}^{\lambda}\frac{\dot{\lambda}}{r}=2kh_3h^4, \tag{6}$$

which implies that we shall have to choose either h_3 or h^4 to vanish, since from Eq. (6) and the static condition $\dot{\lambda}=0$, the left side of Eq. (6) vanishes. At this point, we may check whether these equations will admit asymptotically flat solutions. This is important in assuming that not only do all metric coefficients Φ and λ vanish, but their time and radial derivatives do as well. The demonstration is algebraically very simple, if we assume the simple algorithm of defining the KR field products by

$$a_B=h_Bh^B, \tag{7}$$

where ($B=1,2,3,4$), without the Einstein summation convention. Since the left-hand sides of all the above ECKR equations vanish, at $\rightarrow\pm\infty$ limit on both sides of the wormhole throat. Therefore, the system of equations which asymptotically vanishes on the left-hand side yields four algebraic equations in $h_B(\pm\infty)$, which yields the asymptotic KR vector field h_B solution

$$h_2h^4=h_1h^4=h_2h^3=h_1h^2=h_3h^1=0. \tag{8}$$

Here, $H_{ijk}=\partial_{[i}B_{jk]}$ is the strength of the KR field B_{jk}. The KR field strength components H_{012}, H_{013}, H_{023} and H_{123} are denoted by h_1,h_2,h_3 and h_4, respectively. Since it is easy to prove that all h_B vanish asymptotically, the remaining equations are fulfilled. The wormhole spacetime [3,4] line element used is

$$ds^2 = e^{\Phi(r,t)}dt^2 - \frac{dr^2}{(1-\frac{b(r)}{r})} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \quad (9)$$

This is the MT wormhole [3,4], where $b(r)$ is the throat shape parameter. These last two constraints do not mean that there are no torsional wormholes at all. Cosmological torsional KR wormholes like Kerr wormholes can be found in ECKR theory of gravity.

3 Traversable wormholes from conformal KR vector fields

Ovgun [20] recently investigated traversable wormholes from dark energy (DE) and dark matter (DM). From Chaplygin gas it is possible to show that the DE and DM avoid collapse of the throat and maintain wormhole stable. This is done in a dynamical universe, conformally spatial to a spherically symmetric wormhole. In this section, we show that by applying this conformal geometry to the Friedman spatially flat cosmology, a shape function of the wormhole throat $b(r)$ is obtained in terms of the product of two components of the KR vector. Furthermore, we conclude that even in the absence of KR fields, one may obtain a traversable wormhole. Nevertheless, as $r \to \infty$, the wormhole stability is due to the KR vector fields. Comparing line elements (5) and (9), which consists in embedding wormholes into the spherically symmetric metric, one obtains the expression

$$e^{\lambda} = \left[1 - \frac{b(r)}{r}\right]^{-1}, \quad (10)$$

which is expression (6). By comparing this expression with the more general KR expression

$$\frac{e^{2\lambda}}{r} = kh_3h_4. \quad (11)$$

In the approximation. $r \to \infty$ implies that

$$\left[1 + \frac{b}{r}\right] = kh_3h_4r. \quad (12)$$

which leads to a relation between the wormhole shape and the components of the KR vector fields as

$$b(r) = -r(1 - 2kh_3h_4r). \quad (13)$$

Before the end of this section, we note that the ECKR wormhole can be embedded in the dynamical Friedman metric for a wormhole as

$$ds^2 = R^2(\eta)\left[d\eta^2 - \left(1 - Kr^2 - \frac{b}{r}\right)^{-1} - r^2d\Omega^2\right], \quad (14)$$

as in the case of a spatially flat cosmological model, where $K = 0$. Now let us check whether this solution is asymptotically flat. This is simple.

4 Static black hole trapped into the wormhole with a KR string loop torsional ring singularity

In this section, we show that there exists a static black hole trapped inside the ER bridge, where KR field components are dependent not only on the radial coordinate but also on the angular coordinate θ. This implies the existence of a KR field ring singularity around the ER bridge. Now let us assume that we are closer to the wormhole throat, where we have to solve the set of differential equations [18,19]

$$h_{,2} + \cot\theta h = 0, \quad (15)$$

where $h = h^4(r,\theta)$ is the only non-vanishing KR field. Based on the previous section, we may achieve this goal by changing the above equations, taking into account the presence of KR fields of the type $h(r,\theta)$. Here, $r = x_1$ and $\theta = x_2$. Therefore, taking the generalized KR component $h^4 = \cos\theta + f(r)$ and separating variables (r,θ), the expression (15) reduces to

$$(f + \cos\theta)_{,r} + \left[\frac{(\Phi, r + \lambda_{,r})}{2} + \frac{2}{r}\right](f + \cos\theta) = 0. \quad (16)$$

Using the same condition $\Phi = -\lambda$ in this expression, one obtains

$$(f + \cos\theta)_{,r} + \left[\frac{2}{r}\right](f + \cos\theta) = 0, \quad (17)$$

whose solution is $f(r) = r^{-2}$. Meanwhile, the general solution is

$$h^4 = \frac{1}{r^2} + \cos\theta, \quad (18)$$

which in turn yields a toroidal singularity black hole induced by the torsionful KR fields.

5 Open torsion strings inside no hairy black holes and asymptotic flatness

Recently, Pereira et al. [21] built a Kerr black hole [22] in Einstein's teleparallel gravity [1] and computed the Lense–Thirring effect. It is interesting to note that the effects of tensorial and axial parts of torsion cancel in the equation of motion, which implies no effect of torsion outside the static Schwarzschild black hole. Moreover, in this section we consider a teleparallel Schwarzschild solution, but in a completely different context. First, we use the method of Cartan's calculus of differential forms instead of the tetrad method used by Pereira et al. [21]. Secondly, we use a system of a cosmic torsion string, not a spinning string [23], passing through the center of a static black hole as given in Vilenkin and Shellard [7]. The use of differential forms and the computation of torsion 2-forms allows us to compute the torsion flux integrals in the teleparallel frame. Since it is possible to compute the relation between the torsion forms and the spin 3-forms, is also possible to show that the torsion string inside the black hole is polarized along the cosmic string itself. A similar result was obtained just for the system of cosmic strings and spin-polarized matter around it by Garcia de Andrade [23]. It is also important to point out that the problem of torsion singularities inside the black hole is also computed, and we found that some of the components of torsion are singular on the Schwarzschild radius $r = 2Gm$ while other components of torsion form are finite. Therefore, to determine whether the singularity is on the Schwarzschild radius or at the center of the black hole as happens in the curvature case in Einstein's general relativity, one has to compute a torsion invariant analogous to the case of the Kretschmann curvature scalar [24] $R^{ijkl}R_{ijkl} = \frac{2m}{r^3}$, where R_{ijkl} is the Riemann curvature tensor and $i, j = 0, 1, 2, 3$. It is shown from this computation that the true torsion singularity is at the center of the black hole, which is also the locus of the torsion cosmic strings. We further discuss the torsion singularities inside the black hole and compute the torsion components by using the teleparallel condition. In what follows, the torsion flux is computed along with the spin polarization direction of the spins inside the black hole. It is also interesting to note that this system may be used in the case, for example, of neutron stars with torsion strings inside in order to detect torsion, as it is well known that this system possesses about 10^{40} polarized spins [26]. Just to give an idea of how large this figure is, the ferromagnetic material which possesses the highest spin density on Earth [25] is above 10^{23}. **As in previous sections, we address the problem of asymptotic flatness. Obviously, the Schwarzschild metric is asymptotically flat. Nevertheless, here we also show below by computing the torsion invariant, which will vanish, and torsion hair does not appear in these black holes with cosmic strings.** Let us consider the spacetime metric describing the black hole with a cosmic string torsion inside

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2, \quad (19)$$

where now the solid angle $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + (1-8G\mu)sin^2\theta\mathrm{d}\phi^2$, where μ is the string mass and m is the black hole mass. In terms of the Cartan exterior differential form, the metric (1) can be expressed as

$$\mathrm{d}s^2 = (\mathrm{e}^0)^2 - (\mathrm{e}^1)^2 - (\mathrm{e}^2)^2 - (\mathrm{e}^3)^2, \quad (20)$$

where the basis 1-forms ω^i are given by

$$\mathrm{e}^0 = \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}}\mathrm{d}t, \quad (21)$$

$$\mathrm{e}^1 = \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}}\mathrm{d}r, \quad (22)$$

$$\mathrm{e}^2 = r\mathrm{d}\theta, \quad (23)$$

$$\mathrm{e}^3 = (1 - 8G\mu)^{\frac{1}{2}}\sin\theta\mathrm{d}\phi. \quad (24)$$

From the Cartan structural equations,

$$T^i = \mathrm{d}\omega^i + \omega^i_j \wedge \omega^j, \quad (25)$$

where ω^i_j is the connection 1-form. The curvature 2-form is

$$R^i_j = R^i_{jkm}(\Gamma)\mathrm{e}^k \wedge \mathrm{e}^m = \mathrm{d}\omega^i_j + \omega^i_k \wedge \omega^k_j. \quad (26)$$

Here, $\wedge$ is the exterior product symbol, and $R^i_{jkl}(\Gamma)$ are the components of the Riemann–Cartan geometry curvature tensor. Rewriting the metric (1) using the differential forms, one obtains

$$\mathrm{d}s^2 = \eta_{ij}\omega^i\omega^j, \quad (27)$$

where $\eta_{ij} = \mathrm{diag}(+1, -1, -1, -1)$ is the Minkowski metric. Using the teleparallel condition $R^i_{jkl}(\Gamma) = 0$ in Eq. (8), we note that the constraint

$$\omega^i_j = 0 \quad (28)$$

fulfills the teleparallel condition. Here, we add this stronger teleparallel condition which was also added by Letelier [26] in the construction of torsion loops in teleparallel spacetimes. Using condition (10) in Eq. (7), one obtains the torsion 2-form in the form

$$T^i = de^i = T^i_{jk}\mathrm{e}^j \wedge \mathrm{e}^k, \quad (29)$$

where T^i_{jk} are the components of the Cartan torsion tensor. Applying this simple expression to the expressions for the basis 1-forms of the black hole torsion string system above, one obtains the components of the torsion tensor as

$$T^0_{10} = \frac{\frac{Gm}{r^2}}{\left(1 - \frac{2Gm}{r}\right)^{\frac{1}{2}}}, \quad (30)$$

$$T^2_{12} = \frac{1}{r}\left(1 - \frac{2Gm}{r}\right), \tag{31}$$

$$T^3_{23} = \frac{\cot\theta}{r}. \tag{32}$$

Then, one notes that from expression (31), the string term does not contribute to torsion components unless on a rotating string. From these last expressions one notes that some components of torsion are singular at the Schwarzschild radius whereas others are singular at the center of the black Hole. To determine which locus of singularity represents the true singular behavior and which represents simply a coordinate singularity, we compute the torsion invariant $T_{ijk}T^{ijk}$, which is analogous to the curvature scalar invariant in Riemannian geometry. In teleparallel gravity this is the most fundamental tensor invariant, as there is no way to build the Riemann–Cartan curvature invariant since this vanishes by the definition of teleparallelism. Therefore, the scalar torsion invariant is

$$T^{ijk}T_{ijk} = -2\left[\frac{\left(\frac{Gm}{r^2}\right)^2}{\left(1-\frac{2Gm}{r}\right)} + \frac{\left(1-\frac{2Gm}{r}\right)}{r^2} + (1-8G\mu)\frac{ctg^2\theta}{r^2}\right]. \tag{33}$$

This expression clearly shows that at the Schwarzschild radius, torsion invariant $T^{ijk}T_{ijk}$ also vanishes at the center of the black hole. Therefore, there is no simple way to detect torsion singularities. The only thing we can say so far is that torsion is confined inside the black hole. In John Wheeler's terminology, the black hole would have no torsion hair! In the next section, we compute torsion flux, in order to gain a better idea of the behavior of torsion inside the black holes. Using the mathematical apparatus of the previous section, we compute the torsion flux according to Anandan [27] as

$$\int_\Sigma T^i = \int e^i, \tag{34}$$

where Σ is the surface across which the torsion flux is computed, and the second integral is the line integral due to the Stokes theorem. It is easy to observe from the expressions for the torsion forms T^i that the only non-vanishing torsion flux is given by the integral

$$\int \Sigma T^i = \int e^3 = (1-8G\mu)^{\frac{1}{2}}\sin\theta \int d\phi = 2\pi(1-8G\mu)^{\frac{1}{2}}\sin\theta. \tag{35}$$

From this formula, there is no torsion flux along the cosmic string, where θ takes the values 0 or 2π. There is torsion flux along the direction around the cosmic torsion string in the form of a torsion vortex in the superfluid case. Note also that for a torsion string of mass $\mu = \frac{1}{8G}$, there is no torsion flux around this direction and no torsion flux at all. We could imagine this case, as the system of black hole and torsion strings would completely shield the effects of torsion. Let us now use Cartan's equation, relating the spin 3-form to the torsion 2-form to investigate the polarization effect of the cosmic torsion string on the spinning particles of the spinning fluids of black holes. There is some controversy as to whether the teleparallel theories should possess spin density, but here we adopt the point of view that spin density is allowed in teleparallel gravity [3,4]. From Cartan's equation

$$s_{ij} = \frac{-1}{8\pi G}[T^k \wedge e^l \epsilon_{ijkl}]. \tag{36}$$

Since the only non-vanishing component of the torsion form is

$$T^3 = \frac{\cot\theta}{r} e^2 \wedge e^3, \tag{37}$$

we can easily obtain the spin component we wish by simply substituting formula (37) into expression (35), which yields

$$s_{12} = \frac{-1}{8\pi G}\left[\frac{\cot\theta}{r}\omega^2\wedge\omega^3\wedge\omega^0\epsilon_{1230}\right]. \tag{38}$$

This shows that the polarization of the spinning particles around the torsion string inside the black holes occurs along the torsion string direction in analogy with the magnetic field which orients the magnetized particles along the magnetic field direction. Note that from expression (8), the spin density seems to be singular at $r = 0$. This does not imply a Dirac delta distribution, since off the torsion string there is matter inside the black hole in the form of spinning fluid in teleparallel gravity.

6 Morris–Thorne wormhole and the asymptotic flatness in teleparallelism

In this section, we shall give an example of an MT wormhole in teleparallel gravity that is not asymptotically flat, so it is not actually a wormhole, since topologically it does not join two flat regions of spacetime. To obtain this result, we start to write the MT metric as

$$ds^2 = e^{\Phi(r)}dt^2 - \left[1 - \left(\frac{b(r)}{r}\right)\right]^{-1} dr^2 - r^2 d\Omega^2, \tag{39}$$

where $\Phi(r)$ is the redshift function and $b(r)$ is the throat shape function. Here we are concerned with studying the functions of differentiation, continuity, and asymptotic flatness of the wormhole in teleparallel torsion. We shall also demonstrate that in an example where torsion is constant, one is left with a non-flat asymptotic topology. Nevertheless, we note that, in general, asymptotic flatness can be obtained and this is a pathology. In this case all torsion components asymptotically vanish, and no torsion is found away from the wormhole throat. In this section we examine the conditions of the stability of the MT wormhole endowed with teleparallel geometry. We have a series of conditions to the wormhole formation, as follows.

1. The throat shape function $b(r)$ should be continuous and

differentiable.

2. Moreover, it should obey the relation

$$\left(1-\frac{b(r)}{r}\right) \geq 0 \rightarrow \frac{b}{r} \leq 1. \tag{40}$$

Also, there is the following expression: Let us start by choosing a natural frame of e^i when the torsion 2-form is given by the second structural equation of Cartan's calculus of exterior differential forms.

$$T^i = de^i + \omega^i{}_k \wedge e^k, \tag{41}$$

where $\omega^i{}_k$ is the spin coefficient including contortion [2]. Here, to simplify computations, we consider the othonormal frame where $\omega^i{}_k$ vanishes, which reduces torsion 2-forms to

$$T^i = de^i. \tag{42}$$

From the MT wormhole we may add cosmic strings and express the frame for this metric as

$$e^0 = e^{\frac{\Phi(r)}{2}} dt, \tag{43}$$

$$e^1 = \left[1-\left(\frac{b(r)}{r}\right)\right]^{-\frac{1}{2}} dr, \tag{44}$$

$$e^2 = r d\theta, \tag{45}$$

and

$$e^3 = [1-8G\mu(r)]^{\frac{1}{2}} r \sin\theta d\phi. \tag{46}$$

By performing the Poincaré exterior differential d on these frame components of the MT wormhole, we obtain the torsion 2-form for this metric. It is important to note that this metric is a solution of Einstein gravity of GR. Therefore, the T^i for the MT wormhole with cosmic strings are

$$T^0 = \frac{\Phi_{,r}}{2}\left[1-\frac{b}{r}\right]^{\frac{1}{2}} e^1 \wedge e^0, \tag{47}$$

$$T^2 = \frac{1}{r}\left[1-\frac{b}{r}\right] e^1 \wedge e^2, \tag{48}$$

and

$$T^3 = \left[1-\frac{b}{r}\right]^{\frac{1}{2}}\left[\left[\frac{1}{2}\frac{dln(1-8G\mu)}{dr}\right] + \frac{1}{r}(1-8G\mu)^{\frac{1}{2}}]e^2 \wedge e^3 + \frac{\cot\theta}{r} e^1 \wedge e^3\right]. \tag{49}$$

The torsion component T^1 vanishes everywhere in the wormhole manifold due to the skew-symmetric character of the wedge product. Now let us examine whether this metric in teleparallelism is asymptotically flat or not. Then, it is simple to start examining the component T^0. This yields

$$T^0{}_{10} = \Phi_{,r}\left[1-\frac{b}{r}\right]^{-\frac{1}{2}}. \tag{50}$$

Thus. let us compute the tensor component $T^0{}_{10}(r)$ at $r \rightarrow \pm\infty$. A little observation yields

$$T^0{}_{10}(\pm\infty) \approx \Phi_{,r}. \tag{51}$$

Since for asymptotic flatness we must have vanishing torsion far away from the throat, to derive the redshift factor Φ, we impose the condition that $T^0{}_{10}$ vanishes From this asymptotically, yielding

$$\Phi = \text{constant}, \tag{52}$$

which with a scale transformation yields Let us now assume that the component could be constant in general. Therefore,

$$\Phi_{,r}\left(1-\frac{b}{r}\right)^{-\frac{1}{2}} = \text{constant} = c_0 \tag{53}$$

implies that asymptotically

$$c_0 = T^0{}_{10} \approx \Phi_{,r}, \tag{54}$$

which by integration yields

$$\Phi(r) = T^0{}_{10} r. \tag{55}$$

Note that if we consider the string mass density μ as the axion-like particle (ALP) scalar, one may say that in the case of torsional wormholes without asymptotic flatness, one obtains a hairy wormhole as in Duncan et al.'s [27] hairy wormholes in KR spacetime manifold. Now let us to proceed to check whether the other torsion components vanish as well. From this computation one needs the other torsion components, which can be obtained from the above MT torsion T^i as

$$T^2{}_{12} = \frac{1}{r}\left[1 - \frac{b}{r}\right]^{-\frac{1}{2}}. \tag{56}$$

This component asymptotically vanishes and is singular at $r \to 0$, which is the string axis. The remaining components of torsion are

$$T^3{}_{13}(\pm\infty) \approx \frac{\cos\theta}{r} \to 0, \tag{57}$$

as $r \to \pm\infty$. Therefore, we note that again this torsion component does not introduce a hair away from the throat. We also note that the third present torsion component diverges at the origin. The component $T^3{}_{23}$ reads

$$T^3{}_{23}(\pm\infty) \approx \frac{1}{2}\frac{d ln(1 - 8G\mu)}{\mathrm{d}r} \to 0, \tag{58}$$

where we imposed that this last component also vanishes, not to have any torsion hair away from the throat. Therefore, we conclude that in order for this component to vanish, $\mu_{,r}$ vanishes, which implies that the cosmic string tension must be constant. Therefore, to obtain a torsion hair-free wormhole, the cosmic string must have constant tension. Note that in the special solution for $T^3{}_{23}$ to be constant, a tension of

$$8G\mu = 1 - \mathrm{e}^{T^3{}_{23}r}, \tag{59}$$

which must diverge far away from the throat. But note that this is unphysical, since the string tension must remain finite and in general $G\mu << 1$. Then the only way to obtain a physical solution is to consider $T^3{}_{23}$ negative, which asymptotically would fulfill the maximum $G\mu \to 1$. In recently investigating the cosmic string Friedmann universe, we noted that $G\mu \leq 10^{-7}$ induced by LIGO a gravitational wave of amplitude $\sim 10^{-17}$, which are typical for gravitational waves of cosmological origin [28].

7 Conclusions

In this paper we discuss whether the criteria for the existence of asymptozic flat wormholes in two kinds of torsional throats—for the case of Einstein–Cartan–Kalb–Ramond gravity and teleparallel gravity—apply if the MT metric is modified to include axionic cosmic strings. Embedding of topological defects in the dynamical evolution of the Friedmann model and the wormhole metric is always possible. Torsion strings immersed in static black holes are shown to induce spin polarization of black hole spinning fluid along the torsion string direction itself. This is analogous to the polarization of the magnetic moment in magnetic materials in condensed matter physics. In the near future, other types of torsion defects besides torsion strings may be constructed inside teleparallel black holes. Here, we showed that asymptotic wormholes and black holes have no torsion hair generated by comic strings far away from the throat of wormholes and far away from axionic cosmic strings. The addition of cosmic string dynamics to the wormhole seems to be an important ingredient to check whether there is an asymptotic or non-asymptotic flatness in torsion theories of gravity. We finally would like to call attention to the fact that the concept of hair in black holes is associated with magnetic fields, whereas this paper only referred to the presence of dynamical cosmic strings.

Acknowledgements I am very grateful to P.S. Letelie (in memoriam), M. Visser, J.G. Pereira, and Jeeva Anandam (in memoriam) for helpful discussions on the subject of this paper.

Data Availability Statement Data will be made available on reasonable request. [Author's comment: The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.]

Code Availability Statement Code/software will be made available on reasonable request. [Author's comment: The code/software generated during and/or analysed during the current study is available from the corresponding author on reasonable request.]

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Funded by SCOAP^3.

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